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Research article

Perfect hypercomplex algebras: Semi-tensor product approach

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Abstract: The set of associative and commutative hypercomplex numbers, called the perfect hypercomplex algebras (PHAs) is investigated. Necessary and sufficient conditions for an algebra to be a PHA via semi-tensor product (STP) of matrices are reviewed. The zero sets are defined for non-invertible hypercomplex numbers in a given PHA, and characteristic functions are proposed for calculating zero sets. Then PHA of various dimensions are considered. First, classification of 2-dimensional PHAs are investigated. Second, all the 3-dimensional PHAs are obtained and the corresponding zero sets are calculated. Finally, 4- and higher dimensional PHAs are also considered.

Keywords: perfect hypercomplex algebra (PHA); zero-set; semi-tensor product (STP) of matrices

1. Introduction

Hypercomplex numbers (HNs) are generalization of complex numbers. A class of HNs endowed with addition and product forms a special vector space over $\mathbb{R}$, called a hypercomplex algebra (HA). HAs have various applications including signal and image processing [17], dealing with differential operators [1, 2], designing neural networks [3], etc.

It was proved by Weierstrass that the only finite field extension of real numbers ($\mathbb{R}$) is complex numbers ($\mathbb{C}$) [14]. HA can be considered as an extension of real numbers ($\mathbb{R}$) to finite dimensional algebras. We call such extension finite algebra extension of real numbers.

In this paper we consider only a particular class of finite-dimensional algebras over $\mathbb{R}$, which are commutative, associative and unital. Throughout, the following is assumed:

Assumption 1: $\boldsymbol{H}_n$ is the set of n-dimensional PHAs, that is, the set of n-dimensional commutative associative unital algebras over $\mathbb{R}$.

In addition to complex numbers, hyperbolic numbers, dual numbers, and Tessarine quaternion are also PHAs.

STP of matrices is a generalization of conventional matrix product. It has been becoming a necessary tool in the study of finite value systems, such as, Boolean networks [13, 19, 12, 15, 21], finite games[4, 20], and fuzzy systems

[9, 16]. In additionally, it is also a powerful tool to deal with multi-linear mappings. Constrained least square solutions to Sylvester equations have been obtained via STP method in [8]. In [5], STP was used to investigate finite algebra extensions of $\mathbb{R}$. (In fact, the extensions of any $\mathbb{F}$ with $Char(F) = 0$ have been discussed there.) In [10], STP was used to investigate general Boolean-type algebras. A key issue in these approaches is to define a matrix, called product structure matrix of a certain algebra. Using STP, associativity, commutativity, and some other properties of a finite algebra extension can be verified via its product matrix.

In this paper, this STP approach is used to investigate PHAs. First the formulas for verifying whether a HA is associative and commutative are reviewed. Then the zero set is defined as the set of non-invertible numbers. A characteristic function is proposed to calculate (or describe) the zero set. Then the PHA of dimensions 2, 3, or 4 are constructed separately, and higher dimensional cases are also discussed. Their zero-sets, which are of measure zero, are calculated. Analytic functions and some other properties of PHAs are then discussed.

Before ending this section, we give a list of notations:

1. $\ltimes$: STP of matrices.
2. Col(A) (Row(A)): the set of columns (rows) of A; Col$_i$(A) (Row$_i$(A)): the i-th column (row) of A.
3. δ_k^i: the i-th column of identity matrix I_k.

2. Preliminaries

2.1. Semi-tensor product of matrices

Since STP is a fundamental tool in our construction, this section will give a brief survey for STP. We refer to [7] for more details.

Definition 2.1. *Let $A \in \mathbb{R}_{m\times n}$ and $B \in \mathbb{R}_{p\times q}$, $t = \mathrm{lcm}(n, p)$ be the least common multiple of n and p. Then the STP of A and B, denoted by $A \ltimes B$, is defined as*

$$A \ltimes B := (A \otimes I_{t/n})\left(B \otimes I_{t/p}\right), \tag{2.1}$$

where $\otimes$ is Kronecker product.

It is easy to see that STP is a generalization of conventional matrix product. That is, when $n = p$, it degenerates to the conventional matrix product, i.e., $A \ltimes B = AB$.

One of the most important advantages of STP is that it keeps most properties of conventional matrix product available, including association, distribution, etc. In the following we introduce some additional properties of STP, which will be used in the sequel.

Define a swap matrix $W_{[m,n]} \in \mathcal{L}_{mn\times mn}$ as follows:

$$W_{[m,n]} := \left[I_n \otimes \delta_m^1, I_n \otimes \delta_m^2, \cdots, I_n \otimes \delta_m^m\right]. \tag{2.2}$$

Proposition 2.2. *Let $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$ be two column vectors. Then*

$$W_{[m,n]} x \ltimes y = y \ltimes x. \tag{2.3}$$

The following proposition shows how to "swap" a vector with a matrix:

Proposition 2.3. *Let $x \in \mathbb{R}^t$ be a column vector, and A be an arbitrary matrix. Then*

$$x \ltimes A = (I_t \otimes A) \ltimes x. \tag{2.4}$$

Throughout this paper the default matrix product is assumed to be STP, and the symbol $\ltimes$ is omitted if there is no possible confusion.

