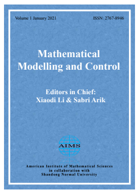
MMC, 2(4): 185–199
DOI:10.3934/mmc.2022019
Received: 19 May 2022
Revised: 09 November 2022
Accepted: 14 November 2022
Published: 19 December 2022

Mathematical Modelling and Control

http://www.aimspress.com/journal/mmc



Research article

Analysis of a chaotic system using fractal-fractional derivatives with exponential decay type kernels

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Abstract: In this article, we introduce and analyze a novel fractal-fractional chaotic system. We extended the memristor-based chaotic system to the fractal-fractional mathematical model using Atangana-Baleanu–Caputo and Caputo-Fabrizio types of derivatives with exponential decay type kernels. We established the uniqueness and existence of the solution through Banach's fixed theory and Schauder's fixed point. We used some new numerical methods to derive the solution of the considered model and study the dynamical behavior using these operators. The numerical simulation results presented in both cases include the two and three-dimensional phase portraits and the time-domain responses of the state variables to evaluate the efficacy of both kernels.

Keywords: fractal-fractional derivative; memristor-based chaotic system; Schauder's fixed point; Banach fixed theory

1. Introduction

The role of mathematical modeling is significant in different fields of science. Mathematical modeling has been identified as a helpful research area for investigating many aspects of circuit problems [1–3]. Many researchers have used integer-order derivatives, but integer order fails to describe the desired behavior of circuit problems. This way, most researchers have been using fractional derivative operators to obtain more realistic results of physical problems [4–6]. It is seen that models involving non-integer order differential equations and integrals are more natural than to the ordinary derivative model [7–9].

Applications of fractional calculus to many problems have been studied extensively in previously published papers [10, 11]. All of these mentioned papers analyzed engineering problems and infectious diseases to decrease the transmission of infectious diseases. Researchers have proved that fractional derivatives can describe such a circuit problem efficiently and provide better results as compared to ordinary derivatives [12–16]. Therefore, we want to study a memristor-based chaotic system in fractional type derivatives. This paper considers two fractional derivatives: Atangana-Baleanu–Caputo and Caputo-Fabrizio derivatives. In [17], the author proposed a circuit integrator model as

$$\begin{cases} {}_0^C\mathscr{D}_\zeta^\alpha[\mathscr{X}(\zeta)] = \mathscr{Y}(\zeta), \\ {}_0^C\mathscr{D}_\zeta^\alpha[\mathscr{Y}(\zeta)] = -\frac{1}{3}\left(\mathscr{X}(\zeta) + \frac{3}{2}\left(\mathscr{Z}^2(\zeta) - 1\right)\mathscr{Y}(\zeta)\right), \\ {}_0^C\mathscr{D}_\zeta^\alpha[\mathscr{Z}(\zeta)] = -\frac{3}{5}\mathscr{Z}(\zeta) - \mathscr{Y}(\zeta) + \mathscr{Z}(\zeta)\mathscr{Y}(\zeta). \end{cases} \tag{1.1}$$

For solving nonlinear fractional differential equations, numerical techniques have been considered effective mathematical tools [16, 18]. An Adam-Bashforth numerical method has been used in many fields of applied science such as engineering and epidemiology problems [19–21]. However, some numerical scheme are not fully applied to differential equations with local and non-local operators, as they were designed only for integer-order derivatives [19]. In this paper, we study the system (1.1) under ABC and CF types of fractal-fractional derivatives with order α.

2. Analysis of the fractal-fractional model (1.1) in ABC sense

In this section, we study the chaotic system (1.1) under ABC type fractal-fractional derivatives with order α as follows:

$$\begin{cases} {}_0^{ABC}\mathscr{D}_\zeta^{\alpha,\gamma}[\mathscr{X}(\zeta)] = \mathscr{Y}(\zeta), \\ {}_0^{ABC}\mathscr{D}_\zeta^{\alpha,\gamma}[\mathscr{Y}(\zeta)] = -\frac{1}{3}\left(\mathscr{X}(\zeta) + \frac{3}{2}\left(\mathscr{Z}^2(\zeta) - 1\right)\mathscr{Y}(\zeta)\right), \\ {}_0^{ABC}\mathscr{D}_\zeta^{\alpha,\gamma}[\mathscr{Z}(\zeta)] = -\frac{3}{5}\mathscr{Z}(\zeta) - \mathscr{Y}(\zeta) + \mathscr{Z}(\zeta)\mathscr{Y}(\zeta). \end{cases} \tag{2.1}$$

The following are the circuital equations related to the system (2.1):

$$\begin{cases} {}_0^{ABC}\mathscr{D}_\zeta^{\alpha,\gamma}[\mathscr{X}(\zeta)] = \frac{1}{C_0}\left[-\left(\frac{\mathcal{R}_2}{\mathcal{R}_1\mathcal{R}_3}\right)\mathscr{Y}\right], \\ {}_0^{ABC}\mathscr{D}_\zeta^{\alpha,\gamma}[\mathscr{Y}(\zeta)] = \frac{1}{C_0}\left[\left(\frac{\mathcal{R}_7}{\mathcal{R}_8\mathcal{R}_5}\right)\mathscr{Y} - \left(\frac{1}{\mathcal{R}_4}\right)\mathscr{X} - \left(\frac{1}{\mathcal{R}_6}\right)\mathscr{Z}^2\mathscr{Y}\right], \\ {}_0^{ABC}\mathscr{D}_\zeta^{\alpha,\gamma}[\mathscr{X}(\zeta)] = \frac{1}{C_0}\left[\left(\frac{\mathcal{R}_{12}}{\mathcal{R}_{13}\mathcal{R}_9}\right)\mathscr{Y}\mathscr{Z} - \left(\frac{1}{\mathcal{R}_{10}}\right)\mathscr{Y} - \left(\frac{1}{\mathcal{R}_{11}}\right)\mathscr{Z}\right], \end{cases} \tag{2.2}$$

where parameters are given as follows: $\mathcal{R}_1 = \mathcal{R}_3 = \mathcal{R}_2 = \mathcal{R}_5 = \mathcal{R}_9 = \mathcal{R}_7 = \mathcal{R}_{10} = \mathcal{R}_{13} = \mathcal{R}_{12} = 10K\Omega, \mathcal{R}_6 = \mathcal{R}_8 = 10.62K\Omega, \mathcal{R}_4 = 11.62K\Omega, \mathcal{R}_{11} = 11.49K\Omega$.

Let $\mathscr{P}(\zeta)$ be a continuous function in (a, b) [22], then the ABC fractal-fractional derivative of $\mathscr{P}(\zeta)$ with order $0 < \alpha \leq 1$ is given by

$${}_a^{ABC}\mathscr{D}_\zeta^{\alpha,\gamma}[\mathscr{P}(\zeta)] = \frac{\mathcal{B}(\alpha)}{1-\alpha}\frac{d}{d\zeta^\gamma}\int_a^\zeta \mathscr{P}(\xi)E_\alpha\left[-\alpha\frac{(\zeta-\xi)^\alpha}{1-\alpha}\right]d\xi, \tag{2.3}$$

where $\mathcal{B}(\alpha)$ is normalization function such that $\mathcal{B}(0) = \mathcal{B}(1) = 1$ and $\gamma \leq 1$.

Let $0 < \alpha \leq 1; \gamma \leq 1$; then, the fractal-fractional integral is given by

$$
\begin{aligned}
& {}_{0}^{ABC}\mathscr{I}_{\zeta}^{\alpha,\gamma}[\mathscr{P}(\zeta)] \\
= \quad & \tfrac{\gamma(1-\alpha)\zeta^{\gamma-1}}{\mathcal{B}(\alpha)}\mathscr{P}(\zeta) + \tfrac{\alpha\gamma}{\mathcal{B}(\alpha)\Gamma(\alpha)}\int_0^{\zeta}\xi^{\gamma-1}\mathscr{P}(\xi)(\zeta-\xi)^{\alpha-1}\,d\xi.
\end{aligned} \tag{2.4}
$$

3. Existence of the solution

In this section, we prove that the model (2.1) has at least one solution.

