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Research article

Signorini problem as a variational limit of obstacle problems in nonlinear elasticity

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Abstract: An energy functional for the obstacle problem in linear elasticity is obtained as a variational limit of nonlinear elastic energy functionals describing a material body subject to pure traction load under a unilateral constraint representing the rigid obstacle. There exist loads pushing the body against the obstacle, but unfit for the geometry of the whole system body-obstacle, so that the corresponding variational limit turns out to be different from the classical Signorini problem in linear elasticity. However, if the force field acting on the body fulfils an appropriate geometric admissibility condition, we can show coincidence of minima. The analysis developed here provides a rigorous variational justification of the Signorini problem in linear elasticity, together with an accurate analysis of the unilateral constraint.

Keywords: calculus of variations; Signorini problem; linear elasticity; nonlinear elasticity; finite elasticity; Gamma-convergence; asymptotic analysis; unilateral constraint

1. Introduction

In its original formulation (see [54]) the Signorini problem in linear elastostatics consists in finding the equilibrium configuration of an elastic body Ω resting on a frictionless rigid support $E \subset \partial\Omega$ in its natural configuration and subject to body forces and surface forces acting on $\partial\Omega \setminus E$; precisely, if $\mathbf{u} : \Omega \to \mathbb{R}^3$ denotes the displacement field of the body, $\mathbb{C}$ represents the classical linear elasticity tensor and $\mathbb{E}$ denotes the linear strain tensor, we assume that

$$\mathcal{Q}(\mathbf{x},\mathbb{E}) := \frac{1}{2}\,\mathbb{E}^T\mathbb{C}(\mathbf{x})\,\mathbb{E}$$

is the corresponding strain energy density (see [27]) and that the body is subject to a load system of forces $\mathbf{f}:\Omega\to\mathbb{R}^3$ and $\mathbf{g}:\partial\Omega\setminus E\to\mathbb{R}^3$ such that

$$\mathcal{L}(\mathbf{u}) := \int_\Omega \mathbf{f}\cdot\mathbf{u}\,d\mathbf{x} + \int_{\partial\Omega\setminus E}\mathbf{g}\cdot\mathbf{u}\,d\mathcal{H}^2 \tag{1.1}$$

is the load potential, where $\mathcal{H}^2$ is the two-dimensional Hausdorff measure. Assuming that $\mathcal{H}^2(E)>0$, the variational formulation of the Signorini problem consists in finding a minimizer of the functional

$$\mathcal{E}(\mathbf{u}) := \int_\Omega \mathcal{Q}(\mathbf{x},\mathbb{E}(\mathbf{u}))\,d\mathbf{x} - \mathcal{L}(\mathbf{u}) \tag{1.2}$$

among all $\mathbf{u}$ in the Sobolev space $H^1(\Omega;\mathbb{R}^3)$ such that $\mathbf{u}\cdot\mathbf{n}\geq 0$ $\mathcal{H}^2$-a.e. on E, where $\mathbf{n}$ is the inward unit vector normal to $\partial\Omega$. A classical result (see [22]) states that a solution of (1.2) exists if the following condition is verified: Every infinitesimal rigid displacement $\mathbf{v}$ fulfills $\mathcal{L}(\mathbf{v})\leq 0$ if $\mathbf{v}\cdot\mathbf{n}\geq 0$ $\mathcal{H}^2$-a.e. on E and $\mathcal{L}(\mathbf{v})=0$ if and only if $\mathbf{v}\cdot\mathbf{n}\equiv 0$ $\mathcal{H}^2$-a.e. on E. Moreover if E is planar, that is $E\subset\partial\Omega\cap\{x_3=0\}$, and if $\mathcal{L}(\mathbf{e}_3)<0$, $\mathbf{f}\in C^{0,\alpha}(\overline{\Omega};\mathbb{R}^3)$ and $\mathbf{g}\in L^2(\partial\Omega\setminus E;\mathbb{R}^3)$ then a minimizer of (1.2) exists if and only if the above condition holds (see [22, Theorem XXXII] and [10]): In particular if Ω is the cylinder

$$\Omega := \{\mathbf{x} : (x_1 - ax_3)^2 + x_2^2 < R^2,\ 0 < x_3 < H\},$$

$$E := \{\mathbf{x} : x_1^2 + x_2^2 < R^2,\ x_3 = 0\},$$

with $a\geq 0$, $R>0$, $H>0$, and $\mathbf{f}=-\mathbf{e}_3$, $\mathbf{g}=0$, then a minimum is attained if and only if $aH<2R$ that is $0\leq\vartheta:=\arctan a<\arctan 2R/H$ where ϑ is the inclination of the cylinder with respect to the x_3-axis.

More recent formulations of constrained problems in the calculus of variations use the notion of capacity (see Section 2 for details) leading to consider more general geometries since any set of null capacity has null $\mathcal{H}^2$ measure (see [57, Theorem 4]) but there exist sets of null $\mathcal{H}^2$ measure and strictly positive capacity (see [1, Theorem 5.4.1]). Indeed, a proper generalization of the latter case is to assume that the set $E\subset\{x_3\geq 0\}$ has positive capacity and accordingly modify the obstacle condition by requiring $x_3+u_3(\mathbf{x})\geq 0$ on E up to a set of null capacity (shortly, q.e. on E): The existence of minimizers for this general setting was proved by [12, Theorem 4.5]. Although the original obstacle formulation given in [54] may look different from the generalized notion exploited in this work, it can be shown (see Remark 2.3) that if the set $E\subset\partial\Omega$ is regular in an appropriate sense (see Remark 2.3) then the two frameworks coincide.

Like in the analysis of many problems in linear elastostatics, it is quite natural to ask whether there exists a sequence of functionals in finite elasticity whose minimizing sequences converge to a minimizer of (1.2), possibly under suitable compatibility conditions on the functional $\mathcal{L}$: Such an approach provides a variational justification of the linearized theory and could help finding other reasonable models rigorously deduced.

In this paper we show sharp conditions on $\mathcal{L}$ entailing that a wide class of energy functionals in finite elasticity fulfill this variational property in the context of obstacle problems; in addition we also show examples of loads leading to the failure of this convergence. In this perspective, denoting by $\mathbf{y} : \Omega \to \mathbb{R}^3$ the deformation field and by $h > 0$ an adimensional parameter, we introduce a family of energy functionals defined by

$$\mathcal{F}_h(\mathbf{y}) := h^{-2} \int_\Omega \mathcal{W}(\mathbf{x}, \nabla\mathbf{y}(\mathbf{x}))\, d\mathbf{x} - h^{-1}\mathcal{L}(\mathbf{y} - \mathbf{x}) \tag{1.3}$$

where $\mathcal{L}$ is defined as in (1.1) and $\mathcal{W} : \Omega \times \mathbb{R}^{3\times3} \to [0, +\infty]$ is the strain energy density. For every $\mathbf{x} \in \Omega$, the function $\mathcal{W}(\mathbf{x}, \cdot)$ is assumed to be frame indifferent and attaining its minimum value 0 at rigid deformations only. We also assume that $\mathcal{W}$ is C^2-regular in a neighborhood of rigid deformations and satisfies a natural coercivity condition, see (2.22).

According to a standard approach in the deduction of linearized theories in continuum mechanics, if $\mathbf{y}_h$ is a minimizing sequence of $\mathcal{F}_h$ (see (2.40)) in a class $\mathcal{A}_h$ of deformations satisfying a suitable obstacle constraint, we aim to investigate whether $\mathcal{F}(\mathbf{y}_h)$ converges, as h goes to 0, to the minimum of $\mathcal{E}$ (with $\mathbb{C} = D^2\mathcal{W}(\mathbf{x}, \mathbf{I})$) among displacements fulfilling $u_3(\mathbf{x}) + x_3 \geq 0$ q.e. on E. Since here the aim is the deduction of the Signorini problem in linear elasticity, it is natural to assume that the unilateral constraint in nonlinear approximating problems takes the form $x_3 + h^{-1}(\mathbf{y}_{h,3} - x_3) \geq 0$ on E that is

$$\mathbf{y}_{h,3} \geq (1 - h)x_3 \quad \text{on } E. \tag{1.4}$$

We define the functionals $\mathcal{G}_h$ coupling the energies $\mathcal{F}_h$ with the unilateral constraint due to rigid obstacle:

$$\mathcal{G}_h(\mathbf{y}) = \begin{cases} \mathcal{F}_h(\mathbf{y}), & \text{if } \mathbf{y}_3 \geq (1 - h)x_3 \text{ on } E, \\ +\infty, & \text{else,} \end{cases} \tag{1.5}$$

where E, the portion of the elastic body sensitive to the obstacle, has an horizontal projection with non negligible capacity. Moreover we have to assume that

$$\mathcal{L}(\mathbf{y} - \mathbf{x}) \leq 0 \tag{1.6}$$

for every deformation $\mathbf{y} = \mathbf{y}(\mathbf{x})$ fulfilling (1.4) and such that

$$\int_\Omega \mathcal{W}(\mathbf{x}, \nabla\mathbf{y})\, d\mathbf{x} = 0. \tag{1.7}$$

On the contrary if $\mathbf{y}$ satisfied (1.4) and (1.7) but $\mathcal{L}(\mathbf{y} - \mathbf{x}) > 0$ then

$$\mathcal{G}_h(\mathbf{y}) = -h^{-1}\mathcal{L}(\mathbf{y} - \mathbf{x}) \to -\infty \text{ as } h \to 0^+.$$

Under our assumptions on $\mathcal{W}$, Eq (1.7) holds true if and only if $\mathbf{y}$ is rigid deformation, i.e., $\mathbf{y}(\mathbf{x}) = \mathbf{R}\mathbf{x}+\mathbf{c}$ for some $\mathbf{R} \in SO(3)$ and $\mathbf{c} \in \mathbb{R}^3$, while (1.4) is fulfilled by these $\mathbf{y}$ if and only if $(\mathbf{R}\mathbf{x})_3 + c_3 \geq (1 - h)x_3$ on E, a condition which is satisfied for every $h > 0$ if and only if

$$c_3 \geq -((\mathbf{R} - \mathbf{I})\mathbf{x})_3 \quad \text{on } E. \tag{1.8}$$

Thus, due to (1.6) we have to assume this geometrical compatibility between load and obstacle

$$\mathcal{L}((\mathbf{R} - \mathbf{I})\mathbf{x} + \mathbf{c}) \leq 0, \quad \forall\, \mathbf{R} \in SO(3),\ \forall \mathbf{c} \in \mathbb{R}^3 \text{ verifying (1.8).} \tag{1.9}$$

Nevertheless though (1.9) entails the compatibility assumptions of Theorem 4.5 in [12] and though compatibility assumptions of Theorem 4.5 in [12] entail the existence of minimizers of the Signorini functional in linear elasticity, these compatibility assumptions alone do not warrant the equiboundedness from below of approximating functionals $\mathcal{G}_h$ (see Example 3.6 below). In the main result of this paper (Theorem 2.4) we show that if $\mathcal{L}$ satisfies the necessary condition (1.9) together with $\mathcal{L}(\mathbf{e}_3) < 0$ and

$$\mathcal{L}\,(\,(\mathbf{Rx}-\mathbf{x})_1\,\mathbf{e}_1 + (\mathbf{Rx}-\mathbf{x})_2\,\mathbf{e}_2\,) \le 0, \quad \forall\,\mathbf{R} \in SO(3), \tag{1.10}$$

then, under some capacitary assumptions on E (see (2.13)), we have

$$\lim_{h\to 0}(\inf \mathcal{G}_h) = \min \mathcal{G}, \tag{1.11}$$

where

$$\mathcal{G}(\mathbf{u}) := \begin{cases} \displaystyle\int_\Omega \mathcal{Q}(\mathbf{x}, \mathbb{E}(\mathbf{u}))\,d\mathbf{x} - \max_{\mathcal{S}_{\mathcal{L},E}} \mathcal{L}(\mathbf{Ru}), & \text{if } x_3 + u_3 \ge 0 \text{ on } E, \\ +\infty, & \text{else}, \end{cases} \tag{1.12}$$

$$\mathcal{Q}(\mathbf{x}, \mathbf{F}) := \frac{1}{2}\,\mathbf{F}^T\,D^2\mathcal{W}(\mathbf{x}, \mathbf{I})\,\mathbf{F}, \qquad \mathbf{F} \in \mathbb{R}^{3\times 3}, \quad \mathbf{x} \in \Omega,$$

$$\mathcal{S}_{\mathcal{L},E} \;=\; \left\{ \mathbf{R} \in SO(3) :\; \mathcal{L}\,((\mathbf{R}-\mathbf{I})\,\mathbf{x}) \;-\; \min_{\mathbf{x}\in E_{ess}}\,((\mathbf{Rx})_3 - x_3)\mathcal{L}(\mathbf{e}_3) = 0 \right\}.$$

Along this paper E_{ess} denotes the essential part of E with respect to the capacity (see (2.9)), a closed canonical subset of $\overline{E}$ such that $E \setminus E_{ess}$ has null capacity.

Under the hypotheses detailed previously we will show (see Lemma 3.8) that either $\mathcal{S}_{\mathcal{L},E} \equiv \{\mathbf{I}\}$ or $\mathcal{S}_{\mathcal{L},E} = \{\mathbf{R} \in SO(3) : \mathbf{Re}_3 = \mathbf{e}_3\}$. If $\mathcal{S}_{\mathcal{L},E} \equiv \{\mathbf{I}\}$ then clearly $\mathcal{G} \equiv \mathcal{E}$, hence in this case the minimum of Signorini problem in linearized elasticity is the limit of the $\inf \mathcal{G}_h$ but, quite surprisingly, the second alternative is much more subtle and indeed we are able to exhibit examples such that

$$\min \mathcal{G} < \min \mathcal{E}, \tag{1.13}$$

namely a gap between $\lim_{h\to 0}(\inf \mathcal{G}_h)$ and $\min \mathcal{E}$ may appear (see Section 5). However the coincidence of minimizers of $\mathcal{G}$ and $\mathcal{E}$ may hold true even if $\mathcal{S}_{\mathcal{L},E}$ is not reduced to the identity matrix: In particular, if Ω is contained in the upper half-space, E is either $\overline{\Omega}$ or $\partial\Omega$, the load

$$\mathcal{L}(\mathbf{v}) := \int_\Omega f\,v_3\,d\mathbf{x} + \int_{\partial\Omega} g\,v_3\,d\mathcal{H}^2$$

satisfies condition (1.9) and $\mathcal{L}(\mathbf{e}_3) < 0$, then $\mathcal{L}(\mathbf{v}) = \mathcal{L}(\mathbf{Rv})$ for every $\mathbf{R} \in \mathcal{S}_{\mathcal{L},E}$ hence $\min \mathcal{G} = \min \mathcal{E}$, say the energy of minimizing sequences for $\mathcal{G}_h$ converges to the minimum energy of $\mathcal{E}$. On the other hand, it is always possible to rotate the external forces in such a way that $\mathcal{G}_\mathbf{R}$ and $\mathcal{E}_\mathbf{R}$ (the functionals obtained replacing the load functional $\mathcal{L}$ with $\mathcal{L}_\mathbf{R}$ defined by $\mathcal{L}_\mathbf{R}(\mathbf{v}) := \mathcal{L}(\mathbf{Rv})$) have the same minimum as shown in Theorem 5.5.

For several contributions facing issues strictly connected with the context of the present paper we refer to [3–6, 8, 9, 11, 13–19, 21, 23, 25–29, 32–43, 46–53, 55, 56].

2. Notation and main results

We set $a^+ := \max\{a, 0\}$, $a^- := \max\{-a, 0\}$ for every $a \in \mathbb{R}$; notations $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$ represent generic points in $\mathbb{R}^3$; $\mathbf{e}_j$, $j = 1, 2, 3$ denote the unitary basis vectors of $\mathbb{R}^3$, $\mathbb{R}^{3\times3}$ is the set of 3×3 real matrices, endowed with the Euclidean norm $|\mathbf{F}| = \sqrt{\mathrm{Tr}(\mathbf{F}^T\mathbf{F})}$. $\mathbb{R}^{3\times3}_{\mathrm{sym}}$ (resp. $\mathbb{R}^{3\times3}_{\mathrm{skew}}$) denotes the subset of symmetric (resp. skew-symmetric) matrices. For every $\mathbf{F} \in \mathbb{R}^{3\times3}$ we define $\mathrm{sym}\,\mathbf{F} := \frac{1}{2}(\mathbf{F} + \mathbf{F}^T)$, $SO(3)$ will denote the special orthogonal group and for every $\mathbf{R} \in SO(3)$ there exist $\vartheta \in [0, 2\pi]$ and $\mathbf{a} \in \mathbb{R}^3$, $|\mathbf{a}| = 1$ such that the following Euler-Rodrigues representation formula holds

$$\mathbf{R}\mathbf{x} = \mathbf{x} + \sin\vartheta\,(\mathbf{a} \wedge \mathbf{x}) + (1 - \cos\vartheta)\,(\mathbf{a} \wedge (\mathbf{a} \wedge \mathbf{x})), \qquad \forall\, \mathbf{x} \in \mathbb{R}^3. \tag{2.1}$$

For every compact set $K \subset \mathbb{R}^N$ we define the capacity of K by setting (see [1, Definition 2.2.1])

$$\mathrm{cap}\,K = \inf\left\{\|w\|^2_{H^1(\mathbb{R}^N)} : w \in C^\infty_0(\mathbb{R}^N),\ w \geq 1 \text{ on } K\right\}. \tag{2.2}$$

If $G \subset \mathbb{R}^N$ is open we define (see [1, Definition 2.2.2])

$$\mathrm{cap}\,G := \sup\{\mathrm{cap}\,K : K \text{ compact},\ K \subset G\} \tag{2.3}$$

and, since (see [1, Proposition 2.2.3])

$$\mathrm{cap}\,K = \inf\{\mathrm{cap}\,G : G \text{ open},\ K \subset G\}, \qquad \forall\, K \text{ compact}, \tag{2.4}$$

we may extend the above definitions to an arbitrary set by setting (see [1, Definition 2.2.4])

$$\mathrm{cap}\,E := \inf\{\mathrm{cap}\,G : G \text{ open},\ E \subset G\}, \qquad \forall\, E \subset \mathbb{R}^N. \tag{2.5}$$

A straightforward consequence of (2.3) and (2.5) is that

$$\mathrm{cap}\,E_1 \leq \mathrm{cap}\,E_2, \qquad \forall\, E_1 \subset E_2 \subset \mathbb{R}^N. \tag{2.6}$$

On the other hand, for every $E \subset \mathbb{R}^N$ the Bessel capacity is defined as (see [1, Definition 2.3.3])

$$\mathrm{Cap}\,E := \inf\left\{\|f\|^2_{L^2(\mathbb{R}^N)} :\ f \geq 0,\ \text{a.e. on}\,\mathbb{R}^N,\ (f * g_1)(\mathbf{x}) \geq 1\ \forall\,\mathbf{x} \in E\right\}, \tag{2.7}$$

where $g_1 \in L^1(\mathbb{R}^3)$ is the first-order Bessel kernel in $\mathbb{R}^N$ defined as the inverse Fourier transform of $(1 + |\boldsymbol{\xi}|^2)^{-1/2}$, say

$$g_1(\mathbf{x}) := (2\pi)^{-N} \int_{\mathbb{R}^3} (1 + |\boldsymbol{\xi}|^2)^{-1/2} e^{i\mathbf{x}\cdot\boldsymbol{\xi}}\, d\boldsymbol{\xi} = \frac{1}{2\pi} \int_0^\infty t^{-(N+1)/2} e^{-\pi|\mathbf{x}|^2/t} e^{-t/(4\pi)}\, dt.$$

Notice that since $f \geq 0$ a.e. we have that $f * g_1$ is everywhere defined if we allow it to take the value $+\infty$ (see [1, Definition 2.3.1]) and that $f * g_1$ is l.s.c. by Proposition 2.3.2 of [1]. Thus inequality $(f * g_1)(\mathbf{x}) \geq 1$ for every $\mathbf{x} \in E$ appearing in formula (2.7) has a precise meaning.

