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*Research article*

# The Pauli problem and wave function lifting: reconstruction of quantum states from physical observables†

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**Abstract:** In this paper we consider the problem to determine a unique quantum state from given distribution of position (density) and momentum density of particles, namely the so called Pauli problem in quantum physics. In the first part, we will review the method of wave function lifting developed in [4, 5] to construct a complex wave function $\psi \in H^s(\mathbf{R}^d)$, $s = 1, 2$, associated to given density and momentum density. The second part is focused on the dynamical version of the wave function lifting, namely we study the relation between solutions to quantum fluid models and wave functions solving the nonlinear Schrödinger equation. The uniqueness of the lifted wave function is essentially related to the structure of vacuum regions of the position density.

**Keywords:** quantum hydrodynamics; Pauli problem; quantum state reconstruction

## 1. Introduction

In quantum physics, the state of a particle is characterised by a complex wave function $\psi$ (quantum state) in a suitable Hilbert space $\mathcal{H}$. However, it is impossible to directly observe a complex quantity in experiments, instead physical observable quantities of a particle are given by the expectation values of operators with respect to the quantum state, such as the probability distribution of position and momentum of particles. It is natural to ask whether it is possible to uniquely determine a quantum state from an appropriate set of physical observables, namely the so called Pauli problem, originally from a footnote in Pauli's article in *Handbuch der Physik* [21], see also for related researches [9, 25].

In this paper, we will partly answer the Pauli problem, by collecting the method of wave function lifting to reconstruct a complex wave function $\psi$ from given distribution of position (density) $\rho$ and momentum $J$, which is formally equivalent to knowing the expectations of all powers of the position and momentum operators.

The relationship between quantum states $\psi$ and physical observable quantities $(\rho, J)$ was early interpreted by Madelung [17] to obtain a hydrodynamic formulation of quantum mechanics. For a given complex wave function $\psi \in H^1_x(\mathbf{R}^d)$, it can be formally decomposed in terms of its amplitude and phase,

$$\psi = \sqrt{\rho}e^{\frac{i}{\hbar}S}, \tag{1.1}$$

where $\rho = |\psi|^2$, and by defining the phase velocity $v = \nabla S$ and momentum $J = \rho v$, we obtain $(\rho, J)$ as the distribution of the position and momentum densities associated to the quantum state $\psi$. Moreover, if the evolution of a quantum state $\psi = \psi(t, x)$ is governed by a Schrödinger-type equation

$$i\hbar\partial_t\psi + \frac{\hbar^2}{2}\triangle\psi = H\psi, \tag{1.2}$$

where $H$ is a potential function, then by writing (1.2) in its real and imaginary parts, we obtain the Hamilton-Jacobi equation of the density $\rho$ and the phase function $S$,

$$\begin{cases} \partial_t\rho + \mathrm{div}(\rho\nabla S) = 0 \\ \partial_t S + \frac{1}{2}|\nabla S|^2 + H = \frac{\hbar^2}{2}\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}. \end{cases}$$

As a consequence, the position density $\rho$ and the momentum density $J$ associated to $\psi$ formally satisfy the quantum hydrodynamic (QHD) system

$$\begin{cases} \partial_t\rho + \mathrm{div}\, J = 0 \\ \partial_t J + \mathrm{div}\left(\frac{J\otimes J}{\rho}\right) + \rho\partial_x H = \frac{\hbar^2}{2}\rho\nabla\left(\frac{\triangle\sqrt{\rho}}{\sqrt{\rho}}\right), \end{cases} \tag{1.3}$$

and the physical observables $\rho$ and $J$ are also called the hydrodynamic data associated to $\psi$. For the Schrödinger equation (1.2), the natural energy functional is defined by

$$E(t) = \int_{\mathbf{R}^d} \frac{\hbar^2}{2}|\nabla\psi|^2 + H|\psi|^2\, dx, \tag{1.4}$$

which has the corresponding hydrodynamic formulation as

$$E(t) = \int_{\mathbf{R}^d} \frac{\hbar^2}{2}|\nabla\sqrt{\rho}|^2 + \frac{|J|^2}{2\rho} + \rho H\, dx. \tag{1.5}$$

The QHD system is extensively used in physics to describe a compressible, inviscid fluid where quantum effects appear at macroscopic / mesoscopic scales and thus they need to be taken into account also in the hydrodynamical description. It is for instance the case when studying phenomena in superfluidity [15], Bose-Einstein condensation [23] or in the modeling of semiconductor devices at nanoscales [13].

The main problem of the Madelung transformation (1.1) is that the phase function $S$ usually fails to be well-defined when vacuum regions $\{\rho = 0\}$ exist. It is straightforward to see that the phase function $S$ may experience high oscillation or blow-up as the density $\rho$ approaching 0, which leads to lost of information of the phase function at vacuum boundaries. Moreover, even though the velocity field $v = \nabla S$ is irrotational outside vacuum, its circulations on closed curves around a vacuum region are actually quantized [12, 19], namely for any Lipschitz closed curve $\gamma$ strictly away from vacuum points, it should follows

$$\int_0^t v(\gamma(t)) \cdot \dot{\gamma}(t) dt \in 2\pi\hbar\mathbf{Z}. \tag{1.6}$$

In [1, 2] the authors overcome the difficulty of defining the hydrodynamic data $(\rho, J)$ when vacuum regions may exist by means of the polar decomposition approach (see Proposition 7, or the overview [3]). In this way it is possible to consider $(\rho, J)$ of any given wave function, which are defined almost everywhere and satisfy a generalized irrotationality condition.

In this present paper we are going to provide the method of wave function lifting to reconstruct a complex wave function from given distribution of particle density $\rho$, which may contain vacuum regions, and momentum density $J$ under suitable conditions, and discuss the uniqueness of the lifted wave function. More explicitly, we will consider hydrodynamic data $(\rho, J)$ given in the following three cases: 1-dimensional data with general vacuum regions, multi-dimensional quantum vortex data satisfying the quantised vorticity condition (1.6), and solutions of the QHD system (1.3) without vacuum. The first two results are reviews of the wave function lifting arguments in [4, 5], and we discuss further the source of non-uniqueness of the wave function lifting in this paper. The last result is our recent extension of the wave function lifting method to the dynamics of quantum fluids governed by QHD models (1.3).

In our construction of wave function, we find that the only possible source of the non-uniqueness of the lifted wave function is the lost of information at vacuum boundary. Therefore in the cases such that the connected components of non-vacuum regions are strictly separated (namely they can be contained in disjoint open neighbourhoods), as the 1-dimensional problem considered in this paper, the answer to the Pauli problem is usually negative—there are infinitely many wave functions associated to the same hydrodynamic data. However, we also show that those lifted wave functions are only differed by piece-wise constant phase shifts (see Lemma 13). On the other hand, for hydrodynamic data such that the position density $\rho$ is continuous and the non vacuum region is connected (for example the case of pointwise quantum vortices), since there is no problem of crossing vacuum boundaries, the lifted wave function is unique upto constant rotations, which gives a positive answer to the Pauli problem.

We start with the wave function lifting result in dimension $d = 1$ for finite energy hydrodynamic data given by functional (1.5). To consider the momentum density in vacuum regions, let us introduce the hydrodynamic state $(\sqrt{\rho}, \Lambda)$ (see Definition 5 or [1, 2]), and the momentum density is given by $J = \sqrt{\rho}\Lambda$. In our results we further require $\Lambda = 0$ almost everywhere in vacuum regions, which is called the physical compatibility condition of hydrodynamic state. This condition is equivalent to say that the energy density vanishes in large vacuum regions, which is a physically reasonable condition, and is guaranteed for all Schrödinger generated data by Lemma 6.

**Theorem 1.** *Let $(\sqrt{\rho}, \Lambda)$ be a hydrodynamic state. There exists an associated wave function $\psi \in H_x^1(\mathbf{R})$ in the sense of the Definition 5, if and only if $(\sqrt{\rho}, \Lambda)$ satisfies*

*(1) there exists a constant* $0 < M_1 < \infty$ *such that*

$$\hbar\|\sqrt{\rho}\|_{H^1_x(\mathbf{R})} + \|\Lambda\|_{L^2_x(\mathbf{R})} \leq M_1;$$

*(2)* $\Lambda = 0$ *almost everywhere in* $\{\rho = 0\}$.

*Furthermore, we have*

$$\hbar\partial_x\psi = (\hbar\partial_x\sqrt{\rho} + i\Lambda)\phi,$$

$$\|\psi\|_{H^1_x(\mathbf{R})} \leq \hbar^{-1}C(M_1),$$

*where* $\phi \in P(\psi)$ *is a polar factor of* $\psi$ *defined in* (2.1).

*For any piecewise constant phase shift*

$$\Theta = \exp\left(i\sum_j \theta_j \mathbf{1}_{(a_j,b_j)}\right), \quad \theta_j \in [0, 2\pi), \tag{1.7}$$

*where* $\{(a_j, b_j)\}_j$ *is a disjoint decomposition of the non-vacuum region* $\{\rho > 0\}$*, we have* $\Theta\psi \in H^1_x(\mathbf{R})$ *and* $\Theta\psi$ *is also a wave function associated to* $(\sqrt{\rho}, \Lambda)$*. On the other hand, if* $\tilde{\psi}$ *is another wave function associated to* $(\sqrt{\rho}, \Lambda)$*, then there exists a phase shift* $\tilde{\Theta}$ *of form* (1.7) *such that* $\tilde{\psi} = \tilde{\Theta}\psi$*.*

The lifting result of wave functions in space $H^2_x$ is established for $(\rho, J)$ with $L^2_x$ bounded generalised chemical potential $\lambda$ (see Definition 9, in 1-D space we omit the subscript), which is abbreviated as GCP states in this paper.

**Theorem 2.** *Let* $(\sqrt{\rho}, \Lambda)$ *be a hydrodynamic state. There exists a wave function* $\psi \in H^2(\mathbf{R})$*, associated to* $(\sqrt{\rho}, \Lambda)$ *in the sense of Definition 5, if and only if*

- $(\sqrt{\rho}, \Lambda)$ *is a GCP state in the sense of Definition 10;*
- *the kinetic energy density* $k = \frac{\hbar^2}{2}(\partial_x\sqrt{\rho})^2 + \frac{1}{2}\Lambda^2$ *is continuous.*

*Moreover, let us assume*

$$\begin{aligned} \hbar\|\sqrt{\rho}\|_{H^1_x(\mathbf{R})} + \|\Lambda\|_{L^2_x(\mathbf{R})} &\leq M_1, \\ \hbar\|\mathbf{1}_{\{\rho>0\}}\partial_x J/\sqrt{\rho}\|_{L^2_x(\mathbf{R})} + \|\lambda\|_{L^2_x(\mathbf{R})} &\leq M_2, \end{aligned} \tag{1.8}$$

*for some constants* $M_1, M_2 < \infty$*, where* $\lambda$ *is defined in* (2.8)*, it follows that*

$$\|\psi\|_{H^2_x(\mathbf{R})} \leq C(\hbar^{-1}M_1 + \hbar^{-2}M_2). \tag{1.9}$$

In the 2-dimensional case, we additionally assume the continuity of density $\rho$, which is essential to give a proper definition to the point vacuum and can not be inferred by the finite energy assumption and Sobolev embedding in multi-dimension. It is also needed to provide a local lower bound on the density when strictly away from the vacuum. Nevertheless, in the physical and mathematical literature the most widely studied quantum vortex structures (1.6) fall in this framework, see for example [7, 20, 24], and in the Theorem, we formulate the quantised vorticity condition in a distributional differential form (1.12). This result can also be extended to the case of simple vortex lines in 3-dimensional case, see Proposition 19.

**Theorem 3.** *Let* $(\sqrt{\rho}, \Lambda)$ *be a finite energy hydrodynamic state satisfying the bound*

$$\hbar\|\sqrt{\rho}\|_{H^1(\mathbf{R}^2)} + \|\Lambda\|_{L^2(\mathbf{R}^2)} \le M_1. \tag{1.10}$$

*Let us further assume that* $\sqrt{\rho}$ *is continuous with isolated point vacuum (consequently* $V = \{x; \rho(x) = 0\}$ *is at most countable)*

$$V = \{x_{(\alpha)}\}_{\alpha\in\mathcal{A}} \subset \mathbf{R}^2, \quad \inf_{\alpha\neq\beta} |x_{(\alpha)} - x_{(\beta)}| > 0, \tag{1.11}$$

*and the velocity* $v$ *satisfies the quantised vorticity condition*

$$\begin{cases} v \in \mathcal{M}(\mathbf{R}^2), \\ \nabla \wedge v = 2\pi\hbar \sum_{\alpha\in\mathcal{A}} k_\alpha \delta_{x_{(\alpha)}}, \quad k_\alpha \in \mathbf{Z}. \end{cases} \tag{1.12}$$

*Then there exists a wave function* $\psi \in H^1(\mathbf{R}^2)$ *such that*

$$\sqrt{\rho} = |\psi|, \quad \Lambda = \hbar\,\mathrm{Im}(\bar{\phi}\nabla\psi),$$

*where* $\phi \in P(\psi)$ *is a polar factor defined in* (2.1)*. The lifted wave function is unique upto constant rotation* $e^{i\theta}$*,* $\theta \in [0, 2\pi)$.

