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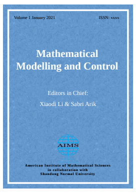

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Research article

An ancient Chinese algorithm for two-point boundary problems and its application to the Michaelis-Menten kinetics

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Abstract: Taylor series method is simple, and an infinite series converges to the exact solution for initial condition problems. For the two-point boundary problems, the infinite series has to be truncated to incorporate the boundary conditions, making it restrictively applicable. Here is recommended an ancient Chinese algorithm called as *Ying Buzu Shu*, and a nonlinear reaction diffusion equation with a Michaelis-Menten potential is used as an example to show the solution process.

Keywords: nonlinear differential equation; ancient Chinese mathematics; shooting method; Ying Buzu Shu

1. Introduction

Nonlinear equations arise always in electroanalytical chemistry with complex and esoteric nonlinear terms [1, 2], though there are some advanced analytical methods to deal with nonlinear problems, for examples, the Gamma function method [3], Fourier spectral method [4], the reproducing kernel method [5], the perturbation method [6], the homotopy perturbation method [7, 8], He's frequency formulation [9–11]and the dimensional method [12], chemists are always eager to have a simple one step method for nonlinear equations. This paper introduces an ancient Chinese algorithm called as the *Ying Buzu* algorithm [13] to solve nonlinear differential equations.

2. Taylor series solution

We first introduce the Taylor series method [14]. Considering the nonlinear differential equation:

$$\frac{d^2u}{dx^2} + F(u) = 0. \tag{0.1}$$

The boundary conditions are

$$\frac{du}{dx}(a) = \alpha, \tag{0.2}$$

$$u(b) = \beta. \tag{0.3}$$

If $u(a)$ is known, we can use an infinite Taylor series to express the exact solution [14]. We assume that

$$u(a) = c. \tag{0.4}$$

From (0.1), we have

$$u''(a) = -F(u(a)) = -F(c),$$

$$u'''(a) = -\frac{\partial F(c)}{\partial u}u'(a) = -\alpha\frac{\partial F(c)}{\partial u}.$$

Other higher order derivatives can be obtained with ease, and its Taylor series solution is

$$\begin{aligned} u(x) =& u(a) + (x-a)u'(a) + \frac{1}{2!}(x-a)^2u''(a) \\ &+ \frac{1}{3!}(x-a)^3u'''(a) + ... + \frac{1}{N!}(x-a)^N u^{(N)}(a), \end{aligned}$$

the constant c can be determined by the boundary condition of (0.3).

3. The *Ying Buzu* algorithm

The *Ying Buzu* algorithm [15, 16] was used to solve differential equations in 2006 [13], it was further developed to He's frequency formulation for nonlinear oscillators [13, 17–23] and Chun-Hui He's algorithm for numerical simulation [24].

As c in (0.4) is unknown, according to the *Ying Buzu* algorithm [13, 15, 16], we can assume two initial guesses:

$$u_1(a) = c_1,\ u_2(a) = c_2. \tag{0.5}$$

where c_1 and c_2 are given approximate values.

Using the initial conditions given in (0.2) and (0.5), we can obtain the terminal values:

$$u(b, c_1) = \beta_1,\ u(b, c_2) = \beta_2.$$

According to the *Ying Buzu* algorithm [6–12], the initial guess can be updated as

$$u(a)_{est} = c_3 = \frac{c_1(\beta - \beta_2) - c_2(\beta - \beta_1)}{(\beta - \beta_2) - (\beta - \beta_1)},$$

and its terminal value can be calculated as

$$u(b, c_3) = \beta_3.$$

For a given small threshold, ε, $|\beta - \beta_3| \le \varepsilon$, we obtain $u(a) = c_3$ as an approximate solution.

4. Application

Here, we take Michaelis Menten dynamics as an example to solve the equation. Michaelis Menten reaction diffusion equation is considered as follows [25, 26]:

$$\frac{d^2u}{dx^2} - \frac{u}{1+u} = 0. \tag{0.6}$$

The boundary conditions of it are as follows:

$$\frac{du}{dx}(0) = 0, u(1) = 1. \tag{0.7}$$

We assume

$$u(0) = c.$$

From (0.6), we have

$$u''(0) = \frac{c}{1+c},$$
$$u'''(0) = 0, \tag{0.8}$$
$$u^{(4)} = \frac{c}{(1+c)^3}.$$

The 2^{nd} order Taylor series solution is

$$u(x) = u(0) + \frac{u'(0)}{1!}x + \frac{u''(0)}{2!}x^2 = c + \frac{c}{2(1+c)}x^2.$$