2.2. Matrix expression of an algebra

We are interested in algebras over $\mathbb{R}$.

Definition 2.4. *[11]*

(i) An algebra over $\mathbb{R}$ is a pair, denoted by $\mathcal{A} = (V, *)$, where V is a real vector space, $* : V \times V \to V$, satisfying

$$(ax + by) * z = ax * z + by * z,$$
$$x * (ay + bz) = ax * y + bx * z, \quad x, y, z \in V, \ a, b \in \mathbb{R}. \tag{2.5}$$

(ii) An algebra $\mathcal{A} = (V, *)$ is said to be commutative, if

$$x * y = y * x, \quad x, y \in V. \tag{2.6}$$

(iii) An algebra $\mathcal{A} = (V, *)$ is said to be associative, if

$$(x * y) * z = x * (y * z), \quad x, y, z \in V. \tag{2.7}$$

Definition 2.5. *Let $\mathcal{A} = (V, *)$ be an algebra over $\mathbb{R}$, where V is a k-dimensional vector space with $e = \{\mathbf{i}_1, \mathbf{i}_2, \cdots, \mathbf{i}_k\}$ as a set of basis. Denote*

$$\mathbf{i}_i * \mathbf{i}_j = \sum_{s=1}^{k} c_{i,j}^{s} \mathbf{i}_s, \quad i, j = 1, 2, \cdots, k. \tag{2.8}$$

Then the product structure matrix (PSM) of $\mathcal{A}$ is defined as

$$P_{\mathcal{A}} := \begin{bmatrix} c_{1,1}^{1} & c_{1,2}^{1} & \cdots & c_{1,k}^{1} & \cdots & c_{k,k}^{1} \\ c_{1,1}^{2} & c_{1,2}^{2} & \cdots & c_{1,k}^{2} & \cdots & c_{k,k}^{2} \\ \vdots & & \ddots & & \ddots & \vdots \\ c_{1,1}^{k} & c_{1,2}^{k} & \cdots & c_{1,k}^{k} & \cdots & c_{k,k}^{k} \end{bmatrix}. \tag{2.9}$$

Write $x = \sum_{j=1}^{k} x_i \mathbf{i}_j$ in a column vector form as $x = (x_1, x_2, \cdots, x_k)^T$. Similarly, $y = (y_1, y_2, \cdots, y_k)^T$. Then we have the following result.

Theorem 2.6. *In vector form the product of two hypercomplex numbers $x, y \in \mathcal{A}$ is computable via following formula*

$$x * y = P_{\mathcal{A}} x y. \tag{2.10}$$

Using formula (2.10) and the properties of STP yields the following results, which are fundamental for our further investigation.

Theorem 2.7. *[5]*

(i) $\mathcal{A}$ is commutative, if and only if,

$$P_{\mathcal{A}} \left[I_k - W_{[k,k]} \right] = 0. \tag{2.11}$$

(ii) $\mathcal{A}$ is associative, if and only if,

$$P_{\mathcal{A}}^2 = P_{\mathcal{A}} \left(I_k \otimes P_{\mathcal{A}} \right). \tag{2.12}$$

3. Hypercomplex numbers

3.1. Perfect hypercomplex algebra on $\mathbb{R}$

Definition 3.1. *[18] A number p is called a hypercomplex number, if it can be expressed in the form*

$$p = p_0 + p_1 \mathbf{i}_1 + \cdots + p_n \mathbf{i}_n, \tag{3.1}$$

where $p_i \in \mathbb{R}$, $i = 0, 1, \cdots, n$, $\mathbf{i}_i$, $i = 1, 2, \cdots, n$ are called hyperimaginary units.

Remark 3.2. *A hypercomplex number may belong to different algebras, depending on their product structure matrices. A hypercomplex algebra, denoted by $\mathcal{A}$, is an algebra over $\mathbb{R}$ with basis $e = \{\mathbf{i}_0 := \mathbf{1}, \mathbf{i}_1, \cdots, \mathbf{i}_n\}$, where $\mathbf{1}$ is the unit of multiplication.*

Proposition 3.3. *Assume*

$$\mathcal{A} = \{p_0 + p_1 \mathbf{i}_1 + \cdots + p_n \mathbf{i}_n \mid p_0, p_1, \cdots, p_n \in \mathbb{R}\}.$$

Then its product matrix

$$P_{\mathcal{A}} := [M_0, M_1, \cdots, M_n],$$

where $M_i \in \mathbb{R}_{(n+1)\times(n+1)}$, $i = 0, 1, \cdots, n$, satisfy the following conditions:

(i)

$$M_0 = I_{n+1} \tag{3.2}$$

is an identity matrix.