*If we furthermore assume that* $\triangle\rho, \nabla J \in L^1_{loc}(\mathbf{R}^2)$ *and* $(\sqrt{\rho}, \Lambda)$ *satisfy also the bounds*

$$\|\lambda_j\|_{L^2(\mathbf{R}^2)} + \hbar\|\frac{\partial_{x_j} J_j}{\sqrt{\rho}}\|_{L^2(\mathbf{R}^2)} \le M_2, \quad j = 1, 2, \tag{1.13}$$

*where* $\lambda_j$*'s are the generalised chemical potential given by Definition 9, then* $\psi \in H^2(\mathbf{R}^2)$ *and we have*

$$\|\psi\|_{H^2_x(\mathbf{R}^2)} \le C(\hbar^{-1}M_1 + \hbar^{-2}M_2).$$

Our results establish the equivalence between hydrodynamic states and the corresponding wave functions $\psi$ in $H^1_x$ and $H^2_x$, and we also refer to the recent paper [6], where the wave function lifting method of $\psi \in H^1_x$ is considered for general spaces $(X, \mathbf{d}, \nu)$ with metric $\mathbf{d}$ and measure $\nu$. In [6], the velocity $v$ is interpreted as a cotangent vector filed in $L^2(T^*X)$, and the quantised vorticity condition is extended in the integral form

$$\int_0^t v(\gamma(t)) \cdot \dot{\gamma}(t)dt \in 2\pi\hbar\mathbf{Z},$$

for almost every closed curve $\gamma$ on $X$. However, in general spaces usually one can not expect the uniqueness of lifted wave functions.

Let us remind that the previous results do not provide correspondence between the dynamics of Schrödinger equations and the QHD models. For given a hydrodynamic solution $(\rho, J)$, one can construct its associated wave function $\psi(t, \cdot)$ by Theorem 1 to Theorem 3 at each fixed time $t$. But to make $\psi$ a solution to the corresponding Schrödinger equation, more delicate construction is required. In the last result, we restrict us to the non-vacuum case and extend the wave function lifting result to the dynamical case for GCP solutions of the QHD system (1.3), which provides the equivalence of the dynamics of QHD models (1.3) and Schrödinger equations (1.2).

**Theorem 4.** *Let $(\rho, J)$ be a GCP solution (see Definition 12) of* (1.3) *on* $[0, T) \times \mathbf{R}^d$ *with potential* $H \in L_t^\infty L^1_{loc,x}$*, such that* $\rho$ *is strictly positive on any compact set, then there exists a unique (upto constant phase shifts) wave function* $\psi \in L_t^\infty([0, T), H_x^2(\mathbf{R}^d))$ *associated to* $(\rho, J)$ *in the sense of Definition 5 for almost every* $t \in [0, T)$*. Moreover,* $\psi$ *is a weak solution of the Schrödinger equation* (1.2).

If the corresponding Scrödinger equation has satisfactory well-posedness result, as a consequence of Theorem 4 one immediately obtains the well-possedness of the QHD model for non-vacuum solutions. However, this correspondence generally does not hold true if vacuum regions exist. For example, the cubic nonlinear Schrödinger equation $H = |\psi|^2$ has well-known well-posedness theory, while the uniqueness of the corresponding QHD system for initial data with quantized vorticity is still an open problem. If the condition of quantized vorticity is not required, [10] (multi-D) and [18] (1-D) provide the non-uniqueness results of finite energy weak solutions.

The rest of this paper is structured in the following way: In Section 2 we will give the definitions and recall some preliminary results, including the polar decomposition approach. In Section 3 we construct the lifted wave function for 1-dimensional hydrodynamic data (Theorems 1 and 2) which may contain general vacuum regions, while multi-dimensional data satisfying the quantised vorticity condition are considered in Theorem 3 in Section 4. Last, in Section 5 we prove Theorem 4, namely for a given solution of the QHD system (1.3) without vacuum region, there exists a unique (upto constant rotations) lifted wave function, which is also a solution to the corresponding Schrödinger equation (1.2).

## 2. Definitions and preliminaries

Notations and definitions will be fixed in this section and are used through this paper. The standard notation for Lebesgue and Sobolev norms are given by

$$\|f\|_{L_x^p} := (\int_{\mathbf{R}^d} |f(x)|^p dx)^{\frac{1}{p}},$$

$$\|f\|_{W_x^{k,p}} := \sum_{j=0}^{k} \|\nabla^j f\|_{L_x^p},$$

and we denote $H_x^k := H^k(\mathbf{R}^d) = W^{k,2}(\mathbf{R}^d)$. The mixed Lebesgue norm of functions $f : I \to L^r(\mathbf{R}^d)$ is defined as

$$\|f\|_{L_t^q L_x^r} := \left(\int_I \|f(t)\|_{L_x^r}^q dt\right)^{\frac{1}{q}} = \left(\int_I (\int_{\mathbf{R}^d} |f(x)|^r dx)^{\frac{q}{r}} dt\right)^{\frac{1}{q}},$$

where $I \subset [0, \infty)$ is a time interval, and the explicit dependence on $I$ is omitted in this paper. The mixed Sobolev norm $L_t^q W_x^{k,r}$ is defined analogously. Last, often letters as $C$ and $M$ are used to denote positive constants, which may change line to line.

Now we recall the method of polar decomposition to define the hydrodynamic state associated to a given quantum state $\psi$ possibly containing vacuum regions. Given any function $\psi \in H_x^1(\mathbf{R}^d)$, we can define the set of polar factors as

$$P(\psi) := \left\{\phi \in L_x^\infty(\mathbf{R}^d) \mid \|\phi\|_{L_x^\infty} \le 1,\ \sqrt{\rho}\phi = \psi \ a.e.\right\}, \tag{2.1}$$

where $\sqrt{\rho} := |\psi|$.

**Definition 5.** Let $\Omega \subset \mathbf{R}^d$ be an open set. Given a wave function $\psi \in H^1_x(\Omega)$, we say that $(\sqrt{\rho}, \Lambda)$ is the hydrodynamic state associated to $\psi$ if

$$\sqrt{\rho} = |\psi| \quad \text{and} \quad \Lambda = \hbar \operatorname{Im}(\bar{\phi}\nabla\psi), \tag{2.2}$$

where $\phi \in P(\psi)$ is a polar factor of $\psi$, and the momentum density of $\psi$ is defined by

$$J = \sqrt{\rho}\Lambda.$$

In general the polar factor of a wave function $\psi$ is not unique, due to the possible appearance of vacuum regions. Nevertheless, $\Lambda = \hbar \operatorname{Im}(\bar{\phi}\nabla\psi)$ is well defined, as a consequence of the following Sard type lemma (see e.g., Theorem 6.19 in [16] for a general statement).

**Lemma 6.** *Let $\Omega \subset \mathbf{R}^d$ be an open set, $g : \Omega \to \mathbb{R}$ be in $H^1_x(\Omega)$, and*

$$B = g^{-1}(\{0\}) = \{x \in \Omega \ : \ g(x) = 0\}.$$

*Then $\nabla g(x) = 0$ for almost every $x \in B$.*

We stress that, in view of the lemma above, $\Lambda$ satisfy the physical compatibility condition

$$\Lambda = 0 \quad \text{a.e. on } \{\rho = 0\}. \tag{2.3}$$

Furthermore, $(\sqrt{\rho}, \Lambda)$ enjoy the following properties.

**Proposition 7** (Polar factorization [1, 2])**.** *Let $\psi \in H^1_x(\mathbf{R}^d)$ and $(\sqrt{\rho}, \Lambda)$ be a hydrodynamic state associated to $\psi$, then we have $(\sqrt{\rho}, \Lambda) \in H^1_x(\mathbf{R}^d) \times L^2_x(\mathbf{R}^d)$, and*

$$\hbar^2 \operatorname{Re}(\nabla\overline{\psi} \otimes \nabla\psi) = \hbar^2 \nabla\sqrt{\rho} \otimes \nabla\sqrt{\rho} + \Lambda \otimes \Lambda, \quad \text{a.e. in } \mathbf{R}^d. \tag{2.4}$$

*Moreover the following identity is satisfied in the sense of distributions*

$$\nabla \wedge J = 2\nabla\sqrt{\rho} \wedge \Lambda. \tag{2.5}$$

*Remark* 8. The identity (2.5) is called the generalised irrotationality condition introduced by the authors of [1, 2]. Formally it is easy to compute that (2.5) is equivalent to

$$\rho\nabla \wedge v = 0,$$

therefore (2.5) can be seen as an extension of the irrotationality property of the phase velocity $v = \nabla S$ to the case when vacuum regions are present.

By taking the trace of (2.4), we have

$$\hbar^2|\nabla\psi|^2 = \hbar^2|\nabla\sqrt{\rho}|^2 + |\Lambda|^2 = \hbar^2|\nabla\sqrt{\rho}|^2 + \frac{|J|^2}{\rho},$$

therefore $\nabla\sqrt{\rho}$ and $\Lambda$ together characterise the $\dot{H}^1_x$ regularity of $\psi$. Following the energy functional (1.5) and the terminology of hydrodynamic theory, we define the kinetic energy density function to be

$$k = \frac{\hbar^2}{2}|\nabla\psi|^2 = \frac{\hbar^2}{2}|\nabla\sqrt{\rho}|^2 + \frac{1}{2}|\Lambda|^2. \tag{2.6}$$

Let us also recall the space of hydrodynamic functions corresponding to $H^2_x$ wave functions, which is introduced by [4,5]. To define the new space, we start by considering the following chemical potential

$$\mu = -\frac{\hbar^2}{2}\frac{\triangle\sqrt{\rho}}{\sqrt{\rho}} + \frac{1}{2}|v|^2 + H. \tag{2.7}$$

Formally it is possible to interpret $\mu$ as the first variation of the total energy functional (1.5) with respect to the position density [14],

$$\mu = \frac{\delta E}{\delta\rho}.$$

Obviously, the chemical potential $\mu$ cannot be used to carry out a satisfactory mathematical analysis when vacuum regions exist. For this reason it will be more convenient to consider the following "generalized chemical potential" $\lambda_j$.

**Definition 9.** Let

$$(\sqrt{\rho}, \Lambda) \in [H^1_x(\mathbf{R}^d) \cap C_x(\mathbf{R}^d)] \times L^2_x(\mathbf{R}^d; \mathbf{R}^d),$$

such that $\Lambda$ satisfies (2.3) and $\triangle\rho \in L^1_{loc}(\mathbf{R})$. Then we define the generalized chemical potential $\lambda_j$, $j = 1, 2, \ldots, d$, to be the measurable function given by

$$\lambda_j = \begin{cases} -\frac{\hbar^2}{2}\partial^2_{x_j}\sqrt{\rho} + \frac{1}{2}\frac{\Lambda_j^2}{\sqrt{\rho}} & \text{in } \{\rho > 0\}, \\ 0 & \text{elsewhere}, \end{cases} \tag{2.8}$$

where $\Lambda_j$ is the $j$-th component of $\Lambda$.

In what follows we provide the definition of bounded generalized chemical potential hydrodynamic states, namely those states where $\lambda_j$ can be rigorously characterized.

**Definition 10** (GCP states)**.** We say that the pair $(\sqrt{\rho}, \Lambda)$ is a state with bounded generalized chemical potential (GCP state, in short) if:

- $\sqrt{\rho} \in H^1_x(\mathbf{R}^d) \cap C_x(\mathbf{R}^d)$ and $\Lambda \in L^2_x(\mathbf{R}^d; \mathbf{R}^d)$;
- $\Lambda = 0$ a.e. on $\{\rho = 0\}$;
- $\triangle\rho, \nabla J \in L^1_{loc,x}(\mathbf{R}^d)$, where $(\rho, J) = ((\sqrt{\rho})^2, \sqrt{\rho}\Lambda)$;
- the following bounds are satisfied

$$\hbar\|\mathbf{1}_{\{\rho>0\}}\partial_{x_j}J/\sqrt{\rho}\|_{L^2_x(\mathbf{R}^d)} + \|\lambda_j\|_{L^2_x(\mathbf{R}^d)} \le C, \quad j = 1, 2, \ldots, d,$$

  where $\lambda$ is defined as in (2.8).