Consider the model (2.1) as

$$
\begin{cases}
{}_{0}^{ABC}\mathscr{D}_{\zeta}^{\alpha,\gamma}[\mathscr{X}(\zeta)] = \gamma\zeta^{\gamma-1}\mathfrak{X}, \\
{}_{0}^{ABC}\mathscr{D}_{\zeta}^{\alpha,\gamma}[\mathscr{Y}(\zeta)] = \gamma\zeta^{\gamma-1}\mathfrak{Y}, \\
{}_{0}^{ABC}\mathscr{D}_{\zeta}^{\alpha,\gamma}[\mathscr{Z}(\zeta)] = \gamma\zeta^{\gamma-1}\mathfrak{Z},
\end{cases} \tag{3.1}
$$

where

$$
\begin{cases}
\mathfrak{X}(\mathscr{X},\mathscr{Y},\mathscr{Z},\zeta) = \mathscr{Y}(\zeta) \\
\mathfrak{Y}(\mathscr{X},\mathscr{Y},\mathscr{Z},\zeta) = -\frac{1}{3}\left(\mathscr{X}(\zeta) + \frac{3}{2}\left(\mathscr{Z}^2(\zeta) - 1\right)\mathscr{Y}(\zeta)\right) \\
\mathfrak{Z}(\mathscr{X},\mathscr{Y},\mathscr{Z},\zeta) = -\frac{3}{5}\mathscr{Z}(\zeta) - \mathscr{Y}(\zeta) + \mathscr{Z}(\zeta)\mathscr{Y}(\zeta).
\end{cases}
$$

We can write system (3.1) as:

$$
\begin{cases}
{}_{0}^{ABC}\mathscr{D}_{\zeta}^{\alpha,\gamma}[\mathscr{P}(\zeta)] = \gamma\zeta^{\gamma-1}\mathcal{G}(\zeta,\mathscr{P}(\zeta)), \quad \zeta \geq 0, \\
\mathscr{P}(0) = \mathscr{P}_0.
\end{cases} \tag{3.2}
$$

By replacing ${}_{0}^{ABC}\mathscr{D}_{\zeta}^{\alpha}$ with ${}_{0}^{ABC}\mathscr{D}_{\zeta}^{\alpha,\gamma}$ and applying a fractional integral, we get

$$
\begin{aligned}
\mathscr{P}(\zeta) - \mathscr{P}(0) = & \frac{\gamma\zeta^{\gamma-1}(1-\alpha)}{\mathcal{B}(\alpha)}\mathcal{G}(\zeta,\mathscr{P}(\zeta)) + \frac{\alpha\gamma}{\mathcal{B}(\alpha)\Gamma(\alpha)} \\
& \int_0^{\zeta}\xi^{\gamma-1}(\zeta-\xi)^{\gamma-1}\mathcal{G}(\xi,\mathscr{P}(\xi))\,d\xi.
\end{aligned} \tag{3.3}
$$

We define a Banach space $\Omega = \mathscr{F}\times\mathscr{F}\times\mathscr{F}$, for the existence theory, where $\mathscr{F} = \mathbb{C}[0,T]$ with the norm

$\|\mathscr{P}\| = \max_{\zeta\in[0,T]}|\mathscr{X}(\zeta) + \mathscr{Y}(\zeta) + \mathscr{Z}(\zeta)|$.

We define an operator $\mathscr{T}:\Omega\to\Omega$ as follows:

$$
\begin{aligned}
\mathscr{T}\mathscr{P}(\zeta) = & \mathscr{P}(0) + \frac{\gamma\zeta^{\gamma-1}(1-\alpha)}{\mathcal{B}(\alpha)}\mathcal{G}(\zeta,\mathscr{P}(\zeta)) \\
& + \frac{\alpha\gamma}{\mathcal{B}(\alpha)\Gamma(\alpha)}\int_0^{\zeta}\xi^{\gamma-1}(\zeta-\xi)^{\gamma-1}\mathcal{G}(\xi,\mathscr{P}(\xi))\,d\xi.
\end{aligned} \tag{3.4}
$$

The Lipschitz condition is applied to non-linear function $\mathcal{G}(\zeta,\mathscr{P}(\zeta))$ as:

- For each $\mathscr{P}\in\Omega$, there exist constants $\mathcal{M}>0$, and $\mathcal{M}_a$ such that

$$
|\mathcal{G}(\zeta,\mathscr{P}(\zeta))| \leq \mathcal{M}|\mathscr{P}(\zeta)| + \mathcal{M}_a. \tag{3.5}
$$

- For each $\mathscr{P},\mathscr{P}_a\in\Omega$, there exists a constant $\mathcal{M}_a>0$ such that

$$
|\mathcal{G}(\zeta,\mathscr{P}(\zeta)) - \mathcal{G}(\zeta,\mathscr{P}_a(\zeta))| \leq \mathcal{M}_a|\mathscr{P}(\zeta) - \mathscr{P}_a(\zeta)|. \tag{3.6}
$$

Theorem 3.1. *Assume that the condition (3.5) is satisfied. Let $\Xi:[0,T]\times\Omega\to\mathcal{R}$ be a function that is continuous (2.1) must have at least one solution.*

Proof. First, we prove that the operator $\mathscr{T}$ described by Eq. (3.4) is fully continuous. Because Ξ is continuous, $\mathscr{T}$ is also continuous. Let $\mathscr{A} = \{\mathscr{P}\in\Omega : \|\mathscr{P}\|\leq M, M>0\}$. Now for any $\mathscr{P}\in\Omega$, we have

$$
\begin{aligned}
& \|\mathscr{T}\mathscr{P}\| \\
= & \max_{\zeta\in[0,T]}\left|\mathscr{P}(0) + \frac{\gamma\zeta^{\gamma-1}(1-\alpha)}{\mathcal{B}(\alpha)}\mathcal{G}(\zeta,\mathscr{P}(\zeta))\right.
\end{aligned}
$$

$$+\frac{\alpha\gamma}{\mathcal{B}(\alpha)\Gamma(\alpha)}\int_0^{\zeta}\xi^{\gamma-1}(\zeta-\xi)^{\gamma-1}\mathcal{G}(\xi,\mathscr{P}(\xi))\,d\xi\Big|,$$
$$\le\mathscr{P}(0)+\frac{\gamma\zeta^{\gamma-1}(1-\alpha)}{\mathcal{B}(\alpha)}(\mathcal{M}\|\mathscr{P}\|+\mathcal{M}_a)$$
$$+\max_{\zeta\in[0,T]}\frac{\alpha\gamma}{\mathcal{B}(\alpha)\Gamma(\alpha)}\int_0^{\zeta}\xi^{\gamma-1}(\zeta-\xi)^{\gamma-1}|\mathcal{G}(\xi,\mathscr{P}(\xi))|\,d\xi,$$
$$\le\mathscr{P}(0)+\frac{\gamma\zeta^{\gamma-1}(1-\alpha)}{\mathcal{B}(\alpha)}(\mathcal{M}\|\mathscr{P}\|+\mathcal{M}_a)$$
$$+\frac{\alpha\gamma}{\mathcal{B}(\alpha)\Gamma(\alpha)}(\mathcal{M}\|\mathscr{P}\|+\mathcal{M}_a)\zeta_2^{\alpha+\gamma-1}\mathscr{B}(\alpha,\gamma)\le\mathcal{R}.$$

where $\mathscr{B}(\alpha,\gamma)$ represents the beta function. Thus the operator $\mathscr{T}$ is uniformly period,

For eqicontinuity of $\mathscr{T}$, let us take $\mathscr{K}_1<\mathscr{K}_2\le$T. Then consider

$$\|\mathscr{T}\mathscr{P}(\zeta_2)-\mathscr{T}\mathscr{P}_a(\zeta_1)\|$$
$$=\max_{\zeta\in[0,T]}\Big|\frac{\gamma\zeta_2^{\gamma-1}(1-\alpha)}{\mathcal{B}(\alpha)}\mathcal{G}(\zeta_2,\mathscr{P}(\zeta_2))$$
$$+\frac{\alpha\gamma}{\mathcal{B}(\alpha)\Gamma(\alpha)}\int_0^{\zeta}\xi^{\gamma-1}(\zeta_2-\xi)^{\gamma-1}\mathcal{G}(\xi,\mathscr{P}(\xi))\,d\xi$$
$$-\frac{\gamma\zeta_1^{\gamma-1}(1-\alpha)}{\mathcal{B}(\alpha)}\mathcal{G}(\zeta_1,\mathscr{P}(\zeta_1))$$
$$-\frac{\alpha\gamma}{\mathcal{B}(\alpha)\Gamma(\alpha)}\int_0^{\zeta}\xi^{\gamma-1}(\zeta_1-\xi)^{\gamma-1}\mathcal{G}(\xi,\mathscr{P}(\xi))\,d\xi\Big|$$
$$\le\frac{\gamma\zeta_1^{\gamma-1}(1-\alpha)}{\mathcal{B}(\alpha)}(\mathcal{M}\|\mathscr{P}\|+\mathcal{M}_a)+\frac{\alpha\gamma}{\mathcal{B}(\alpha)\Gamma(\alpha)}(\mathcal{M}\|\mathscr{P}\|$$
$$+\mathcal{M}_a)\zeta_1^{\alpha+\gamma-1}\mathscr{B}(\alpha,\gamma)-\frac{\gamma\zeta_2^{\gamma-1}(1-\alpha)}{\mathcal{B}(\alpha)}(\mathcal{M}\|\mathscr{P}\|+\mathcal{M}_a)$$
$$-\frac{\alpha\gamma}{\mathcal{B}(\alpha)\Gamma(\alpha)}(\mathcal{M}\|\mathscr{P}\|+\mathcal{M}_a)\zeta_2^{\alpha+\gamma-1}\mathscr{B}(\alpha,\gamma).$$

when $\zeta_1\to\zeta_2$, then $|\mathscr{T}\mathscr{P}(t_2)-\mathscr{T}\mathscr{P}_a(\zeta_1)|\to 0$, Hence $\mathscr{T}$ is equicontinuous. So, by the Arzela-Ascoli theorem is completely continuous. Thus, by Schauder's fixed point result the proposed model has at least one solution. □