(ii)

$$\mathrm{Col}_1(M_j) = \delta_{n+1}^{j+1}, \quad j = 1, 2, \cdots, n. \tag{3.3}$$

An n-dimensional hypercomplex algebra $\mathcal{A}$ is called a perfect hypercomplex algebra (PHA), denoted by $\mathcal{A} \in \boldsymbol{H}_n$, if it is commutative and associative.

Example 3.4. *Consider $\mathbb{C}$. It is easy to solve its product structure matrix as*

$$P_{\mathbb{C}} = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{bmatrix}. \tag{3.4}$$

A straightforward computation verifies (2.11) and (2.12), hence it is a PHA.

3.2. Invertibility of elements in PHA

Now for a PHA, say, $\mathcal{A} = (V, *)$, if every $0 \neq x \in V$ has its inverse x^{-1} such that $x * x^{-1} = x^{-1} * x = 1$, then $\mathcal{A}$ is a field. Unfortunately, according to Weierstrass, if $\mathcal{A} \neq \mathbb{C}$, it is not a field. Naturally, we are interested in the conditions for an element $x \in \mathcal{A}$ to be invertible.

To answer this we need some new concepts, which are firstly discussed in [5].

Definition 3.5. *(i) Let $A_1, A_2, \cdots, A_r$ be a set of square real matrices. $A_1, A_2, \cdots, A_r$ are said to be jointly non-singular, if their non-trivial linear combination is non-singular. That is, if*

$$\det\left(\sum_{i=1}^{r} c_i A_i\right) = 0,$$

then $c_1 = c_2 = \cdots = c_r = 0$.

(ii) Let $A \in \mathbb{R}_{k \times k^2}$. A is said to be jointly non-singular, if $A = [A_1, A_2, \cdots, A_k]$, where $A_i \in \mathbb{R}_{k \times k}$, $i = 1, 2, \cdots, k$ are jointly non-singular.

Obviously, the following condition is equivalent to the definition of jointly non-singularity of $A \in \mathbb{R}_{k \times k^2}$: $\forall x = (x_1, \cdots, x_k)^T \neq 0$, the matrix $Ax \in \mathbb{R}_{k \times k}$ is non-singular, i.e., the homogeneous polynomial

$$\xi(x_1, \cdots, x_k) = \det(Ax) \neq 0. \tag{3.5}$$

We call $\xi(x_1, \cdots, x_k)$ the characteristic function of A.

Example 3.6. *Consider $\mathbb{C} = \mathbb{R}(i)$. Calculating right hand side of (3.5) for $P_{\mathbb{C}}$, we have*

$$\xi(x_1, x_2) = x_1^2 + x_2^2.$$

Hence, $\xi(x_1, x_2) = 0$, if and only if, $x_1 = x_2 = 0$. It follows that $P_{\mathbb{C}}$ is jointly non-singular.

Summarizing the above arguments, we have the following result.

Proposition 3.7. *Let $\mathcal{A}$ be a finite dimensional algebra over $\mathbb{R}$. Then $\mathcal{A}$ is a field, if and only if,*

(i) $\mathcal{A}$ is commutative, that is, (2.11) holds;
(ii) $\mathcal{A}$ is associative, that is, (2.12) holds;
(iii) Each $0 \neq x \in \mathcal{A}$ is invertible, that is, $P_{\mathcal{A}}$ is jointly invertible.

When $\mathcal{A}$ is not a field, there exist nonzero elements that are not invertible.

Definition 3.8. *Let $\mathcal{A} \in \boldsymbol{H}$. Its zero set is defined by*

$$\mathcal{Z}_{\mathcal{A}} := \{z \in \mathcal{A} \mid \det(P_{\mathcal{A}} z) = 0\}. \tag{3.6}$$

It is clear that

(i) if $\mathcal{A} = \mathbb{C}$, then $\mathcal{Z}_{\mathcal{A}} = \{0\}$;
(ii) if $\mathcal{A} \neq \mathbb{C}$, then $\mathcal{Z}_{\mathcal{A}} \backslash \{0\} \neq \emptyset$.

3.3. Isomorphisms of hypercomplex algebras

With the STP method, in this subsection we give an interpretation of PHA isomorphisms.

Definition 3.9. *Let $\mathcal{A}$ and $\overline{\mathcal{A}}$ be two $n+1$ dimensional hypercomplex algebras. $\mathcal{A}$ and $\overline{\mathcal{A}}$ are called isomorphic, if there exists a bijective mapping $\Psi : \mathcal{A} \to \overline{\mathcal{A}}$, satisfying*

(i)

$$\Psi(1) = 1; \tag{3.7}$$

(ii)

$$\Psi(ax + by) = a\Psi(x) + b\Psi(y), \quad x, y \in \mathcal{A}, a, b \in \mathbb{R}; \tag{3.8}$$

(iii)

$$\Psi(x * y) = \Psi(x) * \Psi(y), \quad x, y \in \mathcal{A}. \tag{3.9}$$

Ψ is called an isomorphism.