Now we extend the hydrodynamic spaces defined above to solutions of the QHD system (1.3). Since we consider hydrodynamic functions with relatively low regularity, we need to introduce the definition of general weak solutions of (1.3). Preliminary, we first rewrite the quantum pressure term as

$$\frac{\hbar^2}{2}\rho\nabla\left(\frac{\triangle\sqrt{\rho}}{\sqrt{\rho}}\right) = \frac{\hbar^2}{4}\operatorname{div}(\rho\nabla^2\log\rho) = \frac{\hbar^2}{4}\nabla\triangle\rho - \hbar^2\operatorname{div}(\nabla\sqrt{\rho}\otimes\nabla\sqrt{\rho}). \tag{2.9}$$

We can now give the definition of weak solutions to the QHD system.

**Definition 11** (Weak solutions)**.** Let $\rho_0, J_0 \in L^1_{loc}(\mathbf{R}^d)$, we say the pair $(\rho, J)$ is a finite energy weak solution to the Cauchy problem for (1.3) with initial data $\rho(0) = \rho_0$, $J(0) = J_0$, in the space-time slab $[0,T) \times \mathbf{R}^d$ if there exist $\sqrt{\rho} \in L^2_{loc}(0,T; H^1_{loc}(\mathbf{R}^d))$, $\Lambda \in L^2_{loc}(0,T; L^2_{loc}(\mathbf{R}^d))$ such that

(i) $\rho = (\sqrt{\rho})^2$, $J = \sqrt{\rho}\Lambda$;
(ii) $\forall\ \eta \in C_0^\infty([0,T) \times \mathbf{R}^d)$,

$$\int_0^T \int_{\mathbf{R}^d} \rho \partial_t \eta + J \cdot \nabla \eta \, dxdt + \int_{\mathbf{R}^d} \rho_0(x) \eta(0,x)\, dx = 0; \tag{2.10}$$

(iii) $\forall \zeta \in C_0^\infty([0,T) \times \mathbf{R}^d; \mathbf{R}^d)$,

$$\begin{aligned} \int_0^T \int_{\mathbf{R}^d} & J \cdot \partial_t \zeta + \Lambda \otimes \Lambda : \nabla \zeta - \rho \zeta \partial_x H + \hbar^2 \nabla \sqrt{\rho} \otimes \nabla \sqrt{\rho} : \nabla \zeta \\ & + \frac{\hbar^2}{4} \rho \Delta \operatorname{div} \zeta \, dxdt + \int_{\mathbf{R}^d} J_0(x) \cdot \zeta(0,x)\, dx = 0; \end{aligned} \tag{2.11}$$

(iv) (generalized irrotationality condition) for almost every $t \in (0,T)$

$$\nabla \wedge J = 2 \nabla \sqrt{\rho} \wedge \Lambda, \tag{2.12}$$

holds in the sense of distributions.

The weak solutions of finite energy and GCP weak solutions are given by the following definition.

**Definition 12.** Let $(\rho, J)$ be a weak solution to the QHD system (1.3) as in the Definition 11. We say that $(\rho, J)$ is a finite energy weak solution, if we have $\sup_{t \in [0,T)} E(t) < \infty$, where $E(t)$ is the energy functional defined by (1.5).

Moreover, we say $(\rho, J)$ is a GCP weak solution if for almost every $t \in [0,T)$, the corresponding hydrodynamic state $(\sqrt{\rho}, \Lambda)(t)$ is a GCP state as in Definition 10, and satisfies the uniform bound

$$\hbar \|\partial_t \sqrt{\rho}\|_{L^\infty(0,T;L^2(\mathbf{R}^d))} + \sum_{j=1}^d \|\lambda_j\|_{L^\infty(0,T;L^2(\mathbf{R}^d))} \le I_0 < \infty.$$

## 3. Wave function lifting in 1-dimensional space

In this section we establish the method of wave function lifting in dimension $d = 1$. We first prove Theorem 1, which gives sufficient and necessary conditions for a hydrodynamic data $(\sqrt{\rho}, \Lambda)$ to have an associated wave function $\psi \in H^1(\mathbf{R})$.

*Proof of the lifting part of Theorem 1.* Let $\psi \in H^1_x(\mathbf{R})$ be associated to $(\sqrt{\rho}, \Lambda)$, then by means of the polar factorization in Proposition 7 we have that both conditions *(1)* and *(2)* are fulfilled.

To prove the converse statement, we consider $\delta_n(x) = \frac{1}{n} e^{-|x|^2/2}$, which is locally strictly positive and converges to 0 as $n \to \infty$. We define the following approximating hydrodynamical quantities

$$\sqrt{\rho_n} = \sqrt{\rho} + \delta_n, \quad \Lambda_n = \frac{J}{\sqrt{\rho_n}} = \frac{\sqrt{\rho}}{\sqrt{\rho_n}} \Lambda,$$

then it is straightforward to check that the sequence $\{(\sqrt{\rho_n}, \Lambda_n)\}$ also satisfies a uniform bound as in condition (1),

$$\|\sqrt{\rho_n}\|_{H^1_x(\mathbf{R})} \leq \|\sqrt{\rho}\|_{H^1_x(\mathbf{R})} + \|\delta_n\|_{H^1_x(\mathbf{R})} \leq C(\hbar^{-1}M_1),$$

and since $\sqrt{\rho_n}(x) > \sqrt{\rho}(x)$ for every $x \in \mathbf{R}$, we have $|\Lambda_n(x)| \leq |\Lambda(x)|$ a.e. and consequently

$$\|\Lambda_n\|_{L^2_x(\mathbf{R})} \leq \|\Lambda\|_{L^2_x(\mathbf{R})}.$$

By we have $\sqrt{\rho_n} \to \sqrt{\rho}$ in $H^1_x(\mathbf{R})$. Moreover, the physical compatibility condition (2) holds also for $\Lambda_n$. Hence $\sqrt{\rho_n}(x) \to \sqrt{\rho}(x)$ and $\Lambda_n(x) \to \Lambda(x)$ a.e. in $\mathbf{R}$. Then the dominated convergence theorem yields $\Lambda_n \to \Lambda$ in $L^2_x(\mathbf{R})$. Furthermore, since $\sqrt{\rho_n}(x) > 0$ it is possible to define the approximating velocity field

$$v_n = \frac{\Lambda_n}{\sqrt{\rho_n}}.$$

Notice that, since by definition $\sqrt{\rho_n}$ is uniformly bounded away from zero on compact intervals, we have $v_n \in L^1_{loc}(\mathbf{R})$, hence it makes sense to define the phase

$$S_n(x) = \int_0^x v_n(x)\,dx$$

and consequently the wave function

$$\psi_n(x) = \sqrt{\rho_n}(x) e^{\frac{i}{\hbar} S_n(x)}.$$

Let us remark that $\psi_n$ is uniquely defined, up to a constant phase shift. We now show that the sequence $\psi_n$ has a limit $\psi \in H^1_x(\mathbf{R})$ satisfying $|\psi| = \sqrt{\rho}$, $\mathrm{Im}(\bar{\phi}\partial_x\psi) = \Lambda$, where $\phi \in P(\psi)$. Since $\partial_x\psi_n = e^{\frac{i}{\hbar}S_n}(\partial_x\sqrt{\rho_n} + i\Lambda_n)$, we have

$$\hbar\|\psi_n\|^2_{H^1_x} = \hbar\|\partial_x\sqrt{\rho_n}\|^2_{H^1_x} + \|\Lambda_n\|^2_{L^2_x} \leq C(M_1),$$

thus, up to subsequences, $\psi_n \rightharpoonup \psi$ in $H^1$. On the other hand, we also have $\sqrt{\rho_n} \to \sqrt{\rho}$ in $H^1$, $\Lambda_n \to \Lambda$ in $L^2$ and moreover $e^{\frac{i}{\hbar}S_n} \rightharpoonup \phi$ weakly* in $L^\infty_x$, for some $\phi \in L^\infty_x$. It is straightforward to check that $\phi \in P(\psi)$, indeed we have $\psi_n = \sqrt{\rho_n}e^{\frac{i}{\hbar}S_n} \rightharpoonup \sqrt{\rho}\phi = \psi$ in $L^2_x(\mathbf{R})$. Furthermore

$$\hbar\partial_x\psi_n = e^{\frac{i}{\hbar}S_n}\left(\hbar\partial_x\sqrt{\rho_n} + i\Lambda_n\right) \rightharpoonup \phi\left(\hbar\partial_x\sqrt{\rho} + i\Lambda\right), \quad \text{in } L^2_x(\mathbf{R})$$

so that $\partial_x\psi = \phi\left(\hbar\partial_x\sqrt{\rho} + i\Lambda\right)$ and hence $(\sqrt{\rho}, \Lambda)$ are the hydrodynamical quantities associated to $\psi$ in the sense of the Definition 5. □

Now we begin to discuss the uniqueness and stability properties of the wave function lifting given in Theorem 1 in $H^1_x(\mathbf{R})$. Actually the wave function associated to a finite energy hydrodynamic data can not be uniquely determined because of the arbitrary phase shifts allowed on each connected component of $\{\rho > 0\}$. In the next lemma, we show that this is the only source of non-uniqueness. Since this lemma is also used in later contents for multi-dimensional cases, we present it here in a general form.

**Lemma 13.** *Let $\Omega \subset \mathbf{R}^d$ be a connected open set and $\psi_1,\ \psi_2 \in H^1_x(\Omega)$ be two wave functions associated to the same hydrodynamic data $(\sqrt{\rho}, \Lambda) \in [H^1_x(\Omega) \cap C^1_x(\Omega)] \times L^2_x(\Omega)$. Furthermore we assume that $\rho > 0$ on $\Omega$.*

*Then there exists a constant phase shift $\theta \in [0, 2\pi)$, such that*

$$\psi_2 = e^{i\theta} \psi_1. \tag{3.1}$$

*Proof.* By our assumption and the polar factorization, we have

$$\psi_j = \sqrt{\rho}\phi_j \quad \textit{and} \quad \nabla\psi_j = (\nabla \sqrt{\rho} + \frac{i}{\hbar}\Lambda)\phi_j, \quad j = 1, 2,$$

where $\phi_j \in P(\psi_j)$, $j = 1, 2$. Since $\rho > 0$ on $\Omega$, the polar factor $\phi_j$ is uniquely defined by

$$\phi_j = \frac{\psi_j}{\sqrt{\rho}},$$

then we can define function $f = \psi_1/\psi_2 \in H^1(\Omega)$. A direct computation shows that

$$\nabla f = \frac{1}{\psi_2^2}\left[ \sqrt{\rho}\phi_2(\nabla \sqrt{\rho} + \frac{i}{\hbar}\Lambda)\phi_1 - \sqrt{\rho}\phi_1(\nabla \sqrt{\rho} + \frac{i}{\hbar}\Lambda)\phi_2 \right] = 0, \quad a.e. \text{ in } \Omega.$$

Since $\Omega$ is open and connected, the function $f = \psi_1/\psi_2$ is a constant $C$ on $\Omega$, which satisfies

$$|C| = \frac{|\psi_1|}{|\psi_2|} = 1,$$

therefore formula (3.1) holds true. □

Before stating the next results, here we collect some elementary properties of Sobolev functions that will be used later. Let $g$ be a function in $H^1_x(a, b)$ and $H^1_x(b, c)$, then $g \in H^1_x(a, c)$ if and only if $g$ is continuous at point $b$, and in this case we have

$$\|\partial_x g\|^2_{L^2_x(a,c)} = \|\partial_x g\|^2_{L^2_x(a,b)} + \|\partial_x g\|^2_{L^2_x(b,c)}.$$

Then let us consider $g \in H^1_x(\Omega)$, where $\Omega \subset \mathbf{R}$ is an open set. The continuity of $g$ allows us to decompose the set $\{x; g(x) \neq 0\}$ into disjoint open intervals, i.e.,

$$\{x; g \neq 0\} = \cup_j (a_j, b_j), \quad g(a_j) = g(b_j) = 0. \tag{3.2}$$

The next lemma shows that we can introduce arbitrary constant phase shift on each component $(a_j, b_j)$, without breaking the $H^1$ regularity of $g$.

