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Research article

A mathematical model for assessing the impact of dual-level toxicity on aquatic biospecies and its optimal control analysis

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Abstract: Ecological models have become paramount for assessing the pesticides effect on the function and structure of aquatic ecosystems. The most paramount concerns are assessments of pesticides/toxicants that have the potential to change from one form to another when they are released into the aquatic ecosystem. Optimal control model is formulated from the nonlinear mathematical model for assessing dual-level toxicity of pesticides effect on aquatic species with the goal to minimizing the pesticides concentration in the aquatic species environment and maximizing the aquatic species population. Two control functions were introduced to represent a policy of not allowing pesticides concentration into the aquatic species environment and the removal of those pesticides that are already in the aquatic environment. The resulting optimal controls are characterized in terms of the optimality system and it was solved quantitatively for different scenarios using both forward and backward sweep iterative method with Runge-Kutta fourth order scheme. The result of the system showed different levels of the aquatic species population stability due to the different levels of the pesticides influx. It was also observed that the degradation of pesticides concentration causes pesticides concentration to vary significantly between the water body and the sediment region with significant level effect on the aquatic species.

Keywords: optimal control; biodegradable pesticides; water; sediments; aquatic biospecies

1. Introduction and Preliminaries

The global increase demand for food security and commercial values of agricultural products have led to upsurge of pesticide usage due to global increased in agricultural activities has undoubtedly reduced crop loss and improved crop yield [1]. Thus, contamination of aquatic environment are usually emanated from off-site movement of pesticides either through leaching, spray drift and runoff [2,3]. The danger of pesticides and other toxicants to aquatic organisms or species have resulted to subject of numerous studies in the past several centuries [4, 5]. Quite numbers of ecotoxicological studies have proved that biological species in both terrestrial and aquatic habitats are negatively affected by pesticides application in their environment [6–8].

There are a lot of evidences that pesticides are major threat to aquatic life and humans [9–11]. More so, Pesticides concentration contaminate various region of biological species habitat such as soil, sediments and water and enter food chain, and these pesticides will finally get to humans through feeding on aquatic organisms and consumption of food products [1]. When some pesticides enter into the aquatic ecosystem they undergo different processes and thereby changing from one form to another such as degradation (photo, biological, microbiological or chemical) to simpler compounds which may be as toxic as the parent compounds or more toxic and more persistent than the parent compounds [12–14]. Some levels of pesticides have been found in surface water, sediments and aquatic environment and along with their attendant degradation products [9, 15, 16].

The increase in cancer reported cases prevalence has linked to different types of pollution such as organochlorine, heavy metals, 3-methycholanthrene (PAH), aromatic hydrocarbons, organic pollutants (e.g. genotoxic persistent organic pollutants, mutagenic PAHs, POPs) and some non-essential metallic elements such as cadmium and arsenic [17].

A thorough knowledge of the links between cancer and pollution in aquatic animals allows using different species as sentinels for the contamination of aquatic ecosystems with oncogenic chemicals for preventing an impact on human health and other biological organisms [14, 17].

The needs for pollution control in both aquatic and terrestrial habitats cannot be overstated. Recently, the United Nation project entitled: From Pollution to Solution - A global assessment of marine litter and plastic pollution provides the needs for the following guidelines, which include governance, legislation, coordination, cooperation, business solution, environmentally sounds technologies and innovations, research and development [18] that need to be observed and implemented in order to combat pollution. The aim of this said assessment is to provide evidence that will enable policymakers and the wider society to understand the magnitude and severity of the risk associated with pesticides pollution and to safeguard human and ecological health. However, pollution in aquatic environments are projected to nearly triple by 2040 without meaningful control in place.

The issue of pollution in our present day society has become a great concern. The needs for everybody in our society to gain the understanding of dangers and control of pollution are of utmost needs and paramount in this study. A novel mathematical model has been developed by authors in [19] to assess the dual-level toxicity effects of biodegradable pesticides to aquatic biological species. However, in this study, we are interested on how the just said assessed pesticides risks should be reduced/eliminated from the aquatic species environment. To do this, an optimal control version of the model in [19] becomes indispensable. Ability to carry out a well-defined and structured optimal control model version in [19] is of utmost important to maintain a sound aquatic ecosystem as well as maintaining human health by avoiding consumption of poisonous aquatic fish or animals. The outcome of this study will also serves as guidelines for government and non-governmental agencies on how different forms of obnoxious pollutants released into our society can be reduced to barest minimum.

The global increase understanding of environmental preservation has call for the need to evaluate the effects of pesticides in aquatic ecosystem. Ecotoxicological risk assessment of pesticides in aquatic habitat is one of the most essential areas for the assessment of pesticide effects for the preservation of biodiversity [20, 21]. The paramount goal of ecotoxicological risk assessment for toxin is to provide knowledge that can be used to protect the components of ecosystems from chemical pressure [22]. With respect to this, ecological models have been widely used as tools for assessing the ecological quagmire for predicting pesticides/toxicants effects on biological population's environment. The authors in [23–25] developed different mathematical models to examine the effects of toxicants/pollutants on biological species. Both the qualitative and the numerical results showed that the total biological species population decreases while the deformed biological species population increases due to increase of toxicants/pollutants influx. Also, study [26] developed a mathematical approach to study the effect of pollutants/toxicants in aquatic environment. It was found that toxicity in the nutrient pool causes increase equilibrium level of resource and fish populations. These mathematical models serve as tools for illustrating the connection as well as assumptions and likelihoods in the extrapolation between endpoints [2, 3].

Many mathematical models have been applied widely to find out the effect of pesticides or toxicants on biological species. Study [23] worked on the external toxicant effect on the population density and deformity of biological species and reported that when emission of the external toxicant increased, the total population density of the biological species will also decreased and the density of the deformed subclass populations increased. A model designed by [25] reported that as the collective rates of emission and formation of toxicant into the biological species environment increased, the densities of the population and its resources did also decreased.

Recently the authors in [19] studied the use of ecological model to evaluate the effects of dual-level toxicity of biodegradable pesticides on aquatic species. It was reported that the aquatic species will completely die with time in the case of pesticides excesses in the aquatic environment, the population of the aquatic population will remain under some certain conditions, the pesticides will persist in the aquatic environment and there will be a periodic oscillation in the population of the aquatic species over time. Study [27] designed a mathematical model to find out the role of rain in confiscation of air pollutants and its long term effects on human population. The result of the model analysis showed that rain serves as a cleaning agent of air pollution. In addition to aforementioned studies, another interesting study was conducted by [7] which proposed and analyzed a model that investigates the effect of pollutant on biological species. From the result of the analysis, it was reported that the population density of the biological species population settled down to its equilibrium level and the magnitude hinged on the equilibrium levels of emission and washout rates of pollutant as well as on the rate of outrider accumulation and its abatement. The rate of outrider accumulation showed to be harmful in affecting the biological species population. The result also established that the survival of the biological species would be jeopardized if the concentration of pollutant increases uncontrollably. Currently, a mathematical model to assess the effect of toxicant in an aquatic ecosystem was carried out by [26] and it was reported that the emission of high concentration of toxicant high than its equilibrium level lead to a total wipe off of the biological species population.

Recently many mathematical models that give insight on model equilibrium analysis, applications and some that were formulated based on optimal control approach of some infectious diseases that are life threatening have been carefully studied [28–31]. Some of their methods of analyses are adopted in this study.

However, various numbers of studies have been conducted on the effect of toxicants or pesticides on aquatic biological population, but based on our investigation; none of the existing models have considered the assessment of minimizing of dual-level pesticides toxicity effects (presence of pesticides in water and sediments) to aquatic populations using a model with time dependent controls. Therefore, in this paper, we proposed and analyzed an optimal control mathematical model that assesses the effect of two pesticides on the population of aquatic species where both pesticides are available in water and sediments with one of the pesticides capable of biodegrading into the other but not vice versa by [19]. The optimal control model will be based on the mathematical model in [19].

The organization of this paper is as follows. In Section 2, we present a nonlinear model that study the impact of the dual-level toxicity (water and sediments) on aquatic species by [19] and carried out review of some mathematical analysis of the formulated model. In Section 3, we formulated the optimal control model and carried out the qualitative analysis. In Section 4, We provide the numerical solutions of the optimal control model. Results of the model are discussed in Section 5 and we conclude in Section 6.

2. Mathematical model

In order to formulate the optimal control model that assesses the effects of biodegradable pesticides on the aquatic population, we will first of all present model [19] upon which its optimal control model will be build. The model [19] has seven compartments, namely: $F(t)$ which stands the population of the aquatic species (e.g. fish, frogs) at time t; $P_{w1}(t)$ stands for the concentration of pesticide 1 in water at time t; $P_{w2}(t)$ is the concentration of pesticide 2 in water at time t; $P_{s1}(t)$ is the concentration of pesticide 1 in sediments at time t; $P_{s2}(t)$ is the concentration of pesticide 2 in sediments at time t; $C_{f1}(t)$ stands the uptake concentration of pesticide 1 in both water and sediments at time t; and $C_{f2}(t)$ represents the uptake concentration of pesticide 2 in both water and sediments at time t.

2.1. Descriptions of model equations

The aquatic population (fish) is increased by its birth rate r_0 and $r(C_{f1}, C_{f2})$ is decrease in birth rate as a result of pesticides concentration uptake. The total aquatic population is decreased by $\frac{r_0 F^2}{K(P_{w1}, P_{w2}, P_{s1}, P_{s2})}$ due to pesticides concentration in the aquatic species environment. The equation is given by

$$\frac{dF}{dt} = r(C_{f1}, C_{f2})F - \frac{r_0 F^2}{K(P_{w1}, P_{w2}, P_{s1}, P_{s2})}.$$

The pesticide 1 in water is increased by concentration of pesticide 1, Q_1, fraction of the dead aquatic species concentration added back to the aquatic environment, π_{w1}, and the conversion of pesticide 1 in sediment to pesticide 1 in water, ξ_{s1}. It is decreased by natural depletion rate, δ_{w1}, the uptake concentration rate, α_{w1}, conversion of pesticide 1 in water to pesticide 1 in sediment rate, ϕ_{w1}, and the rate at which pesticide 1 in water is converted to pesticide 2 in water, θ_{w1}. The equation is given by

$$\begin{aligned}\frac{dP_{w1}}{dt} =& Q_1 - \delta_{w1}P_{w1} - \alpha_{w1}P_{w1}F + \pi_{w1}V_1 FC_{f1}\\ &- \phi_{w1}P_{w1} + \xi_{s1}P_{s1} - \theta_{w1}P_{w1}.\end{aligned}$$

The pesticide 2 in water is increased by concentration of pesticide 2, Q_2, fraction of the dead aquatic species concentration added back to the aquatic environment, π_{w2}, the conversion of pesticide 2 in sediment to pesticide 2 in water, ξ_{s1}, and the rate at which pesticide 1 in water is converted to pesticide 2 in water, θ_{w1}. It is decreased by natural depletion rate, δ_{w2}, the uptake concentration rate, α_{w2}, conversion of pesticide 2 in water to pesticide 2 in sediment rate, ϕ_{w2}. The equation is given by

$$\begin{aligned}\frac{dP_{w2}}{dt} =& Q_2 - \delta_{w2}P_{w2} - \alpha_{w2}P_{w2}F + \pi_{w2}v_2 FC_{f2}\\ &- \phi_{w2}P_{w2} + \xi_{s2}P_{s2} + \theta_{w1}P_{w1}.\end{aligned}$$

