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Research article

Propagation of logarithmic regularity and inviscid limit for the 2D Euler equations†

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Abstract: The aim of this note is to study the Cauchy problem for the 2D Euler equations under very low regularity assumptions on the initial datum. We prove propagation of regularity of logarithmic order in the class of weak solutions with L^p initial vorticity, provided that $p \geq 4$. We also study the inviscid limit from the 2D Navier-Stokes equations for vorticity with logarithmic regularity in the Yudovich class, showing a rate of convergence of order $|\log \nu|^{-\alpha/2}$ with $\alpha > 0$.

Keywords: incompressible Euler equations; vorticity; propagation of regularity; modulus of continuity; inviscid limit

1. Introduction

We consider the Cauchy problem for the two-dimensional incompressible Euler equations in vorticity formulation

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega = 0, \\ u(t,\cdot) = \nabla^{\perp}(-\Delta)^{-1}\omega(t,\cdot), \\ \omega(0,\cdot) = \omega_0, \end{cases} \tag{1.1}$$

where the unknowns are the vorticity $\omega : [0,T]\times\mathbb{T}^2 \to \mathbb{R}$ and the velocity $u : [0,T]\times\mathbb{T}^2 \to \mathbb{R}^2$, while $\omega_0 : \mathbb{T}^2 \to \mathbb{R}$ is a given initial datum with zero average, i.e.,

$$\int_{\mathbb{T}^2} \omega_0(x)\,\mathrm{d}x = 0.$$

The coupling between the velocity and the vorticity in the second line of (1.1) is known as the *Biot-Savart law*, and in particular it implies the incompressibility condition $\operatorname{div} u = 0$. The understanding of the well-posedness of these equations represents one of the classic problems of mathematical fluid dynamics; we refer to [31] for an overview of the available theory. In the two-dimensional case the Euler equations in vorticity formulation are a non-linear and non-local transport equation driven by an incompressible velocity field. Thus, a simple energy estimate for smooth solutions gives that

$$\|\omega(t,\cdot)\|_{L^p} = \|\omega_0\|_{L^p}, \qquad \forall t > 0,$$

for all $1 \le p \le \infty$. It is therefore natural to study weak solutions of (1.1) in an L^p framework. We highlight from the outset that we work on the two-dimensional torus $\mathbb{T}^2$ to reduce technical details, but all the results can be adapted to the whole-space case $\mathbb{R}^2$ requiring $\omega_0 \in L^1 \cap L^p(\mathbb{R}^2)$. The existence of weak solutions with $\omega_0 \in L^p$ with $p > 1$ has been proved by DiPerna and Majda in [23], see also [21] for positive measures Radon measures $\omega_0 \in H^{-1}_{\mathrm{loc}}$ and [36] for $\omega_0 \in L^1$. Uniqueness of weak solutions is known only in the case of bounded vorticity $\omega \in L^\infty$ (see [39]) or slightly less (see [20,40]). However, the uniqueness of weak solutions in L^p with $p < \infty$ is still an open and very challenging problem. Recently several non-uniqueness results have been proved (see [9–11,33,37,38]) but the complete picture is far from being fully understood.

In these notes we investigate two problems related to (1.1). The first one is the *propagation of regularity* for weak solutions of (1.1). Roughly speaking, it means the following: Given a Banach space Y we want to understand whether the information that the initial datum ω_0 belongs to Y implies that the solution $\omega(t,\cdot) \in Y$ for every $t > 0$. In [4] it is shown that if one considers Dini continuous initial data ω_0, the 2D Euler equations (1.1) admits a unique global solution within these critical spaces. In particular, the propagation of the Dini semi-norm provides an L^∞ bound on ∇u with a constant that grows exponentially in time, see also [27]. In the case the velocity field is not Lipschitz there is no propagation in general due to the lack of Lipschitz regularity of the flow. However, one can have a control on the loss of regularity. This is indeed the case for Yudovich's solutions for which the velocity is only log-Lipschitz. In this regards, in [2] it has been shown that if $\omega_0 \in L^\infty \cap W^{\alpha,p}$ then, for all $0 < \beta < \alpha$, the unique bounded solution ω of (1.1) belongs to the space $W^{\beta(t),p}$ where

$$\beta(t) := \beta \exp\left(-\int_0^t V(\tau)\,\mathrm{d}\tau\right), \qquad V(t) := \sup_{0<|x-x'|\le 1} \frac{|u(t,x)-u(t,x')|}{|x-x'|(1-\log|x-x'|)}. \tag{1.2}$$

We point out that the quantity $\int_0^t V(\tau)\,\mathrm{d}\tau$ is finite since u is log-Lipschitz. Later on, in [19] it is showed that if $\omega_0 \in W^{\alpha,p}$ (with $\alpha p \le 2$, $p > 1$ and $\alpha \in (0,2)$) is a *continuous* function, then the unique bounded solution $\omega(t,\cdot) \in W^{\beta,p}$ for any $t \in [0,T]$ and any $0 < \beta < \alpha$. These results have been improved recently in [14], where the authors showed that solutions in the Yudovich class $\omega \in L^\infty([0,T]\times\mathbb{T}^2) \cap C([0,T];L^1(\mathbb{T}^2))$ satisfy the following: let $0 < \alpha \le 1$ and $p \ge 1$, we have that

(i) If $\omega_0 \in W^{\alpha,p}(\mathbb{T}^2)$ then

$$[\omega(t,\cdot)]_{W^{\alpha(t),p}} \lesssim_{\alpha,p} \|\omega_0\|_{L^\infty} + [\omega_0]_{W^{\alpha,p}},$$

for any $t \in [0,T]$, where $\alpha(t) = \frac{\alpha}{1+C\|\omega_0\|_{L^\infty}\alpha p t}$.