**Lemma 14.** *Let $g \in H^1_x(\Omega)$ and let $\Theta \in L^\infty_x(\Omega)$ be a piecewise constant phase shift given by the formula*

$$\Theta = \exp\left( i \sum_j \theta_j \mathbf{1}_{(a_j,b_j)} \right), \quad \theta_j \in [0, 2\pi), \tag{3.3}$$

*where $(a_j, b_j)$'s are the components of $\{g \neq 0\}$ as in (3.2). Then we have $\Theta g \in H^1_x(\Omega)$, and*

$$\partial_x(\Theta g) = \Theta \partial_x g. \tag{3.4}$$

*Proof.* The proof of the lemma follows a standard argument of weak derivative. We take $\eta \in C_c^\infty(\Omega)$ to be a test function and consider the weak derivative $\partial_x(\Theta g)$:

$$\int_\Omega \eta\partial_x(\Theta g)\,dx = -\int_\Omega \Theta g\partial_x\eta\,dx = -\sum_j \int_{a_j}^{b_j} e^{i\theta_j} g\partial_x\eta\,dx.$$

On each interval $(a_j, b_j)$ by integration by parts and $g(a_j) = g(b_j) = 0$, we obtain

$$\int_\Omega \eta\partial_x(\Theta g)\,dx = \sum_j \int_{a_j}^{b_j} e^{i\theta_j}\eta\partial_x g\,dx.$$

Furthermore the vanishing derivative Lemma 6 implies $\partial_x g = 0$ a.e. outside $\{x; g(x) \neq 0\} = \bigcup_j (a_j, b_j)$, therefore we can conclude

$$\int_\Omega \eta\partial_x(\Theta g)\,dx = \int_\Omega \eta\Theta\partial_x g\,dx,$$

namely $\partial_x(\Theta g) = \Theta\partial_x g \in L_x^2(\Omega)$, which finishes the proof of the lemma. □

A direct consequence of Lemmas 13 and 14 is the non-uniqueness of wave function lifting for given hydrodynamic data at $H^1$ level, and the only source of non-uniqueness is the arbitrary phase shifts allowed on the components of the non-vacuum regions. More precisely, let $\psi \in H_x^1(\mathbf{R})$ be a wave function associated to $(\sqrt{\rho}, \Lambda)$ in the sense of the Definition 5, then for any piecewise phase shift function $\Theta$ of the form (3.3), by formula (3.4) it is straightforward to check that $\Theta\psi \in H_x^1(\mathbf{R})$ is another wave function associated to the same hydrodynamic data, whose polar factors are $P(\Theta\psi) = \Theta P(\psi)$.

Another application of Lemma 14 is the following Proposition 15. We can show that even though the lifted wave functions do not enjoy general stability property, there is at least one wave function associated to the limiting hydrodynamic data, which can be attained as a strong limit in $H_x^1(\mathbf{R})$ of a subsequence of associated wave functions with well-chosen phase shift.

**Proposition 15.** *Given a sequence of hydrodynamic data* $\{(\sqrt{\rho_n}, \Lambda_n)\}$ *which converges to* $(\sqrt{\rho_0}, \Lambda_0)$ *in* $H_x^1(\mathbf{R}) \times L_x^2(\mathbf{R})$*, let us assume* $\{\psi_n\}$ *and* $\psi_0$ *are wave functions associated to* $\{(\sqrt{\rho_n}, \Lambda_n)\}$ *and* $(\sqrt{\rho_0}, \Lambda_0)$*, respectively.*

*Then there exists a subsequence* $\psi_{n_k}$ *and a piecewise constant phase shift* $\Theta$ *given by the formula* (3.3) *on the connected components of* $\{\psi_0 \neq 0\}$*, such that*

$$\lim_{n_k\to\infty} \|\psi_{n_k} - \Theta\,\psi_0\|_{H_x^1(\mathbf{R})} = 0.$$

*Proof.* For the case that $\rho_0 \equiv 0$, the proposition is trivial. Therefore we assume $\rho_0$ is not identically 0.

By Lemma 14, for any phase shift $\Theta$ given by (3.3), we have $\Theta\psi_0 \in H_x^1(\mathbf{R})$ with

$$\partial_x(\Theta\psi_0) = \Theta\partial_x\psi_0.$$

By the polar decomposition provided by Proposition 7, we have

$$\psi_n = \sqrt{\rho_n}\phi_n \quad \textit{and} \quad \partial_x\psi_n = (\partial_x\sqrt{\rho_n} + \frac{i}{\hbar}\Lambda_n)\phi_n \tag{3.5}$$

with a polar factor $\phi_n \in P(\psi_n)$. Since $\|\phi_n\|_{L_x^\infty} \le 1$, then up to a subsequence $\{\phi_{n_k}\}$ converges weakly* to some $\tilde{\phi}_0$ in $L_x^\infty(\mathbf{R})$. We define $\tilde{\psi}_0 = \sqrt{\rho_0}\tilde{\phi}_0$. Since $(\sqrt{\rho_n}, \Lambda_n)$ converges strongly to $(\sqrt{\rho_0}, \Lambda_0)$ in $H_x^1(\mathbf{R}) \times L_x^2(\mathbf{R})$, by passing to the limit as $n_k \to \infty$ in (3.5) we obtain

$$\psi_{n_k} = \sqrt{\rho_{n_k}}\phi_{n_k} \rightharpoonup \sqrt{\rho_0}\tilde{\phi}_0 = \tilde{\psi}_0, \quad \text{in } L_x^2(\mathbf{R}),$$

and

$$\partial_x \psi_{n_k} = (\partial_x \sqrt{\rho_{n_k}} + \frac{i}{\hbar}\Lambda_{n_k})\phi_{n_k} \rightharpoonup (\partial_x \sqrt{\rho_0} + \frac{i}{\hbar}\Lambda_0)\tilde{\phi}_0, \quad \text{in } L_x^2(\mathbf{R}).$$

Consequently, $\tilde{\psi}_0$ is a weak limit of $\psi_{n_k}$, with $\tilde{\psi}_0 \in H_x^1(\mathbf{R})$ and

$$\partial_x \tilde{\psi}_0 = (\partial_x \sqrt{\rho_0} + \frac{i}{\hbar}\Lambda_0)\tilde{\phi}_0.$$

Moreover the polar factorization of $\psi_{n_k}$ and $\tilde{\psi}_0$ implies

$$\hbar\|\psi_{n_k}\|^2_{H_x^1(\mathbf{R})} = \hbar\|\sqrt{\rho_{n_k}}\|^2_{H_x^1(\mathbf{R})} + \|\Lambda_{n_k}\|^2_{L_x^2(\mathbf{R})} \to \hbar\|\sqrt{\rho_0}\|^2_{H_x^1(\mathbf{R})} + \|\Lambda_0\|^2_{L_x^2(\mathbf{R})} = \hbar\|\tilde{\psi}_0\|^2_{H_x^1(\mathbf{R})},$$

then by the weak convergence and the convergence of norms we obtain $\psi_{n_k} \to \tilde{\psi}_0$ in $H_x^1(\mathbf{R})$. Since $\tilde{\psi}_0$ is a wave function associated to $(\sqrt{\rho_0}, \Lambda_0)$, by Lemma 13 on each component $(a_j, b_j)$ of $\{\psi_0 \neq 0\}$, there exist a unitary $e^{i\theta_j}$ such that

$$\tilde{\psi}_0 = e^{i\theta_j}\psi_0 \; on \; (a_j, b_j).$$

Let

$$\Theta = e^{i\sum_j \theta_j \mathbf{1}_{(a_j,b_j)}},$$

then we have $\tilde{\psi}_0 = \Theta\,\psi_0$ on $\mathbf{R}$. □

Now we are going to prove Theorem 2, which states the wave function lifting at the $H^2$ level. One should notice that Sobolev regularity for the wave function does not correspond to further regularity for the hydrodynamical quantities. For example if we consider $\psi(x) = x$ on $[-1, 1]$, then $\psi \in C_x^\infty$, but on the other hand $\sqrt{\rho}$ is only Lipschitz, with $\partial_x^2 \sqrt{\rho} = \delta_0$. In fact, the same example is considered in more details in [8], which shows that Sobolev regularity above $\frac{3}{2}$ is not preserved, but regularity below $\frac{3}{2}$ is preserved. Considering a wave function $\psi \in H_x^2$ will require further information on the energy density and on the generalised chemical potential $\lambda$ given by Definition 9 (in 1-D we omit the subscript of $\lambda_1$). Another important fact we should emphasise when considering a wave function lifting $\psi \in H_x^2(\mathbf{R})$ is that $\partial_x\psi$ should be a continuous function. As mentioned before, $\psi$ should have well designed phase shifts on all the connected components of the set $\{\rho > 0\}$, such that $\partial_x\psi$ have no jump discontinuity at vacuum boundaries.

In order to prove Theorem 2, we first state the following lemma, which shows the connection between the hydrodynamic quantities $\lambda$, $\frac{\partial_x J}{\sqrt{\rho}}\mathbf{1}_{\{\rho>0\}}$ and the $H_x^2$ regularity of the associated wave function $\psi$.

**Lemma 16.** (1) *If the hydrodynamic state $(\sqrt{\rho}, \Lambda)$ is given through an associated wave function $\psi \in H_x^2(\mathbf{R})$, then $(\sqrt{\rho}, \Lambda)$ is a GCP state and the identities*

$$\frac{\partial_x J}{\sqrt{\rho}}\mathbf{1}_{\{\rho>0\}} = \hbar\,\mathrm{Im}(\bar{\phi}\partial_x^2\psi), \quad \lambda = -\frac{\hbar^2}{2}\,\mathrm{Re}(\bar{\phi}\partial_x^2\psi), \tag{3.6}$$

*hold true in $L^2(\mathbf{R})$, where $\phi \in P(\psi)$ is a polar factor of $\psi$.*

(2) *Let us assume that $(\sqrt{\rho}, \Lambda)$ is a GCP state and let $\psi \in H^1_x(\mathbf{R})$ be a wave function associated to $(\sqrt{\rho}, \Lambda)$, then $\psi \in H^2_x(a_j, b_j)$, for all connected components $(a_j, b_j)$ of the set $\{\rho > 0\}$ as given in* (3.2)*, and we have*

$$\hbar^2 \partial_x^2 \psi = \left[-2\lambda + i\hbar \frac{\partial_x J}{\sqrt{\rho}}\right]\phi \quad \text{in } L^2_x(a_j, b_j). \tag{3.7}$$

*Proof.* Let us consider $(\sqrt{\rho}, \Lambda)$ given through an associated wave function $\psi \in H^2_x(\mathbf{R})$, then it follows from Theorem 1 that $(\sqrt{\rho}, \Lambda) \in H^1_x(\mathbf{R}) \times L^2_x(\Lambda)$. Moreover, by the definition of the hydrodynamic quantities $\rho = |\psi|^2$, $J = \hbar\,\mathrm{Im}(\bar{\psi}\partial_x\psi)$, we have on the set $\{\rho > 0\} = \{|\psi| > 0\}$

$$\hbar\,\mathrm{Im}(\bar{\phi}\partial_x^2\psi) = \hbar \frac{\partial_x \mathrm{Im}(\bar{\psi}\partial_x\psi)}{|\psi|} = \frac{\partial_x J}{\sqrt{\rho}}.$$

Moreover, by using the definition (2.8) for $\lambda$, on the set $\{|\psi| > 0\}$ we have

$$\begin{aligned}
-\frac{\hbar^2}{2}\mathrm{Re}(\bar{\phi}\partial_x^2\psi) &= -\frac{\hbar^2}{2|\psi|}\left[\partial_x \mathrm{Re}(\bar{\psi}\partial_x\psi) - |\partial_x\psi|^2\right] \\
&= -\frac{\hbar^2}{2\sqrt{\rho}}\left[\frac{1}{2}\partial_x^2\rho - (\partial_x\sqrt{\rho})^2 - \hbar^{-2}\Lambda^2\right] \\
&= -\frac{\hbar^2}{2}\partial_x^2\sqrt{\rho} + \frac{\Lambda^2}{2\sqrt{\rho}} = \lambda.
\end{aligned}$$

Let us emphasize that the previous computations are rigorous since $\psi \in H^2_x(\mathbf{R})$ and $\{|\psi| > 0\}$ is an open set. By Lemma 6 we have $\partial_x^2\psi = 0$ a.e. in $\{\rho = 0\}$, thus identities (3.6) hold almost everywhere on $\mathbf{R}$.