The pesticide 1 in sediments is increased its concentration, Q_3, fraction of the dead aquatic species concentration added back to the aquatic environment, π_{s1}, and the rate at which pesticide 1 in water settled down to form sediment, ξ_{s1}. It is decreased by natural depletion rate, δ_{s1}, the uptake concentration rate, α_{s1}, conversion rate of pesticide 1 in sediments to pesticide 1 in water, ξ_{s1}, and the rate at which pesticide 1 in sediments converted to pesticide 2 in sediments, θ_{s1}. The corresponding equation becomes

$$\begin{aligned}\frac{dP_{s1}}{dt} =& Q_3 - \delta_{s1}P_{s1} - \alpha_{s1}P_{s1}F + \pi_{s1}v_1 FC_{f1} - \xi_{s1}P_{s1}\\ &- \theta_{s1}P_{s1} + \phi_{w1}P_{w1}.\end{aligned}$$

The pesticide 2 in sediments is increased its concentration, Q_4, fraction of the dead aquatic species concentration added back to the aquatic environment, π_{s2}, the rate at which pesticide 1 in sediments is converted to pesticide 2 in sediments, θ_{s1}, and the rate at which pesticide 2 in water settled down to form pesticide 2 sediment, ξ_{s2}. It is decreased by natural depletion rate, δ_{s2}, the uptake concentration rate, α_{s2}, and the rate at which pesticide 2 in sediments changes to pesticide 2 in water, ξ_{s2}. The corresponding equation is given by

$$\begin{aligned}\frac{dP_{s2}}{dt} =& Q_4 - \delta_{s2}P_{s2} - \alpha_{s2}P_{s2}F - \xi_{s2}P_{s2} + \pi_{s2}v_2 FC_{f2}\\ &+ \theta_{s1}P_{s1} + \phi_{w2}P_{w2}.\end{aligned}$$

The uptake concentration 1 is increased by the rate of uptake concentration 1 in water, α_{w1} and the rate of uptake concentration of pesticide 1 in sediment, α_{s1}. It is decreased by the natural depletion rate coefficient of C_{f1}, β_1, and some members of the aquatic population that died as a result of the pesticide 1 toxicity in both water and sediments, v_1. The equation is given by

$$\frac{dC_{f1}}{dt} = -\beta_1 C_{f1} + \alpha_{w1}P_{w1}F + \alpha_{s1}P_{s1}F - v_1 FC_{f1},$$

The uptake concentration 2 is increased by the rate of uptake concentration 2 in water, α_{w2} and the rate of uptake concentration of pesticide 2 in sediment, α_{s2}. It is decreased by the natural depletion rate coefficient of C_{f2}, β_2, and some members of the aquatic population that died as a result of the pesticide 2 toxicity in both water and sediments, v_2. The equation becomes

$$\frac{dC_{f2}}{dt} = -\beta_2 C_{f2} + \alpha_{w2}P_{w2}F + \alpha_{s2}P_{s2}F - v_2 FC_{f2},$$

The combined model equations is given below.

$$\begin{aligned}
\frac{dF}{dt} =& r(C_{f1}, C_{f2})F - \frac{r_0F^2}{K(P_{w1}, P_{w2}, P_{s1}, P_{s2})},\\
\frac{dP_{w1}}{dt} =& Q_1 - \delta_{w1}P_{w1} - \alpha_{w1}P_{w1}F + \pi_{w1}v_1FC_{f1}\\
& - \phi_{w1}P_{w1} + \xi_{s1}P_{s1} - \theta_{w1}P_{w1},\\
\frac{dP_{w2}}{dt} =& Q_2 - \delta_{w2}P_{w2} - \alpha_{w2}P_{w2}F + \pi_{w2}v_2FC_{f2}\\
& - \phi_{w2}P_{w2} + \xi_{s2}P_{s2} + \theta_{w1}P_{w1},\\
\frac{dP_{s1}}{dt} =& Q_3 - \delta_{s1}P_{s1} - \alpha_{s1}P_{s1}F + \pi_{s1}v_1FC_{f1} - \xi_{s1}P_{s1}\\
& - \theta_{s1}P_{s1} + \phi_{w1}P_{w1},\\
\frac{dP_{s2}}{dt} =& Q_4 - \delta_{s2}P_{s2} - \alpha_{s2}P_{s2}F - \xi_{s2}P_{s2} + \pi_{s2}v_2FC_{f2}\\
& + \theta_{s1}P_{s1} + \phi_{w2}P_{w2},\\
\frac{dC_{f1}}{dt} =& -\beta_1C_{f1} + \alpha_{w1}P_{w1}F + \alpha_{s1}P_{s1}F - v_1FC_{f1},\\
\frac{dC_{f2}}{dt} =& -\beta_2C_{f2} + \alpha_{w2}P_{w2}F + \alpha_{s2}P_{s2}F - v_2FC_{f2},
\end{aligned} \tag{2.1}$$

$$\begin{gathered}
F(0) = F_0 \geq 0, P_{w1}(0) = P_{w10} \geq 0, P_{w2}(0) = P_{w20} \geq 0,\\
P_{s1}(0) = P_{s10} \geq 0, P_{s2}(0) = P_{s20} \geq 0, C_{f1}(0) = C_{f10} \geq cF,\\
C_{f2}(0) = C_{f20} \geq cF, c > 0, 0 \leq \pi_{w1}, \pi_{w2}, \pi_{s1}, \pi_{s2} \leq 1,
\end{gathered}$$

The function $r(C_{f1}, C_{f2})$ is the growth rate of the biological populations in the presence of pesticides, $r(0,0) = r_0 > 0$ is the growth rate of biological populations in the absence of pesticide in the population, $c > 0$ is a proportionality constant determining the measure of initial pesticides concentration in the population at $t = 0$. The function $K(P_{w1}, P_{w2}, P_{s1}, P_{s2})$ in model (2.1) denotes the maximum population size of the aquatic species that the environment can support and it decreases when the pesticide concentration increases; $K(P_{w1}, P_{w2}, P_{s1}, P_{s2})$ satisfies the following properties:

$$\begin{aligned}
K(0,0,0,0) = K_0 > 0,\\
\frac{dK}{dt} < 0,
\end{aligned} \tag{2.2}$$

for $P_{w1}, P_{w2}, P_{s1}, P_{s2} > 0$, where K_0 represents the pesticides free carrying capacity of the aquatic biological species.

All the parameters in the model (2.1) are assumed to be positive constants and they are defined in Table 1. The equilibria of the dynamics of the non-linear model (2.1) are considered under two cases as follows:

2.1.1. Case 1: $Q_1 = Q_2 = Q_3 = Q_4 = 0$

This is a case where there is no discharge of pesticides into the aquatic environment. Hence, the model system (2.1) has a non-negative equilibrium, E_1,

$$E_1(F^{++}, P_{w1}^{++}, P_{w2}^{++}, P_{s1}^{++}, P_{s2}^{++}, C_{f1}^{++}, C_{f2}^{++}) = (K_0, 0, 0, 0, 0, 0, 0),$$

The E_1 equilibrium was reported to be a state whereby the population of the aquatic species will thrive and remain perpetually in the absence of pesticides in aquatic environment [19].

Investigating the local asymptotic stability (LAS) of the equilibrium, E_1 was tedious due to the size of the Jacobian of the system evaluated at E_1, it was mathematically intractable to do so. Readers should see [19] for details of the qualitative results.

2.1.2. Case 2: $Q_1 > 0, Q_2 > 0, Q_3 > 0, Q_4 > 0$

Under this case there is a constant influx of pesticides into the aquatic habitat. The model (2.1) has two equilibria, E_2 and E_3,

$$\begin{aligned}
E_2 &= (F^*, P_{w1}^*, P_{w2}^*, P_{s1}^*, P_{s2}^*, C_{f1}^*, C_{f1}^*)\\
&= \left(0, R_1, T, \frac{Q_3 + \phi_{w1}R_1}{\xi_{s1} + \delta_{s1} + \theta_{s1}}, R_3, 0, 0\right),
\end{aligned}$$

where,

$$\begin{aligned}
R_1 =& \frac{\xi_{s1}(Q_1 + Q_3) + Q_1(\delta_{s1} + \theta_{s1})}{(\delta_{w1} + \theta_{w1})(\xi_{s1} + \delta_{s1} + \theta_{s1}) + \phi_{w1}(\delta_{s1} + \theta_{s1})},\\
R_2 =& (\xi_{s1} + \delta_{s1} + \theta_{s1})(\xi_{s2} + \delta_{s2})Q_2 + (\xi_{s1} + \delta_{s1} + \theta_{s1})\xi_{s2}Q_4\\
& + \xi_{s2}\theta_{s1}Q_3,\\
R_3 =& \frac{(\xi_{s1} + \delta_{s1} + \theta_{s1})Q_4 + \theta_{s1}(Q_3 + \phi_{w1}R_1)R_4}{\xi_{s1} + \delta_{s1} + \theta_{s1}}\\
& + \frac{\phi_{w2}(R_2 + PR_1)(\xi_{s1} + \delta_{s1} + \theta_{s1})}{(\xi_{s2} + \delta_{s1})R_2},\\
R_4 =& (\xi_{s1} + \delta_{s1} + \theta_{s1})(\xi_{s2} + \delta_{s2})(\delta_{w2} + \phi_{w2}) - \xi_{s2}\\
& + \phi_{w2}(\xi_{s1} + \delta_{s1} + \theta_{s1}),\\
P =& \xi_{s2}\theta_{s1}\phi_{w1} - (\xi_{s1} + \delta_{s1} + \theta_{s1})(\xi_{s2} + \delta_{s2})\theta_{w1},\\
T =& \frac{R_2 + PR_1}{R_1},
\end{aligned}$$

and E_3 is given as:

$$E_3 = (F^{**}, P_{w1}^{**}, P_{w2}^{**}, P_{s1}^{**}, P_{s2}^{**}, C_{f1}^{**}, C_{f2}^{**}),$$

where,

$$F^{**}=\frac{r(C_{f1}^{**},C_{f2}^{**})K(P_{w1}^{**},P_{w2}^{**},P_{s1}^{**},P_{s2}^{**})}{r_0},$$
$$P_{w1}^{**}=\frac{Q_1+\xi_{s1}P_{s1}^{**}+\pi_{w1}v_1F^{**}C_{f1}^{**}}{\delta_{w1}+\phi_{w1}+\theta_{w1}+\alpha_{w1}F^{**}},$$
$$P_{w2}^{**}=\frac{Q_2+\xi_{s2}P_{s2}^{**}+\pi_{w2}v_2F^{**}C_{f2}^{**}+\theta_{w1}P_{w1}^{**}}{\delta_{w2}+\phi_{w2}+\alpha_{w2}F^{**}},$$
$$P_{s1}^{**}=\frac{Q_3+\phi_{w1}P_{w1}^{**}+\pi_{s1}v_1F^{**}C_{f1}^{**}}{\xi_{s1}+\delta_{s1}+\theta_{s1}+\alpha_{s1}F^{**}},$$
$$P_{s2}^{**}=\frac{Q_4+\phi_{w2}P_{w2}^{**}+\theta_{s1}P_{s1}^{**}+\pi_{s2}v_2F^{**}C_{f2}^{**}}{\xi_{s2}+\delta_{s2}+\alpha_{s2}F^{**}},$$
$$C_{f1}^{**}=\frac{(\alpha_{w1}P_{w1}^{**}+\alpha_{s1}P_{s1}^{**})F^{**}}{\beta_1+v_1F^{**}},$$
$$C_{f2}^{**}=\frac{(\alpha_{w2}P_{w2}^{**}+\alpha_{s2}P_{s2}^{**})F^{**}}{\beta_2+v_2F^{**}},$$

The steady state, E_2 is a steady state where the aquatic population is totally wiped off while the pesticides remain indefinite. The steady state, E_3 is a steady state that shows the continuous existence of the aquatic population in both water and sediments; hence,the aquatic population remains for all time in the presence of pesticides in the environment [19]. The steady state E_2 was reported to be unstable provided the growth rate r_0 remains positive. Hence, orbits cannot approach it with initial conditions in the neigbourhood of E_2. The steady state E_3 is stable under some certain conditions and unstable if otherwise [19]. Readers should check [19] for details of the qualitative results.