A straightforward verification shows the following result immediately.

Proposition 3.10. *Assume $\mathcal{A}$, $\overline{\mathcal{A}} \in \boldsymbol{H}_{n+1}$, with PSMs $P_{\mathcal{A}}$ and $P_{\overline{\mathcal{A}}}$ respectively. $\mathcal{A}$ and $\overline{\mathcal{A}}$ are isomorphic, if and only if, there exists a non-singular matrix T such that*

$$P_{\overline{\mathcal{A}}} = T^{-1} P_{\mathcal{A}} (T \otimes T). \tag{3.10}$$

Proof. (Necessity) Let T be constructed such that

$$\bar{x} = T^{-1} x.$$

Then we have

$$P_{\mathcal{A}} xy = T P_{\overline{\mathcal{A}}} \bar{x}\bar{y}, \quad x, y \in \mathcal{A}. \tag{3.11}$$

The right hand side (RHS) of (3.11) becomes

$$\begin{aligned} RHS_{(3.11)} &= T P_{\overline{\mathcal{A}}} T^{-1} x T^{-1} y \\ &= T P_{\overline{\mathcal{A}}} T^{-1} \left(I_{n+1} \otimes T^{-1} \right) xy. \end{aligned}$$

Since x, y are arbitrary, we have

$$P_{\mathcal{A}} = T P_{\overline{\mathcal{A}}} T^{-1} \left(I_{n+1} \otimes T^{-1} \right).$$

Hence,

$$\begin{aligned} P_{\overline{\mathcal{A}}} &= T^{-1} P_{\mathcal{A}} (I_{n+1} \otimes T) T \\ &= T^{-1} P_{\mathcal{A}} (T \otimes T). \end{aligned}$$

(Sufficiency) If (3.11) holds, it is easy to verify that

$$\bar{x} = T^{-1} x$$

is an isomorphism. □

4. Lower dimensional PHAs

Via PSMs, this section considers some examples of PHAs, and tries to classify some lower dimensional algebras from the viewpoint of isomorphism.

4.1. Structure of $\mathcal{A} \in \boldsymbol{H}_2$

Consider $\mathcal{A} \in \boldsymbol{H}_2$. According to Proposition 3.3, its PSM is

$$P_{\mathcal{A}} = \begin{bmatrix} 1 & 0 & 0 & \alpha \\ 0 & 1 & 1 & \beta \end{bmatrix}. \tag{4.1}$$

Without loss of generality, we may assume any algebra in $\boldsymbol{H}_2$ has its unit as the first basis vector. Therefore isomorphisms from $\mathcal{A}$ to any $\bar{\mathcal{A}} \in \boldsymbol{H}_2$ can be expressed in the following form

$$T = \begin{bmatrix} 1 & s \\ 0 & t \end{bmatrix}, \quad t \neq 0.$$

Using formula (3.10), we have

$$\begin{aligned} P_{\overline{\mathcal{A}}} &= T^{-1} P_{\mathcal{A}} (T \otimes T) \\ &= \begin{bmatrix} 1 & 0 & 0 & \alpha t^2 - s(s + t\beta) \\ 0 & 1 & 1 & 2s + t\beta \end{bmatrix}. \end{aligned} \tag{4.2}$$

If $\beta \neq 0$ in $P_{\mathcal{A}}$, we can always choose an isomorphism such that

$$s = -\frac{1}{2} t\beta$$

to make the entry become zero. Therefore it makes no difference to assume that $\beta = s = 0$, and

$$P_{\overline{\mathcal{A}}} = \begin{bmatrix} 1 & 0 & 0 & \alpha t^2 \\ 0 & 1 & 1 & 0 \end{bmatrix}. \tag{4.3}$$

Since $t \neq 0$, αt^2 has the sign with α. Therefore we may classify $\boldsymbol{H}_2$ by the sign of α.

- If $\alpha = 0$, we have

$$P_{\overline{\mathcal{A}}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}. \tag{4.4}$$

- If $\alpha > 0$, choosing $t = \frac{1}{\sqrt{\alpha}}$, then we have

$$P_{\overline{\mathcal{A}}} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}. \tag{4.5}$$

- If $\alpha < 0$, choosing $t = \frac{1}{\sqrt{|\alpha|}}$ yields

$$P_{\overline{\mathcal{A}}} = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{bmatrix}. \tag{4.6}$$

We conclude that up to isomorphism there are three $\mathcal{A} \in \boldsymbol{H}_2$, respectively

- set of dual numbers ($\mathcal{A}_D$), which corresponds to (4.4);
- set of hyperbolic numbers ($\mathcal{A}_H$), which corresponds to (4.5);
- set of complex numbers ($\mathbb{C}$), which corresponds to (4.6).