In addition it is possible to show that there exist two constants $\alpha, \beta > 0$ such that

$$\alpha\,\mathrm{Cap}\,E \leq \mathrm{cap}\,E \leq \beta\,\mathrm{Cap}\,E, \quad \forall E \subset \mathbb{R}^N, \tag{2.8}$$

(see [1, Definition 2.2.6 and Proposition 2.3.13]).

A property is said to hold quasi-everywhere (q.e. for short) if it holds true outside a set of zero capacity. It is convenient to introduce (see [12]) a canonical representative of the set E, called the *essential part* of E and denoted by E_{ess}, which nevertheless coincides with E itself whenever it is a smooth closed manifold or the closure of an open subset of $\mathbb{R}^N$.

For every set $E \subset \mathbb{R}^3$ we define the essential part E_{ess} of E (with respect to the capacity) by

$$E_{ess} \;:=\; \bigcap \{\, C : \; C \text{ is closed and } \operatorname{cap}(E \setminus C) = 0 \,\}. \tag{2.9}$$

It has been shown in [12] that

$$E_{ess} \;\text{ is a closed subset of } \overline{E}, \tag{2.10}$$

$$\operatorname{cap}(E \setminus E_{ess}) = 0, \tag{2.11}$$

$$\operatorname{cap} E = 0 \;\text{ if and only if } E_{ess} = \emptyset. \tag{2.12}$$

In the following $\operatorname{co} A$, $\operatorname{aff} A$, $\operatorname{ri} A$, $\operatorname{r}\partial A$ and $\operatorname{proj} A$ denote respectively, the *closed convex hull* of the set $A \subset \mathbb{R}^3$ (say, the intersection of all convex sets containing A), the *affine hull* of the set A (say, the smallest affine space containing A), the *relative interior* of A (say, the interior part of A with respect to the affine hull of A), the *relative boundary* of A (say, the boundary of A with respect to the affine hull of A: $\operatorname{ri} \partial A = \overline{A} \setminus \operatorname{ri} A$) and the *projection* of A onto the horizontal plane $\{x_3 = 0\}$.

Throughout the paper we will assume that

$$\operatorname{cap}(\operatorname{proj}(\operatorname{co} E_{ess})) > 0. \tag{2.13}$$

Notice that $\operatorname{cap} E > 0$ does not imply (2.13) whereas the converse is true: Indeed, if $\operatorname{cap} E = 0$ then by (2.12) we get $E_{ess} = \emptyset$ so $\operatorname{proj}(\operatorname{co} E_{ess}) = \emptyset$ and $\operatorname{cap}(\operatorname{proj}(\operatorname{co} E_{ess})) = 0$, a contradiction to (2.13).

In the following Ω will denote the reference configuration of an elastic body and it is always assumed to be a nonempty, bounded, connected, Lipschitz open set in $\mathbb{R}^3$. We need to show that any function in the Sobolev space $H^1(\Omega; \mathbb{R}^3)$ actually has a precise representative defined quasi-everywhere on the whole $\overline{\Omega}$ with respect to the capacity. Indeed, if $\mathbf{u} \in H^1(\Omega; \mathbb{R}^3)$ and $\mathbf{v} \in H^1(\mathbb{R}^3; \mathbb{R}^3)$ is a Sobolev extension of $\mathbf{u}$, it is well known (see [1, Proposition 6.1.3]) that the limit

$$\mathbf{v}^*(\mathbf{x}) \;:=\; \lim_{r \downarrow 0} \frac{1}{|B_r(\mathbf{x})|} \int_{B_r(\mathbf{x})} \mathbf{v}(\xi)\, d\xi \tag{2.14}$$

exists for q.e. $\mathbf{x} \in \mathbb{R}^3$. The function $\mathbf{v}^*$ is called the precise representative of $\mathbf{v}$ and is a quasicontinuous function in $\mathbb{R}^3$, that is to say, for every $\varepsilon > 0$ there exists an open set $V \subset \mathbb{R}^3$ such that $\operatorname{cap} V < \varepsilon$ and $\mathbf{v}^*$ is continuous in $\mathbb{R}^3 \setminus V$. We claim that if $\mathbf{v}_1$, $\mathbf{v}_2$ are two distinct Sobolev extensions of $\mathbf{u}$ then

$$\mathbf{v}_1^*(\mathbf{x}) \;=\; \mathbf{v}_2^*(\mathbf{x}), \qquad \text{q.e. } \mathbf{x} \in \overline{\Omega}. \tag{2.15}$$

The claim is trivial for q.e. $\mathbf{x} \in \Omega$, thus we are left to show (2.15) for q.e. $\mathbf{x} \in \partial\Omega$.

Ler $R > 0$ such that $\overline{\Omega} \subset B_R(0)$ and let $\Omega_R := B_R(0) \setminus \overline{\Omega}$. Since Ω has Lipschitz boundary it is well known (see [2]) that

$$\lim_{r \downarrow 0} \frac{|B_r(\mathbf{x}) \cap \Omega_R|}{|B_r(\mathbf{x})|} = \frac{1}{2} \tag{2.16}$$

and for $\mathcal{H}^2$ a.e. $\mathbf{x} \in \partial\Omega$,

$$\mathbf{u}(\mathbf{x}) = \lim_{r\downarrow 0} \frac{2}{|B_r(\mathbf{x})|} \int_{B_r(\mathbf{x})\cap\Omega_R} \mathbf{v}_1(\xi)\, d\xi = \lim_{r\downarrow 0} \frac{2}{|B_r(\mathbf{x})|} \int_{B_r(\mathbf{x})\cap\Omega_R} \mathbf{v}_2(\xi)\, d\xi, \quad \mathcal{H}^2 \text{ a.e. } \mathbf{x} \in \partial\Omega, \tag{2.17}$$

where we have denoted again with $\mathbf{u}$ the trace of $\mathbf{u}$ on $\partial\Omega$. Hence

$$\frac{1}{|B_r(\mathbf{x})|} \int_{B_r(\mathbf{x})\cap\Omega_R} (\mathbf{v}_1(\xi) - \mathbf{v}_2(\xi))\, d\xi = 0, \quad \mathcal{H}^2 \text{ a.e. } \mathbf{x} \in \partial\Omega,$$

so, by taking account $\partial\Omega \subset \partial\Omega_R$ and by recalling that $\partial\Omega$ is Lipschitz, we may apply Theorem 2.1 of [20] to $\mathbf{v}_1 - \mathbf{v}_2 \in H^1(\Omega_R; \mathbb{R}^3)$ and we get

$$\frac{1}{|B_r(\mathbf{x})|} \int_{B_r(\mathbf{x})\cap\Omega_R} (\mathbf{v}_1(\xi) - \mathbf{v}_2(\xi))\, d\xi = 0, \qquad \text{q.e. } \mathbf{x} \in \partial\Omega.$$

Since $\mathbf{v}_1 = \mathbf{v}_2 = \mathbf{u}$ a.e. in $B_r(\mathbf{x})\setminus\Omega_R$ the claim follows easily by (2.14). Therefore if $\mathbf{u} \in H^1(\Omega; \mathbb{R}^3)$ we may define its precise representative for quasi-every $\mathbf{x}$ on $\overline{\Omega}$ by

$$\mathbf{u}^*(\mathbf{x}) = \lim_{r\downarrow 0} \frac{1}{|B_r(\mathbf{x})|} \int_{B_r(\mathbf{x})} \mathbf{v}(\xi)\, d\xi, \qquad \text{q.e. } \mathbf{x} \in \overline{\Omega}, \tag{2.18}$$

where $\mathbf{v}$ is any Sobolev extension of $\mathbf{u}$.

The function $\mathbf{u}^*$ is pointwise quasi-everywhere defined by (2.18) and is *quasicontinuous* on $\overline{\Omega}$ i.e., for every $\varepsilon > 0$ there exists a relatively open set $V \subset \overline{\Omega}$ such that $\operatorname{cap} V < \varepsilon$ and $\mathbf{u}^*$ is continuous in $\overline{\Omega} \setminus V$.

2.1. The elastic energy density

Let $\mathcal{L}^3$ and $\mathcal{B}^3$ denote respectively the σ-algebras of Lebesgue measurable and Borel measurable subsets of $\mathbb{R}^3$ and let $\mathcal{W} : \Omega \times \mathbb{R}^{3\times 3} \to [0, +\infty]$ be $\mathcal{L}^3 \times \mathcal{B}^9$-measurable satisfying the following assumptions, see also [3, 45]:

$$\mathcal{W}(\mathbf{x}, \mathbf{RF}) = \mathcal{W}(\mathbf{x}, \mathbf{F}), \qquad \forall \mathbf{R} \in SO(3), \ \forall\, \mathbf{F} \in \mathbb{R}^{3\times 3}, \ \text{ for a.e. } \mathbf{x} \in \Omega, \tag{2.19}$$

$$\min_{\mathbf{F}} \mathcal{W}(\mathbf{x}, \mathbf{F}) = \mathcal{W}(\mathbf{x}, \mathbf{I}) = 0, \quad \text{for a.e. } \mathbf{x} \in \Omega \tag{2.20}$$

and as far as it concerns the regularity of $\mathcal{W}$, we assume that there exist an open neighborhood $\mathcal{U}$ of $SO(3)$ in $\mathbb{R}^{3\times 3}$, an increasing function $\omega : \mathbb{R}_+ \to \mathbb{R}$ satisfying $\lim_{t\to 0^+} \omega(t) = 0$ and a constant $K > 0$ such that for a.e. $\mathbf{x} \in \Omega$

$$\begin{aligned} &\mathcal{W}(\mathbf{x}, \cdot) \in C^2(\mathcal{U}), \quad |D^2\mathcal{W}(\mathbf{x}, \mathbf{I})| \le K \ \text{ and} \\ &|D^2\mathcal{W}(\mathbf{x}, \mathbf{F}) - D^2\mathcal{W}(\mathbf{x}, \mathbf{G})| \le \omega(|\mathbf{F} - \mathbf{G}|), \quad \forall\, \mathbf{F}, \mathbf{G} \in \mathcal{U}. \end{aligned} \tag{2.21}$$

Moreover we assume that there exists $C > 0$ such that for a.e. $\mathbf{x} \in \Omega$

$$\mathcal{W}(\mathbf{x}, \mathbf{F}) \ge C(d(\mathbf{F}, SO(3)))^2, \qquad \forall\, \mathbf{F} \in \mathbb{R}^{3\times 3}, \tag{2.22}$$

where $d(\,\cdot\,, SO(3))$ denotes the Euclidean distance function from the set of rotations.

The frame indifference assumption (2.19) implies that there exists a function $\mathcal{V}$ such that

$$\mathcal{W}(\mathbf{x}, \mathbf{F}) = \mathcal{V}(\mathbf{x}, \tfrac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{I})), \qquad \text{for a.e. } \mathbf{x} \in \Omega,\ \forall\, \mathbf{F} \in \mathbb{R}^{3\times 3}. \tag{2.23}$$

By (2.19)–(2.21), for a.e. $\mathbf{x} \in \Omega$, we have $\mathcal{W}(\mathbf{x}, \mathbf{R}) = D\mathcal{W}(\mathbf{x}, \mathbf{R}) = \mathbf{0}$ for any $\mathbf{R} \in SO(3)$. By (2.23), for a.e. $\mathbf{x} \in \Omega$, given $\mathbf{B} \in \mathbb{R}^{3\times 3}$ and $h > 0$, we have

$$\mathcal{W}(\mathbf{x}, \mathbf{I} + h\mathbf{B}) = \mathcal{V}(\mathbf{x}, h\,\mathrm{sym}\mathbf{B} + \tfrac{1}{2}h^2\mathbf{B}^T\mathbf{B})$$

and (2.20), (2.21) together imply

$$\lim_{h\to 0} h^{-2}\mathcal{W}(\mathbf{x}, \mathbf{I} + h\mathbf{B}) = \frac{1}{2}\,\mathrm{sym}\mathbf{B}\, D^2\mathcal{V}(\mathbf{x}, \mathbf{0})\,\mathrm{sym}\mathbf{B} = \frac{1}{2}\,\mathbf{B}^T D^2\mathcal{W}(\mathbf{x}, \mathbf{I})\,\mathbf{B}, \qquad \forall\, \mathbf{B} \in \mathbb{R}^{3\times 3}.$$

Hence, by (2.22) and polar decomposition [27], we obtain, for a.e. $\mathbf{x}\in\Omega$ and every $\mathbf{B}\in\mathbb{R}^{3\times 3}$, eventually as $h \to 0_+$ (since $\det(\mathbf{I} + h\mathbf{B}) > 0$ for small h)

$$\begin{aligned}\frac{1}{2}\,\mathbf{B}^T D^2\mathcal{W}(\mathbf{x}, \mathbf{I})\,\mathbf{B} &= \lim_{h\to 0} h^{-2}\mathcal{W}(\mathbf{x}, \mathbf{I} + h\mathbf{B}) \ge \limsup_{h\to 0} Ch^{-2}\, d^2(\mathbf{I} + h\mathbf{B}, SO(3))\\ &= \limsup_{h\to 0} Ch^{-2}\left|\sqrt{(\mathbf{I} + h\mathbf{B})^T(\mathbf{I} + h\mathbf{B})} - \mathbf{I}\right|^2 = C|\mathrm{sym}\mathbf{B}|^2.\end{aligned}$$

Moreover, as noticed also in [44], by expressing the remainder of Taylor's expansion in terms of the $\mathbf{x}$-independent modulus of continuity ω of $D^2\mathcal{W}(\mathbf{x}, \cdot)$ on the set $\mathcal{U}$ from (2.21), we have

$$\left|\mathcal{W}(\mathbf{x}, \mathbf{I} + h\mathbf{B}) - \frac{h^2}{2}\,\mathrm{sym}\mathbf{B}\, D^2\mathcal{W}(\mathbf{x}, \mathbf{I})\,\mathrm{sym}\mathbf{B}\right| \le h^2\omega(h|\mathbf{B}|)|\mathbf{B}|^2 \tag{2.24}$$

for any small enough h (such that $h\mathbf{B} \in \mathcal{U}$). Similarly, $\mathcal{V}(\mathbf{x}, \cdot)$ is C^2 in a neighborhood of the origin in $\mathbb{R}^{3\times 3}$, with an $\mathbf{x}$-independent modulus of continuity $\eta : \mathbb{R}_+ \to \mathbb{R}$, which is increasing and such that $\lim_{t\to 0^+} \eta(t) = 0$, and we have

$$\left|\mathcal{V}(\mathbf{x}, h\mathbf{B}) - \frac{h^2}{2}\,\mathrm{sym}\mathbf{B}\, D^2\mathcal{V}(\mathbf{x}, \mathbf{0})\,\mathrm{sym}\mathbf{B}\right| \le h^2\eta(h|\mathbf{B}|)|\mathbf{B}|^2 \tag{2.25}$$

for any small enough h.

We mention a general class of energy densities $\mathcal{W}$ (the so called Yeoh materials) fulfilling the assumptions above (2.19)–(2.22) and for which the main result of the present paper (see Theorem 2.4 below) applies.

Example 2.1. For simplicity, we consider the homogeneous case and assume that a standard isochoric-volumetric decomposition of elastic energy density by setting

$$\mathcal{W}(\mathbf{F}) := \begin{cases} \mathcal{W}_{\mathrm{iso}}\left(\dfrac{\mathbf{F}}{(\det\mathbf{F})^{1/3}}\right) + \mathcal{W}_{\mathrm{vol}}(\mathbf{F}), & \text{if } \det\mathbf{F} > 0,\\ +\infty, & \text{if } \det\mathbf{F} \le 0, \end{cases} \tag{2.26}$$

where $\mathcal{W}_{\mathrm{iso}}$ is an energy density of Yeoh type which is defined by choosing

$$\mathcal{W}_{\mathrm{iso}}(\mathbf{F}) := \sum_{k=1}^{3} c_k(|\mathbf{F}|^2 - 3)^k \tag{2.27}$$

with coefficients $c_k > 0$ and $\mathcal{W}_{\rm vol}(\mathbf{F}) = g(\det \mathbf{F})$ for some convex $g \in C^2(\mathbb{R}_+)$ such that

$$\begin{cases} g(t) \geq 0 \text{ for all } t > 0, \quad g(t) = 0 \text{ if and only if } t = 1, \\ g''(1) > 0, \quad \lim_{t\to 0^+} g(t) = +\infty, \\ \text{there exists } \ C' > 0 \text{ and } r \geq 2 \text{ such that } \ g(t) \geq C' t^r, \quad \text{for } t > 0 \text{ sufficiently large.} \end{cases} \tag{2.28}$$

It is easy to check that with this choice the energy density satisfies all assumptions from (2.19) to (2.21) while inequality (2.22) has been proven in [43].

It is worth noticing that when material constants are suitably chosen then also Ogden-type energies may fulfil assumptions (2.19)–(2.22) and we refer to [43] for all details.

2.2. External forces

We introduce a body force field $\mathbf{f} \in L^{6/5}(\Omega, \mathbb{R}^3)$ and a surface force field $\mathbf{g} \in L^{4/3}(\partial\Omega, \mathbb{R}^3)$. From now on, $\mathbf{f}$ and $\mathbf{g}$ will always be understood to satisfy these summability assumptions. The load functional is the following linear functional

$$\mathcal{L}(\mathbf{v}) := \int_\Omega \mathbf{f} \cdot \mathbf{v}\, d\mathbf{x} + \int_{\partial\Omega} \mathbf{g} \cdot \mathbf{v}\, d\mathcal{H}^2(\mathbf{x}), \qquad \mathbf{v} \in H^1(\Omega, \mathbb{R}^3). \tag{2.29}$$

We note that since Ω is a bounded Lipschitz domain, the Sobolev embedding $H^1(\Omega, \mathbb{R}^3) \hookrightarrow L^6(\Omega, \mathbb{R}^3)$ and the Sobolev trace embedding $H^1(\Omega, \mathbb{R}^3) \hookrightarrow L^4(\partial\Omega, \mathbb{R}^3)$ imply that $\mathcal{L}$ is a bounded functional over $H^1(\Omega, \mathbb{R}^3)$ and throughout the paper we denote its norm with $\|\mathcal{L}\|_*$.