The system (2.1) undergoes a Hopf bifurcation phenomenon when the pesticide emission rate passes a critical level (see 1 for more details).

2.2. *Convergence properties of system (2.1)*

In general, system (2.1) is of the form

$$u^{(k+1)}=g(u^{(k)}), \tag{2.3}$$

where g:$\Re^n \to \Re^n$. A solution is a discrete collection of points $u^{(k)}\in\Re^n$, in which the superscript, $k=0,1,2,3,4,...$ are non-negative integer values.

Definition 2.1. *An equilibrium or a fixed point of a discrete dynamical system (2.3) is a vector $u^*\in\Re^n$ such that*

$$g(u^*)=u^*.$$

It is easy to see that every equilibrium of the system gives a constant solution to the discrete dynamical system $u^{(k)}=u^*$ for all k. More so, it is easy to prove that any convergent solution necessarily converges to a fixed point.

Proposition 2.1. *If a solution to a discrete dynamical system converges,* $\lim_{k\to+\infty}u^{(k)}=u^*$*, then the limit u^* is a fixed point.*

Proof: This is a simple consequence of the continuity of g. We get

$$u^*=\lim_{k\to+\infty}u^{(k+1)}=\lim_{k\to+\infty}g(u^{(k)})=g(\lim_{k\to+\infty}u^{(k)})=g(u^*),$$

Th continuity of g is due to the last two equality.

In summary, system (2.1) converges when the system fixed or equilibrium point is

(i) asymptotically stable,
(ii) stable.

However, system (2.1) diverges when the system fixed point is unstable.

3. Optimal control model formulation and analysis

The goal of this work is to propose and analyse an optimal control model that will assists us in minimizing the level of concentration of pesticides entering into the aquatic bio-species population environment. For us to control the rate at which the pesticides P_{w1}, P_{w2}, P_{s1} and P_{s2} are introduced into the aquatic environment and the uptake of their concentration by the fish population, C_{f1} and C_{f2}, we introduce control functions, $u_1(t)$ and $u_2(t)$ respectively into model (2.1). The $u_1(t)$ and $u_2(t)$ are bounded Lebesgue measurable functions, where the control, $u_1(t)$ is a control policy that prevents pesticides from findings their way into the aquatic environment. The effort that maintains such policy is denoted by $1-u_1(t)$. $u_1(t)$ near zero indicates that the policy is "not adequately adhered", it does not prevent introduction of harmful pesticides from entering the aquatic organisms environment, while $u_1(t)$ near 1 indicates an "adequate adherence" policy that prevent harmful pesticides from increasing the concentration of pesticides in aquatic environment. The control, $u_2(t)$ is a case finding control that reduces the concentration of pesticides that already found their ways into the water body.

We also want to maximize the aquatic bio-species. Hence,

$$\begin{aligned} Minimize \quad T(u_1,u_2) = \int_0^U \Big(& C_{f1}(t) + C_{f2}(t) + P_{w1}(t) \\ &+ P_{w2}(t) + P_{s1}(t) + P_{s2}(t) - F(t) \\ &+ \frac{A}{2}u_1^2(t) + \frac{B}{2}u_2^2(t)\Big) dt, \end{aligned} \tag{3.1}$$

subject to:

$$\begin{aligned} \frac{dF}{dt} =& r(C_{f1}, C_{f2})F - \frac{r_0F^2}{K(P_{w1}, P_{w2}, P_{s1}, P_{s2})}, \\ \frac{dP_{w1}}{dt} =& (1 - u_1(t))\, Q_1 - \delta_{w1}P_{w1} - \alpha_{w1}P_{w1}F \\ &+ \pi_{w1}v_1FC_{f1} - \phi_{w1}P_{w1} + \xi_{s1}P_{s1} - \theta_{w1}P_{w1}, \\ \frac{dP_{w2}}{dt} =& (1 - u_1(t))\, Q_2 - \delta_{w2}P_{w2} - \alpha_{w2}P_{w2}F \\ &+ \pi_{w2}v_2FC_{f2} - \phi_{w2}P_{w2} + \xi_{s2}P_{s2} + \theta_{w1}P_{w1}, \\ \frac{dP_{s1}}{dt} =& (1 - u_1(t))\, Q_3 - \delta_{s1}P_{s1} - \alpha_{s1}P_{s1}F + \pi_{s1}v_1FC_{f1} \\ &- \xi_{s1}P_{s1} - \theta_{s1}P_{s1} + \phi_{w1}P_{w1}, \\ \frac{dP_{s2}}{dt} =& (1 - u_1(t))\, Q_4 - \delta_{s2}P_{s2} - \alpha_{s2}P_{s2}F - \xi_{s2}P_{s2} \\ &+ \pi_{s2}v_2FC_{f2} + \theta_{s1}P_{s1} + \phi_{w2}P_{w2}, \\ \frac{dC_{f1}}{dt} =& -\beta_1C_{f1} + (1 - u_2(t))\, \alpha_{w1}P_{w1}F \\ &+ (1 - u_2(t))\, \alpha_{s1}P_{s1}F - v_1FC_{f1}, \\ \frac{dC_{f2}}{dt} =& -\beta_2C_{f2} + (1 - u_2(t))\, \alpha_{w2}P_{w2}F \\ &+ (1 - u_2(t))\, \alpha_{s2}P_{s2}F - v_2FC_{f2}, \\ & F(0) = F_0 \geq 0, P_{w1}(0) = P_{w10} \geq 0, P_{w2}(0) = P_{w20} \geq 0, \\ & P_{s1}(0) = P_{s10} \geq 0, P_{s2}(0) = P_{s20} \geq 0, C_{f1}(0) = C_{f10} \geq cF, \\ & C_{f2}(0) = C_{f20} \geq cF, c > 0, 0 \leq \pi_{w1}, \pi_{w2}, \pi_{s1}, \pi_{s2} \leq 1. \end{aligned} \tag{3.2}$$

All the parameters and the state variables in model (3.2) are assumed to be positive and they are defined in Table 1. In addition, we assumed that the above two cost are nonlinear and follow a quadratic form. The coefficients, A and B are balancing cost factors [32]. Therefore, we seek to find from our objective function, an optimal control pair, u_1^* and u_2^*, such that

$$T(u_1^*, u_2^*) = minT(u_1^*, u_2^*) \in \Omega,$$

where $\Omega = \{(u_1, u_2) \in \{L'(0, U)\}^2 | a_i \leq u_i \leq b_i\}$ and a_i, b_i, $i = 1, 2$, are fixed non-negative constants.

The necessary conditions that the optimal control pair, u_1^* and u_2^* must satisfy is obtained from Pontryagin's Maximum Principle [33].

By constructing Hamiltonian function and adjoint functions, $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7$ as

$$\begin{aligned} L = & C_{f1} + C_{f2} + P_{w1} + P_{w2} + P_{s1} + P_{s2} - F + \tfrac{A}{2}u_1^2(t) + \tfrac{B}{2}u_2^2(t) \\ &+ \lambda_1 \left(r\left(C_{f1}, C_{f2}\right)F - \tfrac{r_0F^2}{K(P_{w1}, P_{w2}, P_{s1}, P_{s2})} \right) \\ &+ \lambda_2 \left((1 - u_1(t))\, Q_1 - \delta_{w1}P_{w1} - \alpha_{w1}P_{w1}F \right. \\ &\left. + \pi_{w1}v_1FC_{f1} - \phi_{w1}P_{w1} + \xi_{s1}P_{s1} - \theta_{w1}P_{w1} \right) \\ &+ \lambda_3 \left((1 - u_1(t))\, Q_2 - \delta_{w2}P_{w2} - \alpha_{w2}P_{w2}F \right. \\ &\left. + \pi_{w2}v_2FC_{f2} - \phi_{w2}P_{w2} + \xi_{s2}P_{s2} + \theta_{w1}P_{w1} \right) \\ &+ \lambda_4 \left((1 - u_1(t))\, Q_3 - \delta_{s1}P_{s1} - \alpha_{s1}P_{s1}F \right. \\ &\left. + \pi_{s1}v_1FC_{f1} - \xi_{s1}P_{s1} - \theta_{s1}P_{s1} + \phi_{w1}P_{w1} \right) \\ &+ \lambda_5 \left((1 - u_1(t))\, Q_4 - \delta_{s2}P_{s2} - \alpha_{s2}P_{s2}F \right. \\ &\left. - \xi_{s2}P_{s2} + \pi_{s2}v_2FC_{f2} + \theta_{s1}P_{s1} + \phi_{w2}P_{w2} \right) \\ &+ \lambda_6 \left(-\beta_1C_{f1} + (1 - u_2(t))\, \alpha_{w1P_{w1}}F \right. \\ &\left. + (1 - u_2(t))\, \alpha_{s1}P_{s1}F - v_1FC_{f1} \right) \\ &+ \lambda_7 \left(-\beta_2C_{f2} + (1 - u_2(t))\, \alpha_{w2}P_{w2}F \right. \\ &\left. + (1 - u_2(t))\, \alpha_{s2}P_{s2}F - v_2FC_{f2} \right), \end{aligned}$$

The partial derivative of the Lagragian function with respect to each variable of the compartment gives the adjoint functions λ_i for $i = 1, 2, ..., 7$ corresponding to the system (4.2) with functions

$$\begin{aligned} r(C_{f1}, C_{f2}) =& r_0 - (r_1C_{f6} + r_2C_{f7}), \\ K(P_{w1}, P_{w2}, P_{s1}, P_{s2}) =& K_0 - \Big(\frac{b_{11}P_{w1}}{1 + b_{12}P_{w2}} + \frac{b_{21}P_{w2}}{1 + b_{22}P_{w2}} \\ &+ \frac{b_{31}P_{s1}}{1 + b_{32}P_{s2}} + \frac{b_{41}P_{s2}}{1 + b_{42}P_{s2}}\Big), \end{aligned}$$