Next, using (3.5), we calculate their characteristic functions.

- For dual number $\mathcal{A}_D$ case

$$\xi_{\mathcal{A}_D}(x_0, x_1) = x_0^2. \tag{4.7}$$

Then its zero set is obtained

$$\mathcal{Z}_{\mathcal{A}_D} = \{x_0 + x_1\mathbf{i} \in \mathcal{A}_D \mid x_0 = 0\}. \tag{4.8}$$

- For hyperbolic numbers $\mathcal{A}_H$ case

$$\xi_{\mathcal{A}_H}(x_0, x_1) = x_0^2 - x_1^2. \tag{4.9}$$

Then

$$\mathcal{Z}_{\mathcal{A}_H} = \{x_0 + x_1\mathbf{i} \in \mathcal{A}_H \mid x_0 = \pm x_1\}. \tag{4.10}$$

- For complex numbers $\mathbb{C}$ case

$$\xi_{\mathbb{C}}(x_0, x_1) = x_0^2 + x_1^2. \tag{4.11}$$

Then

$$\mathcal{Z}_{\mathbb{C}} = \{0\}. \tag{4.12}$$

Remark 4.1. *(i) It is obvious that $\mathcal{A}_D$, $\mathcal{A}_H$, and $\mathbb{C}$ are all PHAs.*

(ii) They have minimum polynomials x_0^2, $x_0^2 - x_1^2$, and $x_0^2 + x_1^2$ respectively. Since only the minimum polynomial of $\mathbf{i}$ is irreducible, only $\mathbb{C}$ is a field.

(iii) It is easy to see that their zero sets are of measure zero. This is always true for all PHAs, since they are zeros of polynomial functions.

4.2. Structure of triternions

Definition 4.2. *An algebra $\mathcal{A}$ of dimension 3 is called a triternion if $\mathcal{A} \in \boldsymbol{H}_3$.*

If a 3-dimensional unital algebra $\mathcal{A}$ over $\mathbb{R}$ is commutative, according to Theorem 2.7 its PSM is

$$P_{\mathcal{A}} = \begin{bmatrix} 1 & 0 & 0 & 0 & a & d & 0 & d & p \\ 0 & 1 & 0 & 1 & b & e & 0 & e & q \\ 0 & 0 & 1 & 0 & c & f & 1 & f & r \end{bmatrix}. \tag{4.13}$$

Next, we consider when $\mathcal{A}$ is associative. According to Theorem 2.7, the necessary and sufficient condition is

$$P_{\mathcal{A}}^2 = P_{\mathcal{A}} \left(I_3 \otimes P_{\mathcal{A}} \right). \tag{4.14}$$

Denote $I = I_3$,

$$A = \begin{bmatrix} 0 & a & d \\ 1 & b & e \\ 0 & c & f \end{bmatrix}, \quad B = \begin{bmatrix} 0 & d & p \\ 0 & e & q \\ 1 & f & r \end{bmatrix}.$$

A direct computation shows that

$$\begin{aligned} &\text{LHS of (4.14)} = (I, A, B, A, aI + bA + cB, \\ &dI + eA + fB, B, dI + eA + fB, pI + qA + rB), \\ &\text{RHS of (4.14)} = (I, A, B, A, A^2, AB, B, BA, B^2). \end{aligned} \qquad (4.15)$$

Then we have the following result:

Theorem 4.3. *$\mathcal{A} \in \boldsymbol{H}_3$, if and only if, $P_{\mathcal{A}}$ has the form of (4.13) with parameters satisfying*

$$\begin{aligned} a &= ce + f^2 - bf - cr, \\ d &= cq - ef, \\ p &= e^2 + fq - bq - er. \end{aligned} \qquad (4.16)$$

Proof. (Necessity) (4.15) shows that a necessary condition for (4.14) is (refer to the 6th and 8th blocks of both sides)

$$AB = BA. \qquad (4.17)$$

Then it is easy to verify that (4.16) provides necessary and sufficient condition for (4.17) to be true.

(Sufficiency) A careful computation shows as long as (4.16) holds, the RHS of (4.14) and the LHS of (4.14), shown in (4.15), are equal.

□

Remark 4.4. *Theorem 4.3 provides an easy way to construct $\mathcal{A} \in \boldsymbol{H}_3$. In fact, the parameters b, c, e, f, q, r can be arbitrarily assigned, and a, d, p can then be obtained by (4.16). Obviously there are uncountably many unital algebras of dimension 3 which are commutative and associative.*

Next, we give a numerical example.