For every $\mathbf{R} \in SO(3)$ we set

$$C_{\mathbf{R}} := \{\mathbf{c} : c_3 \geq -\min_{\mathbf{x}\in E_{ess}} ((\mathbf{R}\mathbf{x})_3 - x_3)\} \tag{2.30}$$

and, as we have observed in the Introduction, we must assume the following geometrical compatibility between load and obstacle

$$\mathcal{L}((\mathbf{R} - \mathbf{I})\mathbf{x} + \mathbf{c}) \leq 0, \qquad \forall \mathbf{R} \in SO(3),\ \forall \mathbf{c} \in C_{\mathbf{R}} \tag{2.31}$$

together with

$$\mathcal{L}\left((\mathbf{R}\mathbf{x} - \mathbf{x})_\alpha\, \mathbf{e}_\alpha\right) \leq 0, \quad \forall\, \mathbf{R} \in SO(3), \tag{2.32}$$

the summation convention over repeated index $\alpha = 1, 2$ being understood all along this paper. It can be shown that condition (2.31) is equivalent to (see Remark 3.4 below)

$$\mathcal{L}(\mathbf{e}_3) \leq 0 = \mathcal{L}(\mathbf{e}_1) = \mathcal{L}(\mathbf{e}_2), \qquad \Phi(\mathbf{R}, E, \mathcal{L}) \leq 0, \quad \forall\, \mathbf{R} \in SO(3), \tag{2.33}$$

where we have set

$$\Phi(\mathbf{R}, E, \mathcal{L}) := \mathcal{L}((\mathbf{R} - \mathbf{I})\mathbf{x}) \; - \; \mathcal{L}(\mathbf{e}_3) \min_{\mathbf{x}\in E_{ess}} \{((\mathbf{R}\mathbf{x})_3 - x_3)\}$$

and from now on we will use (2.33) in place of (2.31). On the other hand Remark 4.5 below will show that also condition (2.32) is in fact unavoidable.

2.3. Energy functionals

If $E \subset \overline{\Omega} \subset \{\mathbf{x}: x_3 \geq 0\}$, the classical Signorini problem in linear elasticity can be described as the minimization of the functional $\mathcal{E} : H^1(\Omega, \mathbb{R}^3) \to \mathbb{R} \cup \{+\infty\}$ defined by

$$\mathcal{E}(\mathbf{u}) := \begin{cases} \displaystyle\int_\Omega Q(\mathbf{x}, \mathbb{E}(\mathbf{u}))\, d\mathbf{x} - \mathcal{L}(\mathbf{u}), & \text{if } \mathbf{u} \in \mathcal{A}, \\ +\infty, & \text{otherwise in } H^1(\Omega, \mathbb{R}^3), \end{cases} \tag{2.34}$$

where $\mathbb{E}(\mathbf{u}) := \operatorname{sym} \nabla \mathbf{u}$, $Q(\mathbf{x}, \mathbf{F}) = \frac{1}{2} \mathbf{F}^T \mathbb{C}\, \mathbf{F}$ with $\mathbb{C} = D^2 \mathcal{W}(\mathbf{x}, \mathbf{I})$ and $\mathcal{A}$ denotes the *set of admissible displacements*, defined by

$$\mathcal{A} := \left\{ \mathbf{u} \in H^1(\Omega; \mathbb{R}^3) : \ u_3^*(\mathbf{x}) + x_3 \geq 0 \ \text{ q.e } \mathbf{x} \in E \right\}. \tag{2.35}$$

The meaning of such constraint is that, if the portion E of the elastic body is contained in $\{x_3 \geq 0\}$ in the reference configuration, then the deformed configuration of E, namely $\{\mathbf{y}(\mathbf{x}) := \mathbf{x} + \mathbf{u}(\mathbf{x}),\ \mathbf{x} \in E\}$, is constrained to remain in $\{y_3 \geq 0\}$.

For every $\mathbf{y} \in H^1(\Omega, \mathbb{R}^3)$ we introduce the set

$$\mathcal{M}(\mathbf{y}) := \operatorname{argmin} \left\{ \int_\Omega |\nabla \mathbf{y} - \mathbf{R}|^2\, d\mathbf{x} : \ \mathbf{R} \in SO(3) \right\}. \tag{2.36}$$

Thus, due to the rigidity inequality of [24], there exists a constant $C = C(\Omega) > 0$ such that for every $\mathbf{y} \in H^1(\Omega, \mathbb{R}^3)$ and every $\mathbf{R} \in \mathcal{M}(\mathbf{y})$

$$\int_\Omega \left(d(\nabla \mathbf{y}, SO(3)) \right)^2 d\mathbf{x} \ \geq\ C \int_\Omega |\nabla \mathbf{y} - \mathbf{R}|^2\, d\mathbf{x}\,, \tag{2.37}$$

where $d(\mathbf{F}, SO(3)) := \min\{|\mathbf{F} - \mathbf{R}| : \mathbf{R} \in SO(3)\}$.

We introduce the *set of admissible deformations* $\mathcal{A}_h$ as

$$\mathcal{A}_h := \{\, \mathbf{y} \in H^1(\Omega, \mathbb{R}^3) : \ y_3^*(\mathbf{x}) - x_3 \geq -h x_3 \ \text{ q.e. } \mathbf{x} \in E \,\} \tag{2.38}$$

and the rescaled finite elasticity functionals $\mathcal{G}_h : H^1(\Omega, \mathbb{R}^3) \to \mathbb{R} \cup \{+\infty\}$ by setting

$$\mathcal{G}_h(\mathbf{y}) = \begin{cases} h^{-2} \displaystyle\int_\Omega \mathcal{W}(\mathbf{x}, \nabla \mathbf{y})\, d\mathbf{x} - h^{-1} \mathcal{L}(\mathbf{y} - \mathbf{x}), & \text{if } \mathbf{y} \in \mathcal{A}_h, \\ +\infty, & \text{otherwise.} \end{cases} \tag{2.39}$$

It is readily seen that, for every $\mathbf{R} \in SO(3)$ and for every $\mathbf{c} \in \mathbb{R}^3$ such that

$$c_3 \geq -\min_{E_{ess}} ((\mathbf{R}\mathbf{x})_3 - x_3)$$

the map $\mathbf{y}(\mathbf{x}) := \mathbf{R}\mathbf{x} + \mathbf{c}$ belongs to $\mathcal{A}_h$ for every $h > 0$. In the sequel we use the short notations $\mathcal{G}_j := \mathcal{G}_{h_j}$ and $\mathcal{A}_j := \mathcal{A}_{h_j}$ whenever $\{h_j\}_{j \in \mathbb{N}}$ is a sequence of strictly positive real numbers such that $h_j \to 0^+$ as $j \to +\infty$.

We say that $(\mathbf{y}_j)_{j\in\mathbb{N}} \subset H^1(\Omega,\mathbb{R}^3)$ is a *minimizing sequence of the sequence of functionals* $\mathcal{G}_j$ if

$$\lim_{j\to+\infty}\left(\mathcal{G}_j(\mathbf{y}_j) - \inf_{H^1(\Omega,\mathbb{R}^3)}\mathcal{G}_j\right) = 0. \tag{2.40}$$

The main focus of the paper is to investigate whether minimizers of (2.34) can be approximated by minimizing sequences of the sequence of functionals $\mathcal{G}_j$, as defined by (2.39) and (2.40).

To this end we introduce the functionals $\mathcal{I}$, $\widetilde{\mathcal{G}}$, $\mathcal{G}$: $H^1(\Omega,\mathbb{R}^3) \to \mathbb{R}\cup\{+\infty\}$ defined by

$$\mathcal{I}(\mathbf{u}) := \min_{\mathbf{b}\in\mathbb{R}^2}\int_\Omega \mathcal{Q}(\mathbf{x},\mathbb{E}(\mathbf{u}) + \frac{1}{2}b_\alpha(\mathbf{e}_\alpha\otimes\mathbf{e}_3 + \mathbf{e}_3\otimes\mathbf{e}_\alpha))\,d\mathbf{x}, \tag{2.41}$$

$$\widetilde{\mathcal{G}}(\mathbf{u}) := \begin{cases} \mathcal{I}(\mathbf{u}) - \max\limits_{\mathbf{R}\in\mathcal{S}_{\mathcal{L},E}}\mathcal{L}(\mathbf{R}\mathbf{u}), & \text{if } \mathbf{u}\in\mathcal{A},\\ +\infty, & \text{otherwise in } H^1(\Omega,\mathbb{R}^3),\end{cases} \tag{2.42}$$

and

$$\mathcal{G}(\mathbf{u}) := \begin{cases} \displaystyle\int_\Omega \mathcal{Q}(\mathbf{x},\mathbb{E}(\mathbf{u})\,d\mathbf{x} - \max_{\mathbf{R}\in\mathcal{S}_{\mathcal{L},E}}\mathcal{L}(\mathbf{R}\mathbf{u}), & \text{if } \mathbf{u}\in\mathcal{A},\\ +\infty, & \text{otherwise in } H^1(\Omega,\mathbb{R}^3),\end{cases} \tag{2.43}$$

where

$$\mathcal{S}_{\mathcal{L},E} = \{\,\mathbf{R}\in SO(3) : \Phi(\mathbf{R},E,\mathcal{L}) = 0\}\,. \tag{2.44}$$

Remark 2.2. It is worth noticing that $\mathcal{G} \le \mathcal{E}$, since $\mathbf{I}\in\mathcal{S}_{\mathcal{L},E}$ and it is straightforward checking that $\mathcal{I}$, $\widetilde{\mathcal{G}}$, $\mathcal{G}$ are all continuous with respect to the strong convergence in $H^1(\Omega;\mathbb{R}^3)$.

Before stating the main result in Theorem 2.4, we show the next Remark with some insight on technicalities implied by precise obstacle formulation in the Sobolev space $H^1(\Omega)$.

Remark 2.3. If $w\in H^1(\Omega)$ then $w^-\in H^1(\Omega)$ too. Moreover, both $(w^-)^*$ and $(w^*)^-$ are quasicontinuous in $\overline{\Omega}$ and $(w^-)^* = (w^*)^- = w^-$ a.e. in Ω. Then, by [30], $(w^-)^* = (w^*)^-$ q.e. in $\overline{\Omega}$. Therefore the condition $(w^-)^* = 0$ q.e. in E_{ess} is equivalent to $w^*\ge 0$ q.e. in E_{ess}.

In particular we claim that

$$(w^-)^* = 0 \ \text{ q.e. in } \overline{\Omega} \tag{2.45}$$

is equivalent to

$$w\ge 0 \ \text{ a.e. in } \Omega \ \text{ and } \ w\ge 0 \ \mathcal{H}^2 \text{ a.e. on } \partial\Omega. \tag{2.46}$$

Indeed if $w\ge 0$ a.e. in Ω then $(w^-)^* = 0$ a.e. in Ω and hence $(w^-)^* = 0$ q.e. in Ω.

If $w\ge 0$ $\mathcal{H}^2$ a.e. on $\partial\Omega$ and v is a Sobolev extension of w^- then

$$\lim_{r\downarrow 0}\frac{1}{|B_r(\mathbf{x})\cap\Omega|}\int_{B_r(\mathbf{x})\cap\Omega} v(\xi)\,d\xi = \lim_{r\downarrow 0}\frac{1}{|B_r(\mathbf{x})\setminus\Omega|}\int_{B_r(\mathbf{x})\setminus\Omega} v(\xi)\,d\xi = 0, \qquad \mathcal{H}^2 \text{ a.e. } \mathbf{x}\in\partial\Omega.$$

and by taking (2.16) into account we get

$$\lim_{r\downarrow 0}\frac{2}{|B_r(\mathbf{x})|}\int_{B_r(\mathbf{x})\cap\Omega} v(\xi)\,d\xi = \lim_{r\downarrow 0}\frac{2}{|B_r(\mathbf{x})|}\int_{B_r(\mathbf{x})\setminus\Omega} v(\xi)\,d\xi = 0, \qquad \mathcal{H}^2 \text{ a.e. } \mathbf{x}\in\partial\Omega.$$

By recalling that Ω is a Lipschitz set, it is easily checked that $\partial\Omega$ is Ahlfors 2-regular, that is there are constants $c_1, c_2 > 0$ such that

$$c_1 r^2 \le \mathcal{H}^2(\partial\Omega \cap B_r(\mathbf{x})) \le c_2 r^2 \tag{2.47}$$

for every $0<r<\mathrm{diam}(\partial\Omega)$ and for every $\mathbf{x} \in \partial\Omega$. Therefore if we choose $R>0$ such that $\overline{\Omega} \subset B_R(0)$ we may apply Proposition 6.1.3. of [1] and Theorem 2.1 of [20] both in $H^1(\Omega)$ and in $H^1(B_R(0) \setminus \overline{\Omega})$ and we get

$$\lim_{r\downarrow 0} \frac{2}{|B_r(\mathbf{x})|} \int_{B_r(\mathbf{x})\cap\Omega} v(\xi)\, d\xi = \lim_{r\downarrow 0} \frac{2}{|B_r(\mathbf{x})|} \int_{B_r(\mathbf{x})\setminus\Omega} v(\xi)\, d\xi = 0, \qquad \text{q.e. } \mathbf{x} \in \partial\Omega,$$

that is (2.46) implies (2.45).

Conversely if (2.45) holds then $(w^-)^* = 0$ a.e. in Ω and $\mathcal{H}^2$ a.e. on $\partial\Omega$. Therefore $w \ge 0$ a.e. in Ω and by recalling again that the negative part of the trace of w and the trace of its negative part coincide $\mathcal{H}^2$ a.e. on $\partial\Omega$ we get $w \ge 0$ $\mathcal{H}^2$ a.e. on $\partial\Omega$ thus proving (2.46) and the claim.

Similarly, again by Theorem 2.1 of [20], if $E_{ess} \subset \partial\Omega$ is Ahlfors 2-regular then the condition $w \ge 0$ q.e. on E is equivalent to $w \ge 0$ $\mathcal{H}^2$ a.e. on E, so the classical framework of [12, 31, 54] is equivalent to ours in this case as it was claimed in the Introduction.

2.4. The convergence result

The convergence result is stated in the next theorem, referring to (2.36) and (2.40).

Theorem 2.4. *Assume* (2.13), (2.19)–(2.22), (2.32)–(2.33) *and* $\mathcal{L}(\mathbf{e}_3) < 0$. *Let* $h_j \to 0^+$ *as* $j \to +\infty$ *and let* $(\overline{\mathbf{y}}_j)_{j\in\mathbb{N}} \subset H^1(\Omega, \mathbb{R}^3)$ *be a minimizing sequence of* $\mathcal{G}_j$. *If* $\mathbf{R}_j \in \mathcal{M}(\overline{\mathbf{y}}_j)$ *for every* $j \in \mathbb{N}$, *then there are* $\overline{\mathbf{c}}_j \in \mathbb{R}^3$ *such that the sequence*

$$\overline{\mathbf{u}}_j(\mathbf{x}) \ := \ h_j^{-1}\mathbf{R}_J^T\Big\{(\overline{\mathbf{y}}_j - \overline{\mathbf{c}}_j - \mathbf{R}_j\mathbf{x})_\alpha\, \mathbf{e}_\alpha \ + \ (\overline{y}_{j,3} - x_3)\mathbf{e}_3\Big\} \tag{2.48}$$

is weakly compact in $H^1(\Omega, \mathbb{R}^3)$. *Therefore up to subsequences,* $\overline{\mathbf{u}}_j \rightharpoonup \overline{\mathbf{u}}$ *in* $H^1(\Omega, \mathbb{R}^3)$ *and also*

$$\mathcal{G}_j(\mathbf{y}_j) \to \widetilde{\mathcal{G}}(\overline{\mathbf{u}}) = \min_{H^1(\Omega,\mathbb{R}^3)} \widetilde{\mathcal{G}} = \min_{H^1(\Omega,\mathbb{R}^3)} \mathcal{G}, \qquad \textit{as } j \to +\infty. \tag{2.49}$$

Remark 2.5. Since $\widetilde{\mathcal{G}} \le \mathcal{G}$ then equality $\min \widetilde{\mathcal{G}} = \min \mathcal{G}$ is equivalent to $\operatorname{argmin} \mathcal{G} \subset \operatorname{argmin} \widetilde{\mathcal{G}}$ with possible strict inclusion (see Remark 5.6), thus in general $\overline{\mathbf{u}}$ may not belong to $\operatorname{argmin} \mathcal{G}$.

Remark 2.6. Conditions (2.32) and (2.33) are compatible. Indeed set

$$\begin{aligned} \Omega &:= \{\mathbf{x} : x_1^2 + x_2^2 < 1,\ 0 < x_3 < 1\}, \\ E &:= \{\mathbf{x} : x_1^2 + x_2^2 < 1,\ x_3 = 0\}, \end{aligned} \tag{2.50}$$

and $\mathbf{f} = \mathbf{0}$, $\mathbf{g} = -\mathbf{e}_3 \mathbf{1}_E$. It is readily seen that $\mathcal{L}(\mathbf{e}_3) < 0 = \mathcal{L}(\mathbf{e}_1) = \mathcal{L}(\mathbf{e}_2)$ and

$$\mathcal{L}((\mathbf{R}\mathbf{x} - \mathbf{x})_\alpha \mathbf{e}_\alpha) = 0, \ \ \Phi(\mathbf{R}, E, \mathcal{L}) = -\pi\sqrt{1 - R_{33}^2} \le 0, \qquad \forall\, \mathbf{R} \in SO(3),$$

thus both (2.32) and (2.33) are fulfilled.

On the other hand condition (2.32) does not entail (2.33). Indeed if Ω and E are as in (2.50), $\mathbf{f} = -\mathbf{e}_3$ $\mathbf{g} = 0$ then is not since if $\overline{\mathbf{R}} = \mathbf{e}_1 \otimes \mathbf{e}_1 - \mathbf{e}_2 \otimes \mathbf{e}_2 - \mathbf{e}_3 \otimes \mathbf{e}_3$ a direct calculation yields

$$\Phi(\overline{\mathbf{R}}, E, \mathcal{L}) = 2 \int_\Omega x_3 \, d\mathbf{x} + |\Omega| \min_{E_{ess}}(-2x_3) = \pi > 0. \tag{2.51}$$

Eventually (2.33) does not imply (2.32), see Remark 4.5.

Example 2.7. Here we show an example where the all the assumptions in Theorem 2.4 concerning the geometry of the material body Ω and its portion E sensitive to the constraint and their compatibility with the loads are fulfilled. Set

$$\Omega := \{\mathbf{x} \in \mathbb{R}^3 : x_1^2 + x_2^2 < 1, \ 0 < x_3 < 1\}, \quad E := \overline{\Omega}, \tag{2.52}$$

$$\mathbf{f} := p\,\mathbf{e}_3, \quad \mathbf{g} \equiv 0, \quad p < 0. \tag{2.53}$$

Then $\mathcal{L}(\mathbf{u}) = \int_\Omega \mathbf{f} \cdot \mathbf{u} \, d\mathbf{x}$, condition (2.33) is satisfied and $\mathcal{S}_{\mathcal{L},E} = \{\mathbf{R} \in SO(3) : \mathbf{R}\mathbf{e}_3 = \mathbf{e}_3\}$.