We therefore state the following theorem:

Theorem 3.1. *There exists an optimal control pair,* u_1^* *and* u_2^* *and corresponding solution,* $F^*, P_{w1}^*, P_{w2}^*, P_{s1}^*, P_{s2}^*, C_{f1}^*, C_{f2}^*$*, that minimizes* $T(u_1, u_2)$ *over* Ω*. Therefore, there exists adjoint functions,* $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7$*, such that*

$$\dot{\lambda}_1 = -\frac{\partial L}{\partial F}$$
$$=1-\lambda_1(r_0-(r_1C^*_{f1}+r_2C^*_{f2})$$
$$+\frac{2r_0F^*}{(K_0-(\frac{b_{11}P^*_{w1}}{1+b_{12}P^*_{w1}}+\frac{b_{21}P^*_{w2}}{1+b_{22}P^*_{w2}}+\frac{b_{31}P^*_{s1}}{1+b_{32}P^*_{s1}}+\frac{b_{41}P^*_{s2}}{1+b_{42}P^*_{s2}}))}$$
$$-\lambda_2\left(\pi_{w1}v_1C^*_{f1}-\alpha_{w1}P^*_{w1}\right)-\lambda_3\left(\pi_{w2}v_2C^*_{f2}-\alpha_{w2}P^*_{w2}\right)$$
$$-\lambda_4\left(\pi_{s1}v_1C^*_{f1}-\alpha_{s1}P^*_{s1}\right)-\lambda_5\left(\pi_{s2}v_2C^*_{f2}-\alpha_{s2}P^*_{s2}\right)$$
$$-\lambda_6\left((1-u^*_2(t))\,\alpha_{w1}P^*_{w1}+(1-u^*_2(t))\,\alpha_{s1}P^*_{s1}-v_1C^*_{f1}\right)$$
$$-\lambda_7\left((1-u^*_2(t))\,\alpha_{w2}P^*_{w2}+(1-u^*_2(t))\,\alpha_{s2}P^*_{s2}-v_2C^*_{f2}\right), \tag{3.3a}$$

$$\dot{\lambda}_2 = -\frac{\partial L}{\partial P_{w1}}$$
$$=-1-\lambda_2\left(-\delta_{w1}-\alpha_{w1}F^*-\phi_{w1}-\theta_{w1}\right)$$
$$-\lambda_3\theta_{w1}-\lambda_4\phi_{w1}-\lambda_6\left(1-u^*_2(t)\right)\alpha_{w1}F^*$$
$$-\lambda_1\left(\frac{F^{*2}(\frac{b_{11}b_{12}P^*_{w1}}{(1+b_{12}P^*_{w1})^2}-\frac{b_{11}}{1+b_{12}P^*_{w1}})r_0}{\left(K_0-\left(\frac{b_{11}P^*_{w1}}{1+b_{12}P^*_{w1}}+\frac{b_{21}P^*_{w2}}{1+b_{22}P^*_{w2}}+\frac{b_{31}P^*_{s1}}{1+b_{32}P^*_{s1}}+\frac{b_{41}P^*_{s2}}{1+b_{42}P^*_{s2}}\right)\right)^2}\right), \tag{3.3b}$$

$$\dot{\lambda}_3 = -\frac{\partial L}{\partial P_{w2}}$$
$$=-1-\lambda_3\left(-\delta_{w2}-\alpha_{w2}F^*-\phi_{w2}\right)$$
$$-\lambda_5\phi_{w2}-\lambda_7\left(1-u^*_2(t)\right)\alpha_{w2}F^*$$
$$-\lambda_1\left(\frac{F^{*2}(\frac{b_{21}b_{22}P^*_{w2}}{(1+b_{22}P^*_{w2})^2}-\frac{b_{21}}{1+b_{22}P^*_{w2}})r_0}{\left(K_0-\left(\frac{b_{11}P^*_{w1}}{1+b_{12}P^*_{w1}}+\frac{b_{21}P^*_{w2}}{1+b_{22}P^*_{w2}}+\frac{b_{31}P^*_{s1}}{1+b_{32}P^*_{s1}}+\frac{b_{41}P^*_{s2}}{1+b_{42}P^*_{s2}}\right)\right)^2}\right), \tag{3.3c}$$

$$\dot{\lambda}_4 = -\frac{\partial L}{\partial P_{s1}}$$
$$=-1-\lambda_2\xi_{s1}-\lambda_4\left(-\delta_{s1}-\alpha_{s1}F^*-\xi_{s1}-\theta_{s1}\right)$$
$$-\lambda_5\theta_{s1}-\lambda_6\left(1-u^*_2(t)\right)\alpha_{s1}F^*$$
$$-\lambda_1\left(\frac{F^{*2}(\frac{b_{31}b_{32}P^*_{s1}}{(1+b_{22}P^*_{s1})^2}-\frac{b_{31}}{1+b_{32}P^*_{s1}})r_0}{\left(K_0-\left(\frac{b_{11}P^*_{w1}}{1+b_{12}P^*_{w1}}+\frac{b_{21}P^*_{w2}}{1+b_{22}P^*_{w2}}+\frac{b_{31}P^*_{s1}}{1+b_{32}P^*_{s1}}+\frac{b_{41}P^*_{s2}}{1+b_{42}P^*_{s2}}\right)\right)^2}\right), \tag{3.3d}$$

$$\dot{\lambda}_5 = -\frac{\partial L}{\partial P_{s2}}$$
$$=-1-\lambda_3\left(\xi_{s2}\right)-\lambda_5\left(-\delta_{s2}-\alpha_{s2}F^*-\xi_{s2}\right)$$
$$-\lambda_7\left(1-u^*_2(t)\right)\alpha_{s2}F^*$$
$$-\lambda_1\left(\frac{F^{*2}(\frac{b_{41}b_{42}P^*_{s2}}{(1+b_{42}P^*_{s2})^2}-\frac{b_{41}}{1+b_{42}P^*_{s2}})r_0}{\left(K_0-\left(\frac{b_{11}P^*_{w1}}{1+b_{12}P^*_{w1}}+\frac{b_{21}P^*_{w2}}{1+b_{22}P^*_{w2}}+\frac{b_{31}P^*_{s1}}{1+b_{32}P^*_{s1}}+\frac{b_{41}P^*_{s2}}{1+b_{42}P^*_{s2}}\right)\right)^2}\right), \tag{3.3e}$$

$$\dot{\lambda}_6 = -\frac{\partial L}{\partial C_{f1}}$$
$$=-1-\lambda_1r_1F^*-\lambda_2\pi_{w1}v_1F^*-\lambda_4\pi_{s1}v_1F^* \tag{3.3f}$$
$$-\lambda_6\left(-\beta_1-v_1F^*\right),$$

$$\dot{\lambda}_7 = -\frac{\partial L}{\partial C_{f2}}$$
$$=-1-\lambda_1r_2F^*-\lambda_3\pi_{w2}v_2F^* \tag{3.3g}$$
$$-\lambda_5\pi_{s2}v_2F^*-\lambda_7\left(-\beta_2-v_2F^*\right),$$

with transversality conditions

$$\lambda_i\left(U\right)=0, i=1,2,...,7, \tag{3.4}$$

The following characterizations implies:

$$u^*_1\left(t\right)=min\left(max\left(a_1,\frac{1}{A}\left(\lambda_2Q_1+\lambda_3Q_2+\lambda_4Q_3+\lambda_5Q_4\right)\right),b_1\right),$$
$$u^*_2\left(t\right)=min\left(max\left(a_2,\frac{1}{B}\left[\lambda_6(\alpha_{w1}P^*_{w1}+\alpha_{s1}P^*_{s1})\right.\right.\right.$$
$$\left.\left.\left.+\lambda_7(\alpha_{w2}P^*_{w2}+\alpha_{s2}P^*_{s2})\right]F^*\right),b_2\right), \tag{3.5}$$

Proof When the Pontryagin's Maximum Principle is applied, we have that

$$\begin{aligned}
&\dot{\lambda}_1=-\frac{\partial L}{\partial F},\lambda_1\left(U\right)=0,\dot{\lambda}_2=-\frac{\partial L}{\partial P_{w1}},\lambda_2\left(U\right)=0,\\
&\dot{\lambda}_3=-\frac{\partial L}{\partial P_{w2}},\lambda_3\left(U\right)=0,\dot{\lambda}_4=-\frac{\partial L}{\partial P_{s1}},\lambda_4\left(U\right)=0,\\
&\dot{\lambda}_5=-\frac{\partial L}{\partial P_{s2}},\lambda_5\left(U\right)=0,\dot{\lambda}_6=-\frac{\partial L}{\partial C_{f1}},\lambda_6\left(U\right)=0,\\
&\dot{\lambda}_7=-\frac{\partial L}{\partial C_{f2}},\lambda_7\left(U\right)=0,
\end{aligned} \tag{3.6}$$

We evaluate the optimal control pair, u_1 and u_2 with corresponding states, we then obtain the results in the stated adjoint system $(3.3a-3.3g)$ and (3.4). We consider the optimality conditions,

$$\frac{\partial L}{\partial u_1}=0,\frac{\partial L}{\partial u_2}=0,$$

and solving for u_1^*, u_2^*, subject to the state variables,the characterizations in (3.5) can be obtained, taking into account the bounds on the control.

Based on this, we have

$$\frac{\partial L}{\partial u_1} = Au_1 - \lambda_2 Q_1 - \lambda_3 Q_2 - \lambda_4 Q_3 - \lambda_5 Q_4$$
$$\implies u_1^*(t) = \frac{1}{A}\left(\lambda_2 Q_1 + \lambda_3 Q_2 + \lambda_4 Q_3 + \lambda_5 Q_4\right),$$

on the set $\{t|0 < u_1^*(t) < 1\}$.

To obtain the optimal control, $u_2^*(t)$, we have

$$\begin{aligned}\frac{\partial L}{\partial u_2} =& Bu_2 - [\lambda_6(\alpha_{w1}P_{w1} + \alpha_{s1}P_{s1}) \\ &+\lambda_7(\alpha_{w2}P_{w2} + \alpha_{s2}P_{s2})]\,F, \\ \implies u_2^*(t) =& \frac{1}{B}\left[\lambda_6(\alpha_{w1}P_{w1}^* + \alpha_{s1}P_{s1}^*)\right. \\ &\left.+\lambda_7(\alpha_{w2}P_{w2}^* + \alpha_{s2}P_{s2}^*)\right]F^*,\end{aligned}$$

on the set $\{t|0 < u_2^*(t) < 1\}$.

We observe that the optimality conditions (taking derivatives of the Hamiltonian function, L with respect to control variables) only hold in the interior of the control set. Hence the proof.