Example 4.5. *Construct $\mathcal{A} \in \boldsymbol{H}_3$ by setting $b = c = f = q = r = 0$ and $e = 1$. Then we have $d = a = 0$ and $p = 1$. The PSM of $\mathcal{A}$ is*

$$P_{\mathcal{A}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}. \qquad (4.18)$$

In fact, when $x \in \mathcal{A}$ is expressed in standard form as

$$x = x_0 + x_1\mathbf{i}_1 + x_2\mathbf{i}_2, \quad x_0, x_1, x_2 \in \mathbb{R},$$

we have

$$\mathbf{i}_1^2 = 0, \quad \mathbf{i}_2^2 = 1,$$
$$\mathbf{i}_1 * \mathbf{i}_2 = \mathbf{i}_2 * \mathbf{i}_1 = \mathbf{i}_1.$$

Then it is easy to calculate that

$$\xi_{\mathcal{A}}(x_0, x_1, x_2) = (x_0 - x_2)(x_0 + x_2)^2. \qquad (4.19)$$

Hence,

$$\mathcal{Z}_{\mathcal{A}} = \{(x_0, x_1, x_2) \in \mathbb{R}^3 \mid x_0 = \pm x_2\}. \qquad (4.20)$$

4.3. Structure of perfect quaternions

This subsection considers some algebras in $\boldsymbol{H}_4$. It seems not easy to provide a general description for algebras in $\boldsymbol{H}_4$. The principle argument is similar to triternions. We give some simple examples.

Example 4.6. *Consider an $\mathcal{A} \in \boldsymbol{H}_4$. Assume*

$$\mathcal{A} = \{p_0 + p_1\mathbf{i}_1 + p_2\mathbf{i}_2 + p_3\mathbf{i}_3 \mid p_0, p_1, p_2, p_3 \in \mathbb{R}\},$$

satisfying

$$\mathbf{i}_1^2, \mathbf{i}_2^2, \mathbf{i}_3^2 \in \{-1, 0, 1\}, \quad \mathbf{i}_1 * \mathbf{i}_2 = \mathbf{i}_2 * \mathbf{i}_1 = \pm\mathbf{i}_3,$$
$$\mathbf{i}_2 * \mathbf{i}_3 = \mathbf{i}_3 * \mathbf{i}_2 = \pm\mathbf{i}_1, \quad \mathbf{i}_3 * \mathbf{i}_1 = \mathbf{i}_1 * \mathbf{i}_3 = \pm\mathbf{i}_2.$$

To save space, we denote

$$P_{\mathcal{A}_i} = [I_4, Q_i].$$

Using MATLAB for an exhausting searching, we get eight PHAs as follows:

- $$Q_1 = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$
- $$Q_2 = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$
- $$Q_3 = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$
- $$Q_4 = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$
- $$Q_5 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$
- $$Q_6 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$
- $$Q_7 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$
- $$Q_8 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

Next, choose some $\mathcal{A} \in \boldsymbol{H}_4$ for further study.

Example 4.7. *Recall Example 4.6.*

(i) Consider $\mathcal{A}_3$:

It is easy to calculate that

$$\begin{aligned} \xi_{\mathcal{A}_3} &= \det(P_{\mathcal{A}_3}x) \\ &= (x_0^2 - x_2^2)^2 + (x_1^2 - x_3^2)^2 \\ &\quad +2(x_0x_1 + x_2x_3)^2 + 2(x_0x_3 + x_1x_2)^2. \end{aligned} \tag{4.21}$$

It follows that

$$\begin{aligned} &\mathcal{Z}_{\mathcal{A}_3} = \left\{(x_0, x_1, x_2, x_3)^T \in \mathbb{R}^4 \mid \right. \\ &\left.(x_0 = x_2) \cap (x_1 = -x_3) \text{ or } (x_0 = -x_2) \cap (x_1 = x_3)\right\}. \end{aligned} \tag{4.22}$$

(ii) Consider $\mathcal{A}_8$:

It is easy to calculate that

$$\begin{aligned} \xi_{\mathcal{A}_8} &= \det(P_{\mathcal{A}_8}x) \\ &= x_0^4 + x_1^4 + x_2^4 + x_3^4 \\ &\quad -2(x_0^2x_1^2 + x_0^2x_2^2 + x_0^2x_3^3 + x_1^2x_2^2 \\ &\quad +x_1^2x_3^2 + x_2^2x_3^2) + 8x_0x_1x_2x_3. \end{aligned} \tag{4.23}$$

It follows that

$$\begin{aligned} &\mathcal{Z}_{\mathcal{A}_8} = \left\{(x_0, x_1, x_2, x_3)^T \in \mathbb{R}^4 \mid \right. \\ &\left.\xi_{\mathcal{A}_8}(x_0, x_1, x_2, x_3) = 0\right\}. \end{aligned} \tag{4.24}$$

4.4. Some other examples

To see there are also $\mathcal{A} \in \boldsymbol{H}_n$, for $n > 4$, such examples are presented as follows. The first example is a set of simplest PHAs, which are called trivial PHAs.