To prove the converse statement, we assume $(\sqrt{\rho}, \Lambda)$ to be a GCP state, and let $\psi \in H^1_x(\mathbf{R})$ be a wave function associated to $(\sqrt{\rho}, \Lambda)$. By using the polar factorization, it follows that

$$\psi = \sqrt{\rho}\phi, \; \partial_x\psi = (\partial_x\sqrt{\rho} + \frac{i}{\hbar}\Lambda)\phi,$$

where $\phi \in P(\psi)$. Since $|\psi| > 0$ on $(a_j, b_j)$, the polar factor $\phi \in P(\psi)$ is uniquely defined, $\phi = \frac{\psi}{|\psi|}$, and a direct computation gives

$$\partial_x\phi = \frac{1}{|\psi|}\left(\partial_x\psi - \partial_x|\psi|\frac{\psi}{|\psi|}\right) = \frac{i\Lambda}{\hbar\sqrt{\rho}}\phi.$$

Using the identities above and the definition of hydrodynamic quantities, we obtain

$$\begin{aligned}
\left[-2\lambda + i\hbar\frac{\partial_x J}{\sqrt{\rho}}\right]\phi &= \left(\hbar^2\partial_x^2\sqrt{\rho} + i\hbar\partial_x\Lambda + i\hbar\partial_x\sqrt{\rho}\frac{\Lambda}{\sqrt{\rho}} - \frac{\Lambda^2}{\sqrt{\rho}}\right)\phi \\
&= \hbar\partial_x\left[(\hbar\partial_x\sqrt{\rho} + i\Lambda)\phi\right] = \hbar^2\partial_x(\partial_x\psi).
\end{aligned}$$

Thus we prove the identity (3.7), and by using the bounds of $\lambda$ and $\frac{\partial_x J}{\sqrt{\rho}}$ given in the Definition 10 of GCP state, it follows that $\psi \in H^2_x(a_j, b_j)$. □

The previous lemma shows that the condition

$$\hbar\|\mathbf{1}_{\{\rho>0\}}\partial_x J/\sqrt{\rho}\|_{L^2_x(\mathbf{R})} + \|\lambda\|_{L^2_x(\mathbf{R})} \leq M_2,$$

on the hydrodynamic state allows to improve on the regularity of the wave function $\psi$ on the connected components of $\{\rho > 0\}$. However, in general $\psi$ is not in $H^2_x(\mathbf{R})$, since $\partial_x\psi$ may possibly experience jump discontinuities at vacuum boundaries, due to the fact that the $H^1$ regularity allows for arbitrary phase shifts on connected components. On the other hand, $H^2$ regularity requires compatibility conditions of the phase shift at vacuum boundaries. We will show that, by additionally assuming the continuity of kinetic energy density $k$, it is possible to provide an appropriate choice of the phase shifts on every connected component, which will enable us to construct another wave function $\tilde{\psi} \in H^2_x(\mathbf{R})$ associated to the same hydrodynamical quantities. More precisely, starting from $\psi$, we will construct a $\tilde{\psi} \in H^2_x(\mathbf{R})$ such that $\tilde{\psi} = \Theta\psi$, where $\Theta$ is a piecewise phase shift of the form

$$\Theta = \exp\left(i\sum_j \theta_j \mathbf{1}_{(a_j,b_j)}\right), \quad \theta_j \in [0, 2\pi). \tag{3.8}$$

The main difficulty of our construction lies in the accumulation points of vacuum boundaries. To overcome this difficulty, we need the next lemma which shows that under the assumption of Theorem 2, the kinetic energy density $k$ vanishes at the accumulation points of vacuum boundaries, and $\sqrt{k}$ is indeed a function in the space $H^1_x(\mathbf{R})$.

**Lemma 17.** *Let $(\sqrt{\rho}, \Lambda)$ be a GCP hydrodynamic state and let us assume that the kinetic energy density $k$ is continuous. If $\tilde{x}$ is an accumulation point of vacuum boundaries $\cup_j\{a_j, b_j\}$, then $k(\tilde{x}) = 0$.*

*Proof.* By using Theorem 1, there exists a $\psi \in H^1_x(\mathbf{R})$ associated to $(\sqrt{\rho}, \Lambda)$, and the kinetic energy density is given by

$$k = \frac{\hbar^2}{2}|\partial_x\psi|^2.$$

In particular, the continuity of $k$ implies that $|\partial_x\psi|$ is also a continuous function. To prove Lemma 17, it is sufficient to show $|\partial_x\psi|$ vanishes at the accumulation points of vacuum boundaries.

Let us consider an accumulation point $\tilde{x}$ of the vacuum boundaries $\cup_j\{a_j, b_j\}$, namely there exists a sequence of intervals $\{(a_{j_n}, b_{j_n})\}_n$ such that $(a_{j_n}, b_{j_n}) \to \tilde{x}$ as $n \to \infty$. On each $(a_{j_n}, b_{j_n})$, we take $x_{j_n} = \frac{1}{2}(a_{j_n} + b_{j_n})$, and we claim $\partial_x\psi(x_{j_n}) \to 0$ as $n \to \infty$. If the claim holds, by the continuity of $|\partial_x\psi|$ we obtain $|\partial_x\psi(\tilde{x})| = 0$.

The claim $\partial_x\psi(x_{j_n}) \to 0$ can be proved by contradiction. Indeed, let us assume that

$$\liminf_n |\partial_x\psi(x_{j_n})| > c > 0.$$

Since $a_{j_n}$ and $b_{j_n}$ are vacuum boundary points, namely $\psi(a_{j_n}) = \psi(b_{j_n}) = 0$, then we have

$$0 = \int_{a_{j_n}}^{b_{j_n}} \partial_x\psi(y)dy.$$

Moreover by the property (2) of Lemma 16, we have $\psi \in H^2_x(a_{j_n}, b_{j_n})$, so we can decompose the integral as

$$0 = \int_{a_{j_n}}^{b_{j_n}} \left[\partial_x\psi(x_{j_n}) + \int_{x_{j_n}}^{y} \partial_x^2\psi(s)ds\right]dy = \partial_x\psi(x_{j_n})l_{j_n} + \int_{a_{j_n}}^{b_{j_n}}\int_{x_{j_n}}^{y} \partial_x^2\psi(s)dsdy,$$

where $l_{j_n} = b_{j_n} - a_{j_n}$. By using the identity (3.7) and the bounds (1.8), it follows that

$$\|\partial_x^2\psi\|_{L_x^2(a_{j_n},b_{j_n})} \le 2\hbar^{-2}\|\lambda\|_{L_x^2} + \hbar^{-1}\|\frac{\partial_x J}{\sqrt{\rho}}\mathbf{1}_{\{\rho>0\}}\|_{L_x^2} \le C(\hbar^{-2}M_2),$$

then we have

$$\left|\int_{a_{j_n}}^{b_{j_n}}\int_{x_{j_n}}^{y}\partial_x^2\psi(s)dsdy\right| \le l_{j_n}^{\frac{3}{2}}\|\partial_x^2\psi\|_{L_x^2(a_{j_n},b_{j_n})} \le l_{j_n}^{\frac{3}{2}}C(\hbar^{-2}M_2).$$

For $n$ large enough one has $|\partial_x\psi(x_{j_n})| > c$ and $l_{j_n}^{\frac{1}{2}}C(\hbar^{-2}M_2) < \frac{c}{2}$ since $l_{j_n} \to 0$ as $n \to \infty$, hence it follows that

$$0 \ge |\partial_x\psi(x_{j_n})l_{j_n}| - \left|\int_{a_{j_n}}^{b_{j_n}}\int_{x_{j_n}}^{y}\partial_x^2\psi(s)dsdy\right| > \frac{c}{2}l_{j_n} > 0,$$

which is a contradiction. □

Now we are ready to prove Theorem 2.

*Proof of Theorem 2.* Let us consider the hydrodynamic data $(\sqrt{\rho}, \Lambda)$ given through an associated wave function $\psi \in H^2(\mathbf{R})$, then by the property (1) of Lemma 16 we know that $(\sqrt{\rho}, \Lambda)$ is a GCP state. Moreover, by Sobolev embedding, $k = \frac{\hbar^2}{2}|\partial_x\psi|^2$ is a continuous function.

Now we assume that $(\sqrt{\rho}, \Lambda)$ is a GCP state, and decompose the non vacuum regions into disjoint intervals,

$$\{\rho > 0\} = \cup_j(a_j, b_j), \quad \rho(a_j) = \rho(b_j) = 0. \tag{3.9}$$

Lemma 16 shows that on all the connected components $(a_j, b_j)$ of $\{\rho > 0\}$ we have $\psi \in H_x^2(a_j, b_j)$, and the identity (3.7) holds true in $L_x^2(a_j, b_j)$. However as discussed before, to obtain a $H^2$ wave function defined on the whole $\mathbf{R}$, we need to overcome the possible jump discontinuity at vacuum boundaries. The additionally assumption on the continuity of energy density allows us to provide an well-designed choice of the phase shifts on every connected component and to construct another wave function $\tilde{\psi} \in H_x^2$ associated to the same hydrodynamical quantities. More precisely, starting from $\psi$, we will construct a piecewise phase shift function $\Theta$ of the form

$$\Theta = \exp\left(i\sum_j \theta_j \mathbf{1}_{(a_j,b_j)}\right), \quad \theta_j \in [0, 2\pi), \tag{3.10}$$

such that $\tilde{\psi} = \Theta\psi$ belongs to $H_x^2(\mathbf{R})$.

Before the construction of $\Theta$, we first give some discussion on vacuum boundaries $\cup_j\{a_j, b_j\}$. Let $\{\mathring{a}_j\}$ denote the isolated vacuum points, namely $\rho(\mathring{a}_j) = 0$ and $\rho > 0$ on $(\mathring{a}_j - \epsilon, \mathring{a}_j + \epsilon) \setminus \{\mathring{a}_j\}$ for some small $\epsilon > 0$. Then we define the set

$$W = \{\rho > 0\} \bigcup_j \{\mathring{a}_j\}.$$

By the continuity of $\rho$ and the definition of $\mathring{a}_j$, it is straightforward to see that $W$ is an open set, consequently it can be represented as disjoint open intervals

$$W = \bigcup_j(\bar{a}_j, \bar{b}_j),$$

where $\bar{a}_j, \bar{b}_j \in \cup_j\{a_j, b_j\} \setminus \cup_j\{\mathring{a}_j\}$. The set $\cup_j\{a_j, b_j\} \setminus \cup_j\{\mathring{a}_j\}$ consists of the following two types of vacuum boundary points. The first type is the boundary of large vacuum intervals, where by definition of $\mathring{a}_j$'s we have $|\psi|^2 = \rho \equiv 0$ on $(\bar{a}_j - \epsilon, \bar{a}_j)$ or on $(\bar{b}_j, \bar{b}_j + \epsilon)$ for some $\varepsilon > 0$, and by the continuity of $|\partial_x\psi|$ we have $|\partial_x\psi(\bar{a}_j)| = |\partial_x\psi(\bar{b}_j)| = 0$. The second type is the accumulation point of vacuum boundaries, where $|\partial_x\psi|$ also vanishes as proved in Lemma 17.

Let us fix an interval $(\bar{a}_j, \bar{b}_j)$ and let $\{\mathring{a}_{j_k}\}_{k=K_1}^{K_2}$ denote the vacuum points in $(\bar{a}_j, \bar{b}_j)$. By our construction, the only possible accumulation points of $\{\mathring{a}_{j_k}\}$ are $\bar{a}_j$ and $\bar{b}_j$, hence we can assume $\mathring{a}_{j_k} < \mathring{a}_{j_{k+1}}$ for all $k$.

The phase shift function $\Theta$ on $(\bar{a}_j, \bar{b}_j)$ is constructed through the following strategy. We start from a fixed $(\mathring{a}_{j_0}, \mathring{a}_{j_1})$ and set $\theta_{j_0} = 0$, namely $\Theta = 1$ on $(\mathring{a}_{j_0}, \mathring{a}_{j_1})$. To extend the definition of $\Theta$ to $(\mathring{a}_{j_1}, \mathring{a}_{j_2})$, we notice that the intervals $(\mathring{a}_{j_k}, \mathring{a}_{j_{k+1}})$ are connected components of the set $\{\rho > 0\}$. Thus by Lemma 16 we have the $H^2$ regularity of $\psi$ on $(\mathring{a}_{j_0}, \mathring{a}_{j_1})$ and $(\mathring{a}_{j_1}, \mathring{a}_{j_2})$, which implies the the left and right side limits $\partial_x\psi(\mathring{a}_{j_1}^-)$ and $\partial_x\psi(\mathring{a}_{j_1}^+)$ exist. On the other hand, the continuity of the kinetic energy density $k$ implies $|\partial_x\psi| = \hbar^{-1}\sqrt{2k}$ is continuous, hence we have $|\partial_x\psi(\mathring{a}_{j_1}^-)| = |\partial_x\psi(\mathring{a}_{j_1}^+)|$. Therefore we can choose a $\theta_{j_1} \in [0, 2\pi)$ such that $(\Theta \cdot \partial_x\psi)(\mathring{a}_{j_1}^-) = e^{i\theta_{j_1}}\partial_x\psi(\mathring{a}_{j_1}^+)$ ($\theta_{j_1} = 0$ if $|\partial_x\psi|$ vanishes at $\mathring{a}_{j_1}$), and we define $\Theta = e^{i\theta_{j_1}}$ on $(\mathring{a}_{j_1}, \mathring{a}_{j_2})$. By repeating this process inductively, we extend the definition of $\Theta$ to the whole $(\bar{a}_j, \bar{b}_j)$. Moreover we have $\Theta\partial_x\psi \in H^1(\mathring{a}_{j_k}, \mathring{a}_{j_k})$ and our construction ensures the continuity of $\Theta\partial_x\psi$ at all point vacuum $\mathring{a}_{j_k}$. Thus by the elementary property of Sobolev functions, it follows that $\Theta\partial_x\psi \in H^1_x(\bar{a}_j, \bar{b}_j)$, and by Lemmas 14 and 16 we have the identity

$$\hbar^2\partial_x(\Theta\partial_x\psi) = \left[\hbar^2\partial_x^2\sqrt{\rho} - \frac{\Lambda^2}{\sqrt{\rho}} + i\hbar\frac{\partial_x J}{\sqrt{\rho}}\right]\Theta\phi \quad in\ L^2(\bar{a}_j, \bar{b}_j), \quad \phi \in P(\psi). \tag{3.11}$$