(ii) If $\omega_0 \in C(\mathbb{T}^2) \cap W^{\alpha,p}(\mathbb{T}^2)$ with $p > 1$ then $\omega(t,\cdot) \in W^{\beta,p}(\mathbb{T}^2)$ for any $0 < \beta < \alpha$ and any $t \in [0,T]$. When $\alpha = 1$ we also have $\omega(t,\cdot) \in W^{1,p'}(\mathbb{T}^2)$ for any $1 \le p' < p$ and any $t \in [0,T]$.

(iii) If $\omega_0 \in W^{\alpha,p}(\mathbb{T}^2)$ with $p > 2/\alpha$ then $\omega(t,\cdot) \in W^{\alpha,p}(\mathbb{T}^2)$ for any $t \in [0,T]$.

In a similar fashion, in [26] the author constructed an example of initial datum $\omega_0 \in L^\infty \cap W^{1,p}$ with $p > 2$ such that the unique bounded solution ω continuously loses *integrability* in time, i.e., $\omega(t,\cdot)$ belongs to the Sobolev space $W^{1,p(t)}$ with $p(t)$ being a decreasing function of time. On the other hand, log-Hölder coefficients of Yudovich's solution of (1.1) are conserved if one assumes that the velocity is Lipschitz, see [15].

For more irregular initial data, in [18] the authors showed that if $\omega_0 \in L^\infty \cap B^s_{p,\infty}$, for some $s > 0$ and $p \ge 1$, the unique solution ω of (1.1) satisfies $\omega(t,\cdot) \in L^\infty \cap B^{s(t)}_{p,\infty}$ with $s(t) := s\exp(-Ct\|\omega_0\|_\infty)$. Here $B^s_{p,\infty}$ denotes the usual Besov space. Moreover, they suggest that the loss may be improved to a polynomial law as in [14] and [18, Remark 3].

Our purpose is to look for a (possibly more general) class of initial data so that the regularity is propagated without any loss. The motivation comes from recent results on the propagation of regularity for the linear transport equation driven by irregular velocity fields, i.e., the equation

$$\begin{cases} \partial_t\theta + b\cdot\nabla\theta = 0, \\ \theta(0,\cdot) = \theta_0. \end{cases} \tag{1.3}$$

It is well known that if the vector field is less regular than Lipschitz, the regularity of the initial data can be immediately lost, see [1]. We refer to [3] for a classical reference in which the case of Besov regularity with Log-Lipschitz field is analyzed. On the other hand, in [12] it has been shown that bounded solutions of (1.3) propagate *regularity of logarithmic order* provided that b is a divergence-free vector field in $L^1([0,T];W^{1,p}(\mathbb{T}^d))$. In detail, for any $\alpha > 0$ we define $H^{\log,\alpha}$ to be the functional space

$$H^{\log,\alpha}(\mathbb{T}^d) := \left\{ f \in L^2(\mathbb{T}^d) : [f]^2_{H^{\log,\alpha}} := \int_{B_{1/3}}\int_{\mathbb{T}^d} \frac{|f(x+h)-f(x)|^2}{|h|^d} \frac{\mathrm{d}x\,\mathrm{d}h}{\log(1/|h|)^{1-\alpha}} < \infty \right\},$$

which is a Banach space endowed with the norm $\|f\|^2_{H^{\log,\alpha}} := \|f\|_{L^2} + [f]^2_{H^{\log,\alpha}}$. The authors of [12] show the following bound on the $[\,\cdot\,]_{H^{\log,\alpha}}$ semi-norm

$$[\theta(t,\cdot)]_{H^{\log,p}} \lesssim_{p,d} \left(\int_0^t \|\nabla b(s,\cdot)\|_{L^p}\,\mathrm{d}s \right)^{\frac{p}{2}} \|\theta_0\|_{L^\infty} + [\theta_0]_{H^{\log,p}}, \tag{1.4}$$

with $\alpha = p > 1$. The same result has been reproduced in [34] using an equivalent Besov-type semi-norm and Littlewood–Paley's theory. Going back to the Euler equations, it is immediate to check that the bound (1.4) holds for solutions in the Yudovich class (as pointed out in Theorem 3.1 below). However, our aim is to go beyond the class of bounded solutions. To do this, we must consider a slightly bigger functional space but still with the same scaling of $H^{\log,\frac{1}{2}}$, namely the space $W^2_{\log,\frac{1}{2}}$ introduced in [5, 6]: We will say that a function $f \in L^2(\mathbb{T}^d)$ belongs to $W^2_{\log,\frac{1}{2}}(\mathbb{T}^d)$ if

$$\sup_{0<h\le 1/2} \frac{1}{|\log h|^{\frac{1}{2}}} \int_{\mathbb{T}^d}\int_{\mathbb{T}^d} K_h(x-y)|u(x)-u(y)|^2 \,\mathrm{d}x\,\mathrm{d}y < \infty,$$

where the kernel K_h is a positive, bounded, smooth and symmetric function defined as

$$K_h(x) = \frac{1}{(|x|+h)^d}, \qquad \text{for } |x| < 1/2,$$

see Section 3 for the precise definition. Then, we can prove that any solution of (1.1) arising from initial data $\omega_0 \in L^p \cap W^2_{\log,\frac{1}{2}}$ with $p \ge 4$ satisfies $\omega(t,\cdot) \in L^p \cap W^2_{\log,\frac{1}{2}}$. The result is the following:

Theorem 1.1. *Let* $\omega_0 \in L^4 \cap W^2_{\log,\frac{1}{2}}(\mathbb{T}^2)$ *and let* $\omega \in L^\infty([0,T]; L^4(\mathbb{T}^2))$ *be any weak solution of* (1.1) *starting from* ω_0. *Then, there exists a constant* $C > 0$ *such that* ω *satisfies*

$$[\omega(t,\cdot)]_{W^2_{\log,\frac{1}{2}}} \le [\omega_0]_{W^2_{\log,\frac{1}{2}}} + C\sqrt{t}\|\omega_0\|^{\frac{1}{2}}_{L^2}\|\omega_0\|_{L^4}, \qquad \textit{for any } t \in [0,T]. \tag{1.5}$$

Remarkably, Theorem 1.1 holds in a class in which uniqueness is not known, but all the solutions in $L^\infty([0,T]; L^4(\mathbb{T}^2))$ are renormalized in the sense of DiPerna-Lions [22], as shown in [30]. Moreover, in addition to improving the result in terms of integrability of the initial datum ω_0, the proof is substantially different from the one in [12] being more Eulerian in spirit. In particular, we do not use the Lagrangian representation of the solutions (which still holds as a consequence of the renormalization property) and we use a commutator estimate proved in [6, Proposition 13]. We observe here *en passant* that the Littlewood-Paley approach of [34] could also work under the hypothesis of Theorem 1.1, but we preferred not to follow this route in order to make the proof as simple as possible.