In view of the boundary condition of (0.7), we have

$$u(1) = c + \frac{c}{2(1+c)} = 1, \tag{0.9}$$

solving c from (0.9) results in

$$c = 0.7808.$$

So we obtain the following approximate solution

$$u(x) = 0.7808 + 0.2192x^2.$$

Similarly the fourth order Taylor series solution is

$$u(x) = c + \frac{c}{2!(1+c)}x^2 + \frac{c}{4!(1+c)^3}x^4.$$

Incorporating the boundary condition, $u(1) = 1$, we have

$$c + \frac{c}{2!(1+c)} + \frac{c}{4!(1+c)^3} = 1. \tag{0.10}$$

We use the *Ying Buzu* algorithm to solve c, and write (0.10) in the form

$$R(c) = c + \frac{c}{2(1+c)} + \frac{c}{24(1+c)^3} - 1.$$

Assume the two initial solutions are

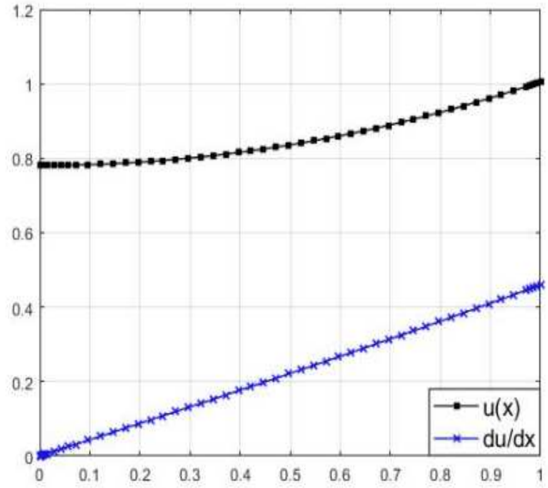
$$c_1 = 0.8,\ c_2 = 0.5.$$

We obtain the following residuals

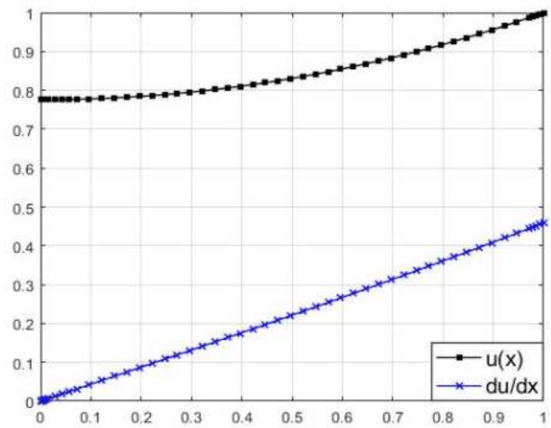
$$R_1(0.8) = 0.0279,\ R_2(0.5) = -0.3271.$$

By the *Ying Buzu* algorithm, c can be calculated as

$$c = \frac{R_2c_1 - R_1c_2}{R_2 - R_1} = \frac{0.0279 \times 0.5 + 0.3271 \times 0.8}{0.0279 + 0.3271} = 0.7764.$$



(a) The second order Taylor series solution.



(b) The fourth order Taylor series solution.

Figure 1. Taylor series solution.

The exact solution of (0.10) is

$$c = 0.7758.$$

The 4^{th} order Taylor series solution is

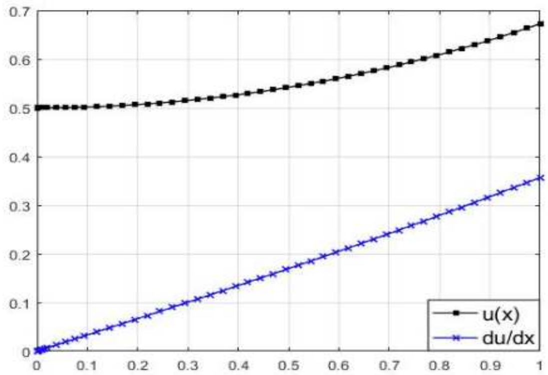
$$u(x) = 0.7758 + 0.2192x^2 + 0.0057x^4.$$

Figure 1 shows the Taylor series solutions, which approximately meet the requirement of the boundary condition at $x = 1$.