4. Uniqueness of the solution

Theorem 4.1. *Let (3.6) hold. If* $C<1$, *where* $C=\left(\frac{\gamma T^{\gamma-1}(1-\alpha)}{\mathcal{B}(\alpha)}+\frac{\alpha\gamma}{\mathcal{B}(\alpha)\Gamma(\alpha)}T^{\alpha+\gamma-1}\mathscr{B}(\alpha,\gamma)\right)\mathcal{M}_a$, then the considered model has a unique solution

Proof. For $\mathscr{P},\mathscr{P}_a\in\Omega$, we have

$$\|\mathscr{T}\mathscr{P}-\mathscr{T}\mathscr{P}_a\|$$
$$=\max_{\zeta\in[0,T]}\Big|\frac{\gamma\zeta^{\gamma-1}(1-\alpha)}{\mathcal{B}(\alpha)}(\mathcal{G}(\zeta,\mathscr{P}(\zeta))-\mathcal{G}(\zeta,\mathscr{P}_a(\zeta)))$$
$$+\frac{\alpha\gamma}{\mathcal{B}(\alpha)\Gamma(\alpha)}\int_0^{\zeta}\xi^{\gamma-1}(t-\xi)^{\gamma-1}(\mathcal{G}(\xi,\mathscr{P}(\xi))$$
$$-\mathcal{G}(\xi,\mathscr{P}_a(\xi)))d\xi\Big|$$
$$\le\frac{\gamma T^{\gamma-1}(1-\alpha)}{\mathcal{B}(\alpha)}+\frac{\alpha\gamma}{\mathcal{B}(\alpha)\Gamma(\alpha)}T^{\alpha+\gamma-1}\mathscr{B}(\alpha,\gamma)\|\mathscr{P}-\mathscr{P}_a\|$$
$$\le C\|\mathscr{P}-\mathscr{P}_a\|.$$

As a result, $\mathscr{T}$ is a contraction. So, the model has a unique solution according to the Banach contraction principle. □

5. Numerical procedure

Toufik and Atangana [23, 24] presented a numerical scheme. In this section, we illustrate the numerical scheme in details:

$$\begin{cases}{}_0^{ABC}\mathscr{D}_{\zeta}^{\alpha,\gamma}[\mathscr{P}(\zeta)]=\mathcal{G}(\zeta,\mathscr{P}(\zeta)), & \zeta\ge 0,\\ \mathscr{P}(0)=\mathscr{P}_0.\end{cases}\tag{5.1}$$

Using the fundamental theorem of fractional calculus Eq. (5.1) can be written as:

$$\mathscr{P}(\zeta)-\mathscr{P}(0)=\frac{\gamma\zeta^{\gamma-1}(1-\alpha)}{\mathcal{B}(\alpha)}\mathcal{G}(\zeta,\mathscr{P}(\zeta))+\frac{\gamma\alpha}{\mathcal{B}(\alpha)\Gamma(\alpha)}\int_0^{\zeta}\xi^{\gamma-1}(\zeta-\xi)^{\alpha-1}\mathcal{G}(\xi,\mathscr{P}(\xi))\,d\xi.\tag{5.2}$$

At a point $\zeta=\zeta_{\mathscr{K}+1}$, for $\mathscr{K}=0,1,2,...$, equation Eq.

(5.2) becomes

$$\begin{aligned}&\mathscr{P}(\zeta_{\mathscr{K}+1})-\mathscr{P}(0)\\ =&\frac{\gamma(1-\alpha)\zeta^{\gamma-1}}{\mathcal{B}(\alpha)}\mathcal{G}(\zeta_{\mathscr{K}},\mathscr{P}(\zeta_{\mathscr{K}}))\\ &+\frac{\alpha\gamma}{\mathcal{B}(\alpha)\Gamma(\alpha)}\int_0^{\zeta_{\mathscr{K}+1}}\xi^{\gamma-1}(\zeta_{\mathscr{K}+1}-\xi)^{\alpha-1}\mathcal{G}(\xi,\mathscr{P}(\xi))\,d\xi,\end{aligned}\tag{5.3}$$

$$\begin{aligned}&\mathscr{P}_{\mathscr{K}+1}=\mathscr{P}(\zeta_{\mathscr{K}+1})\\ =&\mathscr{P}(0)+\frac{(1-\alpha)\gamma\zeta^{\gamma-1}}{\mathcal{B}(\gamma\alpha)}\mathcal{G}(t_{\mathscr{K}},\mathscr{P}(\zeta_{\mathscr{K}}))+\frac{\alpha}{\mathcal{B}(\gamma\alpha)\Gamma(\alpha)}\\ &\sum_{i=0}^{\mathscr{K}}\int_{\zeta_i}^{\zeta_{i+1}}\xi^{\gamma-1}(\zeta_{\mathscr{K}+1}-\xi)^{\alpha-1}\mathcal{G}(\xi,\mathscr{P}(\xi))\,d\xi.\end{aligned}\tag{5.4}$$

We can approximated the function $\mathcal{G}(\xi,\mathscr{P}(\xi))$ in the interval $[\zeta_i,\zeta_{i+1}]$.

$$\begin{aligned}q_{\mathscr{K}}=&\mathcal{G}(\xi,\mathscr{P}(\xi))\\ =&\frac{\xi-\zeta_{i-1}}{\zeta_i-\zeta_{i-1}}\mathcal{G}(\zeta_i,\mathscr{P}_i)+\frac{\xi-\zeta_i}{\zeta_{i-1}-\zeta_i}\mathcal{G}(\zeta_{i-1},\mathscr{P}_{i-1}).\end{aligned}\tag{5.5}$$

Substituting the above value in Eq. (5.4), we get

$$\begin{aligned}&\mathscr{P}_{\mathscr{K}+1}\\ =&\mathscr{P}(0)+\frac{(1-\alpha)\gamma\zeta^{\gamma-1}}{\mathcal{B}(\gamma\alpha)}\mathcal{G}(\zeta_{\mathscr{K}},\mathscr{P}(\zeta_{\mathscr{K}}))\\ &+\frac{\gamma\alpha}{\mathcal{B}(\alpha)\Gamma(\alpha)}\sum_{i=0}^{\mathscr{K}}\Bigg(\frac{\mathcal{G}(\zeta_i,\mathscr{P}(\zeta_i))}{n}\int_{\zeta_i}^{\zeta_{i+1}}\xi^{\gamma-1}(\zeta-\zeta_{i-1})\\ &(\zeta_{\mathscr{K}+1}-\xi)^{\alpha-1}\,d\xi-\frac{\mathcal{G}(\zeta_{i-1},\mathscr{P}(\zeta_{i-1}))}{n}\\ &\int_{\zeta_i}^{\zeta_{i+1}}\xi^{\gamma-1}(\zeta-\zeta_{i-1})(\zeta_{\mathscr{K}+1}-\xi)^{\alpha-1}\,d\xi\Bigg).\end{aligned}\tag{5.6}$$

After substituting $n=\zeta_i-\zeta_{i-1}$ and simplification, we get

$$\begin{aligned}&\mathscr{P}_{\mathscr{K}+1}\\ =&\mathscr{P}_0+\frac{(1-\alpha)\gamma\zeta^{\gamma-1}}{\mathcal{B}(\gamma\alpha)}\mathcal{G}(\zeta_{\mathscr{K}},\mathscr{P}(\zeta_{\mathscr{K}}))+\frac{\gamma\alpha}{\mathcal{B}(\alpha)}\\ &\sum_{i=0}^{\mathscr{K}}\Bigg[\frac{n^\alpha\mathcal{G}(\zeta_i,\mathscr{P}(\zeta_i))}{\Gamma(\alpha+2)}((\mathscr{K}-i+1)^\alpha(\mathscr{K}+2-i+\alpha))\\ &(\mathscr{K}-i)^\alpha(\mathscr{K}-i+2+2\alpha)-\frac{n^\alpha\mathcal{G}(\zeta_{i-1},\mathscr{P}(\zeta_{i-1}))}{\Gamma(\alpha+2)}\\ &\Big((\mathscr{K}-i+1)^{\alpha+1}-(\mathscr{K}-i)^\alpha(\mathscr{K}-i+1+\alpha)\Big)\Bigg].\end{aligned}\tag{5.7}$$