To this point we have constructed the phase shift function $\Theta$ on $W = \cup_j(\bar{a}_j, \bar{b}_j)$, and we extend $\Theta$ to the whole $\mathbf{R}$ by defining $\Theta = 1$ on $W^c$, thus $\Theta$ has the form (3.10) as required, and we define $\tilde{\psi} = \Theta\psi$. By Lemma 14 we have $\partial_x\tilde{\psi} = \Theta\partial_x\psi$, and $\tilde{\psi} \in H^1_x(\mathbf{R})$ is also a wave function associated to $(\sqrt{\rho}, \Lambda)$ which satisfies

$$\hbar\partial_x\tilde{\psi} = (\hbar\partial_x\sqrt{\rho} + i\Lambda)\Theta\phi. \tag{3.12}$$

Now we prove $\partial_x\tilde{\psi} = \Theta\partial_x\psi \in H^1_x(\mathbf{R})$. Let $\eta \in C_c^\infty(\mathbf{R})$ be an arbitrary test function, then we have

$$\int_{\mathbf{R}} \eta\,\partial_x(\Theta\partial_x\psi)dr := -\int_{\mathbf{R}} (\Theta\partial_x\psi)\partial_x\eta\,dr.$$

By the vanishing derivative Lemma 6, it follows that $\partial_x\psi = 0$ a.e. outside the set $W$, therefore

$$\int_{\mathbf{R}} \eta\,\partial_x(\Theta\partial_x\psi)dr = -\int_W (\Theta\partial_x\psi)\partial_x\eta\,dx = -\sum_j \int_{\bar{a}_j}^{\bar{b}_j} (\Theta\partial_x\psi)\partial_x\eta\,dx.$$

Since $\Theta\partial_x\psi \in H^1_x(\bar{a}_j, \bar{b}_j)$, we can write

$$\int_{\bar{a}_j}^{\bar{b}_j} (\Theta\partial_x\psi)\partial_x\eta\,dx = -\int_{\bar{a}_j}^{\bar{b}_j} \eta\,\partial_x(\Theta\partial_r\psi)dx + \eta(\bar{b}_j)(\Theta\partial_x\psi)(\bar{b}_j) - \eta(\bar{a}_j)(\Theta\partial_x\psi)(\bar{a}_j),$$

where the boundary terms vanish since $\partial_x\psi(\bar{a}_j) = \partial_x\psi(\bar{b}_j) = 0$ as we have shown before. By using the identity (3.11) and the bounds for GCP states in Definition 10, we obtain

$$\left|\hbar^2\int_{\mathbf{R}} \eta\,\partial_x(\Theta\partial_x\psi)dx\right| \le \|\eta\|_{L^2_x(\mathbf{R})}\left(\|\hbar^2\partial_x^2\sqrt{\rho} - \frac{\Lambda^2}{\sqrt{\rho}}\|_{L^2_x(W)} + \hbar\|\frac{\partial_r J}{\sqrt{\rho}}\|_{L^2_x(W)}\right) \le C(M_2)\|\eta\|_{L^2_x(\mathbf{R})}.$$

Thus we conclude $\partial_x\tilde{\psi} = \Theta\partial_x\psi \in H^1_x(\mathbf{R})$. Moreover again by the Sard type Lemma 6, we have $\partial_x(\Theta\partial_x\psi) = 0$ almost everywhere on the set $\{\rho = 0\} = \{\psi = 0\}$, therefore the identity

$$\hbar^2\partial_x^2\tilde{\psi} = \hbar^2\partial_x(\Theta\partial_x\psi) = \left[\hbar^2\partial_x^2\sqrt{\rho} - \frac{\Lambda^2}{\sqrt{\rho}} + i\hbar\frac{\partial_x J}{\sqrt{\rho}}\right]\mathbf{1}_{\{\rho>0\}}\Theta\phi \tag{3.13}$$

holds true in $L^2_x(\mathbf{R})$, and the bound (1.9) for the wave function $\tilde{\psi}$ follows directly. □

## 4. Wave function lifting with quantum vortices

One of the main obstacles for multi-dimensional problems is the topology of the vacuum region $\{\rho = 0\}$. First, while in one space dimension the energy estimate and the Sobolev embedding yield the position density to be continuous, which consequently implies that $\{\rho = 0\}$ is a closed set, in the multidimensional cases we loose this topological information and in general it is not possible to characterize the structure of the vacuum region with the same methods of the 1-D case. Moreover, for $d \geq 2$ an extra structural property of the quantum vortex discussed in Section 1 needs to be taken into account. Therefore we first focus on the wave function lifting method in two-dimensional space, and we assume that the position density $\rho = \rho(x)$, $x \in \mathbf{R}^2$, is continuous such that $V = \{x; \rho(x) = 0\}$ consists of isolated points, namely there exists an at most countable set $\mathcal{A}$ of indices, such that

$$V = \{x_{(\alpha)}\}_{\alpha\in\mathcal{A}} \subset \mathbf{R}^2, \quad \inf_{\alpha\neq\beta}|x_{(\alpha)} - x_{(\beta)}| > 0. \tag{4.1}$$

Moreover a quantization condition for the vorticity is required, as explained in the discussion of quantum vortices presented in Section 1. More precisely, let $v = J/\rho$ be the velocity field, which is well-defined almost everywhere, then the quantised vorticity condition (1.6) is restated in the sense of distribution as

$$\begin{cases} v \in \mathcal{M}(\mathbf{R}^2), \\ \nabla \wedge v = 2\pi\hbar \sum_{\alpha\in\mathcal{A}} k_\alpha \delta_{x_{(\alpha)}}, \quad k_\alpha \in \mathbf{Z}, \end{cases} \tag{4.2}$$

where $\mathcal{M}(\mathbf{R}^2)$ is the space of measures, and $\delta_{x_{(\alpha)}}$ is the Dirac-delta function supported at $x_{(\alpha)}$ for $\alpha \in \mathcal{A}$. This condition is connected to the Bohr-Sommerfeld quantization rule in quantum theory.

*Remark* 18. A straightforward consequence of the wave function lifting is that the quantized vorticity condition (4.2) implies the generalized irrotationality condition

$$\nabla \wedge J = 2\nabla\sqrt{\rho} \wedge \Lambda, \quad \text{for a.e. } x \in \mathbf{R}^2. \tag{4.3}$$

More precisely, given a hydrodynamic state $(\sqrt{\rho}\Lambda)$ satisfying the assumptions of Theorem 3, then we know that there exists an associated wave function $\psi \in H^1_x(\mathbf{R}^2)$, then as a consequence of the polar factorization Proposition 7, we further have that the hydrodynamic state $(\sqrt{\rho}, \Lambda)$ satisfy the identity (4.3), where $J = \sqrt{\rho}\Lambda$. On the other hand, the reverse implication from (4.3) to (4.2) does not hold, see e.g., [6].

Now we are at the point to prove the wave function lifting proposition in the two dimensional case. We will use the notion of difference quotients (see [11]), namely for any function $f \in L^1_{loc}(\mathbf{R}^2)$ and direction $\mathbf{h} \in \mathbf{R}^2$, $|\mathbf{h}| > 0$, we define

$$D^{\mathbf{h}} f(x) = \frac{f(x + \mathbf{h}) - f(x)}{|\mathbf{h}|}. \tag{4.4}$$

As $|\mathbf{h}| \to 0$, the difference quotient $D^{\mathbf{h}} f$ will approximate the derivative of $f$ in the direction $\mathbf{h}$.

We remark that in the cases of quantum vortices, the non-vacuum region is a connected set, therefore as a direct consequence of Lemma 13, the lifted wave function is unique upto constant rotation.

*Proof of Theorem 3.* We restrict to the simplest case, when the vacuum is a single point, and without loss of generality we assume $V = \{0\}$. In this case the quantised vorticity condition becomes

$$\begin{cases} v \in \mathcal{M}(\mathbf{R}^2), \\ \nabla \wedge v = 2k\hbar\pi\delta_0, \quad k \in \mathbf{Z}. \end{cases} \tag{4.5}$$

For isolated vacuum points, the proof follows essentially the same idea.

To construct a wave function we take a sequence of smooth mollifier $\{\chi^\epsilon\}$ such that $supp(\chi^\epsilon) \subset B^\epsilon(0)$, where $B_r(x)$ is the ball centred at $x$ with radius $r$. We define

$$v^\epsilon(x) = v * \chi^\epsilon(x) = \langle v(\cdot), \chi^\epsilon(x - \cdot)\rangle,$$

so that

$$\nabla \wedge v^\epsilon = (\nabla \wedge v) * \chi^\epsilon = 2k\hbar\pi\delta_0 * \chi^\epsilon = 2k\pi\chi^\epsilon.$$

Let us fix an arbitrary $x_0 \notin B^\epsilon(0)$, we can take a piecewise smooth curve $\gamma(x)$ connecting $x_0$ and $x$, strictly away from $B^\epsilon(0)$. In this way, for any $x \notin B^\epsilon(0)$, the approximating polar factor $\phi^\epsilon$ can be defined as

$$\phi^\epsilon(x) = \exp\left(\frac{i}{\hbar}\int_{\gamma(x)} v^\epsilon(y) \cdot d\vec{l}(y)\right)$$

and we extend the definition of $\phi^\epsilon$ continuously to $x \in B^\epsilon(0)$ by setting $\phi^\epsilon(x) = \phi^\epsilon(\epsilon x/|x|)$.

We claim that the above definition of $\phi^\epsilon$ is independent of the choice of curve $\gamma$. Indeed, for any two curves $\gamma_1$ and $\gamma_2$ connecting $x_0$ and $x$, we can patch them into a piecewise smooth closed curve $\tilde{\gamma}$. Denote $\Omega$ the domain inside $\tilde{\gamma}$, then we have two cases: $B^\epsilon(0) \subset \Omega$ or $B^\epsilon(0) \cap \Omega = \emptyset$. In either case by Stokes' theorem

$$\int_{\gamma_1} v^\epsilon(y) \cdot d\vec{l}(y) - \int_{\gamma_2} v^\epsilon(y) \cdot d\vec{l}(y) = \int_{\tilde{\gamma}} v^\epsilon(y) \cdot d\vec{l}(y) = \int_\Omega \nabla \wedge v^\epsilon(y) dy = 2k\hbar\pi \ or \ 0,$$

where $k$ is the integer given in (4.5), thus we have

$$\exp\left(\frac{i}{\hbar}\int_{\gamma_1} v^\epsilon(y) \cdot d\vec{l}(y)\right) = \exp\left(\frac{i}{\hbar}\int_{\gamma_2} v^\epsilon(y) \cdot d\vec{l}(y)\right).$$

The polar factors $\{\phi^\epsilon\}$ have uniform $L^\infty_x$ bound 1, therefore up to subsequences we have $\phi^\epsilon \overset{*}{\rightharpoonup} \phi$ in $L^\infty_x(\mathbf{R}^2)$ with $\|\phi\|_{L^\infty_x} \le 1$. The lifted wave function is defined by $\psi = \sqrt{\rho}\phi$. Obviously $\psi \in L^2_x(\mathbf{R}^2)$, then we are going to show $\psi \in H^1_x(\mathbf{R}^2)$.

For this purpose, we only need to show the $L^2$ norm of the difference quotient $D^{\mathbf{h}}\phi$ defined in (4.4) has a uniform bound, namely

$$\|D^{\mathbf{h}}\psi\|_{L^2_x(\mathbf{R}^2)} \le C$$

for some constant $C$ and all $\mathbf{h} \in \mathbf{R}^2$ with $|\mathbf{h}| \neq 0$ small. We have

$$\|D^{\mathbf{h}}\psi\|_{L^2_x(\mathbf{R}^2)} = \|D^{\mathbf{h}}\psi\|_{L^2_x(B_{|\mathbf{h}|}(0))} + \|D^{\mathbf{h}}\psi\|_{L^2_x(B_{|\mathbf{h}|}(0)^c)}.$$

For the part near the vacuum point 0, we simply bound it by

$$\begin{aligned}\|D^{\mathbf{h}}\psi\|_{L^2_x(B_{|\mathbf{h}|}(0))} &= \frac{1}{|\mathbf{h}|}\|\psi(\cdot + h) - \psi(\cdot)\|_{L^2_x(B_{|\mathbf{h}|}(0))} \\ &\leq \frac{2}{|\mathbf{h}|}\|\sqrt{\rho}\|_{L^\infty_x(B_1(0))}|B_{|\mathbf{h}|}(0)|^{\frac{1}{2}} = 2\sqrt{\pi}\|\sqrt{\rho}\|_{L^\infty_x(B_1(0))},\end{aligned}$$

which is uniformly bounded since $\sqrt{\rho}$ is continuous.