Indeed it is readily seen that $\mathcal{L}(\mathbf{e}_3) = p|\Omega| < 0 = \mathcal{L}(\mathbf{e}_1) = \mathcal{L}(\mathbf{e}_2)$; moreover if $\mathbf{R} \in SO(3)$ and we denote its entries as R_{ij} $i, j = 1, 2, 3$ then, taking into account $p < 0$, we get

$$\begin{aligned} \Phi(\mathbf{R}, E, \mathcal{L}) &= \pi (R_{11} + R_{22} - 2) - p\pi \min_{\overline{\Omega}} \{R_{31}x_1 + R_{32}x_2 + (R_{33} - 1)x_3\} \\ &= \frac{\pi p}{2}(R_{33} - 1) + p\pi \sqrt{R_{31}^2 + R_{32}^2} + p\pi(1 - R_{33})^+ \\ &= \frac{\pi p}{2}(1 - R_{33}) + p\pi \sqrt{1 - R_{33}^2} \le 0 \end{aligned} \tag{2.54}$$

and $\Phi(\mathbf{R}, E, \mathcal{L}) = 0$ if and only if $R_{33} = 1$ that is $\mathbf{R}\mathbf{e}_3 = \mathbf{e}_3$ as claimed.

Both conditions (2.13) and (2.32) are trivially fulfilled.

We emphasize that the above claims still hold true if the assumption on E in (3.21) is weakened by allowing any $E \subset \Omega$ such that E fulfills $\mathrm{co}\, E_{ess} = \overline{\Omega}$.

3. Properties of admissible loads

This section makes explicit the properties of admissible loads by exploiting the conditions stated by (2.32) and (2.33).

Lemma 3.1. *Assume that* (2.32) *holds. Then*

$$\mathcal{L}((\mathbf{a} \wedge \mathbf{x})_\alpha \, \mathbf{e}_\alpha) = 0 \ \textit{ and } \ \mathcal{L}((\mathbf{a} \wedge (\mathbf{a} \wedge \mathbf{x}))_\alpha \, \mathbf{e}_\alpha) \le 0 \quad \forall \, \mathbf{a} \in \mathbb{R}^3. \tag{3.1}$$

Proof. By the Euler-Rodrigues formula (2.32) entails

$$\mathcal{L}\left(\frac{\sin\vartheta}{\vartheta} (\mathbf{a} \wedge \mathbf{x})_\alpha \mathbf{e}_\alpha + \frac{1 - \cos\vartheta}{\vartheta} (\mathbf{a} \wedge (\mathbf{a} \wedge \mathbf{x}))_\alpha \mathbf{e}_\alpha \right) \le 0 \tag{3.2}$$

for every $\vartheta \in (0, 2\pi)$ and by letting $\vartheta \to 0^+$ we get $\mathcal{L}((\mathbf{a} \wedge \mathbf{x})_\alpha \mathbf{e}_\alpha) \le 0$ for every $\mathbf{a} \in \mathbb{R}^3$ hence $\mathcal{L}((\mathbf{a} \wedge \mathbf{x})_\alpha \mathbf{e}_\alpha) = 0$ for every $\mathbf{a} \in \mathbb{R}^3$. The second inequality in (3.1) follows by the previous one. □

Remark 3.2. It is worth noticing that, by inserting $\mathbf{a} = \mathbf{e}_1$ or $\mathbf{a} = \mathbf{e}_2$, the condition (3.1) entails $\mathcal{L}(x_3\mathbf{e}_2) = 0$ and $\mathcal{L}(x_3\mathbf{e}_1) = 0$ respectively.

Lemma 3.3. *Assume* (2.13) *and* (2.31). *Then*

(1) $\mathcal{L}(\mathbf{e}_1) = \mathcal{L}(\mathbf{e}_2) = 0$ *and* $\mathcal{L}(\mathbf{e}_3) \le 0$;
(2) $\mathcal{L}(\mathbf{e}_3 \wedge \mathbf{x}) = 0$;
(3) $\mathcal{L}(\mathbf{e}_3 \wedge (\mathbf{e}_3 \wedge \mathbf{x})) \le 0$;
(4) there exists $\mathbf{x}_{\mathcal{L}} \in \operatorname{ri}\operatorname{co} E_{ess}$ *such that* $\mathcal{L}(\mathbf{a} \wedge (\mathbf{x} - \mathbf{x}_{\mathcal{L}})) = 0 \;\; \forall \mathbf{a} \in \mathbb{R}^3$.

Proof. By choosing $\mathbf{R} = \mathbf{I}$ in (2.31) we get $\mathcal{L}(\mathbf{c}) \le 0$ for every $\mathbf{c} \in C_{\mathbf{I}} = \{\mathbf{c} \in \mathbb{R}^3 : c_3 \ge 0\}$. Since $c_1\,\mathbf{e}_1 + c_2\,\mathbf{e}_2 \in C_{\mathbf{I}}$ for every $c_1, c_2 \in \mathbb{R}$, we get $\mathcal{L}(\mathbf{e}_1) = \mathcal{L}(\mathbf{e}_2) = 0$. Moreover $c_3\,\mathbf{e}_3 \in C_{\mathbf{I}}$ for $c_3 \ge 0$ entails $\mathcal{L}(\mathbf{e}_3) \le 0$. Thus (1) is proved and (2.31) entails

$$\Phi(\mathbf{R}, E, \mathcal{L}) := \mathcal{L}((\mathbf{R}-\mathbf{I})\mathbf{x}) - \mathcal{L}(\mathbf{e}_3) \min_{\mathbf{x}\in E_{ess}} \{((\mathbf{R}\mathbf{x})_3 - x_3)\} \le 0, \qquad \forall\, \mathbf{R} \in SO(3), \tag{3.3}$$

that is by the Euler-Rodrigues formula

$$\begin{aligned} \varphi_{\mathbf{a}}(\vartheta) := {} & \mathcal{L}(\sin\vartheta\,(\mathbf{a}\wedge\mathbf{x}) + (1-\cos\vartheta)\,\mathbf{a}\wedge(\mathbf{a}\wedge\mathbf{x})) \\ & - \min_{\mathbf{x}\in E_{ess}} \Big(\sin\vartheta\,(\mathbf{a}\wedge\mathbf{x})_3 + (1-\cos\vartheta)\,\mathbf{a}\wedge(\mathbf{a}\wedge\mathbf{x})_3\Big)\mathcal{L}(\mathbf{e}_3) \le 0, \\ & \forall \mathbf{a} \in \mathbb{R}^3, |\mathbf{a}| = 1,\; \forall\,\vartheta \in [0, 2\pi]. \end{aligned} \tag{3.4}$$

If $\mathbf{a} = \mathbf{e}_3$ then $\mathbf{R}\mathbf{e}_3 = \mathbf{e}_3$ and (3.4) reads

$$\varphi(\vartheta) := \mathcal{L}(\sin\vartheta\,(\mathbf{e}_3\wedge\mathbf{x}) + (1-\cos\vartheta)\,\mathbf{e}_3\wedge(\mathbf{e}_3\wedge\mathbf{x})) \le 0, \qquad \forall\,\vartheta \in [0, 2\pi]. \tag{3.5}$$

By $\varphi(0) = \varphi(2\pi) = 0$ and $\varphi(\vartheta) \le 0$ we get $0 \ge \varphi'(0) = \mathcal{L}(\mathbf{e}_3 \wedge \mathbf{x}) = \varphi'(2\pi) \ge 0$. Thus (2) is proved. By (2) and (3.5) we get $\mathcal{L}(\mathbf{e}_3 \wedge (\mathbf{e}_3 \wedge \mathbf{x})) \le 0$, say (3). In order to show (4), first we notice that $\mathcal{L}(\mathbf{e}_1) = \mathcal{L}(\mathbf{e}_2) = 0$ entail for every $\boldsymbol{\xi} \in \mathbb{R}^3$

$$\begin{aligned} \mathcal{L}(\mathbf{a}\wedge\boldsymbol{\xi}) &= \mathcal{L}\big(\textstyle\sum_{j=1}^3 a_j\mathbf{e}_j\wedge\boldsymbol{\xi}\big) = \textstyle\sum_{j=1}^3 a_j\,\mathcal{L}(\mathbf{e}_j\wedge\boldsymbol{\xi}) \\ &= a_1\,\mathcal{L}(-\xi_3\mathbf{e}_2 + \xi_2\mathbf{e}_3) + a_2\,\mathcal{L}(\xi_3\mathbf{e}_1 - \xi_1\mathbf{e}_3) + a_3\,\mathcal{L}(-\xi_2\mathbf{e}_1 + \xi_1\mathbf{e}_2) \\ &= \mathbf{a}\cdot(\xi_2\mathbf{e}_1 - \xi_1\mathbf{e}_2)\mathcal{L}(\mathbf{e}_3) = (\mathbf{a}\wedge\boldsymbol{\xi})_3\mathcal{L}(\mathbf{e}_3), \end{aligned} \tag{3.6}$$

moreover, $\mathcal{L}(\mathbf{e}_3 \wedge \mathbf{x}) = 0$ entails

$$\mathcal{L}(\mathbf{a}\wedge\mathbf{x}) = a_1\,\mathcal{L}(\mathbf{e}_1\wedge\mathbf{x}) + a_2\,\mathcal{L}(\mathbf{e}_2\wedge\mathbf{x}), \qquad \forall\,\mathbf{a} \in \mathbb{R}^3. \tag{3.7}$$

Let us assume first that $\mathcal{L}(\mathbf{e}_3) < 0$. In this case we can set

$$\widetilde{x}_1 = -\frac{\mathcal{L}(\mathbf{e}_2\wedge\mathbf{x})}{\mathcal{L}(\mathbf{e}_3)}, \qquad \widetilde{x}_2 = \frac{\mathcal{L}(\mathbf{e}_1\wedge\mathbf{x})}{\mathcal{L}(\mathbf{e}_3)}, \tag{3.8}$$

hence, by (3.6)–(3.8),

$$\mathcal{L}(\mathbf{a}\wedge\widetilde{\mathbf{x}}) = (\mathbf{a}\wedge\widetilde{\mathbf{x}})_3\,\mathcal{L}(\mathbf{e}_3) = (a_1\widetilde{x}_2 - a_2\widetilde{x}_1)\,\mathcal{L}(\mathbf{e}_3) = a_1\,\mathcal{L}(\mathbf{e}_1\wedge\mathbf{x}) + a_2\,\mathcal{L}(\mathbf{e}_2\wedge\mathbf{x}) = \mathcal{L}(\mathbf{a}\wedge\mathbf{x})$$

say

$$\mathcal{L}(\mathbf{a}\wedge\mathbf{x}) = \mathcal{L}(\mathbf{a}\wedge\widetilde{\mathbf{x}}), \qquad \forall\,\mathbf{a}\in\mathbb{R}^3,\ \forall\,\widetilde{\mathbf{x}}\in\{(\widetilde{x}_1,\widetilde{x}_2,z) : z\in\mathbb{R}\}. \tag{3.9}$$

Since $\varphi_{\mathbf{a}}(0) = \varphi_{\mathbf{a}}(2\pi) = 0$ and $\varphi_{\mathbf{a}}(\vartheta) \le 0$ for every $\mathbf{a}\in\mathbb{R}^3$, $|\mathbf{a}| = 1$ and for every $\vartheta\in[0,2\pi]$, (3.4) entails

$$0 \ \ge\ \limsup_{\vartheta\to 0_+}\frac{\varphi_{\mathbf{a}}(\vartheta)}{\vartheta} = \mathcal{L}(\mathbf{a}\wedge\mathbf{x}) - \min_{\mathbf{x}\in E_{ess}}(\mathbf{a}\wedge\mathbf{x})_3\,\mathcal{L}(\mathbf{e}_3), \qquad \forall\,|\mathbf{a}| = 1, \tag{3.10}$$

$$0 \ \le\ \liminf_{\vartheta\to 2\pi^-}\frac{\varphi_{\mathbf{a}}(\vartheta)}{\vartheta-2\pi} = \mathcal{L}(\mathbf{a}\wedge\mathbf{x}) - \max_{\mathbf{x}\in E_{ess}}(\mathbf{a}\wedge\mathbf{x})_3\,\mathcal{L}(\mathbf{e}_3), \qquad \forall\,|\mathbf{a}| = 1. \tag{3.11}$$

Hence

$$\max_{\mathbf{x}\in E_{ess}}(\mathbf{a}\wedge\mathbf{x})_3\,\mathcal{L}(\mathbf{e}_3) \ \le\ \mathcal{L}(\mathbf{a}\wedge\mathbf{x}) \ \le\ \min_{\mathbf{x}\in E_{ess}}(\mathbf{a}\wedge\mathbf{x})_3\,\mathcal{L}(\mathbf{e}_3),$$

so, by (3.6), (3.8) and (3.9),

$$\max_{\mathbf{x}\in E_{ess}}(\mathbf{a}\wedge\mathbf{x})_3\,\mathcal{L}(\mathbf{e}_3) \ \le\ (\mathbf{a}\wedge\widetilde{\mathbf{x}})_3\,\mathcal{L}(\mathbf{e}_3) \ \le\ \min_{\mathbf{x}\in E_{ess}}(\mathbf{a}\wedge\mathbf{x})_3\,\mathcal{L}(\mathbf{e}_3). \tag{3.12}$$

By taking account of $\mathcal{L}(\mathbf{e}_3) < 0$, we find

$$\min_{\mathbf{x}\in E_{ess}}(\mathbf{a}\wedge\mathbf{x})_3 \ \le\ (\mathbf{a}\wedge\widetilde{\mathbf{x}})_3 \ \le\ \max_{\mathbf{x}\in E_{ess}}(\mathbf{a}\wedge\mathbf{x})_3, \qquad \forall\,\mathbf{a}\in\mathbb{R}^3 :\ |\mathbf{a}| = 1,$$

hence, by linearity and by homogeneity,

$$\min_{\mathbf{x}\in\mathrm{co}\,E_{ess}}(\mathbf{a}\wedge\mathbf{x})_3 \ \le\ (\mathbf{a}\wedge\widetilde{\mathbf{x}})_3 \ \le\ \max_{\mathbf{x}\in\mathrm{co}\,E_{ess}}(\mathbf{a}\wedge\mathbf{x})_3, \qquad \forall\,\mathbf{a}\in\mathbb{R}^3. \tag{3.13}$$

By subtracting $(\mathbf{a}\wedge\widetilde{\mathbf{x}})_3$ on each term of inequality (3.13) we get

$$\min_{\mathbf{y}\in\,\mathrm{co}\,E_{ess}-\widetilde{\mathbf{x}}}(\mathbf{a}\wedge\mathbf{y})_3 \ \le\ 0 \ \le\ \max_{\mathbf{y}\in\,\mathrm{co}\,E_{ess}-\widetilde{\mathbf{x}}}(\mathbf{a}\wedge\mathbf{y})_3$$

for every $\mathbf{a}\in\mathbb{R}^3$ and for every $\widetilde{\mathbf{x}}\in\{(\widetilde{x}_1,\widetilde{x}_2,z) : z\in\mathbb{R}\}$.

We claim that at least one of the above inequalities is strict for every $\mathbf{a}\in\mathbb{R}^3$ such that $a_1^2 + a_2^2 \ne 0$. Indeed, if by contradiction there was $\overline{\mathbf{a}} = (\overline{a}_1,\overline{a}_2,\overline{a}_3)$ with $\overline{a}_1^2 + \overline{a}_2^2 \ne 0$ such that

$$\min_{\mathbf{x}\in\,\mathrm{co}\,E_{ess}-\widetilde{\mathbf{x}}}(\overline{\mathbf{a}}\wedge\mathbf{x})_3 \ =\ \max_{\mathbf{x}\in\,\mathrm{co}\,E_{ess}-\widetilde{\mathbf{x}}}(\overline{\mathbf{a}}\wedge\mathbf{x})_3 = 0, \qquad \forall\,\widetilde{\mathbf{x}}\in\{(\widetilde{x}_1,\widetilde{x}_2,z),\ z\in\mathbb{R}\},$$

then

$$(\,\mathrm{co}\,E_{ess}-\widetilde{\mathbf{x}}\,) \ \subset\ \{\,\mathbf{x} :\ \overline{a}_1x_2 - \overline{a}_2x_1 = 0\,\}.$$

Since the plane $\{\overline{a}_1x_2 - \overline{a}_2x_1 = 0\}$ is orthogonal to $\{x_3 = 0\}$ we obtain

$$\mathrm{cap}\,(\,\mathrm{proj}\{\,\mathbf{x}\in\mathbb{R}^3 : \overline{a}_1x_2 - \overline{a}_2x_1 = 0\,\}) = 0,$$

hence

$$0 = \mathrm{cap}\Big(\,\mathrm{proj}\,(\,\mathrm{co}\,E_{ess}-\widetilde{\mathbf{x}}\,)\Big) = \mathrm{cap}\Big(\,\mathrm{proj}(\mathrm{co}\,E_{ess})\Big)$$

which contradicts (2.13).

Without loss of generality we can proceed by assuming that the first inequality is strict, say

$$\min_{\mathbf{x}\in \operatorname{co} E_{ess}-\widetilde{\mathbf{x}}} (\mathbf{a}\wedge\mathbf{x})_3 \ < \ 0$$

for every $\mathbf{a} \in \mathbb{R}^3$ such that $a_1^2 + a_2^2 \neq 0$ and for every $\widetilde{\mathbf{x}} \in \{(\widetilde{x}_1, \widetilde{x}_2, z),\ z \in \mathbb{R}\}$. Hence, by setting $T := \operatorname{proj}(\operatorname{co} E_{ess} - \widetilde{\mathbf{x}})$, we get

$$\min_{\mathbf{x}\in T} (\mathbf{a}\wedge\mathbf{x})_3 \ < \ 0 \tag{3.14}$$

for every $\mathbf{a} \in \mathbb{R}^3$ such that $a_1^2 + a_2^2 \neq 0$. For every $(a_1, a_2) \in \mathbb{R}^2$ such that $a_1^2 + a_2^2 \neq 0$ we set now

$$\Gamma(a_1, a_2) \ := \ \{ (x_1, x_2) \in \mathbb{R}^2 :\ a_1x_2 - a_2x_1 \geq \min_{(x_1,x_2)\in T} (a_1x_2 - a_2x_1) \}$$

then $\{ \Gamma(a_1, a_2) :\ a_1^2 + a_2^2 = 1 \}$ is the set of half-planes supporting T. Since T is closed and convex, we get

$$T \ = \bigcap_{a_1^2+a_2^2=1} \Gamma(a_1, a_2).$$

By (3.14), we get

$$\operatorname{dist}(\,(0,0)\,,\,\partial\Gamma(a_1, a_2)\,) = \left| \min_{\mathbf{x}\in T} (a_1x_2 - a_2x_1) \right| > 0.$$

Hence we deduce the existence of $(\widetilde{a}_1, \widetilde{a}_2) :\ \widetilde{a}_1^2 + \widetilde{a}_2^2 = 1$ such that

$$\min_{\widetilde{a}_1^2+\widetilde{a}_2^2=1} \operatorname{dist}(\,(0,0)\,,\,\partial\Gamma(a_1, a_2)) = |\min_{\mathbf{x}\in T} (\widetilde{a}_1x_2 - \widetilde{a}_2x_1)| > 0,$$

so $(0,0) \in \operatorname{ri} T$ that is $(\widetilde{x}_1, \widetilde{x}_2, 0) \in \operatorname{ri}\operatorname{proj}(\operatorname{co} E_{ess})$.