For simplicity, we define

$$\begin{cases}\mathfrak{X}(\mathscr{X},\mathscr{Y},\mathscr{Z},\zeta)=\mathscr{Y}(\zeta)\\ \mathfrak{Y}(\mathscr{X},\mathscr{Y},\mathscr{Z},\zeta)=-\frac{1}{3}\left(\mathscr{X}(\zeta)+\frac{3}{2}\left(\mathscr{Z}^2(\zeta)-1\right)\mathscr{Y}(\zeta)\right)\\ \mathfrak{Z}(\mathscr{X},\mathscr{Y},\mathscr{Z},\zeta)=-\mathscr{Y}(t)-\frac{3}{5}\mathscr{Z}(\zeta)+\mathscr{Z}(\zeta)\mathscr{Y}(\zeta).\end{cases}$$

The numerical scheme for the non-linear integrator circuit under the ABC fractal-fractional derivative is given below:

$$\begin{aligned}\mathscr{X}_{\mathscr{K}+1}=&\mathscr{X}_0+\frac{(1-\alpha)\gamma t^{\gamma-1}}{\mathcal{B}(\gamma\alpha)}\mathfrak{X}(t_{\mathscr{K}},\mathscr{X}(\zeta_{\mathscr{K}}))\\ &+\frac{\gamma\alpha}{\mathcal{B}(\alpha)}\sum_{i=0}^{\mathscr{K}}\Bigg[\frac{n^\alpha\mathfrak{X}(\zeta_i,\mathscr{X}(\zeta_i))}{\Gamma(\alpha+2)}((\mathscr{K}+1-i)^\alpha(\mathscr{K}-i+2+\alpha))\\ &(\mathscr{K}-i)^\alpha(\mathscr{K}+2-i+2\alpha)-\frac{n^\alpha\mathfrak{X}(\zeta_{i-1},\mathscr{X}(\zeta_{i-1}))}{\Gamma(\alpha+2)}\\ &\Big((\mathscr{K}-i+1)^{\alpha+1}-(\mathscr{K}-i)^\alpha(\mathscr{K}+1-i+\alpha)\Big)\Bigg].\end{aligned}\tag{5.8}$$

$$\begin{aligned}\mathscr{Y}_{\mathscr{K}+1}=&\mathscr{Y}_0+\frac{(1-\alpha)\gamma\zeta^{\gamma-1}}{\mathcal{B}(\gamma\alpha)}\mathfrak{Y}(\zeta_{\mathscr{K}},\mathscr{Y}(\zeta_{\mathscr{K}}))\\ &+\frac{\gamma\alpha}{\mathcal{B}(\alpha)}\sum_{i=0}^{\mathscr{K}}\Bigg[\frac{n^\alpha\mathfrak{Y}(\zeta_i,\mathscr{Y}(\zeta_i))}{\Gamma(\alpha+2)}((\mathscr{K}+1-i)^\alpha(\mathscr{K}-i+2+\alpha))\\ &(\mathscr{K}-i)^\alpha(\mathscr{K}+2-i+2\alpha)-\frac{n^\alpha\mathfrak{Y}(\zeta_{i-1},\mathscr{Y}(\zeta_{i-1}))}{\Gamma(\alpha+2)}\\ &\Big((\mathscr{K}-i+1)^{\alpha+1}-(\mathscr{K}-i)^\alpha(\mathscr{K}-i+1+\alpha)\Big)\Bigg].\end{aligned}\tag{5.9}$$

$$\begin{aligned}\mathscr{Z}_{\mathscr{K}+1}=&\mathscr{P}_0+\frac{(1-\alpha)\gamma\zeta^{\gamma-1}}{\mathcal{B}(\gamma\alpha)}\mathfrak{Z}(\zeta_{\mathscr{K}},\mathscr{Z}(\zeta_{\mathscr{K}}))\\ &+\frac{\gamma\alpha}{\mathcal{B}(\alpha)}\sum_{i=0}^{\mathscr{K}}\Bigg[\frac{n^\alpha\mathfrak{Z}(\zeta_i,\mathscr{Z}(\zeta_i))}{\Gamma(\alpha+2)}((\mathscr{K}-i+1)^\alpha(\mathscr{K}+2-i+\alpha))\\ &(\mathscr{K}-i)^\alpha(\mathscr{K}-i+2+2\alpha)-\frac{n^\alpha\mathfrak{Z}(\zeta_{i-1},\mathscr{Z}(\zeta_{i-1}))}{\Gamma(\alpha+2)}\\ &\Big((\mathscr{K}-i+1)^{\alpha+1}-(\mathscr{K}-i)^\alpha(\mathscr{K}+1-i+\alpha)\Big)\Bigg].\end{aligned}\tag{5.10}$$

Now, we present some numerical results displaying complex behaviors of the novel fractal-fractional system (2.1). Figures 1 to 4 present the graphical results for the ABC fractal-fractional model (2.1) by considered the above novel numerical scheme with the initial conditions

$\mathscr{X}(0) = 0.2$, $\mathscr{Y}(0) = 0.2$, and $\mathscr{Z}(0) = 0.3$. The simulation results in both cases include the time-domain responses of the state variables and the two- and three-dimensional phase portraits. The plots in Figure 2 and 4 describe the dynamical behavior of $\mathscr{X}(\zeta)$, $\mathscr{Y}(\zeta)$, and $\mathscr{Z}(\zeta)$ for different fractal-fractional order in the sense of the ABC fractal operator with $\mathcal{R}_1 = \mathcal{R}_3 = \mathcal{R}_2 = \mathcal{R}_5 = \mathcal{R}_9 = \mathcal{R}_7 = \mathcal{R}_{10} = \mathcal{R}_{12} = \mathcal{R}_{13} = 20K\Omega, \mathcal{R}_6 = \mathcal{R}_8 = 11.62K\Omega, \mathcal{R}_4 = 12.62K\Omega, \mathcal{R}_{11} = 15.49K\Omega$. The simulation results in for the two- and three-dimensional phase portraits are displayed in the plots of Figures 1 and 3. In Figure (2)(a-c), we can observe that when the order of the ABC fractal-derivative decreases, the dynamics of the ABC fractal-fractional system (2.1) decrease as well. This means that the ABC has an acceleration effect in the system process when the fractional order decreases. The figures mentioned above confirm the effectiveness of the proposed numerical scheme in exhibiting both chaotic and non-chaotic behaviors of the novel fractal fractional dynamical system (2.1).

6. Analysis of the chaotic system (1.1) under the Caputo-Fabrizo fractal-fractional derivative

In this section, we study the fractal-fractional chaotic system (1.1) under the CF derivative with the fractional order α as follows:

$$\begin{cases} {}_{0}^{CF}\mathscr{D}_{\zeta}^{\alpha,\gamma}[\mathscr{X}(\zeta)] = \mathscr{Y}(\zeta), \\ {}_{0}^{CF}\mathscr{D}_{\zeta}^{\alpha,\gamma}[\mathscr{Y}(\zeta)] = -\frac{1}{3}\left(\mathscr{X}(\zeta) + \frac{3}{2}\left(\mathscr{Z}^2(\zeta) - 1\right)\mathscr{Y}(\zeta)\right), \\ {}_{0}^{CF}\mathscr{D}_{\zeta}^{\alpha,\gamma}[\mathscr{Z}(\zeta)] = -\frac{3}{5}\mathscr{Z}(\zeta) - \mathscr{Y}(\zeta) + \mathscr{Z}(\zeta)\mathscr{Y}(\zeta). \end{cases} \tag{6.1}$$