Then we consider the norm away from the vacuum. For any relatively compact open set $U \subset B_{|\mathbf{h}|}(0)^c$ and for any $\epsilon < |\mathbf{h}|$, we have explicit representation

$$\begin{aligned}\phi^\epsilon &= \exp\left(\frac{i}{\hbar}\int_\gamma v^\epsilon(y) \cdot d\vec{l}(y)\right), \\ \nabla\phi^\epsilon &= \frac{i}{\hbar}v^\epsilon\phi^\epsilon.\end{aligned}$$

By the continuity of $\sqrt{\rho}$, we have $\inf_{x\in U}\{\sqrt{\rho}(x)\} = \alpha > 0$ and the velocity field $v = \Lambda/\sqrt{\rho} \in L^2_x(U)$. Moreover, $v^\epsilon \to v$ in $L^2_x(U)$ and $\phi = * - \lim_{\varepsilon\to 0}\phi^\varepsilon$, then it follows

$$\nabla\phi^\epsilon = \frac{i}{\hbar}v^\epsilon\phi^\epsilon \rightharpoonup \frac{i}{\hbar}v\phi = \nabla\phi \quad in\ L^2_x(U). \tag{4.6}$$

Thus we obtain that in $L^2_x(U)$

$$\nabla\psi = (\nabla\sqrt{\rho} + \frac{i}{\hbar}\sqrt{\rho}v)\phi = (\nabla\sqrt{\rho} + \frac{i}{\hbar}\Lambda)\phi$$

with uniform $L^2_x$ bound

$$\hbar\|\nabla\psi\|_{L^2_x(U)} = \hbar\|\nabla\sqrt{\rho}\|_{L^2_x(U)} + \|\Lambda\|_{L^2_x(U)} \leq M_1.$$

By using a sequence of relatively compact sets $U$ invading $B^c$, we get

$$\|\nabla\psi\|_{L^2_x(B_h(0)^c)} \leq \hbar^{-1}M_1,$$

and we conclude $\psi \in H^1_x(\mathbf{R}^2)$. Furthermore, the above argument implies that for a.e. $x \in \mathbf{R}^2$ we have

$$\nabla\psi = (\nabla\sqrt{\rho} + \frac{i}{\hbar}\Lambda)\phi, \tag{4.7}$$

which proves the polar factorization (2.2).

Further let us assume the condition (1.13), then we are able to show that $\psi \in H^2(\mathbf{R}^2)$. The proof uses difference quotients as in the previous case, but here it requires further small technicalities since $\nabla\psi \notin L^\infty_x$.

Let us consider a relatively compact set $U$, strictly away from the vacuum point 0. By (4.6) and (4.7) it follows that for $j = 1, 2$,

$$[(\hbar^2\partial_{x_j}^2\sqrt{\rho} - \frac{\Lambda_j^2}{\sqrt{\rho}}) + i\hbar\frac{\partial_{x_j}J_j}{\sqrt{\rho}}]\phi = (\hbar^2\partial_{x_j}^2\sqrt{\rho} + i\hbar\partial_{x_j}\Lambda_j + i\hbar\partial_{x_j}\sqrt{\rho}\,v_j - \Lambda_j v_j)\phi$$
$$= \hbar\partial_{x_j}\left[\hbar(\partial_{x_j}\sqrt{\rho} + \Lambda_j)\phi\right] = \hbar^2\partial_{x_j}(\partial_{x_j}\psi)$$

in the sense of distribution. Therefore the condition (1.13) and Definition 9 of $\lambda_j$ imply $\partial_{x_j}^2\psi \in L_x^2(U)$ satisfying the uniform bound

$$\|\partial_{x_j}^2\psi\|_{L_x^2(U)} \leq \hbar^{-2}M_2. \tag{4.8}$$

Now we again take the smooth mollifier $\{\chi^\epsilon\}$ such that $supp(\chi^\epsilon) \subset B^\varepsilon(0)$, and we define the smooth approximating wave function $\psi^\varepsilon = \psi * \chi^\varepsilon$. It follows (4.8) that for any relatively compact open set $U$ such that $dist(U, 0) > \varepsilon$, we have

$$\|\partial_{x_j}^2\psi^\varepsilon\|_{L_x^2(U)} \leq \|\partial_{x_j}^2\psi\|_{L_x^2(U^\varepsilon)} \leq \hbar^{-2}M_2, \tag{4.9}$$

where $U^\varepsilon$ is the $\varepsilon-$neighbourhood of the set $U$. For $\partial_{x_j}\psi^\varepsilon$, $j = 1, 2$, by using the polar factorization we can decompose it as $\partial_{x_j}\psi^\varepsilon = |\partial_{x_j}\psi^\varepsilon|\phi_j^\varepsilon$, where the polar factor $\phi_j^\varepsilon = \partial_{x_j}\psi^\varepsilon/|\partial_{x_j}\psi^\varepsilon|$ when $|\partial_{x_j}\psi^\varepsilon| \neq 0$. In this case direct computation shows

$$\partial_{x_j}\phi_j^\varepsilon = i\frac{\mathrm{Im}(\partial_{x_j}\overline{\psi^\varepsilon}\partial_{x_j}^2\psi^\varepsilon)}{|\partial_{x_j}\psi^\varepsilon|^2}\phi_j^\varepsilon, \tag{4.10}$$

and by the same argument as polar factorization we have

$$|\partial_{x_j}^2\psi^\varepsilon|^2 = \left(\partial_{x_j}|\partial_{x_j}\psi^\varepsilon|\right)^2 + \left(\frac{\mathrm{Im}(\partial_{x_j}\overline{\psi^\varepsilon}\partial_{x_j}^2\psi^\varepsilon)}{|\partial_{x_j}\psi^\varepsilon|}\right)^2. \tag{4.11}$$

To apply the theory of difference quotients, instead of $\partial_{x_j}\psi^\varepsilon$ and $\partial_{x_j}\psi$, we will consider the $L^\infty$ cut-off functions $\{f_{j,n}^\varepsilon\}$ and $f_{j,n}$ defined by

$$f_{j,n}^\varepsilon = \min\{|\partial_{x_j}\psi^\varepsilon|, n\}\phi_j^\varepsilon, \quad j = 1, 2,$$
$$f_{j,n} = \min\{|\partial_{x_j}\psi|, n\}\phi_j, \quad \phi_j \in P(\partial_{x_j}\psi).$$

By using the fact

$$|f_{j,n}^\varepsilon - f_{j,n}| \leq |\partial_{x_j}\psi^\varepsilon - \partial_{x_j}\psi|,$$

and $\partial_{x_j}\psi^\varepsilon \to \partial_{x_j}\psi$ in $L_x^2(\mathbf{R}^2)$, it follows $f_{j,n}^\varepsilon \to f_{j,n}$ in $L^2(\mathbf{R}^2)$ as $\varepsilon \to 0$.

Now we consider the difference quotient of $f_{j,n}$ in the direction $e_j$, where

$$e_1 = (1, 0) \quad and \quad e_2 = (0, 1),$$

and we let $\mathbf{h}_j = |\mathbf{h}|\, e_j$ for $|\mathbf{h}| \neq 0$ small, and we have

$$\|D^{\mathbf{h}_j}f_{j,n}\|_{L_x^2(\mathbf{R}^2)} = \|D^{\mathbf{h}_j}f_{j,n}\|_{L_x^2(B_{|\mathbf{h}|}(0))} + \|D^{\mathbf{h}_j}f_{j,n}\|_{L_x^2(B_{|\mathbf{h}|}(0)^c)}.$$

For the part near the vacuum, it can be simply controlled by

$$\|D^{\mathbf{h}_j} f_{j,n}\|_{L^2_x(B_{|\mathbf{h}|}(0))} \le 2\sqrt{\pi}\|f_{j,n}\|_{L^\infty_x} \le 2\sqrt{\pi}n.$$

The part away from the vacuum is treated again by considering a relatively compact open sets $U \subset B_{|\mathbf{h}|}(0)^c$. By definition $|f^\varepsilon_{j,n}| \le |\partial_{x_j}\psi^\varepsilon|$, and since $\partial^2_{x_j}\psi^\varepsilon \in L^2_x(U)$ for all $\varepsilon < |\mathbf{h}|$, the $x_j$–derivative of the truncated function $f^\varepsilon_{j,n}$ also belongs to $L^2_x(U)$. Moreover by using (4.10) and (4.11), for $|\partial_j\psi^\varepsilon| > n$ we have

$$|\partial_{x_j} f^\varepsilon_{j,n}|^2 = |\partial_{x_j}(n\phi^\varepsilon_j)|^2 = \left(n\frac{\operatorname{Im}(\partial_{x_j}\overline{\psi^\varepsilon}\partial^2_{x_j}\psi^\varepsilon)}{|\partial_{x_j}\psi^\varepsilon|^2}\right)^2 \le \left(\frac{\operatorname{Im}(\partial_{x_j}\overline{\psi^\varepsilon}\partial^2_{x_j}\psi^\varepsilon)}{|\partial_{x_j}\psi^\varepsilon|}\right)^2 \le |\partial^2_{x_j}\psi^\varepsilon|^2,$$

and $\partial_{x_j} f^\varepsilon_{j,n} = \partial^2_{x_j}\psi^\varepsilon$ in the case $|\partial_{x_j}\psi^\varepsilon| \le n$. Therefore by using (4.9), we have the uniform bound

$$\|\partial_{x_j} f^\varepsilon_{j,n}\|_{L^2_x(U)} \le \|\partial^2_{x_j}\psi^\varepsilon\|_{L^2_x(U)} \le \hbar^{-2}M_2.$$

By passing to the limit as $\varepsilon \to 0$, we obtain

$$\|\partial_{x_j} f_{j,n}\|_{L^2_x(U)} \le \hbar^{-2}M_2 \tag{4.12}$$

for any relatively compact open set $U$ strictly away from 0, hence it also works on $B_{|\mathbf{h}|}(0)^c$. Therefore the theory of difference quotient shows $\partial_{x_j} f_{j,n} \in L^2(\mathbf{R}^2)$ and we have the bound

$$\|\partial_{x_j} f_{j,n}\|_{L^2_x(\mathbf{R}^2)} \le \hbar^{-2}M_2 + 2\sqrt{\pi}n.$$

Notice that even though the bound of $\|\partial_{x_j} f_{j,n}\|_{L^2}$ obtained above depends on $n$, as long as we have $\partial_{x_j} f_{j,n} \in L^2(\mathbf{R}^2)$, a zero measure set has no effect on the $L^2$ norm of $\partial_{x_j} f_{j,n}$. Thus by choosing a sequence of open set $U$ invading $\mathbf{R}^2 - \{0\}$ in (4.12), we obtain that

$$\|\partial_{x_j} f_{j,n}\|_{L^2_x(\mathbf{R}^2)} \le \hbar^{-2}M_2, \quad j = 1, 2.$$

On the other hand, it is straightforward to see that $f_{j,n}$ converges to $\partial_{x_j}\psi$ strongly as $n \to \infty$ by construction. Therefore we conclude $\partial^2_{x_j}\psi \in L^2_x(\mathbf{R}^2)$ for $j = 1, 2$, with the bound

$$\|\partial^2_{x_j}\psi\|_{L^2_x(\mathbf{R}^2)} \le \hbar^{-2}M_2.$$

It implies $\triangle\psi \in L^2_x(\mathbf{R}^2)$ and we have

$$\|\triangle\psi\|_{L^2_x(\mathbf{R}^2)} \le 2\hbar^{-2}M_2,$$

then by standard argument we can conclude that $\psi \in H^2_x(\mathbf{R}^2)$ satisfies the bound

$$\|\psi\|_{H^2_x(\mathbf{R}^2)} \le C(\hbar^{-1}M_1 + \hbar^{-2}M_2).$$

□

The wave function lifting argument can also be extended to a simple example of 3D hydrodynamic data, where we assume planar symmetry of the data and the vorticity to be concentrated in a finite number of vortex lines. This is a typical example of the vortex structure of quantum flows in 3D space [22]. More precisely, we consider hydrodynamic data $(\rho(x), J(x))$, $x = (x_1, x_2, x_3) \in \mathbf{R}^3$ and $J \in \mathbf{R}^3$, of the following form:

$$\rho(x) = \rho_1(x_1, x_2)\rho_2(x_3) \quad \textit{and} \quad J(x) = (J_1(x_1, x_2)\rho_2(x_3), 0) = \sqrt{\rho}\Lambda, \tag{4.13}$$

where $\rho, \rho_j \in \mathbf{R}$ and $J_1 \in \mathbf{R}^2$. The data $(\rho, J)$ is essentially a product of a 2D data $(\rho_1, J_1)$ and a 1D data $(\rho_2, 0)$. We assume $(\rho_1, J_1) = ((\sqrt{\rho_1})^2, \sqrt{\rho_1}\Lambda_1)$ has pointwise vacuum and quantised vorticity, namely we assume $\rho_1$ to be continuous with vacuum structure (1.11), and the velocity field $v_1 = J_1/\rho_1$ to satisfy the quantized vorticity condition (1.12). The regularity of $(\rho_1, J_1)$ is characterised by the bounds (1.10) and (1.13), and we assume $\sqrt{\rho_2} \in H^s_x(\mathbf{R})$ with $s = 1$ or 2.