4. Numerical simulations

In this section, we carried out the simulation of the optimal control version of system (2.1). The optimality of the two control strategy is obtained by analyzing the optimality system which consist of seven ordinary differential equations from the state and adjoint equations. The optimality system is analyzed with the aid of an iterative method and a Runge-Kutta fourth order scheme. The state system, with an intial values, is analyzed forward in time, with a guess for the controls over the simulated time while the adjoint equations with values at final time t_f is analyzed backward in time using the current iteration values of the state equations. Then the controls are updated by using convex combination of the previous controls and the value from the characterization. The implicit functions in model (2.1)are given as:

$$\begin{aligned}r(C_{f1}, C_{f2}) =& r_0 - (r_1 C_{f6} + r_2 C_{f7}), \\ K(P_{w1}, P_{w2}, P_{s1}, P_{s2}) =& K_0 - (\frac{b_{11}P_{w1}}{1 + b_{12}P_{w2}} + \frac{b_{21}P_{w2}}{1 + b_{22}P_{w2}} \\ &+ \frac{b_{31}P_{s1}}{1 + b_{32}P_{s2}} + \frac{b_{41}P_{s2}}{1 + b_{42}P_{s2}}),\end{aligned}$$

with numerical values:
$b_{11} = 0.02, b_{12} = 1, b_{21} = 0.01, b_{22} = 2, b_{31} = 0.005,$
$b_{32} = 4, b_{41} = 0.0025, b_{42} = 8, r_1 = 0.0009, r_2 = 0.0006.$

In this simulation, we make use of the parameter values in Table 3.3. For the simulation, the weights parameters are chosen to be $A = 0.2$; $B = 0.3$. We chose our initial conditions of the model state variables as: $F(0) = 100; P_{w1}(0) = 1.3675; P_{w2}(0) = 15.6841; P_{s1}(0) = 1.2356; p_{s2}(0) = 12.6578; C_{f1}(0) = 0.0817; C_{f2}(0) = 1.4550$.

We also vary some parameters that will inform important decision about the control strategies we setup in our model system (4.1). The parameters to vary include:
$\theta_{w1}, \theta_{s1}, \phi_{w1}, \phi_{w2}, \xi_{w1}, \xi_{w2}, \pi_{w1}, \pi_{w2}, Q_1, Q_2, Q_3, Q_4, \delta_{w1}, \nu_1$. These parameter symbols have been defined previously in Table 1.

The parameters values of model (2.1) are given in Table 2.

We shall make use of the parameter values in Table 2 for all the simulations. The unit of the emission rate, Q is measure in moles/Litre and K_0 is measure in metre square (m^2).

5. Results

The increase of pesticides emission rate causes the aquatic species population to decrease. The aquatic initial population was varied from 500 to 3000. It was observed that the aquatic population achieved equilibrium at a higher population level than the initial carrying capacity in the presence of dual-level pesticides emission rates. Varying the initial value of the aquatic species, the aquatic population still achieved equilibrium at a higher population level than the initial carrying capacity in the presence of higher dual-level pesticides emission rates.

Interestingly, we saw from the foregoing that for a high value of the θ_{w1}, which allows for more bio-degradation of pesticide 2, control 1 had to be sustained much longer than control 2 unlike for low θ_{w1} value. However, we observed that for high value of θ_{s1}, at the initial stage, control 2 was needed for effective clear up of the aquatic environment, before control 1 now gradually aided clean up of the environment. In summary, control 1 is needed for high

θ_{w1} than for high θ_{s1} whereas control 2 is needed for high θ_{s1} than for high θ_{w1}.

We found that for a high rate of ϕ_{w1} which allows the bio-degradation of pesticide 1 to settled down to sediment, control 1 was required for a long period of time while control 2 later kicked in. Control 2 was needed at a high rate of ϕ_{w2} that implies the settled down of pesticide 2 to sediment than control 1. In summary, control 1 is needed for high value of ϕ_{w1} than for high value of ϕ_{w2} whereas control 2 is needed for high value of ϕ_{w2} than for high value of ϕ_{w1}.

In particular, we saw from the simulations that for a high value of ξ_{w1}, which allows for more bio-degradation of pesticide 2 in sediment to water, control 2 had to be sustained much longer than control 1 unlike for low ξ_{w1} value. Moreover, we also observed that for high value of ξ_{s1}, at the initial stage, control 2 was needed for effective cleanup of the aquatic environment, before control 1 now gradually aided the implementation of the formulated policy. In summary, control 2 is needed for high value of ξ_{w1} than for high value of ξ_{s1} whereas control 1 is needed for high value of ξ_{s1} than for high value of ξ_{w1}.

We observed from the simulations that for a high value of π_{w1}, which allows for more decayed of dead aquatic species adding more concentration to pesticide 1 in water, control 2 had to be sustained much longer than control 1 unlike for low π_{w1} value. In addition, we also observed that for high value of π_{s1}, at the initial stage, control 1 was needed for effective policy implementation, before control 2 now gradually aided cleanup of the environment. In summary, control 2 is needed for high π_{w1} than for high π_{s1} whereas control 1 is needed for high π_{s1} than for high π_{w1}.

Specifically, we found that for a high value of Q_1, which allows for more pesticide 1 into water body, control 1 had to be sustained much longer than control 2 unlike for low Q_1 value. More so, we also observed that for high values of Q_2, at the initial stage, control 2 was needed for effective clear up of the aquatic environment, before control 1 later aided clean up of the aquatic environment. In summary, control 1 is needed for high Q_1 and high Q_2 whereas control 2 is needed for gradual aiding of clean up of the aquatic environment.

We also found out that for a high value of Q_3, which allows for more pesticide 1 into sediment, control 1 had to be sustained much longer than control 2 unlike for low Q_3 value. More so, we also observed that for high values of Q_4, at the initial stage, control 2 was needed for effective clean up of the aquatic environment, before control 1 later aided clean up of the aquatic environment. In summary, control 1 is needed for high Q_3 and high Q_4 whereas control 2 is needed for gradual aiding of clean up of the aquatic environment.

More importantly, we observed that for a high value of δ_{w1}, which allows for more natural depletion rate of pesticide 1 in water body, control 2 was needed much longer than control 1 unlike for low δ_{w1} value. Finally, we observed that for a high value of v_1, which allows for more natural depletion of the uptake concentration 1, control 1 is needed but control 2 is required for much longer.

5.1. Summary of the results

Control 1 is needed more when:

1. There is a bio-degradation of pesticide 1 in water to pesticide 2 in water.
2. Pesticide 1 in water settled down to sediment.
3. The dead aquatic species decayed to add extra concentration to pesticide 2 in water.
4. There are emission rates of pesticides 1 and 2 in water and the pesticides 1 and 2 in sediment.

Control 2 is needed more when:

1. There is a bio-degradation of pesticide 1 in sediment to pesticide 2 in sediment.
2. Pesticide 2 in water settled down to sediment.
3. pesticide 1 in sediment biodegraded to water body.
4. pesticide 2 in sediment biodegraded to water body.
5. The dead aquatic species decayed to add more concentration to pesticide 1 in water.
6. There is a natural depletion rate of pesticide 1 in water.
7. There is a natural depletion rate of the uptake concentration 1.

6. Discussion

An optimal control analysis of a mathematical model for assessing the impact of dual-level toxicity on aquatic biospecies was designed for minimizing dual-level biodegradable pesticides toxicity effects to aquatic populations where two pesticides are present in both water body and sediment and one of the pesticides is capable of biodegrading into the other pesticide but not vice versa is examined in this study.

The goal of the model is to minimize the level of concentration of pesticides that are entering into the aquatic species' environment. Two u_1 and u_2 that represent control policy that will prevents pesticides from finding their ways into the aquatic species population's environment and the case finding that reduces the concentration of pesticides that had already found their ways and those that have entered the environment as a result of weak control policy respectively was considered. A detailed qualitative analysis was carried out on the model equations and the control functions equations were obtained. A step by step method of solving the optimal control model numerically algorithm was designed and implemented with the use of MATLAB Version 7.5 using ode45 function. In particular, a Forward-Backward Sweep method using the fourth-order Runge-Kutta method was adopted. The objective function (3.1) and the system (3.2) converge as the two control functions, $u_1(t)$ and $u_2(t)$ approach $u_1^*(t)$ and $u_2^*(t)$ respectively.

One of the interesting results showed that the increase in populations of the aquatic species lead to a decrease of pesticides concentration. This result is applicable to a situation when the pesticides concentration in the aquatic species environment become difficult to remove, then the manual control measure is to increase the aquatic species populations so that the pesticides concentration can be minimized. In the model assumption, it was assumed that some aquatic species populations that died as a result of up-taking pesticides concentration decayed to add more concentration to the aquatic species populations' environment. But in our simulation, a significant difference was not observed at different values of the decay parameter values.

Finally, we observed the simulation of the varying degradation parameters values of the model equation and we found out that degradation of pesticides concentration causes pesticides concentration to vary significantly between the water body and the sediment region.

7. Conclusions

The optimal control model puts into consideration the control of pesticides concentration in the aquatic species population's environment was thoroughly formulated and analyzed for possible ecological advice on how pesticides or toxicants risk can be minimized in an ecological setting. The optimal control model was qualitatively analyzed and the two control functions equations were obtained. Numerical simulation was carried out on the optimal control model and the control functions and the pesticides concentration variable were varied at different values. In our simulations, it was found that some biodegradable parameters of the model had significant effects on the aquatic population while few of them show less effects on the aquatic population. On the whole, the result showed that pesticides concentration entering the environment and those that are already in the environment can be control if the law enforcement agencies can implement and monitor the control policies proposed in this work.

Conflict of interest

The authors have no conflicts of interest to declare that are relevant to the content of this article.

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Table 1. Description of the model parameters.

Parameter	Interpretation
δ_{w1}	Natural depletion rate coefficient of P_{w1}
δ_{w2}	Natural depletion rate coefficient of P_{w2}
δ_{s1}	Natural depletion rate coefficient of P_{s1}
δ_{s2}	Natural depletion rate coefficient of P_{s2}
α_{w1}	Rate of uptake of P_{w1} by the aquatic species
α_{w2}	Rate of uptake of P_{w2} by the aquatic species
α_{s1}	Rate of uptake of P_{s1} by the aquatic species
α_{s2}	Rate of uptake of P_{s2} by the aquatic species
Q_1	Emission rate of pesticide 1 in water
Q_2	Emission rate of pesticide 2 in water
Q_3	Emission rate of pesticide 1 in sediment
Q_4	Emission rate of pesticide 2 in sediment
v_1	Depletion rate coefficient of C_{f1} due to decay of some members of F
v_2	Depletion rate coefficient of C_{f2} due to decay of some members of F
π_{w1}	Fraction of C_{f1} that re-enter the environment as a result of P_{w1} uptake
π_{w2}	Fraction of C_{f2} that re-enter the environment as a result of P_{w2} uptake
π_{s1}	Fraction of C_{f1} that re-enter the environment as a result of P_{s1} uptake
π_{s2}	Fraction of C_{f2} that re-enter the environment as a result of P_{s2} uptake
β_1	Natural depletion rate coefficient of C_{f1}
β_2	Natural depletion rate coefficient of C_{f2}
θ_{w1}	Degradation coefficient of P_{w1} to P_{w2}
θ_{s1}	Degradation coefficient of P_{s1} to P_{s2}
ξ_{s1}	Rate at which P_{s1} enters the water body, from the sediments due to disturbance on the water body
ξ_{s2}	Rate at which P_{s2} enters the water body, from the sediments due to disturbance on the water body
ϕ_{w1}	Rate at which P_{w1} settle down to sediments
ϕ_{w2}	Rate at which P_{w2} settle down to sediments