The circuital equations associated with the system equations are given below:

$$\begin{cases} {}_{0}^{CF}\mathscr{D}_{\zeta}^{\alpha,\gamma}[\mathscr{X}(\zeta)] = \frac{1}{C_0}\left[-\left(\frac{\mathcal{R}_2}{\mathcal{R}_1\mathcal{R}_3}\right)\mathscr{Y}\right], \\ {}_{0}^{CF}\mathscr{D}_{\zeta}^{\alpha,\gamma}[\mathscr{Y}(\zeta)] = \frac{1}{C_0}\left[\left(\frac{\mathcal{R}_7}{\mathcal{R}_8\mathcal{R}_5}\right)\mathscr{Y} - \left(\frac{1}{\mathcal{R}_4}\right)\mathscr{X} - \left(\frac{1}{\mathcal{R}_6}\right)\mathscr{Z}^2\mathscr{Y}\right], \\ {}_{0}^{CF}\mathscr{D}_{\zeta}^{\alpha,\gamma}[\mathscr{Z}(\zeta)] = \frac{1}{C_0}\left[\left(\frac{\mathcal{R}_{12}}{\mathcal{R}_{13}\mathcal{R}_9}\right)\mathscr{Y}\mathscr{Z} - \left(\frac{1}{\mathcal{R}_{10}}\right)\mathscr{Y} - \left(\frac{1}{\mathcal{R}_{11}}\right)\mathscr{Z}\right], \end{cases} \tag{6.2}$$

where the parameters values are can be considered as follows: $\mathcal{R}_1 = \mathcal{R}_3 = \mathcal{R}_2 = \mathcal{R}_5 = \mathcal{R}_9 = \mathcal{R}_7 = \mathcal{R}_{12} = \mathcal{R}_{10} = \mathcal{R}_{13} = 10K\Omega, \mathcal{R}_6 = \mathcal{R}_8 = 10.62K\Omega, \mathcal{R}_4 = 11.62K\Omega, \mathcal{R}_{11} = 11.49K\Omega$.

Let $\mathscr{P}(\zeta) \in \mathcal{H}(a,b)$ [25]; then, the Caputo-Fabrizo fractal fractional derivative can be defined as

$${}_{a}^{CF}\mathscr{D}_{\zeta}^{\alpha,\gamma}[\mathscr{P}(\zeta)] = \frac{\mathcal{B}(\alpha)}{1-\alpha}\int_a^{\zeta}\frac{d\mathscr{P}(\xi)}{d\xi^{\gamma}}\exp\left[-\alpha\frac{(\zeta-\xi)}{1-\alpha}\right]d\xi, \tag{6.3}$$

where, $\gamma \leq 1$ and $0 < \alpha \leq 1$ $\mathcal{B}(0) = \mathcal{B}(1) = 1$.

Here, we develop the numerical scheme for the fractional model (6.2) by using the methods described in [23, 26]. We contemplate the following equation:

$$\begin{cases} {}_{0}^{CF}\mathscr{D}_{\zeta}^{\alpha,\gamma}[\mathscr{P}(\zeta)] = \mathcal{G}(\zeta, \mathscr{P}(\zeta)), \quad \zeta \geq 0, \\ \mathscr{P}(0) = \mathscr{P}_0, \end{cases} \tag{6.4}$$

where $\mathcal{G}(\zeta, \mathscr{P}(\zeta))$ represents the right hand sides of the equations in the model (1.1). $\mathscr{P}(\zeta)$ stands for $\mathscr{X}(\zeta), \mathscr{Y}(\zeta)$ and $\mathscr{Z}(\zeta)$.

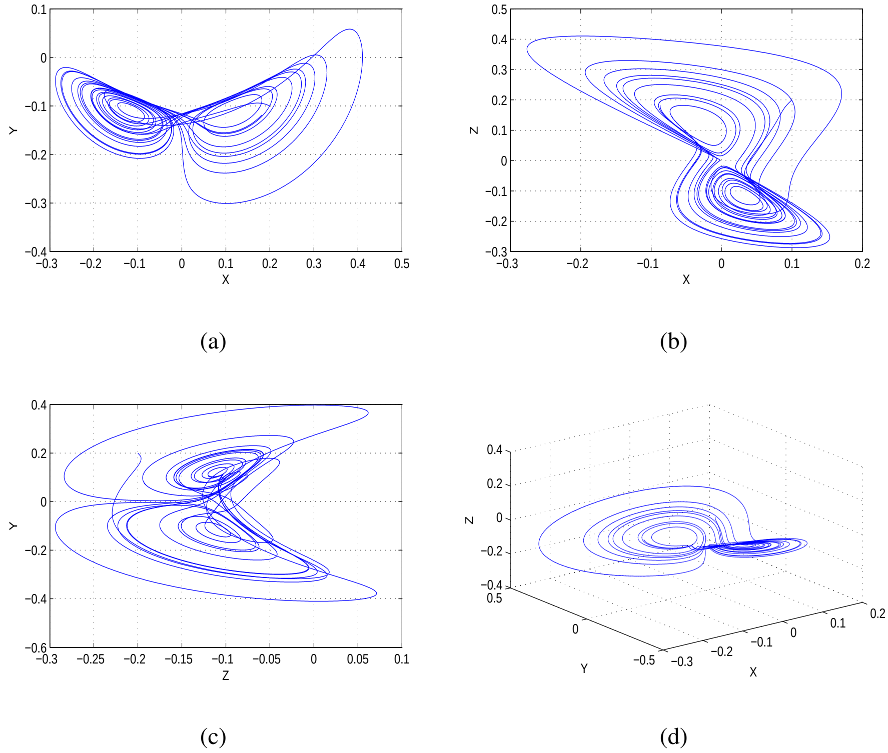

Figure 1. Dynamical behavior of the chaotic attractor for $\mathscr{X}(0) = 0.2, \mathscr{Y}(0) = 0.2$ and $\mathscr{Z}(0) = 0.3$, as obtained for $\gamma = 1$ and $\alpha = 0.95$ by using the ABC fractal-fractional derivative.

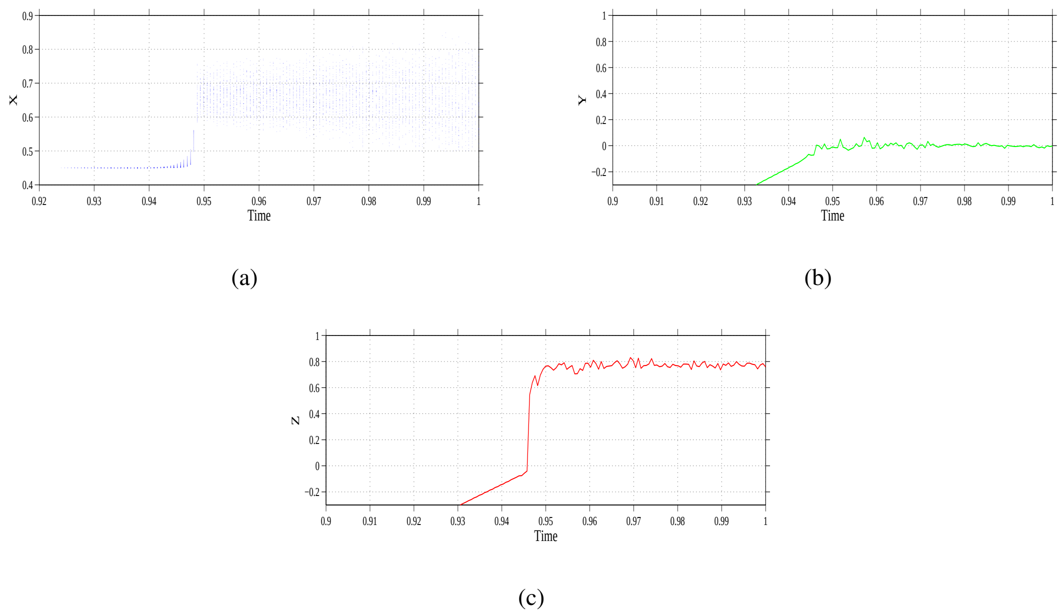

Figure 2. Numerical simulation of (2.1) for $\gamma = 1$ and different values of $\alpha = 0.95$.