Finally, we believe that Theorem 1.1 may be generalized to a larger class of interpolation spaces which ultimately includes $H^{\log,\alpha}$ and, more in general, logarithmic Besov spaces $B^{\log,b}_{2,q}$. A comprehensive approach to these spaces may be found in [24]. We plan to address this issue in a forthcoming paper.

Our second interest is the inviscid limit from the 2D incompressible Navier-Stokes equations

$$\begin{cases} \partial_t\omega^\nu + u^\nu\cdot\nabla\omega^\nu = \nu\Delta\omega^\nu, \\ u^\nu(t,\cdot) = \nabla^\perp(-\Delta)^{-1}\omega^\nu(t,\cdot), \\ \omega^\nu(0,\cdot) = \omega_0^\nu, \end{cases} \tag{NS}$$

where $\nu > 0$ is the kinematic viscosity of the fluid, and ω_0^ν is a zero-average initial datum (possibly depending on ν) such that

$$\omega_0^\nu \to \omega_0, \qquad \text{in } L^p(\mathbb{T}^2).$$

We recall that in two dimensions solutions of the Navier-Stokes equations (NS) are regular if the initial datum is square integrable, i.e., for $\omega_0^\nu \in L^2(\mathbb{T}^2)$. Here we look for rates of convergence as the viscosity $\nu \to 0$ when the spatial domain is the torus $\mathbb{T}^2$, so that no boundary layers have to be considered. In this regards, the case of smooth initial data $u_0 \in H^s$ (with $s > 2$) was analyzed by Masmoudi in [32] who showed that

$$\|u^\nu(t,\cdot) - u(t,\cdot)\|_{H^{s'}} \lesssim \begin{cases} \nu t, & \text{if } s' \leq s-2, \\ (\nu t)^{(s-s')/2}, & \text{if } s-2 \leq s' \leq s-1, \end{cases} \tag{1.6}$$

together with the implications on the vorticity side ($\omega_0 \in H^s$ with $s > 1$). We also mention the recent work [25] in which the authors found a rate of convergence of order ν *uniform in time* for time-quasi-periodic solutions of (1.1) with a small time-quasi-periodic external force.

For Yudovich's solutions, the following rate of convergence for the velocity field was proved by Chemin in [16]

$$\sup_{t\in[0,T]} \|u^\nu(t,\cdot) - u(t,\cdot)\|_{L^2} \leq (4\nu T)^{\frac{1}{2}\exp(-CT\|\omega_0\|_{L^\infty})} \|\omega_0\|_{L^2\cap L^\infty} e^{1-\exp(-CT\|\omega_0\|_{L^\infty})}, \tag{1.7}$$

for some constant $C > 0$. We also refer to [35] for a log improvement of this rate. Concerning rates of convergence for the vorticity, a similar power law rate has been showed in [18] under the additional assumption $\omega_0 \in B^s_{p,\infty}$ with $s > 0$ and $p \geq 1$. In particular, in [18] the authors prove the following rate

$$\|\omega^\nu(t,\cdot) - \omega(t,\cdot)\|_{L^p} \lesssim (\nu t)^{\frac{s(t)}{1+s(t)}}, \qquad s(t) := \exp(-Ct\|\omega_0\|_\infty). \tag{1.8}$$

It is important to point out that this rate has been obtained by combining losing estimates for the Besov regularity (i.e., $\omega^\nu(t,\cdot) \in B^{s(t)}_{p,\infty}$) together with the rate for the velocity (1.7). Notice that (1.7) and (1.8) are not uniform in time and they actually deteriorate exponentially fast. In this regards, the improvement of the losing estimate proved in [14] implies that the rate (1.8) holds with an exponent that deteriorates polynomially in time, i.e., $\tilde{s}(t) = \frac{s}{1+Ctsp}$, as pointed out in [18, Remark 3]. In [17] we proved a related result for Yudovich's solutions but with a different approach: without requiring any kind of regularity on the initial data, we showed that it is possible to obtain the rate of convergence

$$\|\omega^\nu(t,\cdot) - \omega(t,\cdot)\|_{L^p} \lesssim \max\{\phi(\nu), \nu^\beta\}, \tag{1.9}$$

where β depends on $\|\omega_0\|_\infty, T$, and p, while $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function with $\phi(0) = 0$. In detail, the function ϕ is a modulus of continuity depending on the initial datum ω_0 such that

$$\|\omega_0(\cdot + h) - \omega_0\|_{L^p} \leq \phi(|h|), \qquad \text{for } |h| \text{ small enough.}$$

Notice that the regularity of the initial datum is somehow encoded in the function ϕ and it can be made quantitative assuming some regularity on ω_0. Our purpose here is somehow between [17] and [18]: We aim to prove a rate which does not deteriorate in time requiring some regularity on the initial datum. To this end, we look for a regularity class for the initial datum that is rougher than Besov and which provides an explicit form for the function ϕ appearing in (1.9). Once again, the inviscid limit for advection-diffusion equations with Sobolev velocity fields (see [7, 12, 34]) and the propagation of logarithmic regularity suggest to look for a logarithmic rate of convergence.