Table 2. Description of the model parameters' values

Parameter	Value	Source
Q_1, Q_2, Q_3, Q_4	$0.05, 0.02, 0.05, 0.08$	Assumed
δ_{w1}	0.08	[6]
δ_{w2}	0.06	[26]
δ_{s1}	0.08	[19]
δ_{s2}	0.06	[19]
α_{w1}	0.0002	[26]
α_{w2}	0.0002	[6]
α_{s1}	0.0001	[19]
α_{s2}	0.0001	[19]
V_1	0.0002	[6]
V_2	0.001	[19]
π_{w1}	0.0003	[26]
π_{w2}	0.0002	[19]
π_{s1}	0.0001	[19]
π_{s2}	0.001	[19]
β_1	0.08	[26]
β_2	0.07	[19]
θ_{w1}	0.023	[19]
θ_{s1}	0.020	[19]
ξ_{s1}	0.002	[19]
ξ_{s2}	0.006	[19]
ϕ_{w1}	0.0002	[19]
ϕ_{w2}	0.0001	[19]
r_0	0.4, 0.9	[19]
K_0	1000	[19]

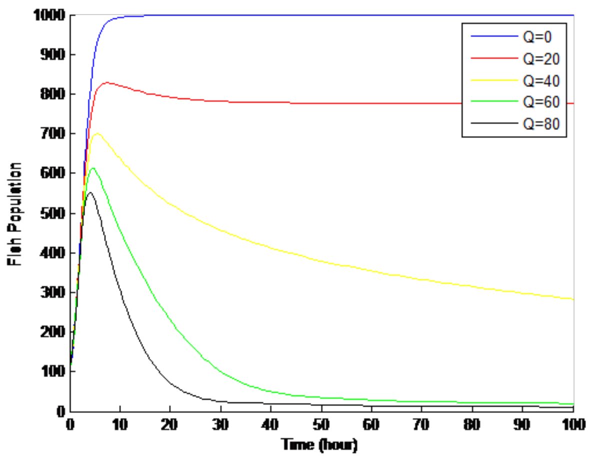

Figure 1. Growth of aquatic population over time.

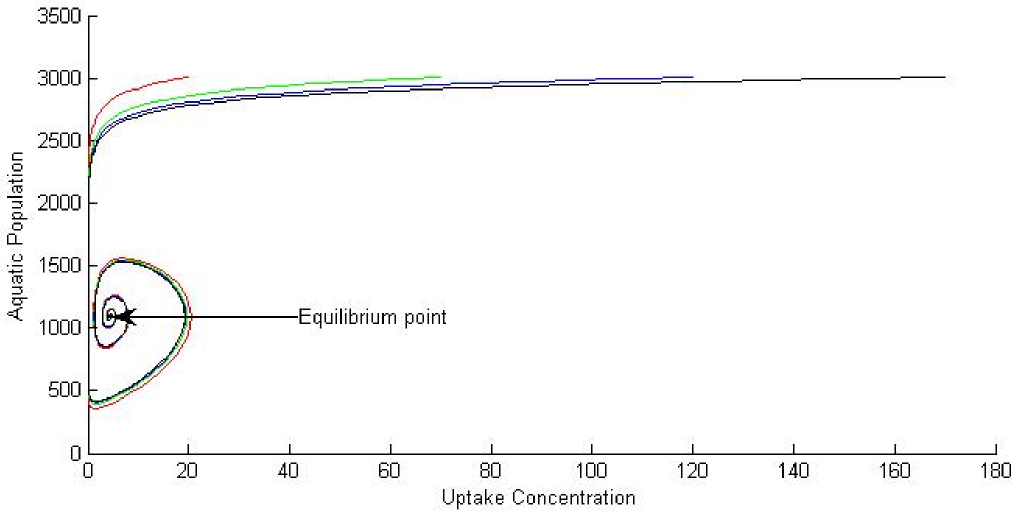

Figure 2. Spiral sink phase plots of the aquatic population with concentrations uptake at different initial values of aquatic population.

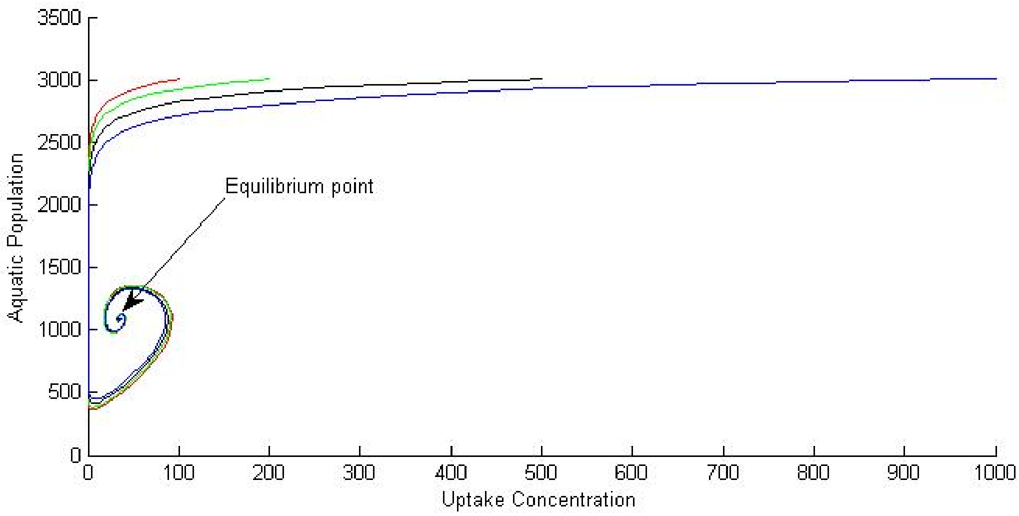

Figure 3. Spiral sink phase plots of the aquatic population with concentrations uptake at different initial values of aquatic population.

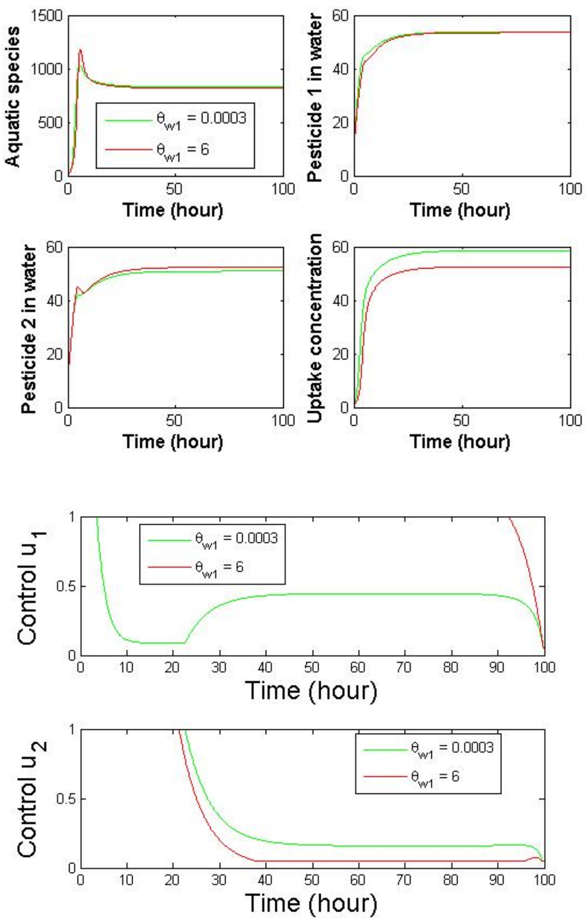

Figure 4. Varying values of degradation coefficient parameter of P_{w1} to P_{w2}.

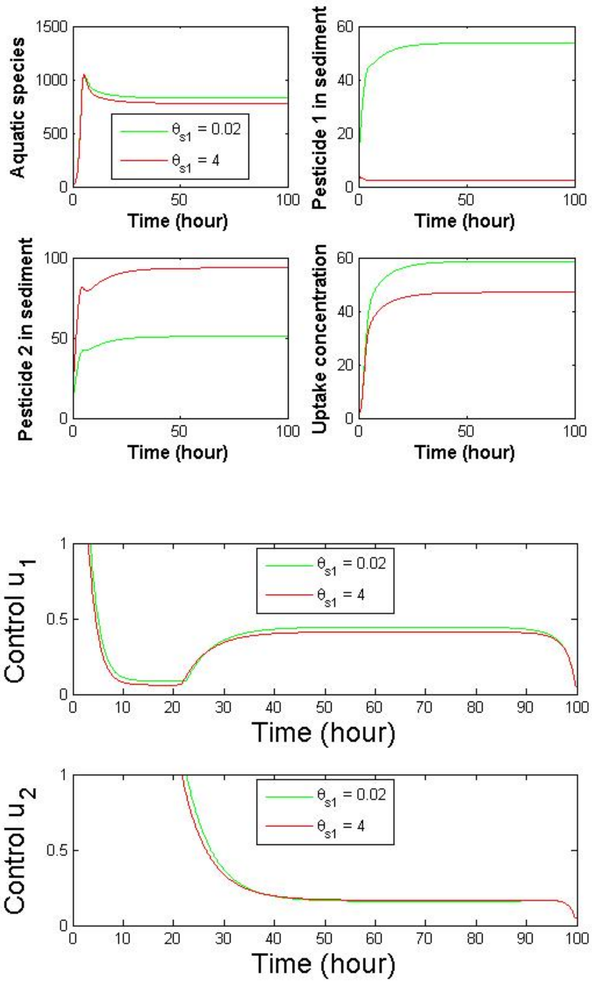

Figure 5. Varying values of degradation coefficient parameter of P_{s1} to P_{s2}.

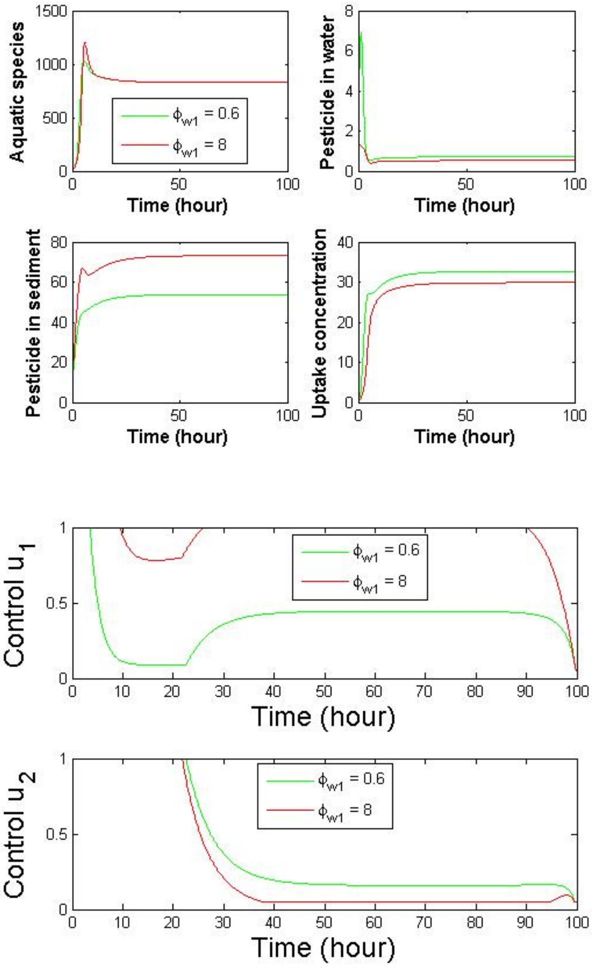

Figure 6. Varying value of biodegradable parameter rate at which P_{w1} settle down to sediments.