We are left to show that there exists $\widetilde{x}_3$ such that $\mathbf{x}_{\mathcal{L}} := (\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3) \in \operatorname{ri}\operatorname{co} E_{ess}$. To this aim it is readily seen that by taking account of $\operatorname{cap}(\operatorname{proj}(\operatorname{co} E_{ess})) > 0$, we get $\operatorname{aff}(\operatorname{proj}(\operatorname{co} E_{ess}))) = \{x_3 = 0\}$ so there exists $\overline{r} > 0$ such that

$$\left\{(x_1, x_2) : |x_1 - \widetilde{x}_1|^2 + |x_2 - \widetilde{x}_2|^2 < \overline{r}^2\right\} \subset \operatorname{proj}(\operatorname{co} E_{ess})).$$

Let now

$$J := \{z : (x_1, x_2, z) \in \operatorname{co} E_{ess}\} \neq \emptyset$$

and assume that $(x_1, x_2, z) \in (\operatorname{co} E_{ess}) \setminus (\operatorname{ri}\operatorname{co} E_{ess})$ for every $z \in J$. Then

$$B_r(x_1, x_2, z) \cap \operatorname{co} E_{ess} \neq \emptyset, \qquad B_r(x_1, x_2, z) \cap (\,(\operatorname{aff}\operatorname{co} E_{ess})\setminus(\operatorname{co} E_{ess})\,) \neq \emptyset$$

for every $z \in J$ and for every $r > 0$, therefore by recalling that $\operatorname{aff}\operatorname{proj}\operatorname{co} E_{ess} = \{x_3 = 0\}$

$$\operatorname{proj} B_r(x_1, x_2, z) \cap \operatorname{proj}\operatorname{co} E_{ess} \neq \emptyset,$$

$$\operatorname{proj} B_r(x_1, x_2, z) \cap (\{x_3 = 0\} \setminus \operatorname{proj}\operatorname{co} E_{ess}) \neq \emptyset$$

for every $r > 0$. This is a contradiction since

$$\operatorname{proj} B_r(x_1, x_2, z) \subset \left\{(x_1, x_2) : |x_1 - \widetilde{x}_1|^2 + |x_2 - \widetilde{x}_2|^2 < \overline{r}^2\right\} \subset \operatorname{proj}(\operatorname{co} E_{ess})$$

for some $r > 0$ thus (4) is proved in this case since by construction $\mathcal{L}(\mathbf{a} \wedge (\mathbf{x} - \mathbf{x}_{\mathcal{L}}) = 0$ for every $\mathbf{a} \in \mathbb{R}^3$. Eventually, if $\mathcal{L}(\mathbf{e}_3) = 0$, (2.33) reduces to

$$\mathcal{L}((\mathbf{R} - \mathbf{I})\mathbf{x}) \leq 0, \qquad \forall\, \mathbf{R} \in SO(3),$$

say $\sin\vartheta\, \mathcal{L}(\mathbf{a} \wedge \mathbf{x}) + (1 - \cos\vartheta)\mathcal{L}(\mathbf{a} \wedge (\mathbf{a} \wedge \mathbf{x})) \leq 0$ for all $\mathbf{a} \in \mathbb{R}^3$, thus, by repeating the analysis made on (3.5), we get

$$\mathcal{L}(\mathbf{a} \wedge \mathbf{x}) = 0, \qquad \forall\, \mathbf{a} \in \mathbb{R}^3,$$

and, since $\operatorname{ri}\operatorname{proj}\operatorname{co} E_{ess} \neq \emptyset$ due to (2.13), by exploiting identity (3.6) with $\boldsymbol{\xi} = \widetilde{\mathbf{x}}$ we obtain, for whatever choice of $\widetilde{\mathbf{x}} \in \operatorname{ri}\operatorname{proj}\operatorname{co} E_{ess}$

$$\mathcal{L}(\mathbf{a} \wedge (\mathbf{x} - \widetilde{\mathbf{x}})) = -\mathcal{L}(\mathbf{a} \wedge \widetilde{\mathbf{x}}) = -(\mathbf{a} \wedge \widetilde{\mathbf{x}})_3\, \mathcal{L}(\mathbf{e}_3) = 0, \qquad \forall\, \mathbf{a} \in \mathbb{R}^3,$$

that is (4) is proven also in this case. □

Remark 3.4. Conditions (2.31) and (2.33) are equivalent as claimed in Subsection 2.2.

Indeed, as it has been pointed out in the proof of Lemma 3.3, condition (2.31) implies that $\mathcal{L}(\mathbf{e}_3) \leq 0$, $\mathcal{L}(\mathbf{e}_1) = \mathcal{L}(\mathbf{e}_2) = 0$ and

$$\Phi(\mathbf{R}, E, \mathcal{L}) := \mathcal{L}((\mathbf{R} - \mathbf{I})\mathbf{x}) - \mathcal{L}(\mathbf{e}_3) \min_{\mathbf{x} \in E_{ess}} ((\mathbf{R}\mathbf{x})_3 - x_3) \leq 0, \qquad \forall\, \mathbf{R} \in SO(3). \tag{3.15}$$

Conversely if the latter condition holds and $\mathcal{L}(\mathbf{e}_3) \leq 0$, $\mathcal{L}(\mathbf{e}_1) = \mathcal{L}(\mathbf{e}_2) = 0$, then

$$\mathcal{L}((\mathbf{R} - \mathbf{I})\mathbf{x} + \mathbf{c}) = \mathcal{L}((\mathbf{R} - \mathbf{I})\mathbf{x} + c_3\mathbf{e}_3) \leq 0$$

for every $\mathbf{c} \in \mathbb{R}^3$ such that $c_3 \geq -\min_{\mathbf{x} \in E_{ess}}((\mathbf{R}\mathbf{x})_3 - x_3)\}$.

Remark 3.5. We emphasize that conditions (1) and (4) in Lemma 3.3 together with (2.13) and $\mathcal{L}(\mathbf{e}_3) < 0$ coincide with conditions (4.9)–(4.11) of Theorem 4.5 of [12], which provides the solution to Signorini problem in linear elasticity.

The whole set of conditions (1)–(4) appearing in the claim of Lemma 3.3 together with condition (2.13) on the set E is not equivalent to admissibility of the loads as expressed by (2.33): This phenomenon is made explicit by subsequent Example 3.6.

Example 3.6. Let $\Omega = \{\mathbf{x} : x_1^2 + x_2^2 < 1,\ 0 < x_3 < H\}$, $E = E_{ess} = \{(x_1, x_2, 0) : x_1^2 + x_2^2 \leq 1\}$ and $\mathcal{L}(\mathbf{v}) = \int_\Omega p\, v_3\, d\mathbf{x}$ with $p < 0$, say $\mathbf{f} = p\,\mathbf{e}_3$, $\mathbf{g} = \mathbf{0}$.

Then E fulfills (2.13), since $\operatorname{cap} E > 0$ and $\operatorname{proj}(\operatorname{co} E_{ess}) = E_{ess} \subset \overline{\Omega} \cap \{x_3 = 0\}$; moreover all claims (1)–(4) of Lemma 3.3 hold true: Indeed

$$\mathcal{L}(\mathbf{e}_1) = \mathcal{L}(\mathbf{e}_2) = 0, \quad \mathcal{L}(\mathbf{e}_3) = p|\Omega| < 0,$$

$$\mathcal{L}(\mathbf{e}_3 \wedge \mathbf{x}) = \int_\Omega (-\mathbf{e}_3) \cdot (\mathbf{e}_3 \wedge \mathbf{x}) d\mathbf{x} = 0,$$

$$\mathcal{L}(\mathbf{e}_3 \wedge (\mathbf{e}_3 \wedge \mathbf{x}) = \int_\Omega (-\mathbf{e}_3) \cdot (\mathbf{e}_3 \wedge (\mathbf{e}_3 \wedge \mathbf{x})) d\mathbf{x} = 0,$$

eventually, by choosing $\mathbf{x}_{\mathcal{L}} = (0,0,0) \in \operatorname{ri}(\operatorname{co} E_{ess}) = E$ and by taking the symmetry of Ω into account, we get

$$\mathcal{L}(\mathbf{a} \wedge (\mathbf{x} - \mathbf{x}_{\mathcal{L}})) \;=\; \mathcal{L}(\mathbf{a} \wedge \mathbf{x}) \;=\; \int_\Omega (-\mathbf{e}_3) \cdot (\mathbf{a} \wedge \mathbf{x})\, d\mathbf{x} \;=\; \mathbf{a} \cdot \int_\Omega (-x_2 \mathbf{e}_1 + x_1 \mathbf{e}_2)\, d\mathbf{x} \;=\; 0.$$

Nevertheless, condition (2.33) is violated, since we can consider the π radians rotation around axis $\mathbf{e}_1$ which keeps E above the horizontal plane (obstacle boundary) but capsizes the body below the horizontal plane, namely $\widetilde{\mathbf{R}} \in SO(3)$ defined by $\widetilde{\mathbf{R}}\mathbf{x} = x_1 \mathbf{e}_1 - x_2 \mathbf{e}_2 - x_3 \mathbf{e}_3$. Thus

$$\mathcal{L}((\widetilde{\mathbf{R}} - \mathbf{I})\mathbf{x}) \;-\; \mathcal{L}(\mathbf{e}_3) \min_{E_{ess}} ((\widetilde{\mathbf{R}}\mathbf{x})_3 - x_3) = -2p \int_\Omega x_3\, d\mathbf{x} - p|\Omega| \min_{E_{ess}} (-2x_3) = -p\pi H^2 \;>\; 0.$$

The assumptions in Lemma 3.3 and in Theorem 2.4 cannot be weakened by assuming only $\operatorname{cap} E > 0$ in place of (2.13) as it is shown in the next example, thus proving that Theorem 2.4 is a sharp result with respect to the sets E subject to the constraint that are admissible.

Example 3.7. Choose $\mathbf{f} = -\mathbf{e}_3,\ \mathbf{g} = \mathbf{0}$ and

$$\Omega = \{\mathbf{x} : x_1^2 + x_2^2 < 1,\ x_2 > 0,\ 0 < x_3 < 1\},$$

$$E \;=\; E_{ess} \;=\; \{(x_1, x_2, x_3) \in \overline{\Omega} :\ x_2 = 0\}.$$

It is readily seen that condition (2.32) is fulfilled since $\mathcal{L}((\mathbf{R}\mathbf{x} - \mathbf{x})_\alpha \mathbf{e}_\alpha) = 0$ moreover, since $\mathcal{L}(\mathbf{e}_3) < 0 = \mathcal{L}(\mathbf{e}_1) = \mathcal{L}(\mathbf{e}_2)$,

$$\begin{aligned}
\Phi(\mathbf{R}, E, \mathcal{L}) \;&= -\tfrac{2}{3} R_{32} + \tfrac{\pi}{2}(1 - R_{33}) + \min\nolimits_{E_{ess}} \{R_{31}x_1 + R_{32}x_2 + (R_{33} - 1)x_3\}\, |\Omega| \\
&= -\pi |R_{31}| - \pi (R_{32})^- - \frac{2}{3} R_{32} + \frac{\pi}{2}\,(R_{33} - 1) \\
&= -\pi |R_{31}| - \frac{\pi}{2} |R_{32}| + \left(\frac{\pi}{2} - \frac{2}{3}\right) R_{32} + \frac{\pi}{2}\,(R_{33} - 1) \le 0
\end{aligned}$$

for every $\mathbf{R} \in SO(3)$, then also condition (2.33) is satisfied. Nevertheless it can be easily checked that $\operatorname{cap} E > 0$ but $\operatorname{cap}(\operatorname{proj}(\operatorname{co} E_{ess})) = 0$, thus E does not fulfil (2.13).

If there was $\overline{\mathbf{x}} \in \operatorname{ri} \operatorname{co} E_{ess}$ such that $\mathcal{L}(\mathbf{a} \wedge (\mathbf{x} - \overline{\mathbf{x}})) = 0$ for every $\mathbf{a} \in \mathbb{R}^3$ then we get

$$\mathcal{L}(\mathbf{a} \wedge \overline{\mathbf{x}}) = \mathcal{L}(\mathbf{a} \wedge \mathbf{x}) = -\int_\Omega \mathbf{e}_3 \cdot (\mathbf{a} \wedge \mathbf{x})\, d\mathbf{x} = -\int_\Omega (a_1 x_2 - a_2 x_1)\, d\mathbf{x} = -\frac{2}{3} a_1;$$

then, since $\mathcal{L}(\mathbf{e}_1) = \mathcal{L}(\mathbf{e}_2) = 0$, we could apply (3.6) and find

$$-\frac{2}{3} a_1 \;=\; \mathcal{L}(\mathbf{a} \wedge \overline{\mathbf{x}}) \;=\; (\mathbf{a} \wedge \overline{\mathbf{x}})_3\, \mathcal{L}(\mathbf{e}_3) = -\pi (a_1 \overline{x}_2 - a_2 \overline{x}_1), \qquad \forall\, \mathbf{a} \in \mathbb{R}^3,$$

hence $\overline{x}_1 = 0,\ \overline{x}_2 = \frac{2}{3\pi}$, thus

$$\overline{\mathbf{x}} \notin \operatorname{ri} \operatorname{co} E_{ess} = \{(x_1, x_2, x_3) \in \overline{\Omega} :\ x_2 = 0\},$$

a contradiction. Thus claim (4) of Lemma 3.3 cannot hold true in this case.

Moreover if we set $\mathbf{w}_k(\mathbf{x}) := -k\,(\,x_2\,\mathbf{e}_3 - x_3\,\mathbf{e}_2\,)$, then $\mathbb{E}(\mathbf{w}_k) = \mathbf{0},\ w_{k,3} + x_3 \equiv x_3 \geq 0$ on E whence $\mathbf{w}_k \in \mathcal{A}$ and it is readily seen that

$$\mathcal{L}(\mathbf{R}\mathbf{v}) = \mathcal{L}(\mathbf{v}), \qquad \forall\, \mathbf{R} \in \mathcal{S}_{\mathcal{L},E}, \tag{3.16}$$

hence

$$\mathcal{G}(\mathbf{w}_k) = -\mathcal{L}(\mathbf{w}_k) = \int_\Omega \mathbf{e}_3 \cdot \mathbf{w}_k\, d\mathbf{x} = -k\int_\Omega x_2\, d\mathbf{x} = -\frac{2}{3}k \to -\infty, \quad \text{as } k \to +\infty$$

that is $\inf_{\mathcal{A}} \mathcal{G} = -\infty$ so the convergence of the energies claimed in Theorem 2.4 fails to be true in this case thus showing sharpness of condition (2.13).

Lemma 3.8. *Assume that* (2.13)*,* (2.33) *hold and that* $\mathcal{L}(\mathbf{e}_3) < 0$*. Then*

$$\textit{either} \quad \mathcal{S}_{\mathcal{L},E} = \{\,\mathbf{I}\,\} \quad \textit{or} \quad \mathcal{S}_{\mathcal{L},E} = \{\,\mathbf{R} \in SO(3) : \mathbf{R}\mathbf{e}_3 = \mathbf{e}_3\,\}. \tag{3.17}$$

Proof. First, we prove the inclusion

$$\mathcal{S}_{\mathcal{L},E} \subset \{\,\mathbf{R} \in SO(3) : \mathbf{R}\mathbf{e}_3 = \mathbf{e}_3\,\}.$$

Indeed if $\mathbf{R}$ belongs to $\mathcal{S}_{\mathcal{L},E}$ and $\mathbf{a}$ is a rotation axis of $\mathbf{R}$ with $|\mathbf{a}| = 1$, then

$$\begin{aligned} \varphi_{\mathbf{a}}(\vartheta) \ &:= \Phi(\mathbf{R}, E, \mathcal{L}) = \mathcal{L}\left(\sin\vartheta(\mathbf{a}\wedge\mathbf{x}) + (1-\cos\vartheta)(\mathbf{a}\wedge(\mathbf{a}\wedge\mathbf{x}))\right) \\ &- \min_{E_{ess}}\left\{\sin\vartheta(\mathbf{a}\wedge\mathbf{x})_3 + (1-\cos\vartheta)(\mathbf{a}\wedge(\mathbf{a}\wedge\mathbf{x}))_3\right\}\mathcal{L}(\mathbf{e}_3) \\ &= 0 \end{aligned} \tag{3.18}$$

for every $\vartheta \in [0, 2\pi]$. By arguing now as in the proof of (4) of Lemma 3.3, we get

$$\begin{aligned} 0 \ &\geq \ \lim_{\vartheta\to0^+} \frac{\varphi_{\mathbf{a}}(\vartheta)}{\vartheta} = \mathcal{L}(\mathbf{a}\wedge\mathbf{x}) - \min_{\mathbf{x}\in E_{ess}}(\mathbf{a}\wedge\mathbf{x})_3\,\mathcal{L}(\mathbf{e}_3) \\ &\geq \ \lim_{\vartheta\to2\pi^-} \frac{\varphi_{\mathbf{a}}(\vartheta)}{\vartheta - 2\pi} = \mathcal{L}(\mathbf{a}\wedge\mathbf{x}) - \max_{\mathbf{x}\in E_{ess}}(\mathbf{a}\wedge\mathbf{x})_3\,\mathcal{L}(\mathbf{e}_3) \\ &\geq \ 0 \end{aligned} \tag{3.19}$$

and, since $\mathcal{L}(\mathbf{e}_3) < 0$, we get

$$\min_{\mathbf{x}\in E_{ess}}(\mathbf{a}\wedge\mathbf{x})_3 = \max_{\mathbf{x}\in E_{ess}}(\mathbf{a}\wedge\mathbf{x})_3 \tag{3.20}$$

that is the function

$$\mathbf{x} \to (\mathbf{a}\wedge\mathbf{x})_3 = a_1x_2 - a_2x_1$$

is constant on E_{ess} hence it is constant in $\mathrm{co}(E_{ess})$ thus $\mathrm{cap\,proj\,co}\,E_{ess} > 0$ entails $a_1 = a_2 = 0$ that is $\mathbf{R}\mathbf{e}_3 = \mathbf{e}_3$ as claimed.