Our second main result is the following:

Theorem 1.2. *Let $\omega_0 \in L^\infty \cap H^{\log,\alpha}(\mathbb{T}^2)$ for some $\alpha > 0$. Let ω and ω^ν be, respectively, the unique bounded solutions of the Euler and Navier-Stokes equations arising from ω_0. Then, there exists a constant $C > 0$ depending on $\alpha, T, \|\omega_0\|_{H^{\log,\alpha}}$, and $\|\omega_0\|_{L^\infty}$ such that*

$$\sup_{t\in(0,T)} \|\omega^\nu(t,\cdot) - \omega(t,\cdot)\|_{L^2} \leq \frac{C}{|\log \nu|^{\alpha/2}}. \tag{1.10}$$

The strategy of the proof relies on the stochastic Lagrangian representation of solutions of the Navier-Stokes equations as in [17], i.e., the solution ω^ν has an explicit formula in terms of the stochastic flow of u^ν which is given by

$$\omega^\nu(t,x) = \mathbb{E}[\omega_0^\nu(X_{t,0}^\nu(x))],$$

with $\mathbb{E}$ denoting the expectation, and $X_{t,s}^\nu$ is the solution of the stochastic differential equation

$$\begin{cases} \mathrm{d}X_{t,s}^\nu(x,\xi) = u^\nu(s, X_{t,s}^\nu(x,\xi))\,\mathrm{d}s + \sqrt{2\nu}\,\mathrm{d}W_s(\xi), & s \in [0,t), \\ X_{t,t}^\nu(x,\xi) = x. \end{cases}$$

The rate (1.9) is then obtained exploiting the estimate on the difference quotients of functions belonging to $H^{\log,\alpha}$ (see Theorem 2.2 below) and the convergence in the zero-noise limit of the stochastic flow X^ν towards the deterministic flow of the limit solution. We remark once again that our result provides a rate of convergence for a class of initial data more irregular than the one considered in [18]. Moreover, it is worth noting that the constant C in (1.10) diverges for $T \to \infty$, but the rate of convergence is independent of T.

2. Functional setting

In this section we fix the notations and we recall some preliminary results. We start by recalling the definitions and some properties of the spaces of functions with derivatives of logarithmic order from [12, 13]. For any $\alpha > 0$ we define the space

$$H^{\log,\alpha}(\mathbb{T}^d) := \left\{ f \in L^2(\mathbb{T}^d) : \int_{B_{1/3}} \int_{\mathbb{T}^d} \frac{|f(x+h) - f(x)|^2}{|h|^d} \frac{\mathrm{d}x\,\mathrm{d}h}{\log(1/|h|)^{1-\alpha}} < \infty \right\},$$

and the corresponding semi-norm

$$[f]_{H^{\log,\alpha}}^2 := \int_{B_{1/3}} \int_{\mathbb{T}^d} \frac{|f(x+h) - f(x)|^2}{|h|^d} \frac{\mathrm{d}x\,\mathrm{d}h}{\log(1/|h|)^{1-\alpha}}. \tag{2.1}$$

The space $H^{\log,\alpha}(\mathbb{T}^d)$ is a Banach space endowed with the norm

$$\|f\|_{H^{\log,\alpha}}^2 := \|f\|_{L^2}^2 + [f]_{H^{\log,\alpha}}^2. \tag{2.2}$$

Using the Fourier representation, the following characterization is shown to hold in [12]:

$$\|f\|_{H^{\log,\alpha}}^2 \sim_{\alpha,d} \sum_{k \in \mathbb{Z}^d} \log(2 + |k|)^\alpha |\hat{u}(k)|^2. \tag{2.3}$$

We now give a precise statement for the theorem on the propagation of regularity to which we referred in the introduction, see [12, Corollary 1.2].

Theorem 2.1. *Let $b \in L^1([0,T]; W^{1,p}(\mathbb{T}^d))$ be a divergence-free vector field for some $p > 1$. Then, any solution $\theta \in L^\infty([0,T] \times \mathbb{T}^d)$ of (1.3) satisfies*

$$[\theta(t,\cdot)]_{H^{\log,p}} \lesssim_{p,d} \left(\int_0^t \|\nabla b(s,\cdot)\|_{L^p}\,\mathrm{d}s \right)^{\frac{p}{2}} \|\theta_0\|_{L^\infty} + [\theta_0]_{H^{\log,p}}, \tag{2.4}$$

for any $t \in [0,T]$.

For any $f \in H^{\log,\alpha}(\mathbb{T}^d)$ we define for any $x \in \mathbb{T}^d$ the function

$$L_\alpha f(x) := \left(\int_{B_{1/3}} \frac{|f(x+h) - f(x)|^2}{|h|^d} \frac{\mathrm{d}h}{\log(1/|h|)^{1-\alpha}} \right)^{\frac{1}{2}}, \tag{2.5}$$

and it follows that $L_\alpha f \in L^2(\mathbb{T}^d)$ with

$$\|L_\alpha f\|_{L^2} = [f]_{H^{\log,\alpha}}. \tag{2.6}$$

The following estimate on the difference quotients holds, see [13, Theorem 1.11].

Theorem 2.2. *Let $\alpha > 0$ be fixed. For any $f \in H^{\log,\alpha}(\mathbb{T}^d)$ it holds*

$$|f(x) - f(y)| \leq C(d,\alpha) \log(1/(|x-y|))^{-\alpha/2} \left(L_\alpha f(x) + L_\alpha f(y) \right), \tag{2.7}$$

for any $x, y \in \mathbb{T}^d$ such that $|x-y| < 1/36$.