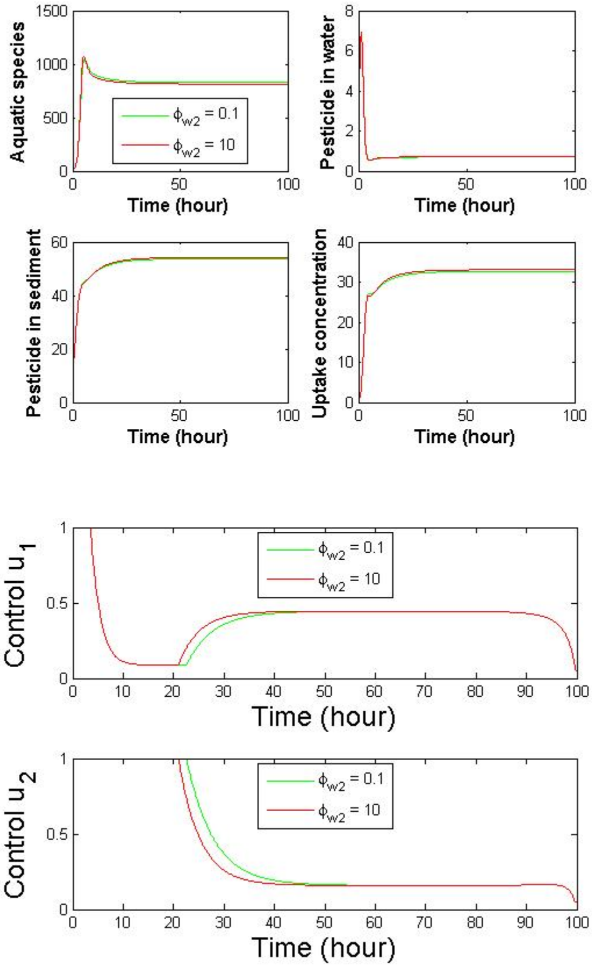

Figure 7. Varying value of biodegradable parameter rate at which P_{w2} settle down to sediments.

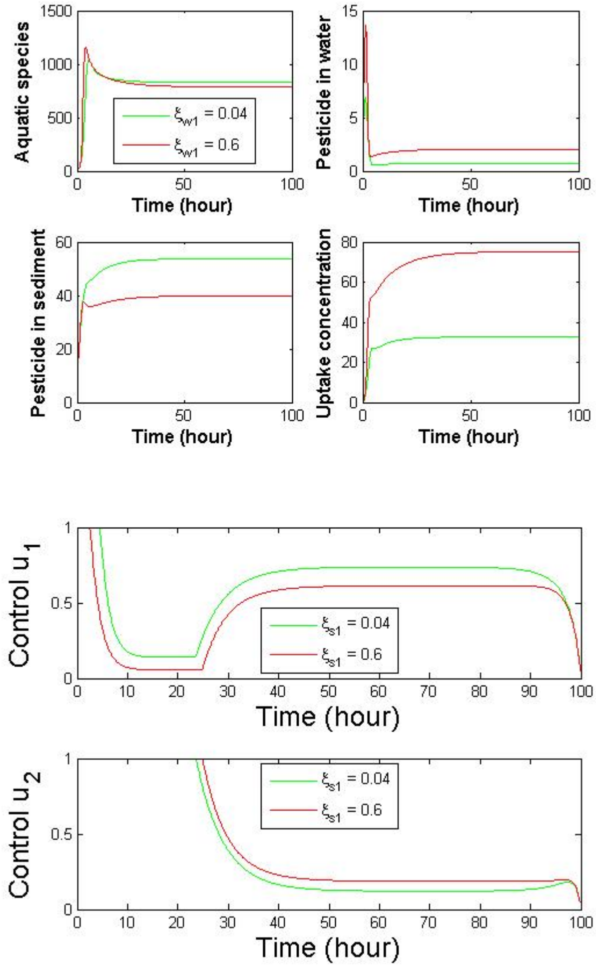

Figure 8. Varying value of biodegradable parameter rate at which P_{s1} enters the water body from the sediments.

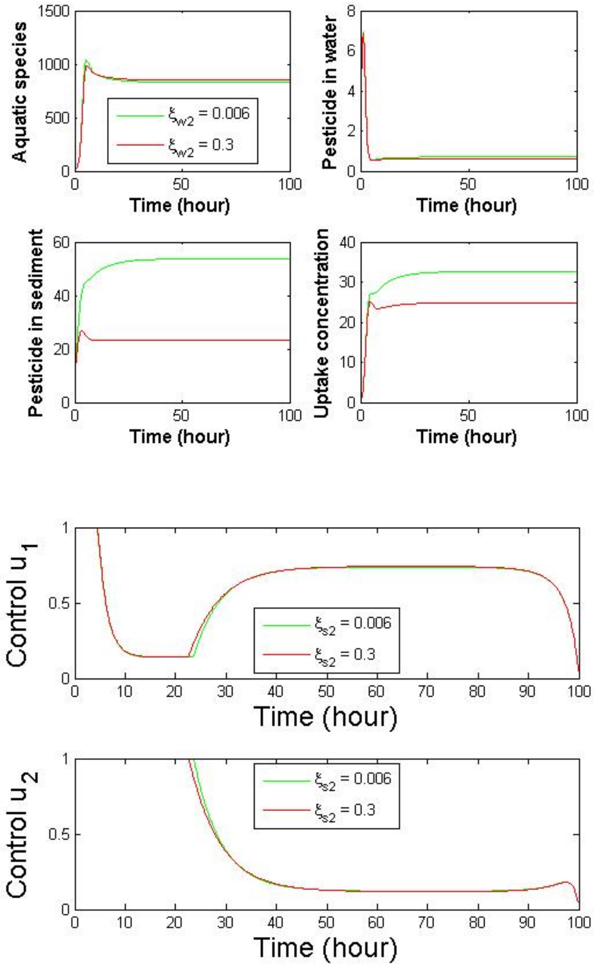

Figure 9. Varying value of biodegradable parameter rate at which P_{s2} enters the water body from the sediments.

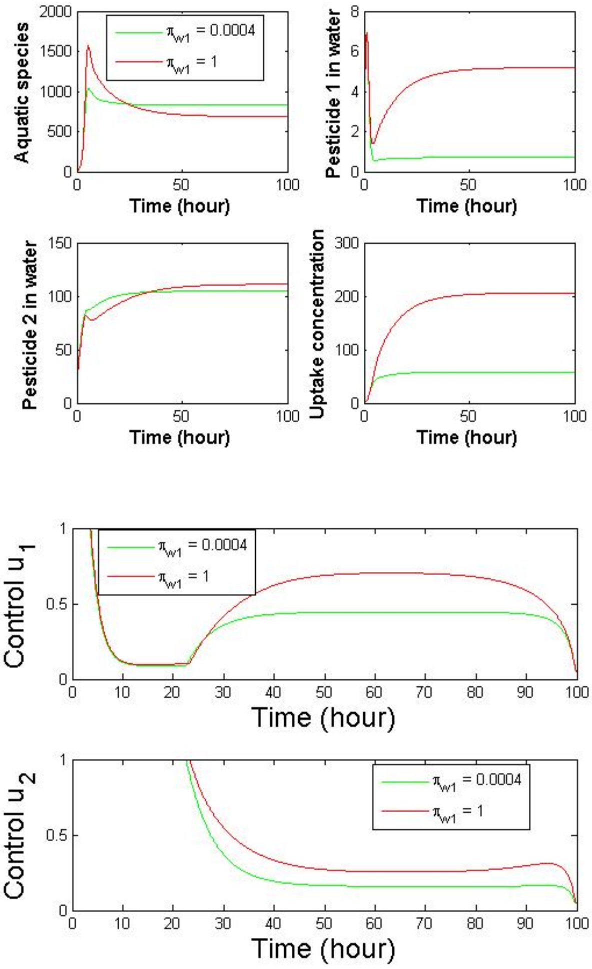

Figure 10. Plot of varying value of parameter rate π_{w1}.

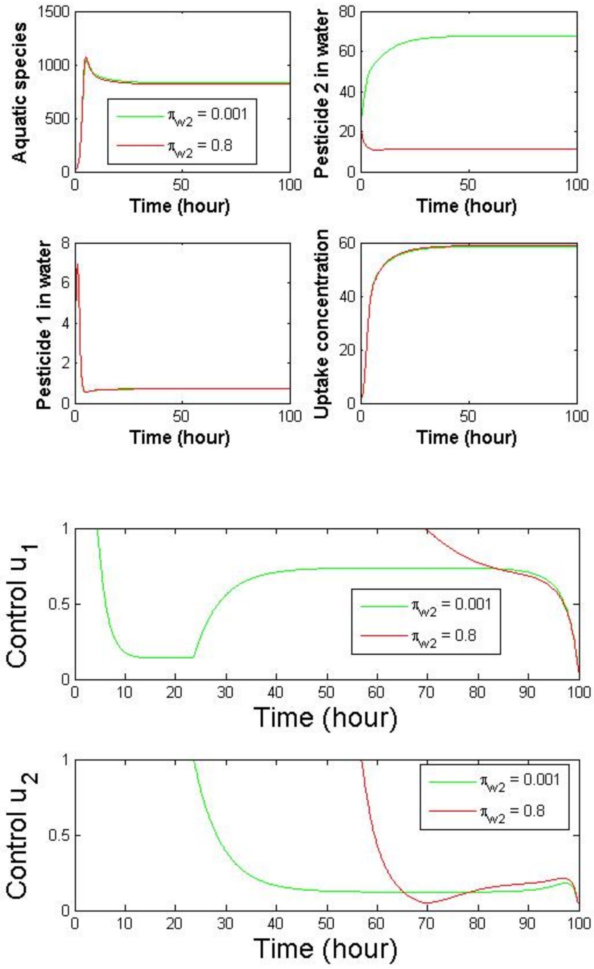

Figure 11. Plot of varying value of parameter rate π_{w2}.