Under the above assumptions, we can state the following proposition as an extension of the wave function lifting argument to 3D data.

**Proposition 19.** *Let $(\rho, J)$ be a hydrodynamic data as in* (4.13)*, with $(\rho_1, J_1)$ satisfying the conditions* (1.11) *and* (1.12)*. Furthermore let us assume that $(\rho_1, J_1)$ satisfy the mass and energy bounds* (1.10) *and $\sqrt{\rho_2} \in H^1_x(\mathbf{R})$. Then there exists a wave function $\psi \in H^1_x(\mathbf{R}^3)$ such that*

$$\sqrt{\rho} = |\psi|, \quad \Lambda = \hbar\,\mathrm{Im}(\bar{\phi}\nabla\psi),$$

*where $\phi \in P(\psi)$ is a polar factor defined in* (2.1)*.*

*If we furthermore assume the bounds* (1.13) *on $(\rho_1, J_1)$ and $\sqrt{\rho_2} \in H^2(\mathbf{R})$, then $\psi \in H^2_x(\mathbf{R}^3)$ and*

$$\|\psi\|_{H^2_x(\mathbf{R}^3)} \leq C(\hbar^{-1}M_1 + \hbar^{-2}M_2).$$

## 5. Wave function lifting for non vacuum solutions of QHD models

In the last section we will extend the wave function lifting method to GCP weak solutions of QHD system (1.3) given by Definition 10. For existence results of GCP solutions, we refer to [4, 5] for polynomial potential $H = \rho^{\gamma-1}$, $\gamma > 1$. We also restrict ourselves to the case of non-vacuum solutions to avoid the difficulty involving the structure of vacuum regions and the dynamics of system (1.3).

In the previous results, the lifted wave function $\psi$ associated to given hydrodynamic data $(\rho, J)$ is constructed at every fixed time, which can be rotated by any constant phase shift. However, to recover the dynamics of the corresponding Schrödinger equation, the evolution of the phase function should also be considered in the scheme of wave function lifting, which is solved by a space-time gradient equation involving the velocity $v$ and the chemical potential $\mu$, as we will see in the next lemma.

**Lemma 20** (Phase function)**.** *Let $(\rho, J)$ be a GCP solution of* (1.3) *on $[0, T) \times \mathbf{R}^d$ with potential $H \in L^\infty_t L^1_{loc,x}$, such that ρ is strictly positive on any compact set. Then there exists a unique (upto constant phase shifts in space-time) phase function $S \in (L^\infty_t H^1_{loc,x} \cap W^{1,\infty}_t L^1_{loc,x})([0, T) \times \mathbf{R}^d)$, which can be explicitly written as*

$$\begin{aligned} S(t, x) =& S_* + \int_U \int_{l(0,x^*)} v_0(y) \cdot d\vec{l}_y dx^* \\ &+ \int_U \int_{l(x^*,x)} v(t, y) \cdot d\vec{l}_y dx^* - \int_U \int_0^t \mu(s, x_*) ds dx^*, \end{aligned} \tag{5.1}$$

*where* $S_* \in [0, 2\pi)$ *and* $U = (-\frac{1}{2}, \frac{1}{2})^d$.

*Proof.* By Lemma 16, for $(\rho, J)$ generated by a wave function $\psi$ such that $|\psi| = \sqrt{\rho}$ is strictly positive on any compact set, we have

$$v = \mathrm{Im}\left(\frac{\nabla\psi}{\psi}\right), \quad \mu = -\frac{\hbar^2}{2}\frac{\triangle\sqrt{\rho}}{\sqrt{\rho}} + \frac{1}{2}|v|^2 + H = -\frac{\hbar^2}{2}\mathrm{Re}\left(\frac{\triangle\psi}{\psi}\right) + H,$$

which imply $v \in L^\infty_t([0,T), L^2_{loc,x}(\mathbf{R}^d))$ and $\mu \in L^\infty_t([0,T), L^1_{loc,x}(\mathbf{R}^d))$. On the other hand, by Madelung transformation (1.1), the phase function $S$, formally given by $S = \frac{1}{i}\log\left(\frac{\psi}{|\psi|}\right)$, of $\psi$ should satisfy a space-time gradient equation

$$\begin{cases} \nabla S = v, \quad \partial_t S = -\mu, \\ S(0,0) = S_*. \end{cases} \tag{5.2}$$

Therefore we should define the phase function $S$ to be the unique solution of the space-time gradient Eq (5.2). For locally strictly positive density, the generalised irrotationality condition (2.5) reduces to

$$\nabla \wedge v = 0$$

in the sense of distribution, and the equation for momentum density in (1.3) can be written as

$$\partial_t v = \frac{\hbar^2}{2}\nabla\left(\frac{\triangle\sqrt{\rho}}{\sqrt{\rho}}\right) - \frac{1}{2}\nabla|v|^2 - \nabla H = -\nabla\mu, \tag{5.3}$$

which exactly combine to the solvability condition of the gradient Eq (5.2).

To rigorously define the phase function $S$ in hydrodynamic functions, we again use the smooth mollifiers $\{\chi^\epsilon\}$ and define

$$v^\epsilon(x) = v * \chi^\epsilon(x), \quad \mu^\epsilon(x) = \mu * \chi^\epsilon(x),$$

which also satisfy the "irrotationality" conditions

$$\nabla \wedge v^\varepsilon = 0, \quad \partial_t v^\varepsilon = -\nabla\mu^\varepsilon. \tag{5.4}$$

Then we define the approximating phase function to be

$$S^\varepsilon(t, x; x^*) = S_* + \int_{l(0,x^*)} v_{\varepsilon,0}(y) \cdot d\vec{l_y} + \int_{l(x^*,x)} v^\varepsilon(t,y) \cdot d\vec{l_y} - \int_0^t \mu^\varepsilon(s, x_*) ds,$$

where $x_* \in \mathbf{R}^d$ is an arbitrary point, and the curve $l(x, y)$ of integration is given by

$$l(x,y) = \cup_{j=1}^d \{(y_1, \ldots, y_{j-1}, z, x_{j+1}, \ldots, x_d); z \in (\min\{x_j, y_j\}, \max\{x_j, y_j\})\},$$

namely the piecewise straight lines parallel to frame axises connecting $x$ and $y$. It is straightforward to check that the approximating phase function $S^\varepsilon$ solves the mollification of Eq (5.2), namely

$$\begin{cases} \nabla S^\varepsilon = vS^\varepsilon, \quad \partial_t S^\varepsilon = -\mu^\varepsilon, \\ S^\varepsilon(0,0) = S^\varepsilon_*. \end{cases}$$

However, for weak solutions $(\rho, J)$, the velocity $v$ and the chemical potential $\mu$ are merely Lebesgue functions in $L_t^\infty L_{loc,x}^1$, therefore we can not directly take the limit $\varepsilon \to 0$ in $S^\varepsilon(t, x; x^*)$. On the other hand, due to the irrotationality condition (5.4), the choice of $x^*$ has no influence on the definition of $S^\varepsilon(t, x; x^*)$, thus we can take the average of $S^\varepsilon(t, x; x^*)$ with respect to $x_* \in U = (-\frac{1}{2}, \frac{1}{2})^d$, which allows us to give the hydrodynamic definition of the approximating phase function $S^\varepsilon$ independent of $x_*$ as

$$\begin{aligned} S^\varepsilon(t, x) =& S_* + \int_U \int_{l(0,x^*)} v_{\varepsilon,0}(y) \cdot d\vec{l}_y dx^* \\ &+ \int_U \int_{l(x^*,x)} v^\varepsilon(t, y) \cdot d\vec{l}_y dx^* - \int_U \int_0^t \mu^\varepsilon(s, x_*) ds dx^*. \end{aligned}$$

By take the limit $\varepsilon \to 0$, we obtain the explicit formulation of the phase function $S$ as (5.1). The regularity of $S$ is a direct consequence of (5.1) and $v \in L_t^\infty([0, T), L_{loc,x}^2(\mathbf{R}^d))$, $\mu \in L_t^\infty([0, T), L_{loc,x}^1(\mathbf{R}^d))$. □

*Remark* 21. If we replace the local positivity of $\rho$ and the irrotationality condition $\nabla \wedge v = 0$ with the quantised vorticity condition (4.2), and assume that the velocity and the chemical potential satisfy

$$v \in L_t^\infty([0, T), L_{x,loc}^1(\mathbf{R}^d)), \quad \mu \in \mathcal{M}([0, T) \times \mathbf{R}^d),$$

where $\mathcal{M}([0, T) \times \mathbf{R}^d)$ is the space of Radon measures, then we can extend Lemma 20 to the cases of quantum vortex. However, so far assumptions above for $v$ and $\mu$, and the structure of quantum vortex, are not proven to be preserved along general GCP weak solutions, due to the lack of estimates of $v$ and $\mu$ when vacuums exist. Therefore we restrict ourselves to the case of non-vacuum.

Now we can proof the main theorem of this section.

*Proof of Theorem 4.* Following the assumption of Theorem 4, we can define a wave function

$$\psi = \sqrt{\rho} e^{\frac{i}{\hbar} S},$$

where $S$ is the phase function of $(\rho, J)$ given by Lemma 20. By Eq (5.2), it is straightforward to compute that $\psi$ is an wave function associated to $(\rho, J)$ in the sense

$$\rho = |\psi|^2, \quad J = \hbar \operatorname{Im}(\bar{\psi} \nabla \psi).$$

Therefore by Theorem 2 or Theorem 3, we have $\psi \in L_t^\infty([0, T), H_x^2(\mathbf{R}^d))$. Moreover, by the continuity equation of $\rho$ and (5.2), we can compute that

$$\hbar \partial_t \psi = (\hbar \partial_t \sqrt{\rho} + i \sqrt{\rho} \partial_t S) e^{\frac{i}{\hbar} S} = -\left[ \hbar \frac{\operatorname{div} J}{2 \sqrt{\rho}} + i \sqrt{\rho} \mu \right] e^{\frac{i}{\hbar} S}.$$

Recall that

$$\begin{aligned} \operatorname{div} J &= \hbar \operatorname{Im}(\bar{\psi} \triangle \psi), \\ \mu &= -\frac{\hbar^2}{2} \operatorname{Re}\left( \frac{\triangle \psi}{\psi} \right) + H, \end{aligned}$$

then it follows that

$$\begin{aligned}\hbar\partial_t\psi &= \left[-\hbar^2\frac{\mathrm{Im}(\bar{\psi}\triangle\psi)}{2\rho} + \frac{i\hbar^2}{2}\mathrm{Re}\left(\frac{\triangle\psi}{\psi}\right) - iH\right]\sqrt{\rho}e^{\frac{i}{\hbar}S} \\ &= \left[-\frac{\hbar^2}{2}\mathrm{Im}\left(\frac{\triangle\psi}{\psi}\right) + \frac{i\hbar^2}{2}\mathrm{Re}\left(\frac{\triangle\psi}{\psi}\right) - iH\right]\psi \\ &= \frac{i\hbar^2}{2}\triangle^2\psi - iH\psi,\end{aligned}$$

which shows $\psi$ is a solution of the corresponding Schrödinger equation. The uniqueness of the lifted wave function is a direct consequence of Lemma 13 and the local positivity of $|\psi|$. □

## 6. Conclusions

In this paper, we consider the Pauli problem to construct a quantum state from given distribution of position density $\rho$ and momentum density $J$ of particles, and the uniqueness of the associated quantum state.

In the following cases: 1-dimensional data with general vacuum regions, multi-dimensional quantum vortex data satisfying the quantised vorticity condition (1.6), and solutions of the QHD system (1.3) without vacuum, we provide the wave function lifting method to construct a complex wave function $\psi$ as a quantum state associated to given $(\rho, J)$ in the sense of

$$\rho = |\psi|^2, \quad J = \mathrm{Im}(\bar{\psi}\nabla\psi),$$

where $(\rho, J)$ belongs to the finite energy space or the GCP space (see Definition 10), and the wave function $\psi$ has corresponding regularity in $H^1_x$ or $H^2_x$.

Concerning the uniqueness problem, we prove that in general the lifted wave function is not unique if the connected components of the non-vacuum region $\{\rho > 0\}$ are strictly separated by open neighbourhoods. However, we also show that the only resource of non-uniqueness is the lost of information on vacuum boundaries, and any two lifted wave functions associated to the same $(\rho, J)$ are only differed by a piecewise constant phase shift function of the form

$$\Theta = \exp\left(i\sum_j \theta_j \mathbf{1}_{\Omega_j}\right), \quad \theta_j \in [0, 2\pi),$$

where $\Omega_j$'s are the connected components of $\{\rho > 0\}$.

## Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

## Conflict of interest

The authors declare no conflicts of interest.

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