We notice that $\mathbf{I} \in \mathcal{S}_{\mathcal{L},E}$ and, the other hand, if $\mathcal{S}_{\mathcal{L},E} \not\equiv \{\,\mathbf{I}\,\}$ then there is

$$\widetilde{\mathbf{R}} \in \mathcal{S}_{\mathcal{L},E} \subset \{\,\mathbf{R} \in SO(3) : \mathbf{R}\mathbf{e}_3 = \mathbf{e}_3\,\}$$

such that $\widetilde{\mathbf{R}} \neq \mathbf{I}$ and

$$\widetilde{\mathbf{R}}\mathbf{x} = \mathbf{x} + \sin\widetilde{\vartheta}\,(\mathbf{e}_3\wedge\mathbf{x}) + (1-\cos\widetilde{\vartheta})\,(\mathbf{e}_3\wedge(\mathbf{e}_3\wedge\mathbf{x}))$$

for every $\mathbf{x} \in \mathbb{R}^3$ and for some suitable $\widetilde{\vartheta} \in (0, 2\pi)$. By taking (2) of Lemma 3.3 into account, we get

$$\begin{aligned} 0 &= \Phi(\widetilde{\mathbf{R}}, E, \mathcal{L}) = \mathcal{L}\Big((\widetilde{\mathbf{R}} - \mathbf{I})\,\mathbf{x}\Big) - \mathcal{L}(\mathbf{e}_3) \min_{\mathbf{x}\in E_{ess}} \big((\widetilde{\mathbf{R}} - \mathbf{I})\mathbf{x})\big)_3 \\ &= \mathcal{L}\Big((\widetilde{\mathbf{R}} - \mathbf{I})\,\mathbf{x}\Big) = \sin\widetilde{\vartheta}\, \mathcal{L}(\mathbf{e}_3 \wedge \mathbf{x}) + (1 - \cos\widetilde{\vartheta})\, \mathcal{L}(\mathbf{e}_3 \wedge (\mathbf{e}_3 \wedge \mathbf{x})) \\ &= (1 - \cos\widetilde{\vartheta})\, \mathcal{L}(\mathbf{e}_3 \wedge (\mathbf{e}_3 \wedge \mathbf{x})), \end{aligned}$$

thus $\mathcal{L}(\mathbf{e}_3 \wedge (\mathbf{e}_3 \wedge \mathbf{x})) = 0$.

Therefore for any other $\mathbf{R} \in SO(3)$, $\mathbf{R} \neq \mathbf{I}$ such that $\mathbf{R}\mathbf{e}_3 = \mathbf{e}_3$ there is $\vartheta \in (0, 2\pi)$ such that

$$\mathbf{R}\mathbf{x} = \mathbf{x} + \sin\vartheta\, (\mathbf{e}_3 \wedge \mathbf{x}) + (1 - \cos\vartheta)\, (\mathbf{e}_3 \wedge (\mathbf{e}_3 \wedge \mathbf{x})), \qquad \forall\, \mathbf{x} \in \mathbb{R}^3,$$

thus, by taking again (2) of Lemma 3.3 into account, we get

$$\begin{aligned} &\mathcal{L}\Big((\mathbf{R} - \mathbf{I})\,\mathbf{x}\Big) - \mathcal{L}(\mathbf{e}_3) \min_{\mathbf{x}\in E_{ess}} \big((\mathbf{R} - \mathbf{I})\mathbf{x})\big)_3 \\ &\qquad = \mathcal{L}\Big((\mathbf{R} - \mathbf{I})\,\mathbf{x}\Big) = \sin\vartheta\, \mathcal{L}(\mathbf{e}_3 \wedge \mathbf{x}) + (1 - \cos\vartheta)\, \mathcal{L}(\mathbf{e}_3 \wedge (\mathbf{e}_3 \wedge \mathbf{x})) \\ &\qquad = (1 - \cos\vartheta)\, \mathcal{L}(\mathbf{e}_3 \wedge (\mathbf{e}_3 \wedge \mathbf{x})) = 0 \end{aligned}$$

that is $\mathbf{R}$ belongs to $\mathcal{S}_{\mathcal{L},E}$ thus concluding the proof of the lemma. □

Remark 3.9. *It is possible to show that both alternatives in Lemma 3.8 can actually occur. Indeed in Example 2.7 we have exhibited an example in which* $\mathcal{S}_{\mathcal{L},E} = \{\mathbf{R} \in SO(3) : \mathbf{R}\mathbf{e}_3 = \mathbf{e}_3\}$ *and we show here that also the other alternative may occur. Indeed set*

$$\Omega := \{\mathbf{x}\in\mathbb{R}^3 : x_1^2 + x_2^2 < 1,\ 0 < x_3 < 1\}, \quad E := \overline{\Omega}, \tag{3.21}$$

$$\mathbf{f} := -\mathbf{e}_3, \quad \mathbf{g} = \mathbf{1}_{\partial_l\Omega}\mathbf{n}, \tag{3.22}$$

where $\partial_l\Omega$ *is the lateral boundary of* Ω *and* $\mathbf{n}$ *the unit outward vector normal to* $\partial_l\Omega$*. If* $\mathbf{R} \in SO(3)$ *and we denote its entries as* R_{ij} $i, j = 1, 2, 3$, *then*

$$\mathcal{L}\left((\mathbf{R}\mathbf{x} - \mathbf{x})_\alpha\, \mathbf{e}_\alpha\right) = \sum_{i=1,2} (R_{ii} - 1) \int_{\partial_l\Omega} x_i^2\, d\mathcal{H}^2 \leq 0$$

that is condition (2.32) *is satisfied. Moreover since*

$$\mathcal{L}(\mathbf{e}_3) = -|\Omega| < 0 = \mathcal{L}(\mathbf{e}_1) = \mathcal{L}(\mathbf{e}_2)$$

and

$$\begin{aligned} \Phi(\mathbf{R}, E, \mathcal{L}) &= \mathcal{L}\left(\textstyle\sum_{j=1}^3 (\mathbf{R}\mathbf{x} - \mathbf{x})_j \mathbf{e}_j\right) + \pi\,(R_{11} + R_{22} - 2) + \pi \min_{\overline{\Omega}} \{R_{31}x_1 + R_{32}x_2 + (R_{33} - 1)x_3\} \\ &= -\frac{\pi}{2}(R_{33} - 1) - \pi\sqrt{R_{31}^2 + R_{32}^2} - \pi(1 - R_{33})^+ \sum_{i=1,2} (R_{ii} - 1) \int_{\partial_l\Omega} x_i^2\, d\mathcal{H}^2 \\ &= -\frac{\pi}{2}(1 - R_{33}) - \pi\sqrt{1 - R_{33}^2} + \sum_{i=1,2} (R_{ii} - 1) \int_{\partial_l\Omega} x_i^2\, d\mathcal{H}^2 \leq 0 \end{aligned} \tag{3.23}$$

and equality holds if and only if $R_{11} = R_{22} = R_{33} = 1$ *then condition* (2.33) *is satisfied and* $\mathcal{S}_{\mathcal{L},E} \equiv \{\mathbf{I}\}$ *as claimed.*

Lemma 3.10. *Assume* (2.13), (2.33) *and* $\mathcal{L}(\mathbf{e}_3) < 0$. *Let* $\mathbf{R}_j \in SO(3)$ *be a sequence of rotations such that* $\mathbf{R}_j\mathbf{e}_3 \neq \mathbf{e}_3$, $\mathbf{R}_j\mathbf{e}_3 \to \mathbf{e}_3$ *as* $j \to +\infty$. *Then*

$$\limsup_{j\to+\infty} \frac{\Phi(\mathbf{R}_j, E, \mathcal{L})}{|\mathbf{R}_j\mathbf{e}_3 - \mathbf{e}_3|\,|\mathcal{L}(\mathbf{e}_3)|} \;<\; 0. \tag{3.24}$$

Proof. $\Phi(\mathbf{R}_j, E, \mathcal{L}) \le 0$, by (2.33). Hence the lim sup in (3.24) cannot be strictly positive.

We assume by contradiction

$$\limsup_{j\to+\infty} \frac{\Phi(\mathbf{R}_j, E, \mathcal{L})}{|\mathbf{R}_j\mathbf{e}_3 - \mathbf{e}_3|\,|\mathcal{L}(\mathbf{e}_3)|} \;=\; 0. \tag{3.25}$$

By Euler-Rodrigues formula there are sequences $\mathbf{a}_j \in \mathbb{R}^3$ and $\vartheta_j \in [0, 2\pi]$, such that $|\mathbf{a}_j| = 1$ and

$$\mathbf{R}_j\mathbf{x} \;=\; \mathbf{x} + (\sin\vartheta_j)(\mathbf{a}_j \wedge \mathbf{x}) + (1 - \cos\vartheta_j)((\mathbf{a}_j \wedge (\mathbf{a}_j \wedge \mathbf{x})), \qquad \forall \mathbf{x} \in \mathbb{R}^3, \tag{3.26}$$

thus a direct computation yields

$$|\mathbf{R}_j\mathbf{e}_3 - \mathbf{e}_3| \;=\; \sqrt{a_{1,j}^2 + a_{2,j}^2}\,\sqrt{2\,(1 - \cos\vartheta_j)}. \tag{3.27}$$

By taking account of $\mathbf{R}_j\mathbf{e}_3 \neq \mathbf{e}_3$ and $\mathbf{R}_j\mathbf{e}_3 \to \mathbf{e}_3$ as $j \to +\infty$, we get $\mathbf{a}_j \neq \mathbf{e}_3$, $\vartheta_j \in (0, 2\pi)$ and therefore, up to subsequences, we may assume: that $\mathbf{a}_j \to \mathbf{a}$, $\vartheta_j \to \vartheta \in [0, 2\pi]$, that either $\vartheta \in \{0, 2\pi\}$ or $a_3 = 1$ and that $\mu_j\, a_{i,j} \to \alpha_i$, $i = 1, 2$ with $\alpha_1^2 + \alpha_2^2 = 1$, where we have set

$$\mu_j := (a_{1,j}^2 + a_{2,j}^2)^{-\frac{1}{2}}.$$

For every $\mathbf{x} \in \Omega$ and $\mathbf{v} \in \mathbb{R}^2$, $|\mathbf{v}| = 1$, set

$$\begin{aligned}\mathbf{p}_{\mathbf{v}}(\mathbf{x}) &= \big((x_1 - \widetilde{x}_1)\mathbf{e}_1 + (x_2 - \widetilde{x}_2)\mathbf{e}_2 + (x_3 - \widetilde{x}_3)\mathbf{e}_3\big) \wedge (-v_1\mathbf{e}_2 + v_2\mathbf{e}_1)\\ &= (v_1(x_1 - \widetilde{x}_1) + v_2(x_2 - \widetilde{x}_2))\,\mathbf{e}_3 \;+\; (x_3 - \widetilde{x}_3)(v_1\mathbf{e}_1 + v_2\mathbf{e}_2),\end{aligned} \tag{3.28}$$

where $(\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3) = \mathbf{x}_{\mathcal{L}} \in \operatorname{ri}\operatorname{co} E_{ess}$ is chosen as in the proof of (4) in Lemma 3.3. Hence, by taking account of (1) and (4) of Lemma 3.3, we have

$$0 = \mathcal{L}(\mathbf{p}_{\mathbf{v}}) \;=\; \mathcal{L}((v_1x_1 + v_2x_2)\,\mathbf{e}_3) - (v_1\widetilde{x}_1 + v_2\widetilde{x}_2)\,\mathcal{L}(\mathbf{e}_3) + \mathcal{L}(\,x_3(v_1\mathbf{e}_1 + v_2\mathbf{e}_2))$$

that is

$$(v_1\widetilde{x}_1 + v_2\widetilde{x}_2)\,\mathcal{L}(\mathbf{e}_3) \;=\; \mathcal{L}((v_1x_1 + v_2x_2)\mathbf{e}_3) + \mathcal{L}(\,x_3(v_1\mathbf{e}_1 + v_2\mathbf{e}_2)). \tag{3.29}$$

By (2) of Lemma 3.3 we know $\mathcal{L}(\mathbf{e}_3 \wedge \mathbf{x}) = 0$, then (3.8) entails

$$\mu_j\,\mathcal{L}(\mathbf{a}_j \wedge \mathbf{x}) = a_{1,j}\mu_j\,\mathcal{L}(\mathbf{e}_1 \wedge \mathbf{x}) + a_{2,j}\mu_j\,\mathcal{L}(\mathbf{e}_2 \wedge \mathbf{x}) \to (\alpha_1\widetilde{x}_2 - \alpha_2\widetilde{x}_1)\,\mathcal{L}(\mathbf{e}_3) \tag{3.30}$$

and by (3) of Lemma 3.3 we have

$$0 \ge \mathcal{L}(\mathbf{e}_3 \wedge (\mathbf{e}_3 \wedge \mathbf{x})) = \mathcal{L}(x_3\mathbf{e}_3 - \mathbf{x}) = -\mathcal{L}(x_1\mathbf{e}_1 + x_2\mathbf{e}_2).$$

By taking (3.29) into account and by recalling that either $\vartheta \in \{0, 2\pi\}$ or $a_3 = 1$, we get

$$\mu_j\,\mathcal{L}(\mathbf{a}_j \wedge (\mathbf{a}_j \wedge \mathbf{x}))\sin\frac{\vartheta_j}{2} = \mu_j\,\mathcal{L}((\mathbf{a}_j\cdot\mathbf{x})\,\mathbf{a}_j - \mathbf{x})\sin\frac{\vartheta_j}{2}$$
$$= \mu_j \sin\frac{\vartheta_j}{2}\Big(a_{1,j}\,\mathcal{L}((\mathbf{a}_j\cdot\mathbf{x})\,\mathbf{e}_1) + a_{2,j}\,\mathcal{L}((\mathbf{a}_j\cdot\mathbf{x})\,\mathbf{e}_2) + \mathcal{L}((a_{3,j}^2 - 1)\,x_3\,\mathbf{e}_3) - \mathcal{L}(x_1\mathbf{e}_1 + x_2\mathbf{e}_2)\Big)$$
$$\leq\ \mathcal{L}(\alpha_1 x_3\mathbf{e}_1 + \alpha_2 x_3\mathbf{e}_2 + (\alpha_1 x_1 + \alpha_2 x_2)\,\mathbf{e}_3)\sin\frac{\vartheta}{2} + o(1)$$
$$= (\alpha_1\widetilde{x}_1 + \alpha_2\widetilde{x}_2)\ \sin\frac{\vartheta}{2}\,\mathcal{L}(\mathbf{e}_3) + o(1),$$

that is

$$\limsup_{j\to+\infty}\ \mu_j\,\mathcal{L}(\mathbf{a}_j \wedge (\mathbf{a}_j \wedge \mathbf{x}))\sin\frac{\vartheta_j}{2}\ \leq\ (\alpha_1\widetilde{x}_1 + \alpha_2\widetilde{x}_2)\ \sin\frac{\vartheta}{2}\mathcal{L}(\mathbf{e}_3)\,. \tag{3.31}$$

Let now $\eta \in C([0, 2\pi])$ such that $\eta(\vartheta) = (2(1 - \cos\vartheta))^{-\frac{1}{2}}\sin(\vartheta)$ for every $\vartheta \in (0, 2\pi)$.

By recalling that either $a_3 = 1$ or $\vartheta \in \{0, 2\pi\}$ we get

$$\mu_j(\mathbf{a}_j \wedge \mathbf{x})_3 \longrightarrow \alpha_1 x_2 - \alpha_2 x_1, \quad \mu_j(\mathbf{a}_j \wedge (\mathbf{a}_j \wedge \mathbf{x}))_3\ \sin\frac{\vartheta_j}{2} \longrightarrow (\alpha_1 x_1 + \alpha_2 x_2)\sin\frac{\vartheta}{2}, \tag{3.32}$$

so, by taking (3.30)–(3.32) into account we obtain

$$0\ = \limsup_{j\to+\infty}\ \frac{\Phi(\mathbf{R}_j, E, \mathcal{L})}{|\mathbf{R}_j\mathbf{e}_3 - \mathbf{e}_3|\,|\mathcal{L}(\mathbf{e}_3)|}$$
$$= \limsup_{j\to+\infty}\frac{\mu_j}{|\mathcal{L}(\mathbf{e}_3)|}\Big\{\eta(\vartheta_j)\mathcal{L}(\mathbf{a}_j \wedge \mathbf{x}) + \mathcal{L}\Big(\mathbf{a}_j \wedge (\mathbf{a}_j \wedge \mathbf{x})\Big)\sin\frac{\vartheta_j}{2} +$$
$$- \min_{\mathbf{x}\in E_{ess}}\left\{\eta(\vartheta_j)(\mathbf{a}_j \wedge \mathbf{x})_3 + \sin\frac{\vartheta_j}{2}(\mathbf{a}_j \wedge (\mathbf{a}_h \wedge \mathbf{x}))_3\right\}\mathcal{L}(\mathbf{e}_3)\Big\}$$
$$\leq \min_{\mathbf{x}\in E_{ess}}\Big\{\eta(\vartheta)(\alpha_1 x_2 - \alpha_2 x_1) + (\alpha_1 x_1 + \alpha_2 x_2)\sin\frac{\vartheta}{2}\Big\} +$$
$$-\eta(\vartheta)(\alpha_1\widetilde{x}_2 - \alpha_2\widetilde{x}_1) - (\alpha_1\widetilde{x}_1 + \alpha_2\widetilde{x}_2)\sin\frac{\vartheta}{2}$$
$$= \min_{\mathbf{x}\in co(E_{ess})}\Big\{\eta(\vartheta)(\alpha_1(x_2 - \widetilde{x}_2) - \alpha_2(x_1 - \widetilde{x}_1)) + (\alpha_1(x_1 - \widetilde{x}_1) + \alpha_2(x_2 - \widetilde{x}_2))\sin\frac{\vartheta}{2}\Big\} \leq\ 0,$$

since $(\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3) \in \operatorname{ri}\operatorname{co} E_{ess}$. Therefore the function

$$g(x_1, x_2) := \eta(\vartheta)(\alpha_1(x_2 - \widetilde{x}_2) - \alpha_2(x_1 - \widetilde{x}_1)) + (\alpha_1(x_1 - \widetilde{x}_1) + \alpha_2(x_2 - \widetilde{x}_2))\sin\frac{\vartheta}{2}$$

attains its minimum on $\operatorname{proj}(\operatorname{co} E_{ess})$ at $(\widetilde{x}_1, \widetilde{x}_2) \in \operatorname{ri}(\operatorname{proj}(\operatorname{co} E_{ess}))$ hence, by taking into account of $\operatorname{aff}(\operatorname{proj}(\operatorname{co} E_{ess})) = \{x_3 = 0\}$, we get

$$0 = |\nabla g(\widetilde{x}_1, \widetilde{x}_2)|^2 = \left(\eta(\vartheta)\alpha_1 + \alpha_2\sin\frac{\vartheta}{2}\right)^2 + \left(\alpha_1\sin\frac{\vartheta}{2} - \alpha_2\eta(\vartheta)\right)^2 = 2\left(\eta^2(\vartheta) + \sin^2\frac{\vartheta}{2}\right) > 0,$$

a contradiction. Thus

$$\limsup_{j\to+\infty} \frac{\Phi(\mathbf{R}_j, E, \mathcal{L})}{|\mathbf{R}_j\mathbf{e}_3 - \mathbf{e}_3|\,|\mathcal{L}(\mathbf{e}_3)|} < 0$$

and the proof is achieved. □

Remark 3.11. If cap(proj(co E_{ess})) = 0 then the claim of Lemma 3.10 may be false even if cap $E > 0$. For instance, set for every $j \in \mathbb{N} \setminus \{0\}$

$$\mathbf{R}_j := \mathbf{e}_1 \otimes \mathbf{e}_1 + (1 - j^{-1})(\mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3) + \sqrt{2\,j^{-1} - j^{-2}}\,(\mathbf{e}_3 \otimes \mathbf{e}_2 - \mathbf{e}_2 \otimes \mathbf{e}_3),$$

let

$$\Omega := \{\mathbf{x} : x_1^2 + x_2^2 < 1,\ 0 < x_3 < 1\},$$

$$E := \overline{\Omega} \cap \{x_2 = 0,\ 0 < x_3 < \frac{1}{2}\},$$

and $\mathbf{f} := -\mathbf{e}_3,\ \mathbf{g} = \mathbf{0}$. It is straightforward checking that $\mathbf{R}_j\mathbf{e}_3 \neq \mathbf{e}_3$, $\mathbf{R}_j \to \mathbf{I}$, moreover since $\mathcal{L}(\mathbf{e}_3) < 0 = \mathcal{L}(\mathbf{e}_1) = \mathcal{L}(\mathbf{e}_2)$,

$$\mathcal{L}((\mathbf{R} - \mathbf{I})\mathbf{x}) - \min_{\mathbf{x}\in E_{ess}}((\mathbf{R}\mathbf{x})_3 - x_3)\}\,\mathcal{L}(\mathbf{e}_3) = -\pi\,|R_{31}| \leq 0, \qquad \forall\,\mathbf{R} \in SO(3) \tag{3.33}$$

and

$$\mathcal{L}((\mathbf{R} - \mathbf{I})\mathbf{x})_\alpha\mathbf{e}_\alpha) = 0, \qquad \forall\,\mathbf{R} \in SO(3), \tag{3.34}$$

conditions (2.32) and (2.33) are satisfied. Nevertheless

$$\mathcal{L}((\mathbf{R}_j - \mathbf{I})\mathbf{x}) - \min_{E_{ess}}((\mathbf{R}_j - \mathbf{I})\mathbf{x})_3)\mathcal{L}(\mathbf{e}_3) = 0 \tag{3.35}$$

and the claim of Lemma 3.10 cannot be true in this case.