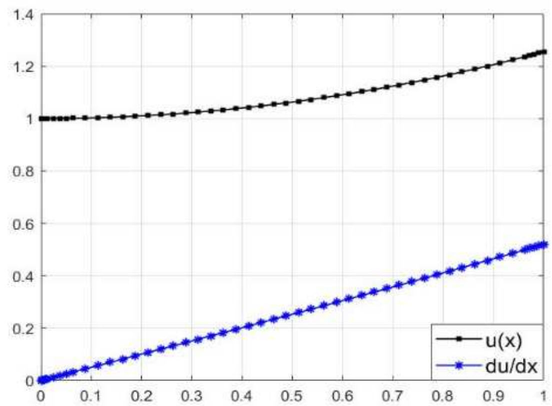
Now we use the *Ying Buzu* algorithm by choosing two initial guesses

$$u_1(0) = 0.5, \ u_2(0) = 1,$$

which lead to $u_1 = 0.6726$ and $u_2 = 1.2550$, respectively, see Figure 2 (a) and (b).



(a) $u_1(0) = 0.5$



(b) $u_2(0) = 1$

Figure 2. The shooting processes with different initial guesses.

It is obvious that the terminal value at $x = 1$ deviates from $u(1) = 1$ for each guess, according to the *Ying Buzu* algorithm, the initial guess can be updated as

$$u_3(0) = \frac{0.5 \times (1 - 1.2550) - 1 \times (1 - 0.6726)}{(1 - 1.2550) - (1 - 0.6726)} = 0.7810. \tag{0.11}$$

The shooting process using (0.11) results in

$$u_3(1) = 1.0058,$$

which deviates the exact value of $u(1) = 1$ with a relative error of 0.5%, see Figure 3.

We can continue the iteration process to obtain a higher accuracy by using two following two guesses $u_1(0) =$

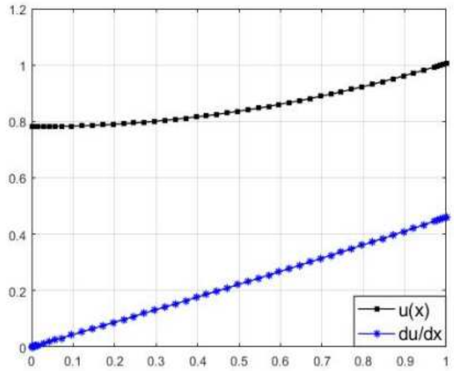


Figure 3. The shooting processes with an updated initial guess of $u(0) = 0.7810$.

$0.5, u_3(0) = 0.7810$:

$$u_4(0) = \frac{0.5 \times (1 - 1.0058) - 0.7810 \times (1 - 0.6726)}{(1 - 1.0058) - (1 - 0.6726)} = 0.7761.$$

Using this updated initial value, the shooting process leads to the result

$$u(1) = 1.0001,$$

so the approximate $u(0) = 0.7761$ has only a relative error of 0.01%.

The above solution process couples the numerical method, and the ancient method can also be solved independently.

We assume that solution is

$$u(x) = c + (1 - c)x^2. \quad (0.12)$$

Equation (0.12) meets all boundary conditions.

The residual equation is

$$R(x) = \frac{d^2u}{dx^2} - \frac{u}{1 + u}.$$

It is easy to find that

$$R(0) = 2(1 - c) - \frac{c}{1 + c}.$$

We choose two guesses:

$$c_1 = 0.5, c_2 = 1.$$

We obtain the following residuals

$$R_1(0) = 2(1 - 0.5) - \frac{0.5}{1 + 0.5} = \frac{2}{3},$$

$$R_2(0) = 2(1 - 1) - \frac{1}{1 + 1} = -\frac{1}{2}.$$

The *Ying Buzu* algorithm leads to the updated result:

$$c = \frac{c_2 R_1(0) - c_1 R_2(0)}{R_1(0) - R_2(0)} = \frac{\frac{2}{3} \times 1 + \frac{1}{2} \times 0.5}{\frac{2}{3} + \frac{1}{2}} = 0.7857.$$

The relative error is 1.2%, and the process can continue if a higher accuracy is still needed.

5. Discussion and Conclusion

The ancient Chinese algorithm provides a simple and straightforward tool to two-point boundary value problems arising in chemistry, and it can be used for fast insight into the solution property of a complex problem.

Conflict of interest

The authors declare that they have no conflicts of interest to this work.

References

1. Y. Liu, C. Wang, S. Li, A fractal Langmuir kinetic equation and its solution structure, *Thermal Science*, **25** (2021),1351–1354.

2. X. Liu, Y. Liu, Z. Wu, Computer simulation of Pantograph delay differential equation, *Thermal Science*, **25** (2021), 1381–1385.

3. K. Wang, G. Wang, Gamma function method for the nonlinear cubic-quintic Duffing oscillators, *J. Low Freq. Noise V. A.*, (2021). https://doi.org/10.1177/14613484211044613.