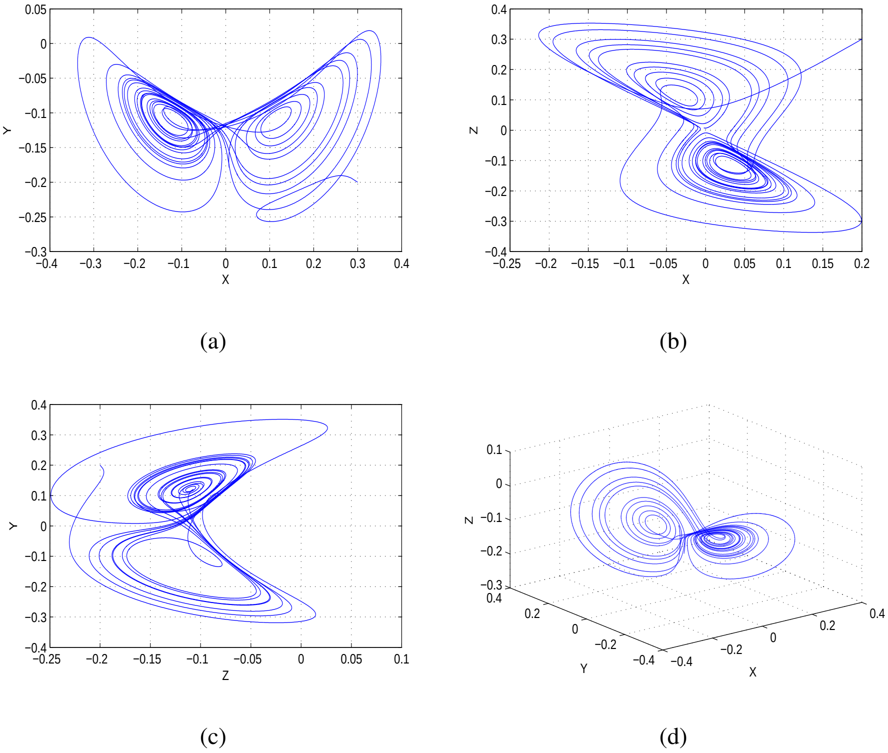

Figure 3. Dynamical behavior of the chaotic attractor for $\mathscr{X}(0) = 0.2, \mathscr{Y}(0) = 0.2$ and $\mathscr{Z}(0) = 0.3$, as obtained for $\gamma = 1$ and $\alpha = 0.85$ by using the ABC fractal-fractional derivative.

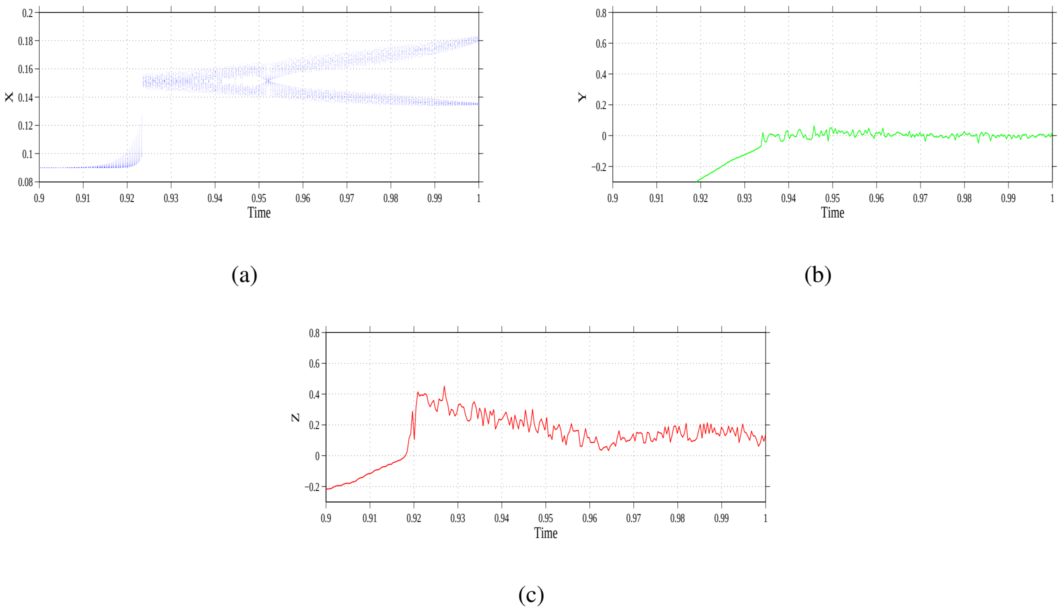

Figure 4. Numerical simulation of (2.1) for $\gamma = 0.85$ and different values of $\alpha = 0.85$.

Equation (6.4) can be expressed by using the fundamental theorem of fractional calculus:

$$\begin{aligned}\mathscr{P}(\zeta) =& \mathscr{P}(0) + \frac{(1-\alpha)\gamma\zeta^{\gamma-1}}{\mathcal{B}(\alpha)}\mathcal{G}(\zeta, \mathscr{P}(\zeta)) \\ &+ \frac{\gamma\alpha}{\mathcal{B}(\alpha)}\int_a^{\zeta}\xi^{\gamma-1}\mathcal{G}(\xi, \mathscr{P}(\xi))\,d\xi.\end{aligned} \tag{6.5}$$

For $\zeta = \zeta_{\mathscr{K}+1}, \mathscr{K} = 0, 1, 2, ...$, we get

$$\begin{aligned}&\mathscr{P}(\zeta_{\mathscr{K}+1}) - \mathscr{P}(0) \\ =&\frac{(1-\alpha)\gamma\zeta^{\gamma-1}}{\mathcal{B}(\alpha)}\mathcal{G}(\zeta_{\mathscr{K}}, \mathscr{P}(\zeta_{\mathscr{K}})) \\ &+ \frac{\gamma\alpha}{\mathcal{B}(\alpha)\Gamma(\alpha)}\int_0^{\zeta_{\mathscr{K}+1}}\xi^{\gamma-1}\mathcal{G}(\xi, \mathscr{P}(\xi))\,d\xi.\end{aligned} \tag{6.6}$$

Now at $\zeta = \zeta_{\mathscr{K}}, \mathscr{K} = 0, 1, 2, ...$, we have:

$$\begin{aligned}\mathscr{P}(\zeta_{\mathscr{K}}) - \mathscr{P}_0 =& \frac{(1-\alpha)\gamma t^{\gamma-1}}{\mathcal{B}(\alpha)}\mathcal{G}(\zeta_{\mathscr{K}-1}, \mathscr{P}(\zeta_{\mathscr{K}-1})) \\ &+ \frac{\gamma\alpha}{\mathcal{B}(\alpha)}\int_0^{\zeta_{\mathscr{K}}}\xi^{\gamma-1}\mathcal{G}(\xi, \mathscr{P}(\xi))\,d\xi.\end{aligned} \tag{6.7}$$

Using the successive terms with difference from the above equations, we get

$$\begin{aligned}&\mathscr{P}(\zeta_{\mathscr{K}+1}) - \mathscr{P}(\zeta_{\mathscr{K}}) \\ =&\frac{(1-\alpha)\gamma\zeta^{\gamma-1}}{\mathcal{B}(\alpha)}\left[\mathcal{G}(\zeta_{\mathscr{K}}, \mathscr{P}(\zeta_{\mathscr{K}})) - \mathcal{G}(\zeta_{\mathscr{K}-1}, \mathscr{P}(\zeta_{\mathscr{K}-1}))\right] \\ &+ \frac{\gamma\alpha}{\mathcal{B}(\alpha)}\int_{\zeta_{\mathscr{K}}}^{\zeta_{\mathscr{K}+1}}\xi^{\gamma-1}\mathcal{G}(\xi, \mathscr{P}(\xi))\,d\xi.\end{aligned} \tag{6.8}$$

Let the function $\mathcal{G}(\xi, \mathscr{P}(\xi))$ be closed in the interval $[\zeta_{\mathscr{K}}, \zeta_{\mathscr{K}+1}]$: then, the function can be approximated as:

$$\begin{aligned}q_{\mathscr{K}} \cong \mathcal{G}(\xi, \mathscr{P}(\xi)) =& \frac{\xi - \zeta_{j-1}}{\zeta_j - \zeta_{j-1}}\mathcal{G}\left(\zeta_j, \mathscr{P}_j\right) \\ &+ \frac{\xi - \zeta_j}{\zeta_{j-1} - \zeta_j}\mathcal{G}\left(\zeta_{j-1}, \mathscr{P}_{j-1}\right).\end{aligned} \tag{6.9}$$

Substituting the above approximation in Eq. (6.8), we get

$$\begin{aligned}&\mathscr{P}_{\mathscr{K}+1} - \mathscr{P}_{\mathscr{K}} \\ =&\frac{(1-\alpha)\gamma\zeta^{\gamma-1}}{\mathcal{B}(\alpha)}\left[\mathcal{G}(\zeta_{\mathscr{K}}, \mathscr{P}(\zeta_{\mathscr{K}})) - \mathcal{G}(\zeta_{\mathscr{K}-1}, \mathscr{P}(\zeta_{\mathscr{K}-1}))\right] \\ &+ \frac{\alpha}{\mathcal{B}(\alpha)}\int_{\zeta_i}^{\zeta_{i+1}}\xi^{\gamma-1}\left(\frac{\mathcal{G}(\zeta_{\mathscr{K}}, \mathscr{P}_{\mathscr{K}})}{\zeta_{\mathscr{K}} - \zeta_{\mathscr{K}-1}}(\xi - \zeta_{\mathscr{K}-1}) \right. \\ &\left. - \frac{\mathcal{G}(\zeta_{\mathscr{K}-1}, \mathscr{P}_{\mathscr{K}-1})}{\zeta_{\mathscr{K}} - \zeta_{\mathscr{K}-1}}(\xi - \zeta_{\mathscr{K}})\right)d\xi.\end{aligned} \tag{6.10}$$