Example 4.8. *Define an $n+1$ dimensional algebra $\mathcal{A}^0_{n+1}$ as follows: Let $\mathbf{i}_k$, $k = 1, 2, \cdots, n$ be its hyperimaginary units. Set*

$$\mathbf{i}_s * \mathbf{i}_t = 0, \quad s, t = 1, 2, \cdots, n.$$

Then it is easy to verify that $\mathcal{A}^0_{n+1} \in \boldsymbol{H}_{n+1}$. Moreover, its PSM, $P_{\mathcal{A}^0_{n+1}}$ can be determined by the following:

$$\mathrm{Col}_i\left(P_{\mathcal{A}^0_{n+1}}\right) = \begin{cases} \delta^i_{n+1}, & i = 1, 2, \cdots, n+1; \\ \delta^{r+1}_{n+1}, & i = r(n+1)+1,\ r = 1, 2, \cdots, n; \\ \mathbf{0}_{n+1}, & \textit{Otherwise}. \end{cases} \tag{4.25}$$

Its characteristic function is

$$\xi_{\mathcal{A}^0_{n+1}} = x_0^{n+1}. \tag{4.26}$$

Hence,

$$\mathcal{Z}_{\mathcal{A}^0_{n+1}} = \{x_0 + x_1\mathbf{i}_1 + \cdots + x_n\mathbf{i}_n \mid x_0 = 0\}. \tag{4.27}$$

If $x \in \mathcal{Z}^c_{\mathcal{A}^0_{n+1}}$, say $x = x_0 + x_1\mathbf{i}_1 + \cdots + x_n\mathbf{i}_n$, $x_0 \neq 0$, then

$$x^{-1} = \frac{1}{x_0} - \sum_{i=1}^{n} \frac{x_i}{x_0^2}\mathbf{i}_i.$$

Next, we give an example for $n = 5$.

Example 4.9. *Consider a hypercomplex algebra $\mathcal{A}$, with PSM as*

$$\begin{aligned} P_{\mathcal{A}} \quad := \quad & \delta_5[1,2,3,4,5,2,0,0,1,0,3,0, \\ & 0,0,0,4,1,0,0,0,5,0,0,0,0], \end{aligned} \tag{4.28}$$

where $\delta_5^0 = \mathbf{0}_5$.

A straightforward computation shows that $\mathcal{A}$ is commutative and associative, hence $\mathcal{A} \in \boldsymbol{H}_5$. Then it is easy to calculate that

$$\xi_{\mathcal{A}}(x) = x_0^3(x_0^2 - 2x_1x_3). \tag{4.29}$$

So its zero set is

$$\mathcal{Z}_{\mathcal{A}} = \{(x_0, x_1, x_2, x_3, x_4)^T \in \mathbb{R}^5 \mid x_0 = 0, \ or\ x_0^2 = 2x_1x_3\}. \tag{4.30}$$

5. Matrices on PHAs

5.1. Hypercomplex matrix

Definition 5.1. *Let $\mathcal{A} \in \boldsymbol{H}_k$.*

(i) An n dimensional vector $\vec{v}$ is called an $\mathcal{A}$-vector of dimension n, if all entries of $\vec{v}$ are hypercomplex numbers of $\mathcal{A}$. The set of such vectors is denoted by $\mathcal{A}^n$, which is a vector space.

(ii) An $m \times n$ matrix A is called an $\mathcal{A}$-matrix of dimension $m \times n$ if all entries of A are hypercomplex numbers in $\mathcal{A}$. The set of such matrices is denoted by $\mathcal{A}_{m\times n}$.

Definition 5.2. *Let $\mathcal{A} \in \boldsymbol{H}_k$ and $A = (a_{i,j})$ be an $n \times n$ matrix with its entries $a_{i,j} \in \mathcal{A}$. The determine of A, denoted by $\det(A)$, is defined as follows:*

$$\det(A) := \sum_{\sigma\in\mathbf{S}_n} sign(\sigma) a_{1,\sigma(1)} * a_{2,\sigma(2)} * \cdots * a_{n,\sigma(n)}. \tag{5.1}$$

A is said to be non-singular if $\det(A) \notin \mathcal{Z}_{\mathcal{A}}$.

Remark 5.3. *Let $A \in \mathcal{A}_{m\times n}$ and $B \in \mathcal{A}_{n\times p}$. Then the transpose of A, denoted by A^T, the trace of A, denoted by $tr(A)$, the product of A and B, and all other operators are defined in the conventional way, if there is no elsewhere stated.*

The following result is obvious.