4. C. Han, Y. Wang, Z. Li, Numerical solutions of space fractional variable-coefficient KdV-modified KdV equation by Fourier spectral method, *Fractals*, (2021). https://doi.org/10.1142/S0218348X21502467.

5. D. Dai, T. Ban, Y. Wang, W. Zhang, The piecewise reproducing kernel method for the time variable fractional order advection-reaction-diffusion equations, *Thermal Science*, **25** (2021), 1261–1268.

6. C. He, D. Tian, G. Moatimid, H. F. Salman, M. H. Zekry, Hybrid Rayleigh-Van der Pol-Duffing Oscillator (HRVD): Stability Analysis and Controller, *J. Low Freq. Noise V. A.*, (2021). http://dx.doi.org/10.1177/14613484211026407.

7. X. Li, C. He, Homotopy perturbation method coupled with the enhanced perturbation method, *J. Low Freq. Noise V. A.*, **38** (2019), 1399–1403.

8. U. Filobello-Nino, H. Vazquez-Leal, B. Palma-Grayeb, The study of heat transfer phenomena by using modified homotopy perturbation method coupled by Laplace transform, *Thermal Science*, **24** (2020), 1105–1115.

9. D. Tian, Q. Ain, N. Anjum, et al. Fractal N/MEMS: from pull-in instability to pull-in stability, *Fractals*, **29** (2021), 2150030.

10. D. Tian, C. He, A fractal micro-electromechanical system and its pull-in stability, *J. Low Freq. Noise V. A.*, **40** (2021), 1380–1386.

11. G. Feng, He's frequency formula to fractal undamped Duffing equation, *J. Low Freq. Noise V. A.*, **40** (2021), 1671–1676.

12. C. He, S. Liu, C. Liu, H. Mohammad-Sedighi, A novel bond stress-slip model for 3-D printed concretes, *Discrete and Continuous Dynamical System*, (2021). http://dx.doi.org/10.3934/dcdss.2021161.

13. J. He, Some asymptotic methods for strongly nonlinear equations, *Int. J. Mod. Phys. B*, **20** (2006), 1141–1199.

14. J. He, Taylor series solution for a third order boundary value problem arising in architectural engineering, *Ain Shams Eng. J.*, **11** (2020), 1411–1414.

15. C. He, A Simple Analytical Approach to a Non-Linear Equation Arising in Porous Catalyst, *International Journal of Numerical Methods for Heat and Fluid-Flow*, **27** (2017), 861–866.

16. C. He, An Introduction an Ancient Chinese Algorithm and Its Modification, *International Journal of Numerical Methods for Heat and Fluid-flow*, **26** (2016), 2486–2491.

17. A. Elias-Zuniga, L. Manuel Palacios-Pineda, I. Jimenez-Cedeno, et al, He's frequency-amplitude formulation for nonlinear oscillators using Jacobi elliptic functions, *J. Low Freq. Noise V. A.*, **39** (2020), 1216–1223.

18. A. Elias-Zuniga, L. Manuel Palacios-Pineda, I. H. Jimenez-Cedeno, et al. Enhanced He's frequency-amplitude formulation for nonlinear oscillators, *Results Phys.*, **19** (2020), 103626.

19. C. Liu, A short remark on He's frequency formulation, *J. Low Freq. Noise V. A.*, **40** (2020), 672–674.

20. Y. Wu, Y. Liu, Residual calculation in He's frequency-amplitude formulation, *J. Low Freq. Noise V. A.*, **40** (2020), 1040–1047.

21. K. Wang, P. Zhang, Investigation of the periodic solution of the time-space fractional Sasa-Satsuma equation arising in the monomode optical fibers, *EPL*, (2021). https://doi.org/10.1209/0295-5075/ac2a62.

22. K. Wang, Generalized variational principle and periodic wave solution to the modified equal width-Burgers equation in nonlinear dispersion media, *Phys. Lett. A*, **419** (2021), 127723.

23. K. Wang, Periodic solution of the time-space fractional complex nonlinear Fokas-Lenells equation by an ancient Chinese algorithm, *Optik*, **243** (2021), 167461.

24. W. Khan, Numerical simulation of Chun-Hui He's iteration method with applications in engineering, *International Journal of Numerical Methods for Heat and Fluid-flow*, (2021). http://dx.doi.org/10.1108/HFF-04-2021-0245.

25. M. Lyons, Understanding the kinetics of catalysed reactions in microheterogeneous thin film electrodes, *Electroanal. Chem.*, **872** (2020), 114278.

26. S. Dhatt, K. Banerjee, Efficacy of quasi-steady-state approximation in Michaelis-Menten kinetics: a stochastic signature, *J. Math. Chem.*, **57** (2019), 1797–1805.