We point out that the previous theorem is proved in [13] in functional spaces that are a generalization of $H^{\log,\alpha}$ in a non-Hilbertian framework. Precisely, one can consider the spaces $X^{\gamma,p}$ which are defined as follows:

Definition 2.3. *Let $p \in (0,\infty)$ and $\gamma \in (0,\infty)$ be fixed. We define*

$$[f]_{X^{\gamma,p}(\mathbb{T}^d)} := \left(\int_{B_{1/3}} \int_{\mathbb{T}^d} \frac{|f(x+h) - f(x)|^p}{|h|^d} \frac{\mathrm{d}x\,\mathrm{d}h}{\log(1/|h|)^{1-p\gamma}} \right)^{1/p}, \tag{2.8}$$

and we set

$$X^{\gamma,p}(\mathbb{T}^d) := \{ f \in L^p(\mathbb{T}^d) : [f]_{X^{\gamma,p}(\mathbb{T}^d)} < \infty \}. \tag{2.9}$$

For $p \geq 1$, the space $X^{\gamma,p}(\mathbb{T}^d)$ is a Banach space endowed with the norm

$$\|f\|_{X^{\gamma,p}} = \|f\|_{L^p} + [f]_{X^{\gamma,p}}, \tag{2.10}$$

and the Gagliardo semi-norm $[f]_{X^{\gamma,p}}$ is lower semicontinuous with respect to the strong topology of L^p. With these notations, we have that $H^{\log,\alpha} = X^{\frac{\alpha}{2},2}$. Moreover, the semi-norms $[\,\cdot\,]_{H^{\log,\alpha}}$ are increasing in α, i.e for any $0 < \alpha \leq \alpha' < \infty$ it holds

$$[f]_{H^{\log,\alpha}} \leq [f]_{H^{\log,\alpha'}},$$

see [13, Proposition 1.3].

3. Propagation of regularity

The aim of this section is to analyze the propagation of logarithmic regularity for weak solutions of the 2D Euler equations. For solutions in the Yudovich class the propagation follows immediately by Theorem 2.1 as shown in the following:

Theorem 3.1. *Let $\omega_0 \in L^\infty \cap H^{\log,p}(\mathbb{T}^2)$ for some $p > 1$. Then, the unique solution $\omega \in L^\infty([0,T] \times \mathbb{T}^2)$ of* (1.1) *satisfies*

$$[\omega(t,\cdot)]_{H^{\log,p}} \lesssim_p t^{\frac{p}{2}} \|\omega_0\|_{L^\infty}^{1+\frac{p}{2}} + [\omega_0]_{H^{\log,p}}, \qquad \textit{for any } \ 0 \leq t \leq T. \tag{3.1}$$

Proof. By Theorem 2.1, for any $0 \leq t \leq T$ the unique bounded solution ω satisfies

$$[\omega(t,\cdot)]_{H^{\log,p}} \lesssim_p \left(\int_0^t \|\nabla u(s,\cdot)\|_{L^p}\,\mathrm{d}s\right)^{\frac{p}{2}} \|\omega_0\|_{L^\infty} + [\omega_0]_{H^{\log,p}}. \tag{3.2}$$

By the properties of the Biot-Savart operator, we have that

$$\|\nabla u(s,\cdot)\|_{L^p} \lesssim_p \|\omega(s,\cdot)\|_{L^p} \leq \|\omega_0\|_{L^\infty}, \qquad \text{for any } \ 0 \leq s \leq T. \tag{3.3}$$

where in the last inequality we used that the spatial domain is $\mathbb{T}^2$ together with the basic energy estimate on ω. Substituting (3.3) in (3.2) the proof is completed. □

We now consider the case of unbounded vorticity. To this end, we need to introduce some preliminary definitions and results from [5, 6, 8] that we adapt to our context.

Definition 3.2. *Let* $0 < \theta < 1$ *and define the semi-norms*

$$[u]_\theta^2 := \sup_{0<h\leq 1/2} \frac{1}{|\log h|^\theta} \int_{\mathbb{T}^d}\int_{\mathbb{T}^d} K_h(x-y)|u(x)-u(y)|^2\,\mathrm{d}x\,\mathrm{d}y, \tag{3.4}$$

where the kernel K_h *is a positive, bounded and symmetric function defined as*

$$K_h(x) = \frac{1}{(|x|+h)^d}, \qquad \text{for } |x| < 1/2, \tag{3.5}$$

independent of h for $|x| \geq 2/3$, *equal to a positive constant outside* $B(0,3/4)$*, and periodized so as to belong in* $C^\infty(\mathbb{T}^d \setminus B(0,3/4))$.

Notice that (3.4) defines a semi-norm because it vanishes if u is a constant. Moreover, the semi-norms are decreasing in θ, i.e.,

$$[u]_\theta \leq [u]_{\theta'}, \ \text{ if } \theta' \leq \theta. \tag{3.6}$$

Correspondingly, we define the space $W^2_{\log,\theta}$ as follows

$$W^2_{\log,\theta} := \{u \in L^2(\mathbb{T}^d) : [u]_\theta < \infty\}, \tag{3.7}$$

which is a Banach space endowed with the norm

$$\|u\|^2_{2,\theta} = \|u\|^2_{L^2} + [u]^2_\theta. \tag{3.8}$$

The following proposition holds, see [6, Proposition 1].

Proposition 3.3. *For any* $s > 0$ *and any* $0 < \theta < 1$ *one has the (compact) embeddings*

$$W^{s,2}(\mathbb{T}^d) \subset W^2_{\log,\theta}(\mathbb{T}^d) \subset L^2(\mathbb{T}^d).$$

In addition, using the Fourier representation it holds

$$\|u\|^2_{L^2} + [u]^2_\theta \sim \sup_h \sum_{k\in\mathbb{Z}^d\setminus\{0\}} \frac{1+\left|\log\left(\frac{1}{|k|}+h\right)\right|}{|\log h|^\theta}|\hat{u}(k)|^2 \leq \sum_{k\in\mathbb{Z}^d} \log(1+|k|)^{1-\theta}|\hat{u}(k)|^2. \tag{3.9}$$

In view of the Proposition 3.3 and the equivalence in (2.3), for $0 < \theta < 1$ we have the inclusion

$$H^{\log,1-\theta}(\mathbb{T}^d) \subset W^2_{\log,\theta}(\mathbb{T}^d). \tag{3.10}$$

We remark that the inclusion in (3.10) is strict. We now recall the following commutator estimate from [6] and adapted to our context, see [6, Proposition 13].