Proposition 5.4. *Assume $A \in \mathcal{A}_{n\times n}$ is non-singular, then there exists a unique $A^{-1} \in \mathcal{A}_{n\times n}$ such that*

$$AA^{-1} = A^{-1}A = I_n.$$

Example 5.5. *Let $\mathcal{A} \in \boldsymbol{H}_3$ with its PSM as in (4.18). $X \in \mathcal{A}_{2\times 2}$, where*

$$\begin{aligned} x_{11} &= 2 - \mathbf{i}_1 - 4\mathbf{i}_2, \\ x_{12} &= 3 + 2\mathbf{i}_1 - 4\mathbf{i}_2, \\ x_{21} &= -3 + 2\mathbf{i}_1 - \mathbf{i}_2, \\ x_{22} &= -2 + \mathbf{i}_1 + 4\mathbf{i}_2. \end{aligned}$$

Then

$$\begin{aligned} \det(X) &= P_{\mathcal{A}} x_{11} x_{22} - P_{\mathcal{A}} x_{12} x_{21} \\ &= -15 + 6\mathbf{i}_1 + 7\mathbf{i}_2 \notin \mathcal{Z}_{\mathcal{A}}. \end{aligned}$$

And

$$\begin{aligned} \tfrac{1}{\det(X)} &= (P_{\mathcal{A}} \det(X))^{-1}\,\delta_3^1 \\ &= -0.0852 - 0.0938\mathbf{i}_1 - 0.0398\mathbf{i}_2. \end{aligned}$$

Finally, we have

$$\begin{aligned} X^{-1} &= \tfrac{1}{\det(X)} \begin{bmatrix} x_{22} & -x_{12} \\ -x_{21} & x_{11} \end{bmatrix} \\ &= \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix}, \end{aligned}$$

where

$$\begin{aligned} y_{11} &= 0.0114 - 0.3125\mathbf{i}_1 - 0.2614\mathbf{i}_2, \\ y_{12} &= 0.0966 + 0.1563\mathbf{i}_1 - 0.2216\mathbf{i}_2, \\ y_{21} &= -0.2955 - 0.1250\mathbf{i}_1 - 0.2045\mathbf{i}_2, \\ y_{22} &= -0.0114 + 0.3125\mathbf{i}_1 + 0.2614\mathbf{i}_2. \end{aligned}$$

The following properties of $\mathcal{A}$-matrices come from classical matrix product with mimic proves.

Proposition 5.6. *(i) Let* $A,\ B \in \mathcal{A}_{n\times n}$*. Then*

$$\det(AB) = \det(A) * \det(B). \tag{5.2}$$

(ii) (Cayley-Hamilton Theorem) Let $A \in \mathcal{A}_{n\times n}$*. The characteristic function of A is*

$$\begin{aligned} p(\lambda) &= \det(\lambda I_n - A) \\ &= \lambda^n + \sum_{i=0}^{n-1} c_i\lambda^i, \quad c_i \in \mathcal{A}, i = 1, 2, \cdots, n. \end{aligned} \tag{5.3}$$

Then

$$p(A) = 0. \tag{5.4}$$

(iii) Assume $A \in \mathcal{A}_{n\times n}$*, and* $P \in \mathcal{A}_{n\times n}\backslash\{\mathcal{Z}_{\mathcal{A}}\}$*. Then*

$$tr(A) = tr(P^{-1}AP). \tag{5.5}$$

5.2. General linear group on PHA's matrices

Definition 5.7. *Given* $\mathcal{A} \in \boldsymbol{H}_k$*. The general linear group on* $\mathcal{A}$*, denoted by*

$$GL(\mathcal{A}, n) = \{A \in \mathcal{A}_{n\times n} \mid \det(A) \notin \mathcal{Z}_{\mathcal{A}}\}, \tag{5.6}$$

with the group product as classical matrix product.

The following result is obvious.

Proposition 5.8. *(i)* $GL(\mathcal{A}, n)$ *is a Lie group of dimension* kn^2*.*

(ii) The Lie algebra of $GL(\mathcal{A}, n)$ *is*

$$gl(\mathcal{A}, n) = (\mathcal{A}_{n\times n}, [\cdot, \cdot]),$$

where the Lie bracket is defined in a conventional way. That is,

$$[A, B] = AB - BA.$$

Note that if $A \in gl(\mathcal{A}, n)$, then $e^A \in GL(\mathcal{A}, n)$, where

$$e^A := \sum_{i=0}^{\infty} \frac{1}{i!}A^i.$$

6. Conclusion

In this paper the perfect hypercomplex algebra is considered. Using STP, necessary and sufficient conditions on its product structure matrix for an algebra to be a PHA are proposed. Based on the matrix expression of homomorphisms between algebras, certain lower dimensional PHAs are classified up to isomorphism. Their characteristic functions and zero sets are discussed. Then the matrices on PHAs are investigated. The general linear group structure of square matrices on PHAs are also discussed.

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Conflict of interest

The authors declared that they have no conflicts of interest to this work.

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