After simplification, we get

$$\begin{aligned}&\mathscr{P}_{\mathscr{K}+1} = \mathscr{P}_0 \\ &+ \left(\frac{(1-\alpha)\gamma\zeta^{\gamma-1}}{\mathcal{B}(\alpha)} + \frac{3(t_{\mathscr{K}} - \zeta_{\mathscr{K}-1})}{2\mathcal{B}(\alpha)}\right)\mathcal{G}(\zeta_{\mathscr{K}}, \mathscr{P}(\zeta_{\mathscr{K}})) \\ &- \left(\frac{(1-\alpha)\gamma\zeta^{\gamma-1}}{\mathcal{B}(\alpha)} + \frac{\gamma\alpha((\zeta_{\mathscr{K}} - \zeta_{\mathscr{K}-1})}{2\mathcal{B}(\alpha)}\right)\mathcal{G}(\zeta_{\mathscr{K}-1}, \mathscr{P}(\zeta_{\mathscr{K}-1})).\end{aligned} \tag{6.11}$$

Now, applying the above numerical scheme for the fractional non-linear integrator circuit gives

$$\begin{aligned}&\mathscr{X}(\zeta_{\mathscr{K}+1}) \\ =&\mathscr{X}_0 + \left(\frac{(1-\alpha)\gamma\zeta^{\gamma-1}}{\mathcal{B}(\alpha)} + \frac{3(\zeta_{\mathscr{K}} - \zeta_{\mathscr{K}-1})}{2\mathcal{B}(\alpha)}\right)\mathscr{Y}(\zeta_{\mathscr{K}}) \\ &- \left(\frac{(1-\alpha)\gamma\zeta^{\gamma-1}}{\mathcal{B}(\alpha)} + \frac{\gamma\alpha((\zeta_{\mathscr{K}} - \zeta_{\mathscr{K}-1})}{2\mathcal{B}(\alpha)}\right)\mathscr{Y}(\zeta_{\mathscr{K}-1}),\end{aligned} \tag{6.12}$$

$$\begin{aligned}&\mathscr{Y}(\zeta_{\mathscr{K}+1}) \\ =&\mathscr{Y}_0 + \left(\frac{(1-\alpha)\gamma\zeta^{\gamma-1}}{\mathcal{B}(\alpha)} + \frac{3(\zeta_{\mathscr{K}} - \zeta_{\mathscr{K}-1})}{2\mathcal{B}(\alpha)}\right) \\ &\left(-\frac{1}{3}\left(\mathscr{X}(\zeta_{\mathscr{K}}) + \frac{3}{2}\left(\mathscr{Z}^2(\zeta_{\mathscr{K}}) - 1\right)\mathscr{Y}(\zeta_{\mathscr{K}})\right)\right) \\ &- \left(\frac{(1-\alpha)\gamma\zeta^{\gamma-1}}{\mathcal{B}(\alpha)} + \frac{\gamma\alpha((\zeta_{\mathscr{K}} - \zeta_{\mathscr{K}-1})}{2\mathcal{B}(\alpha)}\right) \\ &\left(-\frac{1}{3}\left(\mathscr{X}(\zeta_{\mathscr{K}-1}) + \frac{3}{2}\left(\mathscr{Z}^2(\zeta_{\mathscr{K}-1}) - 1\right)\mathscr{Y}(\zeta_{\mathscr{K}-1})\right)\right),\end{aligned} \tag{6.13}$$

$$\begin{aligned}&\mathscr{Z}(\zeta_{\mathscr{K}+1}) \\ =&\mathscr{Z}_0 + \left(\frac{(1-\alpha)\gamma\zeta^{\gamma-1}}{\mathcal{B}(\alpha)} + \frac{3(\zeta_{\mathscr{K}} - \zeta_{\mathscr{K}-1})}{2\mathcal{B}(\alpha)}\right) \\ &\left(-\mathscr{Y}(\zeta_{\mathscr{K}}) - \frac{3}{5}\mathscr{Z}(\zeta_{\mathscr{K}}) + \mathscr{Z}(\zeta_{\mathscr{K}})\mathscr{Y}(\zeta_{\mathscr{K}})\right) \\ &- \left(\frac{(1-\alpha)\gamma\zeta^{\gamma-1}}{\mathcal{B}(\alpha)} + \frac{\gamma\alpha((t_{\mathscr{K}} - \zeta_{\mathscr{K}-1})}{2\mathcal{B}(\alpha)}\right) \\ &\left(-\mathscr{Y}(\zeta_{\mathscr{K}-1}) - \frac{3}{5}\mathscr{Z}(\zeta_{\mathscr{K}-1}) + \mathscr{Z}(\zeta_{\mathscr{K}-1})\mathscr{Y}(\zeta_{\mathscr{K}-1})\right).\end{aligned} \tag{6.14}$$

We present some numerical simulations to study the complex behaviors of the fractal-fractional system (6.1). Figures 5 to 8 present the graphical results for the CF fractal-fractional model (2.1) that were obtained by considering the above novel numerical scheme with the initial conditions $\mathscr{X}(0) = 0.2$, $\mathscr{Y}(0) = 0.2$, and $\mathscr{Z}(0) = 0.3$. For different fractal-fractional orders, we give numerical results for the considered model (6.1) that was developed by using two-phase and three-phase simulations. The plots in Figures 8(a-c) and 6(a-c) shown the dynamical behavior of the $\mathscr{X}(\zeta)$, $\mathscr{Y}(\zeta)$, and $\mathscr{Z}(\zeta)$ for different fractional orders in the sense of the CF fractal operator $\mathcal{R}_1 = \mathcal{R}_3 = \mathcal{R}_2 = \mathcal{R}_5 = \mathcal{R}_9 = \mathcal{R}_7 = \mathcal{R}_{10} = \mathcal{R}_{12} = \mathcal{R}_{13} = 20K\Omega, \mathcal{R}_6 = \mathcal{R}_8 = 12.62K\Omega, \mathcal{R}_4 = 10.62K\Omega, \mathcal{R}_{11} = 16.49K\Omega$. The simulation results for the two- and three-dimensional phase portraits are displayed in the plots of Figures 5 and 7. By decreasing the fractional orders, we can see how the dynamics of the model variables change. We can observe that when the order of the CF fractal-fractional derivative decreases, the dynamics of the system (6.1) decrease as well.

Two newly proposed fractal-fractional operators based on the Mittag-Leffler and exponential functions were applied to extend the memristor-based chaotic system. The one based on the exponential function was developed by Caputo and Fabrizio, while Atangana and Baleanu proposed the one based on the Mittag-Leffler function. We evaluated the model's more complex behavior in the form of a fractal-fractional operator, which is often challenging to achieve by using the integer-order operator. The fractal-fractional operator is thus a better technique for examining the more complicated behavior of the proposed system.

7. Conclusion and discussion

This article has studied the fractal-fractional memristor-based chaotic model by using different types of fractional derivatives. We established results for the uniqueness and existence of the solution of the model. In each case, a numerical scheme has been established to find approximate solutions of the model, which accurately assess the numerical behavior of the system. Finally, numerical simulations were demonstrated. We defined the sensitivity of these fractal-fractional systems by using the results obtained by modifying some parameters in the model. We have seen that as the values of α decreases, the value of $\mathscr{X}(\zeta)$ decreases significantly for the ABC fractal operator as compared to the CF fractal operator, while the value of $\mathscr{Y}(\zeta)$ and $\mathscr{Z}(\zeta)$ increases significantly for the ABC fractal operator as compared to the CF fractal operator. We can find approximate solutions to some other fractal-fractional systems by using these developed numerical schemes.

Conflict of interest

All authors declare no conflicts of interest regarding the publication of this paper.