Proposition 3.4. *Let $a \in W^{1,p}(\mathbb{T}^d)$ be a divergence-free vector field with $1 \le p \le 2$. Then, there exists a constant $C > 0$ depending only on p and d such that for all $g \in L^{p'}(\mathbb{T}^d)$ with $\frac{1}{p} + \frac{1}{p'} = 1$,*

$$\int_{\mathbb{T}^d}\int_{\mathbb{T}^d} \nabla K_h(x-y)(a(x)-a(y))|g(x)-g(y)|^2 \,\mathrm{d}x\,\mathrm{d}y \le C|\log h|^{\frac{1}{2}}\|\nabla a\|_{L^p}\|g\|^2_{L^{2p'}}. \tag{3.11}$$

We can finally prove our first main result.

Theorem 3.5. *Let $\omega_0 \in L^4 \cap W^2_{\log,\frac{1}{2}}(\mathbb{T}^2)$ and let $\omega \in L^\infty([0,T]; L^4(\mathbb{T}^2))$ be any weak solution of* (1.1) *starting from ω_0. Then, there exists a constant $C > 0$ such that ω satisfies the following bound*

$$[\omega(t,\cdot)]_{W^2_{\log,\frac{1}{2}}} \le [\omega_0]_{W^2_{\log,\frac{1}{2}}} + C\sqrt{t}\|\omega_0\|^{\frac{1}{2}}_{L^2}\|\omega_0\|_{L^4}, \qquad \textit{for any } t \in [0,T]. \tag{3.12}$$

Remark 3.6. *The constant in the statement of Theorem 3.5 depends depends on the constant in* (3.11) *of Proposition 3.4 with $d = 2$ and $p = p' = 2$.*

Proof. Let $\omega \in L^\infty([0,T]; L^4(\mathbb{T}^2))$ be a weak solution of (1.1) and let ρ^ε be a family of smooth mollifiers. Then, the function $\omega^\varepsilon = \omega * \rho^\varepsilon$ satisfies the equation

$$\begin{cases} \partial_t \omega^\varepsilon + u\cdot\nabla\omega^\varepsilon = r^\varepsilon, \\ u(t,x) = K * \omega(t,\cdot)(x), \\ \omega^\varepsilon(0,\cdot) = \omega_0 * \rho^\varepsilon, \end{cases} \tag{3.13}$$

where the commutator r^ε is defined as

$$r^\varepsilon := u\cdot\nabla(\omega * \rho^\varepsilon) - (u\cdot\nabla\omega) * \rho^\varepsilon. \tag{3.14}$$

In particular, since $\omega, \nabla u \in L^\infty([0,T]; L^4(\mathbb{T}^2))$ we have that $r^\varepsilon \to 0$ in $L^1([0,T]; L^2(\mathbb{T}^2))$, see [22, Lemma II.1]. Thus, by using the Eq (3.13), the function $|\omega^\varepsilon(t,x) - \omega^\varepsilon(t,y)|^2$ satisfies the equation

$$\begin{aligned} \partial_t|\omega^\varepsilon(t,x) - \omega^\varepsilon(t,y)|^2 + [u(t,x)\cdot\nabla_x + u(t,y)\cdot\nabla_y]|\omega^\varepsilon(t,x) - \omega^\varepsilon(t,y)|^2 \\ = 2(r^\varepsilon(t,x) + r^\varepsilon(t,y))(\omega^\varepsilon(t,x) - \omega^\varepsilon(t,y)). \end{aligned} \tag{3.15}$$

Then, we use (3.15) and $\operatorname{div} u = 0$ to compute

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^2}\int_{\mathbb{T}^2} K_h(x-y)|\omega^\varepsilon(t,x) - \omega^\varepsilon(t,y)|^2 \,\mathrm{d}x\,\mathrm{d}y \\ &\quad = \int_{\mathbb{T}^2}\int_{\mathbb{T}^2} \nabla K_h(x-y)(u(t,x) - u(t,y))|\omega^\varepsilon(t,x) - \omega^\varepsilon(t,y)|^2 \,\mathrm{d}x\,\mathrm{d}y \\ &\quad + 2\int_{\mathbb{T}^2}\int_{\mathbb{T}^2} K_h(x-y)(r^\varepsilon(t,x) + r^\varepsilon(t,y))(\omega^\varepsilon(t,x) - \omega^\varepsilon(t,y)) \,\mathrm{d}x\,\mathrm{d}y. \end{aligned}$$

We integrate in time and we use Proposition 3.4 to obtain that

$$\begin{aligned}
&\int_{\mathbb{T}^2}\int_{\mathbb{T}^2} K_h(x-y)|\omega^\varepsilon(t,x)-\omega^\varepsilon(t,y)|^2\,\mathrm{d}x\,\mathrm{d}y\\
&\qquad\le \int_{\mathbb{T}^2}\int_{\mathbb{T}^2} K_h(x-y)|\omega_0^\varepsilon(x)-\omega_0^\varepsilon(y)|^2\,\mathrm{d}x\,\mathrm{d}y\\
&\qquad+2\int_0^T\int_{\mathbb{T}^2}\int_{\mathbb{T}^2} K_h(x-y)(r^\varepsilon(t,x)+r^\varepsilon(t,y))(\omega^\varepsilon(t,x)-\omega^\varepsilon(t,y))\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\\
&\qquad+Ct|\log h|^{\frac12}\|\nabla u\|_{L^\infty L^2}\|\omega^\varepsilon\|^2_{L^\infty L^4},
\end{aligned}$$

for all $0\le t\le T$. Since any weak solution $\omega\in L^\infty([0,T];L^p(\mathbb{T}^2))$ with $p\ge 2$ is renormalized (see [30]), the following bounds hold