4. Proof of the variational convergence result

This section contains the proof of our main result. We start by showing that sequences of deformations with equibounded energy correspond (up to suitably tuned rotations and translations of the horizontal components) to displacements that are equibounded in H^1.

Lemma 4.1. *(compactness) Assume that E, $\mathcal{L}$ and $\mathcal{W}$ fulfil* (2.13), (2.19)–(2.22), (2.33) *and* $\mathcal{L}(\mathbf{e}_3) < 0$. *If $0 < h_j \to 0^+$ as $j \to +\infty$ then for every $\mathbf{y}_j \in H^1(\Omega;\mathbb{R}^3)$ with $\mathcal{G}_j(\mathbf{y}_j) \leq M < +\infty$ there are $\mathbf{R}_j \in SO(3)$, $\mathbf{c}_j \in C_{\mathbf{R}_j}$ such that*

$$\mathbf{R}_j \to \mathbf{R} \in \mathcal{S}_{\mathcal{L},E} \tag{4.1}$$

and the sequence

$$h_j^{-1}(\mathbf{y}_j - \mathbf{R}_j\mathbf{x} - \mathbf{c}_j)_\alpha\mathbf{e}_\alpha + h_j^{-1}(y_{j,3} - x_3)\mathbf{e}_3$$

is bounded in $H^1(\Omega;\mathbb{R}^3)$.

Proof. Referring to (2.36), we can choose $\mathbf{R}_j \in \mathcal{M}(\mathbf{y}_j)$ in such a way that, up to subsequence and without relabelling, $\mathbf{R}_j \to \mathbf{R}$. Then we define $\mathbf{c}_j = (c_{j,1}, c_{j,2}, c_{j,3})$ by

$$c_{j,\alpha} = |\Omega|^{-1}\int_\Omega (\mathbf{y}_j(\mathbf{x}) - \mathbf{R}_j\mathbf{x})_\alpha \, d\mathbf{x}, \qquad \alpha = 1, 2, \tag{4.2}$$

$$c_{j,3} = -\min_{\mathbf{x}\in E_{ess}} ((\mathbf{R}_j - \mathbf{I})\mathbf{x})_3. \tag{4.3}$$

By the rigidity inequality ([24]) there exists a constant $C = C(\Omega) > 0$ such that

$$\begin{aligned} M \ &\geq\ \mathcal{G}_j(\mathbf{y}_j) \ \geq\ C\,h_j^{-2}\int_\Omega |\nabla\mathbf{y}_j - \mathbf{R}_j|^2\, d\mathbf{x} - h_j^{-1}\mathcal{L}(\mathbf{y}_j - \mathbf{x}) \\ &=\ C\,h_j^{-2}\int_\Omega |\nabla\mathbf{y}_j - \mathbf{R}_j|^2\, d\mathbf{x} - h_j^{-1}\mathcal{L}(\mathbf{y}_j - \mathbf{R}_j\mathbf{x} - \mathbf{c}_j) - h_j^{-1}\mathcal{L}(\mathbf{R}_j\mathbf{x} - \mathbf{x} + \mathbf{c}_j). \end{aligned} \tag{4.4}$$

Thus, by (2.33), $\mathcal{L}(\mathbf{e}_1) = \mathcal{L}(\mathbf{e}_2) = 0$ and the definition of $c_{j,3}$, we get

$$M \ \geq\ C\,h_j^{-2}\int_\Omega |\nabla\mathbf{y}_j - \mathbf{R}_j|^2\, d\mathbf{x} - h_j^{-1}\mathcal{L}(\mathbf{y}_j - \mathbf{R}_j\mathbf{x} - \mathbf{c}_j) \tag{4.5}$$

and Poincaré inequality entails, for every $\varepsilon > 0$,

$$\begin{aligned} h_j^{-1}\sum_{\alpha=1}^{2}\mathcal{L}(\,(\mathbf{y}_j - \mathbf{R}_j\mathbf{x} - \mathbf{c}_j)_\alpha\, \mathbf{e}_\alpha) \quad &\leq\ h_j^{-1} C_P \|\mathcal{L}\|_* \Big(\textstyle\sum_{\alpha=1}^{2}\int_\Omega |\,(\nabla\mathbf{y}_j - \mathbf{R}_j)_\alpha\,|^2\, d\mathbf{x}\Big)^{1/2} \\ &\leq\ \frac{C_P\|\mathcal{L}\|_*^2}{2\varepsilon} + \frac{\varepsilon\, h_j^{-2}\, C_P}{2}\sum_{\alpha=1}^{2}\int_\Omega |(\nabla\mathbf{y}_j - \mathbf{R}_j)_\alpha|^2\, d\mathbf{x}. \end{aligned} \tag{4.6}$$

Estimates (4.5) and (4.6) together with Young inequality provide

$$\begin{aligned} M &\geq h_j^{-2}\Big(C - \frac{\varepsilon\, C_P}{2}\Big)\int_\Omega |\nabla\mathbf{y}_j - \mathbf{R}_j|^2 d\mathbf{x} - \frac{C_P\,\|\mathcal{L}\|_*^2}{2\,\varepsilon} - h_j^{-1}\mathcal{L}((\mathbf{y}_j - \mathbf{R}_j\mathbf{x} - \mathbf{c}_j)_3\,\mathbf{e}_3) \\ &\geq h_j^{-2}\Big(C - \frac{\varepsilon\, C_P}{2}\Big)\int_\Omega |\nabla\mathbf{y}_j - \mathbf{R}_j|^2\, d\mathbf{x} - \frac{C_P\|\mathcal{L}\|_*^2}{2\,\varepsilon} \\ &\quad - h_j^{-1}\,\|\mathcal{L}\|_*\,(\,\|(\mathbf{y}_j - \mathbf{R}_j\mathbf{x} - \mathbf{c}_j)_3\|_{L^2(\Omega)} + \|\nabla(\mathbf{y}_j - \mathbf{R}_j\mathbf{x})_3\|_{L^2(\Omega)}\,) \\ &\geq h_j^{-2}\Big(C - \frac{\varepsilon\, C_P}{2} - \frac{\varepsilon}{2}\Big)\int_\Omega |\nabla\mathbf{y}_j - \mathbf{R}_j|^2\, d\mathbf{x} - \Big(\frac{C_P}{2\,\varepsilon} + \frac{1}{2\,\varepsilon}\Big)\|\mathcal{L}\|_*^2 \\ &\quad - h_j^{-1}\,\|\mathcal{L}\|_*^2\,(\,\|(\mathbf{y}_j - \mathbf{R}_h\mathbf{x} - \mathbf{c}_j)_3\|_{L^2(\Omega)}. \end{aligned} \tag{4.7}$$

By choosing $\varepsilon = C/(C_P + 1)$, we get

$$\begin{aligned} &h_j^{-2}\,\frac{C}{2}\int_\Omega |\nabla\mathbf{y}_j - \mathbf{R}_j|^2\, d\mathbf{x} \\ &\leq\ M\ +\ \frac{(C_P + 1)^2}{2C}\|\mathcal{L}\|_*^2\ +\ h_j^{-1}\,\|\mathcal{L}\|_*^2\,\|(\mathbf{y}_j - \mathbf{R}_j\mathbf{x} - \mathbf{c}_j)_3\|_{L^2(\Omega)}. \end{aligned} \tag{4.8}$$

Thus, if we show that $h_j^{-1}\|(\mathbf{y}_j - \mathbf{R}_j\mathbf{x} - \mathbf{c}_j)_3\|_{L^2(\Omega)}$ is uniformly bounded then, due to estimate (4.8), $\|h_j^{-1}(\nabla\mathbf{y}_j - \mathbf{R}_j)\|_{L^2(\Omega)}$ is equibounded too. So $h_j^{-1}\,(\mathbf{y}_j - \mathbf{R}_j\mathbf{x} - \mathbf{c}_j)$ is uniformly bounded in $H^1(\Omega;\mathbb{R}^3)$ and we set

$$M_1 := \sup_j \|\mathcal{L}\|_* \|\mathbf{y}_j - \mathbf{R}_j\mathbf{x} - \mathbf{c}_j\|_{H^1(\Omega;\mathbb{R}^3)} > 0. \tag{4.9}$$

To this aim we assume by contradiction that, up to subsequences,

$$t_j \ := \ h_j^{-1} \, \| \, (\mathbf{y}_j - \mathbf{R}_j\mathbf{x} - \mathbf{c}_j)_3 \, \|_{L^2(\Omega)} \ \to \ +\infty \tag{4.10}$$

and set $w_j \ := t_j^{-1} h_j^{-1} \, (\mathbf{y}_j - \mathbf{R}_j\mathbf{x} - \mathbf{c}_j)_3$. Then

$$\|w_j\|_{L^2(\Omega)} = 1, \qquad |\nabla \mathbf{y}_j - \mathbf{R}_j|^2 \ = \ \sum_{\alpha=1}^{2} |\, \nabla(\mathbf{y}_j - \mathbf{R}_j\mathbf{x})_\alpha \,|^2 \ + \ h_j^2 t_j^2 \, |\nabla w_j|^2, \tag{4.11}$$

$$\begin{aligned} & C t_j^2 \int_\Omega |\nabla w_j|^2 \, d\mathbf{x} \ - \ t_j \, \mathcal{L}(w_j \, \mathbf{e}_3) \\ \leq \quad & M - C\, h_j^{-2} \sum_{\alpha=1}^{2} \int_\Omega |\, \nabla(\mathbf{y}_j - \mathbf{R}_j\mathbf{x})_\alpha \,|^2 \, d\mathbf{x} \ + \ h_j^{-1} \sum_{\alpha=1}^{2} \mathcal{L}(\, (\mathbf{y}_j - \mathbf{R}_j\mathbf{x} - \mathbf{c}_j)_\alpha \, \mathbf{e}_\alpha \,) \\ \leq \quad & M - C\, h_j^{-2} \sum_{\alpha=1}^{2} \int_\Omega |\, \nabla(\mathbf{y}_h - \mathbf{R}_j\mathbf{x})_\alpha \,|^2 \, d\mathbf{x} \ + \ \frac{C_P \, \|\mathcal{L}\|_*^2}{2\,\varepsilon} \\ + \quad & \frac{h_j^{-2}\, \varepsilon \, C_P}{2} \sum_{\alpha=1}^{2} \int_\Omega |\nabla(\mathbf{y}_j - \mathbf{R}_j\mathbf{x})_\alpha \,|^2 \, d\mathbf{x}, \end{aligned} \tag{4.12}$$

and by choosing $\varepsilon = 2C/C_P$ in (4.12) we get

$$C\, t_j^2 \int_\Omega |\nabla w_j|^2 \, d\mathbf{x} \ - \ t_j \, \mathcal{L}(w_j \, \mathbf{e}_3) \ \leq \ \frac{C_P^2 \, \|\mathcal{L}\|_*^2}{4\, C} + M \tag{4.13}$$

while, by choosing $\varepsilon = C/C_P$, (4.12) yields

$$\begin{aligned} & \frac{1}{2}\, C\, h_j^{-2} \sum_{\alpha=1}^{2} \int_\Omega |\, \nabla(\mathbf{y}_h - \mathbf{R}_j\mathbf{x})_\alpha \,|^2 \, d\mathbf{x} \ + \ C\, t_j^2 \, |\nabla w_j|^2 \ - t_j \, \mathcal{L}(w_h \, \mathbf{e}_3) \\ & \leq \ \frac{C_P^2}{2C} \, \|\mathcal{L}\|_*^2 + M. \end{aligned} \tag{4.14}$$

Thus

$$\frac{1}{2}\, C \, \frac{h_j^{-2}}{t_j^2} \sum_{\alpha=1}^{2} \int_\Omega |\, \nabla(\mathbf{y}_j - \mathbf{R}_j\mathbf{x})_\alpha \,|^2 \, d\mathbf{x} \ \leq \ \frac{1}{t_j} \, \mathcal{L}(w_j \, \mathbf{e}_3) \ + \ \frac{1}{{t_j}^2} \, \frac{C_P^2}{2C} \, \|\mathcal{L}\|_*^2 + \frac{M}{t_j^2}. \tag{4.15}$$

Normalization $\|w_j\|_{L^2} = 1$ entails, for every $\varepsilon > 0$,

$$\begin{aligned} \mathcal{L}(w_j \, \mathbf{e}_3) \ \ & \leq \ \ \|\mathcal{L}\|_* (\, \|w_j\|_{L^2} + \|\nabla w_j\|_{L^2} \,) \ = \ \|\mathcal{L}\|_* (\, 1 + \|\nabla w_j\|_{L^2} \,) \\ & \leq \ \ \|\mathcal{L}\|_* \ + \ \frac{\|\mathcal{L}\|_*^2}{2\,\varepsilon} \ + \ \frac{\varepsilon}{2} \, \|\nabla w_j\|_{L^2}^2, \end{aligned} \tag{4.16}$$

and choosing $\varepsilon = C\, t_j^2$ therein we get, by (4.13),

$$\frac{C}{2} t_j^2 \int_\Omega |\nabla w_j|^2 \ d\mathbf{x} \ \leq \ t_j \, \|\mathcal{L}\|_* + \frac{\|\mathcal{L}\|_*^2}{2C\, t_j} + M, \tag{4.17}$$

thus $\int_\Omega |\nabla w_j|^2\, d\mathbf{x} \to 0$ so by (4.11) $w_j \to w$ in $H^1(\Omega;\mathbb{R}^3)$ with $\nabla w = 0$ a.e. in Ω that is w is a constant function since Ω is a connected open set.

Combining estimates (4.15)–(4.17), we get

$$\frac{1}{2}C\,\frac{h_j^{-2}}{t_j^2}\sum_{\alpha=1}^{2}\int_\Omega |\nabla(\mathbf{y}_j-\mathbf{R}_j\mathbf{x})_\alpha|^2\,d\mathbf{x} \tag{4.18}$$
$$\le \frac{1}{t_j}\left(\|\mathcal{L}\|_*^2 + \frac{\|\mathcal{L}\|_*^2}{2} + \frac{1}{2}\|\nabla w_j\|_{L^p}^2\right) + \frac{1}{{t_j}^2}\,\frac{C_P^2}{2\,C_R\,C}\,\|\mathcal{L}\|_*^2 + \frac{M}{t_j^2},$$

hence

$$\frac{1}{h_j\,t_j}\,\nabla(\mathbf{y}_j-\mathbf{R}_j\mathbf{x})_\alpha \;\to\; 0, \qquad \text{in } L^2(\Omega), \qquad \text{if } \alpha = 1,2 \tag{4.19}$$

and

$$h_j^{-1}t_j^{-1}\left(\mathbf{y}_j-\mathbf{R}_j\mathbf{x}-\mathbf{c}_j\right) \;\to\; w\,\mathbf{e}_3, \qquad \text{q.e. } \mathbf{x}\in E. \tag{4.20}$$

Moreover, by (4.13), we get

$$\mathcal{L}(w_j\,\mathbf{e}_3) \;\ge\; -\frac{C_P^2}{4\,C\,t_j}\,\|\mathcal{L}\|_*^2.$$

Hence, due to $\mathcal{L}(w_j\,\mathbf{e}_3) \to \mathcal{L}(w\,\mathbf{e}_3) = w\,\mathcal{L}(\mathbf{e}_3)$, we have $w\,\mathcal{L}(\mathbf{e}_3)\ge 0$, thus, by taking into account of $\mathcal{L}(\mathbf{e}_3) < 0$, we get $w \le 0$ and eventually, by $\|w_j\|_{L^2} = 1$, we obtain $w < 0$. Then, by (2.33), (4.4) and (4.9), we get

$$0 \le -\Phi(\mathbf{R}_j, E, \mathcal{L}) = \mathcal{L}\left(\mathbf{x} - \mathbf{R}_j\mathbf{x} - \mathbf{c}_j\right) \le (M+M_1)\,h_j. \tag{4.21}$$

Hence, due to $\mathbf{R}_j \to \mathbf{R}$, we have $\Phi(\mathbf{R}, E, \mathcal{L}) = 0$ thus $\mathbf{R}\in\mathcal{S}_{\mathcal{L},E}$ and (4.1) is proven.