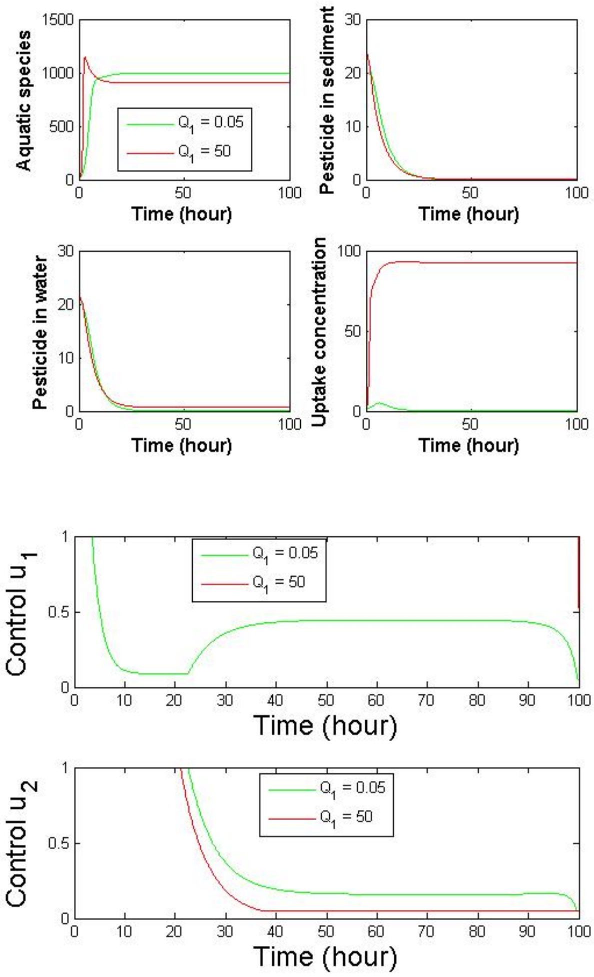

Figure 12. Plot of varying parameter value of emission rate Q_1 of pesticide 1 in water.

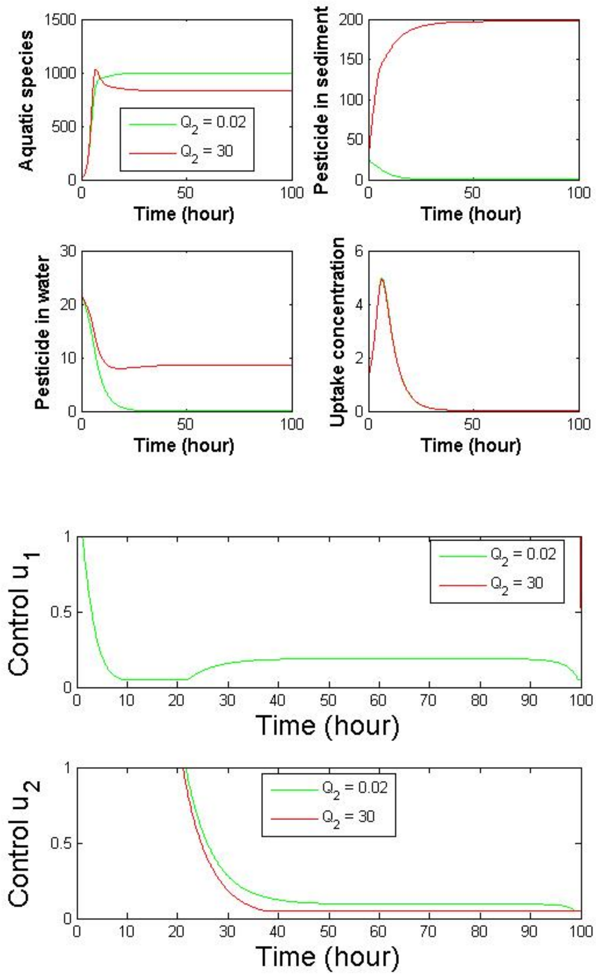

Figure 13. Plot of varying parameter value of emission rate Q_2 of pesticide 2 in water.

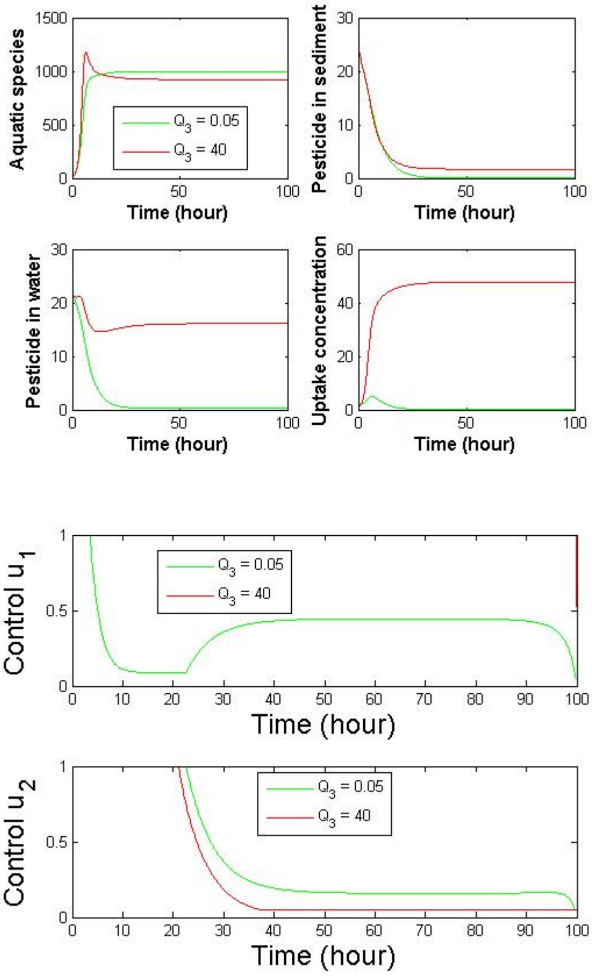

Figure 14. Plot of varying parameter value of emission rate Q_3 of pesticide 1 in sediment.

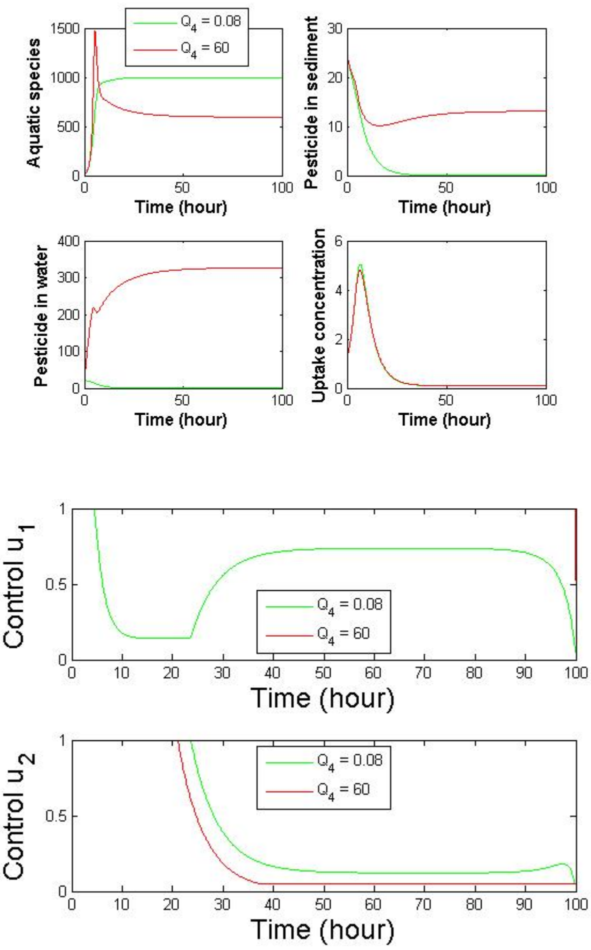

Figure 15. Plot of varying parameter value of emission rate Q_4 of pesticide 2 in sediment.

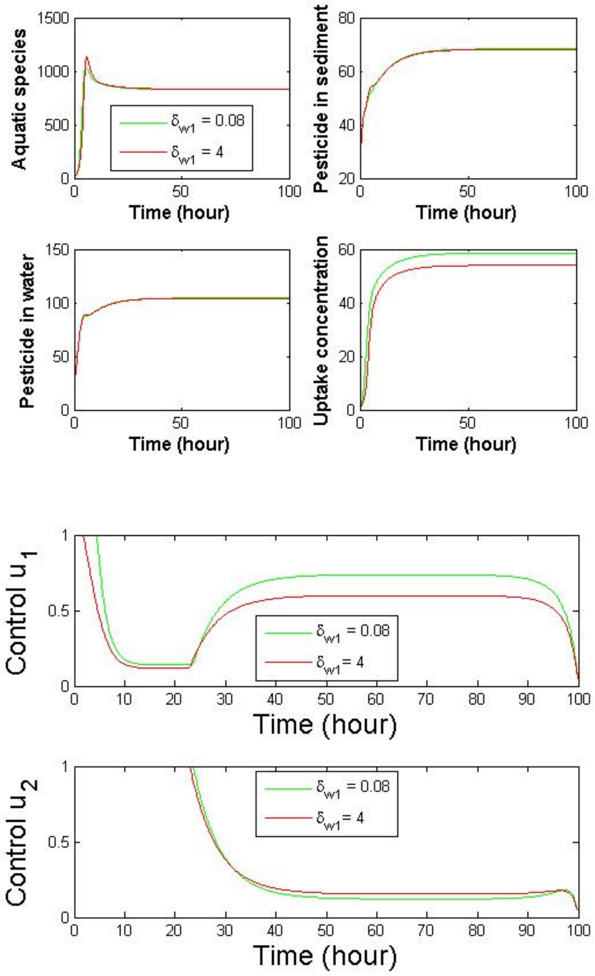

Figure 16. Plot of varying parameter value of the natural depletion rate of pesticide 1 in water, δ_{w1}.

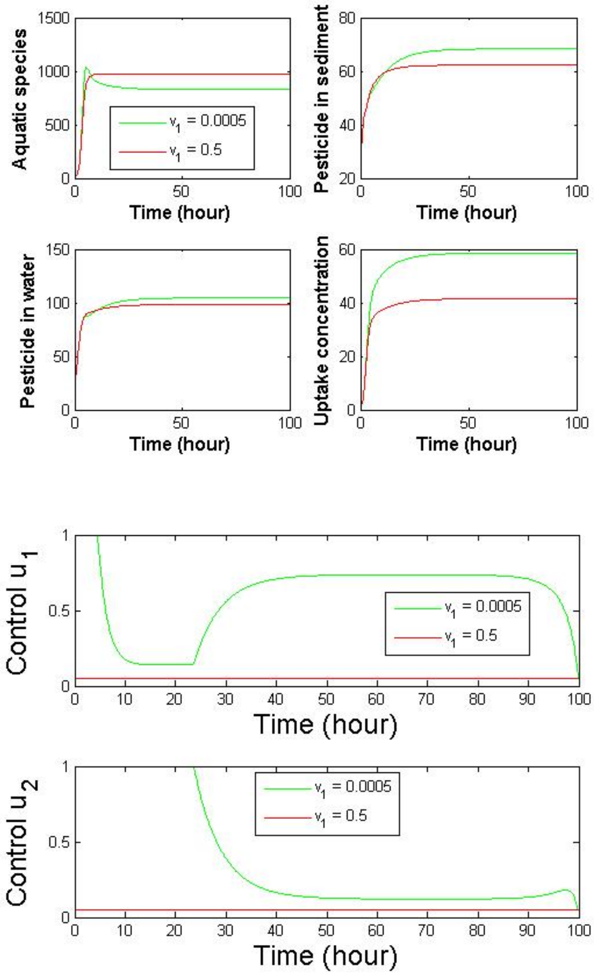

Figure 17. Plot of varying parameter value of the natural depletion rate (v_1) coefficient C_{f1} due to decay of aquatic species.