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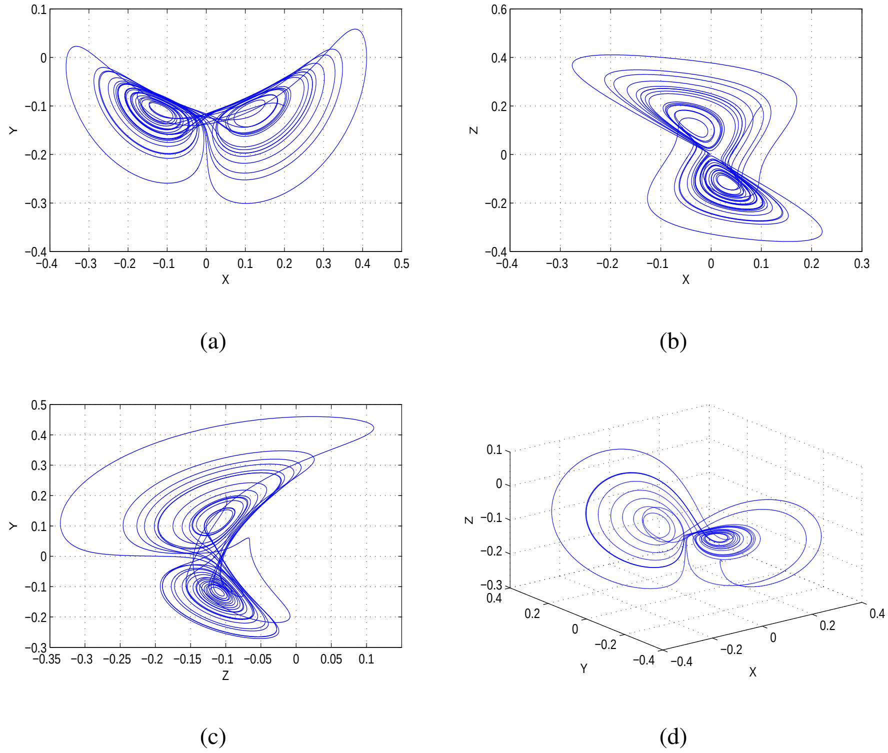

Figure 5. Dynamical behavior of the chaotic attractor for $\mathscr{X}(1) = 0.2, \mathscr{Y}(1) = 0.2$ and $\mathscr{Z}(1) = 0.3$, as obtained for $\gamma = 1$ and $\alpha = 0.95$ by using the CF fractal-fractional derivative.

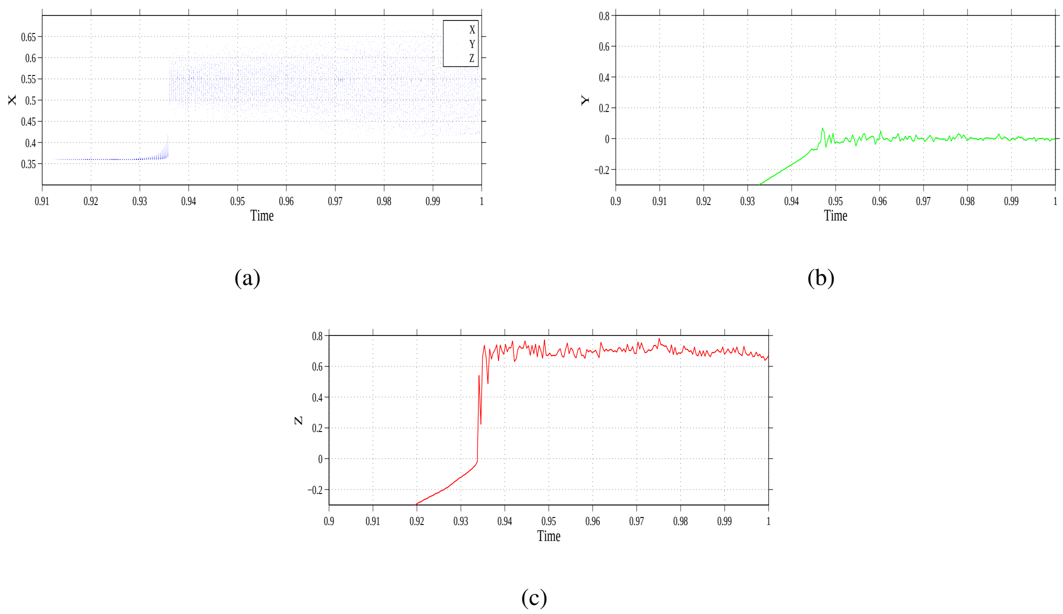

Figure 6. Numerical simulation of (6.2) for $\gamma = 1$ and different values of $\alpha = 0.95$.

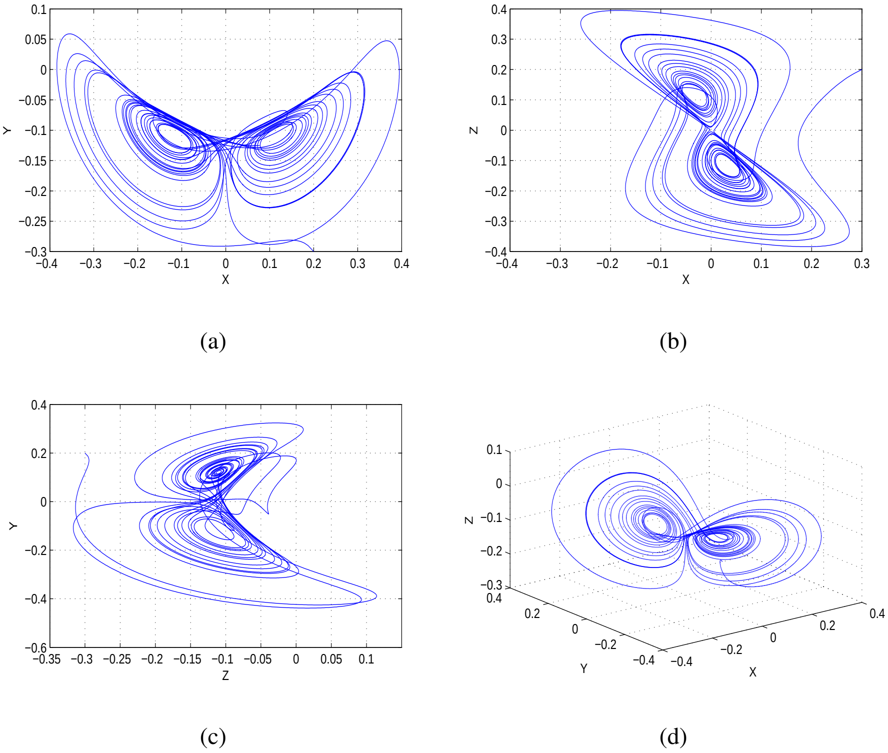

Figure 7. Dynamical behavior of the chaotic attractor for $\mathscr{X}(1) = 0.2$, $\mathscr{Y}(1) = 0.2$ and $\mathscr{Z}(1) = 0.3$, as obtained for $\gamma = 1$ and $\alpha = 0.85$ by using the CF fractal-fractional derivative.

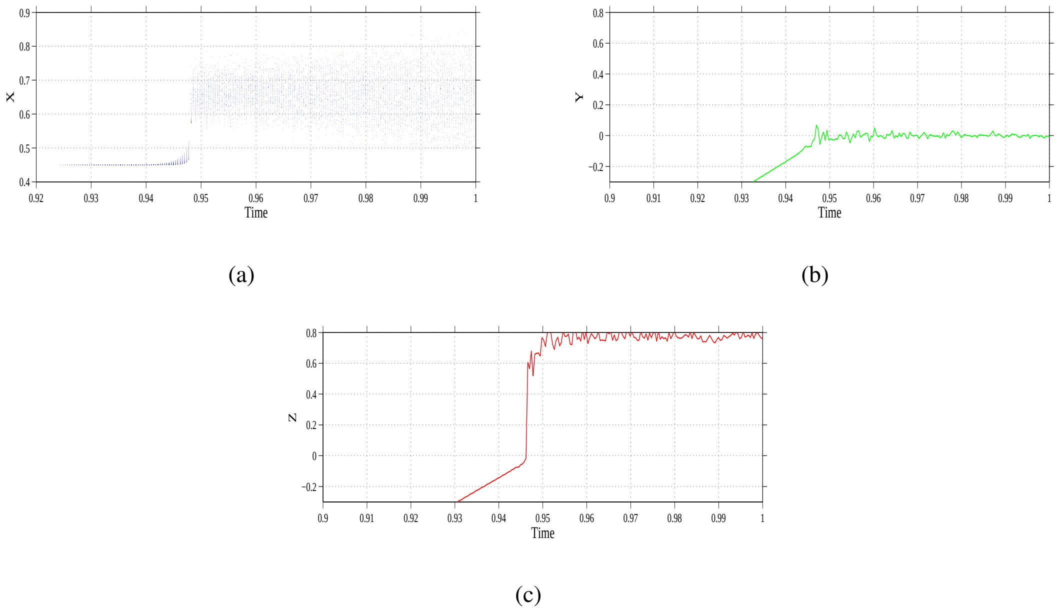

Figure 8. Numerical simulation of (6.2) for $\gamma = 0.95$ and different values of $\alpha = 0.85$.

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