$$\begin{aligned}
&\|\omega^\varepsilon\|_{L^\infty L^4}\le\|\omega_0\|_{L^4},\\
&\|\nabla u\|_{L^\infty L^2}\le C\|\omega\|_{L^\infty L^2}\le C\|\omega_0\|_{L^2},
\end{aligned}$$

and then we get that

$$\begin{aligned}
&\int_{\mathbb{T}^2}\int_{\mathbb{T}^2} K_h(x-y)|\omega^\varepsilon(t,x)-\omega^\varepsilon(t,y)|^2\,\mathrm{d}x\,\mathrm{d}y\\
&\qquad\le \int_{\mathbb{T}^2}\int_{\mathbb{T}^2} K_h(x-y)|\omega_0^\varepsilon(x)-\omega_0^\varepsilon(y)|^2\,\mathrm{d}x\,\mathrm{d}y\\
&\qquad+2\int_0^T\int_{\mathbb{T}^2}\int_{\mathbb{T}^2} K_h(x-y)(r^\varepsilon(t,x)+r^\varepsilon(t,y))(\omega^\varepsilon(t,x)-\omega^\varepsilon(t,y))\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\\
&\qquad+Ct|\log h|^{\frac12}\|\omega_0\|_{L^2}\|\omega_0\|^2_{L^4}.
\end{aligned}$$

Finally, the convergence of the commutator r^ε allows us to take the limit $\varepsilon\to 0$ obtaining that

$$\begin{aligned}
&\int_{\mathbb{T}^2}\int_{\mathbb{T}^2} K_h(x-y)|\omega(t,x)-\omega(t,y)|^2\,\mathrm{d}x\,\mathrm{d}y\\
&\qquad\qquad\le \int_{\mathbb{T}^2}\int_{\mathbb{T}^2} K_h(x-y)|\omega_0(x)-\omega_0(y)|^2\,\mathrm{d}x\,\mathrm{d}y+Ct|\log h|^{\frac12}\|\omega_0\|_{L^2}\|\omega_0\|^2_{L^4},
\end{aligned}$$

and the definition of the semi-norm $[\,\cdot\,]_{\frac12}$ in (3.4) implies that

$$[\omega(t,\cdot)]^2_{\frac12}\le[\omega_0]^2_{\frac12}+Ct\|\omega_0\|_{L^2}\|\omega_0\|^2_{L^4}. \tag{3.16}$$

The proof is complete. □

4. Vanishing viscosity limit

In this last section we study the inviscid limit of the 2D Navier-Stokes equations (NS). Our goal is to provide a logarithmic rate of convergence (in the viscosity) assuming that the initial vorticity belongs to the space $H^{\log,\alpha}$.

We start by introducing the Stochastic Langrangian representation of (NS). Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a given probability space, we define the map

$$X^\nu : [0,T] \times [0,T] \times \mathbb{T}^2 \times \Omega \to \mathbb{T}^2$$

as follows: For $\mathbb{P}$-a.e. $\xi \in \Omega$ and for any $t \in (0,T)$ and any $s \in [0,T]$ we consider a $\mathbb{T}^2$-valued Brownian motion W_s adapted to the backward filtration, i.e., satisfying $W_t = 0$. The map $s \mapsto X^\nu_{t,s}(x,\xi)$ is obtained by solving

$$\begin{cases} \mathrm{d}X^\nu_{t,s}(x,\xi) = u^\nu(s, X^\nu_{t,s}(x,\xi))\,\mathrm{d}s + \sqrt{2\nu}\,\mathrm{d}W_s(\xi), & s \in [0,t), \\ X^\nu_{t,t}(x,\xi) = x. \end{cases} \tag{4.1}$$

For $\mathbb{P}$-a.e. $\xi \in \Omega$ the map $x \in \mathbb{T}^2 \mapsto X^\nu_{t,s}(x,\xi) \in \mathbb{T}^2$ is measure-preserving for any $t \in [0,T]$ and $s \in [0,t]$ (see [29]). Moreover, by the Feynman-Kac formula (see [28,29]), the function

$$\omega^\nu(t,x) = \mathbb{E}[\omega^\nu_0(X^\nu_{t,0}(x))] \tag{4.2}$$

satisfies the advection-diffusion equation

$$\partial_t \omega^\nu + u^\nu \cdot \nabla \omega^\nu - \nu \Delta \omega^\nu = 0,$$

with initial datum ω^ν_0, where we have denoted by $\mathbb{E}[f]$ the expectation, i.e., the average with respect to $\mathbb{P}$. As usual, we will omit the explicit dependence on the parameter $\xi \in \Omega$. Therefore, the couple

$$u^\nu(t,x) := \nabla^\perp(-\Delta)^{-1}\omega^\nu(t,\cdot)(x), \tag{4.3}$$

$$\omega^\nu(t,x) := \mathbb{E}[\omega^\nu_0(X^\nu_{t,0}(x))], \tag{4.4}$$

solves the Cauchy problem for the Navier-Stokes equations (NS). The couple (u^ν, ω^ν) defined by the Eqs (4.3) and (4.4) is the *Lagrangian representation* of solutions to (NS).

We remark that the probability space and the Brownian motion can be arbitrarily chosen. Thus, the Lagrangian representation does not depend on the probability space. Indeed, since u^ν is a smooth function, the Eq (4.1) is satisfied in the strong sense [28,29], namely one can find a solution $X^\nu_{t,\cdot}$ to (4.1) on any given filtered probability space with any given adapted Brownian motions as described above.

Finally, we recall the following theorem proved in [17, Theorem 2.8].

Theorem 4.1. *Let $\omega_0 \in L^\infty(\mathbb{T}^2)$ with $\|\omega_0\|_{L^\infty} = M$. Let (u,ω) and (u^ν, ω^ν) be, respectively, the unique bounded solutions of the Euler and Navier-Stokes equations with the same initial datum ω_0. Denote with X and X^ν the corresponding deterministic and stochastic flows. Then, for any $T > 0$ there exists a constant $\beta(M,T)$ such that*

$$\sup_{s,t\in[0,T]} \mathbb{E}\left[\int_{\mathbb{T}^2} |X^\nu_{t,s}(x) - X_{t,s}(x)|^2\,\mathrm{d}x\right] \le C\nu^{\beta(M,T)}. \tag{4.5}$$

We can now prove our second main result, which we rewrite for the reader's convenience.

Theorem 4.2. *Let $\omega_0 \in L^\infty \cap H^{\log,\alpha}(\mathbb{T}^2)$ for some $\alpha > 0$. Let ω and ω^ν be, respectively, the unique bounded solutions of the Euler and Navier-Stokes equations arising from ω_0. Then, there exists a constant $C > 0$ depending on $\alpha, T, \|\omega_0\|_{H^{\log,\alpha}}$, and $\|\omega_0\|_{L^\infty}$ such that*

$$\sup_{t\in(0,T)} \|\omega^\nu(t,\cdot) - \omega(t,\cdot)\|_{L^2} \le \frac{C}{|\log \nu|^{\alpha/2}}. \tag{4.6}$$

Proof. Let $\varepsilon > 0$ be a parameter that we will fix later. We use the Feynman-Kac formula to write

$$\begin{aligned}\|\omega^\nu(t,\cdot)-\omega(t,\cdot)\|_{L^2}^2 &= \int_{\mathbb{T}^2} |\omega^\nu(t,x)-\omega(t,x)|^2\,\mathrm{d}x\\ &= \int_{\mathbb{T}^2} |\mathbb{E}[\omega_0(X^\nu_{t,0})]-\omega_0(X_{t,0})|^2\,\mathrm{d}x\\ &\le \iint_{\{|X^\nu_{t,0}-X_{t,0}|\le\varepsilon\}} |\omega_0(X^\nu_{t,0})-\omega_0(X_{t,0})|^2\,\mathrm{d}\mathbb{P}\,\mathrm{d}x\\ &+ \iint_{\{|X^\nu_{t,0}-X_{t,0}|>\varepsilon\}} |\omega_0(X^\nu_{t,0})-\omega_0(X_{t,0})|^2\,\mathrm{d}\mathbb{P}\,\mathrm{d}x\\ &:= I + II.\end{aligned}$$

We start by estimating I: if we assume that $\varepsilon < 1/36$, we apply Theorem 2.2 and we have that

$$\begin{aligned}\iint_{\{|X^\nu_{t,0}-X_{t,0}|\le\varepsilon\}} |\omega_0(X^\nu_{t,0})-\omega_0(X_{t,0})|^2\,\mathrm{d}\mathbb{P}\,\mathrm{d}x &\le \frac{C(\alpha)}{|\log\varepsilon|^\alpha}\mathbb{E}\left[\int_{\mathbb{T}^2}\left[L_\alpha\,\omega_0(X^\nu_{t,0})^2 + L_\alpha\,\omega_0(X_{t,0})^2\right]\,\mathrm{d}x\right]\\ &\le \frac{C(\alpha)}{|\log\varepsilon|^\alpha}[\omega_0]^2_{H^{\log,\alpha}},\end{aligned}$$

where in the last line we used (2.6) and the measure preserving property of X^ν and X. To estimate II we use the fact that ω_0 is bounded, Chebishev's inequality and the convergence of the flows in Theorem 4.1 to obtain that

$$\begin{aligned}\iint_{\{|X^\nu_{t,0}-X_{t,0}|>\varepsilon\}} |\omega_0(X^\nu_{t,0})-\omega_0(X_{t,0})|^2\,\mathrm{d}\mathbb{P}\,\mathrm{d}x &\le \frac{C\|\omega_0\|^2_{L^\infty}}{\varepsilon^2}\mathbb{E}\left[\int_{\mathbb{T}^2}|X^\nu_{t,0}(x)-X_{t,0}(x)|^2\,\mathrm{d}x\right]\\ &\le \frac{C\|\omega_0\|^2_{L^\infty}}{\varepsilon^2}\nu^{\beta(M,T)}.\end{aligned}$$

Thus, by defining $\varepsilon := \nu^{\beta(M,T)/4}$, we finally get

$$\begin{aligned}\|\omega^\nu(t,\cdot)-\omega(t,\cdot)\|_{L^2}^2 &\le \frac{C(\alpha)[\omega_0]^2_{H^{\log,\alpha}}}{|\log\varepsilon|^\alpha} + C\|\omega_0\|^2_{L^\infty}\nu^{\beta(M,T)/2}\\ &= \frac{C(\alpha)\alpha(M,T)^{-\alpha}[\omega_0]^2_{H^{\log,\alpha}}}{|\log\nu|^\alpha} + C\|\omega_0\|^2_{L^\infty}\nu^{\beta(M,T)/2}\\ &\le \frac{C(\alpha,T,\|\omega_0\|_{H^{\log,\alpha}},\|\omega\|_{L^\infty})}{|\log\nu|^\alpha},\end{aligned}$$

where in the last line we used that the logarithm converges slower than any power. This concludes the proof. □

Remark 4.3. *An easy interpolation argument implies the convergence of ω^ν towards ω in all L^q spaces with $1 \le q < \infty$ and rate*

$$\sup_{t\in(0,T)}\|\omega^\nu(t,\cdot)-\omega(t,\cdot)\|_{L^q} \le \frac{C}{|\log\nu|^{f(\alpha,q)}}, \qquad \text{with } f(\alpha,q) := \min\left\{\frac{\alpha}{2},\frac{\alpha}{q}\right\}. \tag{4.7}$$

5. Conclusions

In the present paper we proved the propagation of regularity for solutions of the $2D$ Euler equations in the space $W^2_{log,\frac{1}{2}}$. Notably, the propagation holds for solutions which are out of the Yudovich class, namely, out of the class of weak solutions with bounded vorticity. Our method seems flexible enough to deal with other Besov spaces with logarithmic regularity. Furthermore, we also derived rates of convergence for the vanishing viscosity limit in this class of spaces.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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