We notice that either $\mathbf{R}_j\mathbf{e}_3 \ne \mathbf{e}_3$ for j large enough or $\mathbf{R}_j\mathbf{e}_3 = \mathbf{e}_3$ for infinitely many j. In the first case, by taking account of $\mathcal{L}(\mathbf{e}_3) < 0$, Lemma 3.10 entails

$$\limsup_{j\to+\infty}\frac{\Phi(\mathbf{R}_j, E, \mathcal{L})}{|\mathbf{R}_j\mathbf{e}_3-\mathbf{e}_3|\,|\mathcal{L}(\mathbf{e}_3)|} < 0. \tag{4.22}$$

By (4.21) we get $\Phi(\mathbf{R}_j, E, \mathcal{L}) \ge -(M+M_1)h_j$ hence

$$\gamma := \liminf_{j\to+\infty}\frac{h_j}{|\mathbf{R}_j\mathbf{e}_3-\mathbf{e}_3|} > 0 \tag{4.23}$$

and for large enough j (4.23) yields

$$|\mathbf{R}_j\mathbf{e}_3-\mathbf{e}_3| \le \frac{2h_j}{\gamma}. \tag{4.24}$$

Therefore, for every $\mathbf{x}\in\Omega$

$$\frac{|(\mathbf{R}_j\mathbf{x})_3-\mathbf{x}_3|}{h_j\,t_j} \le \frac{|\mathbf{x}||\mathbf{R}_j^T\mathbf{e}_3-\mathbf{e}_3|}{h_j\,t_j} \overset{j\to+\infty}{\longrightarrow} 0, \tag{4.25}$$

hence, by taking into account of $c_{j,3} = -\min\left\{(\mathbf{R}_j\mathbf{x})_3-\mathbf{x}_3 \,:\, \mathbf{x}\in E_{ess}\right\}$, we get for q.e. $\mathbf{x}\in E$

$$\frac{\mathbf{y}_{j,3}^*-\mathbf{x}_3}{h_j\,t_j} = \frac{\mathbf{y}_{j,3}^*-(\mathbf{R}_j\mathbf{x})_3-\mathbf{c}_{j,3}}{h_j\,t_j} + \frac{(\mathbf{R}_j\mathbf{x})_3-\mathbf{x}_3+\mathbf{c}_{j,3}}{h_j\,t_j}$$
$$= \frac{\mathbf{y}_{h_j,3}^*-(\mathbf{R}_j\mathbf{x})_3-\mathbf{c}_{j,3}}{h_j\,t_j} + o(1) \overset{j\to+\infty}{\longrightarrow} w < 0, \tag{4.26}$$

a contradiction since $\mathbf{y}^*_{j,3} \geq (1-h_j)x_3$ for q.e. $\mathbf{x} \in E$, that is $(h_j t_j)^{-1}(\mathbf{y}^*_{j,3} - \mathbf{x}_3) \geq -x_3/t_j \longrightarrow 0$ as $j \to +\infty$ for q.e. $\mathbf{x} \in E_{ess}$. Therefore in this case the sequence t_j is bounded so $h_j^{-1}\left(\mathbf{y}_j - \mathbf{R}_j\mathbf{x} - \mathbf{c}_j\right)$ is equibounded in $H^1(\Omega;\mathbb{R}^3)$ and in particular $h_j^{-1}\left(\mathbf{y}_j - \mathbf{R}_j\mathbf{x} - \mathbf{c}_j\right)_\alpha \mathbf{e}_\alpha$ is equibounded in $H^1(\Omega;\mathbb{R}^3)$.

In the second case we may assume that $\mathbf{R}_j\mathbf{e}_3 = \mathbf{e}_3$ for every j so $c_{j,3} = 0$ for every j. By arguing as in the previous case we may assume that

$$t_j := h_j^{-1} \| (\mathbf{y}_j - \mathbf{R}_j\mathbf{x} - \mathbf{c}_j)_3 \|_{L^2(\Omega)} = h_j^{-1} \|y_{j,3} - x_3\|_{L^2(\Omega)} \to +\infty$$

and by setting $w_j := t_j^{-1}h_j^{-1}(y_{j,3} - x_3)$ we get $w_j \to w < 0$ as before which is again a contradiction, so t_j is a bounded sequence. Eventually we are left to show that, in the first case, $h_j^{-1}(y_{j,3} - x_3)$ is equibounded in $H^1(\Omega;\mathbb{R}^3)$. To this aim let $C > 0$ such that

$$\left\| h_j^{-1}\left(\mathbf{y}_{j,3} - (\mathbf{R}_j\mathbf{x})_3 - \mathbf{c}_{j,3}\right)\right\|_{H^1(\Omega;\mathbb{R}^3)} \leq C$$

for every $j \in \mathbb{N}$ and assume that for every $n \in \mathbb{N}$ there exists j_n such that

$$\left\| h_{j_n}^{-1}(\mathbf{y}_{j_n,3} - \mathbf{x}_3)\right\|_{H^1(\Omega;\mathbb{R}^3)} \geq n. \tag{4.27}$$

Then for every $n > C$ we have $\mathbf{R}_{j_n}\mathbf{e}_3 \neq \mathbf{e}_3$ otherwise $c_{j_n,3} = 0$ and

$$n \leq \left\| h_{j_n}^{-1}(\mathbf{y}_{j_n,3} - \mathbf{x}_3)\right\|_{H^1(\Omega;\mathbb{R}^3)} = \left\| h_{j_n}^{-1}\left(\mathbf{y}_{j_n,3} - (\mathbf{R}_{j_n}\mathbf{x})_3 - \mathbf{c}_{j_n,3}\right)\right\|_{H^1(\Omega;\mathbb{R}^3)} \leq C,$$

a contradiction. By taking account of (4.23) and (4.25) there exists $\widetilde{C} > 0$ such that

$$|\mathbf{R}_{j_n}\mathbf{e}_3 - \mathbf{e}_3| \leq \widetilde{C} h_{j_n}$$

for every $n > C$, hence

$$\begin{aligned}
&\left| h_{j_n}^{-1}(\mathbf{y}_{j_n,3} - \mathbf{x}_3)\right| \\
&\leq \left| h_{j_n}^{-1}(\mathbf{y}_{j_n,3} - (\mathbf{R}_{j_n}\mathbf{x})_3 - \mathbf{c}_{j_n,3})\right| + h_{j_n}^{-1}|(\mathbf{R}_{j_n}\mathbf{x})_3 - x_3| \\
&\leq \left| h_{j_n}^{-1}(\mathbf{y}_{j_n,3} - (\mathbf{R}_{j_n}\mathbf{x})_3 - \mathbf{c}_{j_n,3})\right| + h_{j_n}^{-1}|\mathbf{R}_{j_n}\mathbf{e}_3 - \mathbf{e}_3||\mathbf{x}| \\
&\leq \left| h_{j_n}^{-1}(\mathbf{y}_{j_n,3} - (\mathbf{R}_{j_n}\mathbf{x})_3 - \mathbf{c}_{j_n,3})\right| + \widetilde{C} \sup_\Omega |\mathbf{x}|
\end{aligned} \tag{4.28}$$

thus showing that

$$\left\| h_{j_n}^{-1}(\mathbf{y}_{j_n,3} - \mathbf{x}_3)\right\|_{H^1(\Omega;\mathbb{R}^3)} \leq C + \widetilde{C} \sup_\Omega |\mathbf{x}||\Omega|$$

which contradicts (4.27) and proves that $h_j^{-1}(y_{j,3} - x_3)$ is equibounded in $H^1(\Omega;\mathbb{R}^3)$. □

Remark 4.2. If $\mathrm{cap}(\mathrm{proj}(\mathrm{co}\, E_{ess})) = 0$, the claim of Lemma 4.1 may fail even if $\mathrm{cap}\, E > 0$. Indeed, choose $\Omega, E, \mathbf{f}, \mathbf{g}, \mathbf{R}_j$ as in Remark 3.11 and set $h_j = j^{-1}$. Thus both (2.32) and (2.33) are satisfied but (2.13) is not. It is readily seen that $\mathbf{y}_j(\mathbf{x}) := \mathbf{R}_j\mathbf{x}$ belongs to $\mathcal{A}_j$ since $y_{j,3} = (1 - j^{-1})x_3$ on E and that $\mathcal{G}_j(\mathbf{y}_j) = \pi|\,\Omega|/2$ for every j, but

$$j\,(y_{j,3} - x_3) = j\,(x_2\sqrt{2j^{-1} - j^{-2}} - x_3 j^{-1})$$

is not equibounded in $H^1(\Omega;\mathbb{R}^3)$ as $j \to +\infty$: thus claim of Lemma 4.1 fails in this case.

Lemma 4.3. *Assume that E, $\mathcal{L}$ and $\mathcal{W}$ fulfil conditions* (2.13)*,* (2.19)*–*(2.22)*,* (2.33) *and* $\mathcal{L}(\mathbf{e}_3) < 0$*. Choose* $\mathbf{y}_j$*,* $\mathbf{R}_j$ *as in Lemma 4.1 and set*

$$\mathbf{z}_j(\mathbf{x}) := h_j^{-1}\{(\mathbf{R}_j\mathbf{x})_3 - x_3) - \min_{\mathbf{x}\in E_{ess}}((\mathbf{R}_j\mathbf{x})_3 - x_3)\}\mathbf{e}_3. \tag{4.29}$$

Then there exist $b_1, b_2, b_3 \in \mathbb{R}$ *such that by setting*

$$\mathbf{z}(\mathbf{x}) := (b_1x_1 + b_2x_2 + b_3)\mathbf{e}_3 \tag{4.30}$$

we have, up to subsequences, $\mathbf{z}_j \rightharpoonup \mathbf{z}$ *in* $w^* - W^{1,\infty}(\Omega;\mathbb{R}^3)$*.*

Proof. We may assume that $\mathbf{R}_j\mathbf{e}_3 \neq \mathbf{e}_3$ for infinitely many j otherwise $\mathbf{z}_j \equiv 0$ for j large enough and thesis is obvious. Therefore by Euler-Rodrigues formula, there are sequences $\mathbf{a}_j \in \mathbb{R}^3$ and $\vartheta_j \in (0, 2\pi)$, s.t. $|\mathbf{a}_j| = 1$, $\mathbf{a}_j \neq \mathbf{e}_3$ and

$$\mathbf{R}_j\mathbf{x} = \mathbf{x} + (\sin\vartheta_j)(\mathbf{a}_j \wedge \mathbf{x}) + (1 - \cos\vartheta_j)((\mathbf{a}_j \wedge (\mathbf{a}_j \wedge \mathbf{x})), \qquad \forall \mathbf{x} \in \mathbb{R}^3. \tag{4.31}$$

By recalling (3.27) and (4.1) we have, up to subsequences, $\mathbf{R}_j\mathbf{e}_3 \to \mathbf{e}_3$. Then, up to subsequences, we may assume: that $\mathbf{a}_j \to \mathbf{a}$, $\vartheta_j \to \vartheta \in [0, 2\pi]$, that either $\vartheta \in \{0, 2\pi\}$ or $a_3 = 1$ and that $\mu_j a_{i,j} \to \alpha_i$, $i = 1, 2$ with $\alpha_1^2 + \alpha_2^2 = 1$, where we set $\mu_j := (a_{1,j}^2 + a_{2,j}^2)^{-\frac{1}{2}}$. By recallling (4.24) we may assume that, up to subsequences,

$$h_j^{-1}\min_{\mathbf{x}\in E_{ess}}((\mathbf{R}_j\mathbf{x})_3 - x_3) \to \beta$$

for some $\beta \in \mathbb{R}$. Moreover by exploiting (3.27), (4.31), we get

$$\begin{aligned}
&h_j^{-1}((\mathbf{R}_j\mathbf{x})_3 - \mathbf{x}_3)\\
&= \frac{\mu_j}{\sqrt{2(1-\cos\vartheta_j)}}(\sin\vartheta_j(\mathbf{a}_j \wedge \mathbf{x})_3 + (1-\cos\vartheta_j)((\mathbf{a}_j \wedge (\mathbf{a}_j \wedge \mathbf{x})_3))\frac{|\mathbf{R}_j\mathbf{e}_3 - \mathbf{e}_3|}{h_j}\\
&= \left(\mu_j\eta(\vartheta_j)(\mathbf{a}_j \wedge \mathbf{x})_3 + \mu_j((\mathbf{a}_j \wedge (\mathbf{a}_j \wedge \mathbf{x}))_3 \sin\frac{\vartheta_j}{2}\right)\frac{|\mathbf{R}_j\mathbf{e}_3 - \mathbf{e}_3|}{h_j}
\end{aligned} \tag{4.32}$$

where $\eta \in C([0, 2\pi])$ is such that $\eta(\vartheta) = (2(1 - \cos\vartheta))^{-\frac{1}{2}}\sin\vartheta$ for every $\vartheta \in (0, 2\pi)$.

By arguing as in (3.32) and by taking (4.24) into account we get, up to subsequences,

$$h_j^{-1}((\mathbf{R}_j\mathbf{x})_3 - x_3) \to \lambda\left(\eta(\vartheta)(\alpha_1x_2 - \alpha_2x_1) + (\alpha_1x_1 + \alpha_2x_2)\sin\frac{\vartheta}{2}\right), \quad \forall\, \mathbf{x} \in \Omega \tag{4.33}$$

for some $\lambda \geq 0$. On the other hand $\nabla\mathbf{z}_j = h_j^{-1}(\mathbf{R}_j\mathbf{e}_3 - \mathbf{e}_3)$ and (4.24) entail $\|\nabla\mathbf{z}_j\|_\infty \leq C$ for some $C > 0$ so $\mathbf{z}_j \rightharpoonup \mathbf{z}$ in $w^* - W^{1,\infty}(\Omega : \mathbb{R}^3)$ whenever $b_1 = \lambda(-\alpha_2\eta(\vartheta) + \alpha_1\sin\frac{\vartheta}{2})$, $b_2 = \lambda(\alpha_1\eta(\vartheta) + \alpha_2\sin\frac{\vartheta}{2})$, $b_3 = -\beta$. □

Lemma 4.4. *(**Lower bound**) Assume that E, $\mathcal{L}$, $\mathcal{W}$ fulfil the conditions* (2.13)*–*(2.22)*,* (2.32)*,* (2.33) *and* $\mathcal{L}(\mathbf{e}_3) < 0$*. If* $h_j \to 0^+$ *as* $j \to +\infty$ *then, for every sequence of deformations* $\mathbf{y}_j \in H^1(\Omega;\mathbb{R}^3)$ *such that* $\mathcal{G}_j(\mathbf{y}_j) \leq M < +\infty$ *and for every* $\mathbf{R}_j \in \mathcal{M}(\mathbf{y}_j)$ *there exist* $\mathbf{c}_j \in C_{\mathbf{R}_j}$ *such that by setting*

$$\mathbf{u}_j(\mathbf{x}) := h_j^{-1}\mathbf{R}_j^T\left\{(\mathbf{y}_j - \mathbf{c}_j - \mathbf{R}_j\mathbf{x})_\alpha\,\mathbf{e}_\alpha + (y_{j,3} - x_3)\mathbf{e}_3\right\}, \tag{4.34}$$

there is $\mathbf{u} \in \mathcal{A}$ *such that up to subsequences* $\mathbf{u}_j \rightharpoonup \mathbf{u}$ *weakly in* $H^1(\Omega;\mathbb{R}^3)$ *and*

$$\liminf_{j\to+\infty} \mathcal{G}_j(\mathbf{y}_j) \geq \widetilde{\mathcal{G}}(\mathbf{u}). \tag{4.35}$$

Proof. Due to Lemma 4.1, the sequence defined in (4.34) is equibounded in $H^1(\Omega;\mathbb{R}^3)$ hence there exists $\mathbf{u} \in H^1(\Omega;\mathbb{R}^3)$ such that up to subsequences $\mathbf{u}_j \rightharpoonup \mathbf{u}$ in $H^1(\Omega;\mathbb{R}^3)$. By recalling Lemma A1 of [12] we get, again up to subsequences, $\mathbf{u}_j^*(\mathbf{x}) \to \mathbf{u}^*(\mathbf{x})$ for q.e. $\mathbf{x} \in E$ hence by taking account of

$$u_{j,3}^* = h_j^{-1}(y_{j,3}^* - x_3) \geq h_j^{-1}(x_3 - h_j x_3 - x_3) = -x_3$$

for q.e. $\mathbf{x} \in E$ we get $u_3^* \geq -x_3$ for q.e. $\mathbf{x} \in E$ that is $\mathbf{u} \in \mathcal{A}$.

By taking account of $\mathcal{G}_j(\mathbf{y}_j) \leq M$ and by arguing as in Lemma 4.1 the sequence $h_j^{-1}(\mathbf{y}_j - \mathbf{R}_j - \mathbf{c}_j)$ is bounded in $H^1(\Omega;\mathbb{R}^3)$ hence (4.4) entails

$$0 \;\leq\; \mathcal{L}\Big(\mathbf{x} - \mathbf{R}_j\mathbf{x} - \mathbf{c}_j\Big) \;\leq\; (M + M_1)\, h_j. \tag{4.36}$$

Therefore, by recalling (4.2), (4.3) and that, up to subsequences, $\mathbf{R}_j \to \mathbf{R}$ we get $\mathbf{R} \in \mathcal{S}_{\mathcal{L},E}$. By defining $\mathbf{z}_j$ as in Lemma 4.3 and by setting

$$\mathbf{D}_j := \mathbb{E}(\mathbf{u}_j) + \tfrac{1}{2}h_j\nabla\mathbf{u}_j^T\nabla\mathbf{u}_j,\ \mathbf{F}_j := \mathbb{E}(\mathbf{R}_j^T\mathbf{z}_j) + \tfrac{1}{2}h_j\nabla(R_j^T\mathbf{z}_j)^T\nabla(R_j^T\mathbf{z}_j)$$

a straightforward calculation shows that

$$\nabla\mathbf{y}_j^T\nabla\mathbf{y}_j - \mathbf{I} = 2h_j(\mathbf{D}_j + \mathbf{F}_j). \tag{4.37}$$

If now

$$B_j := \{x \in \Omega :\ \sqrt{h_j}|\nabla\mathbf{u}_j| \leq 1\},$$

we immediately notice that, by Tchebycheff inequality, $|\Omega \setminus B_j| \to 0$ as $j \to +\infty$ and that for large enough j

$$h_j|\mathbf{D}_j| \leq \sqrt{h_j}\Big(\sqrt{h_j}|\nabla\mathbf{v}_j| + \tfrac{1}{2}h_j^{3/2}|\nabla\mathbf{v}_j^T||\nabla\mathbf{v}_j|\Big) \leq 2\sqrt{h_j}, \qquad \text{on } B_j. \tag{4.38}$$

Moreover by Lemma 4.3 there exists $C > 0$ such that

$$h_j|\mathbf{F}_j| \leq Ch_j, \quad \text{in } \Omega \tag{4.39}$$

hence by defining $\mathbf{z}$ as in (4.30) and by taking account of $\mathbf{R}_j \to \mathbf{R} \in \mathcal{S}_{\mathcal{L},E}$, we get

$$\mathbf{F}_j \rightharpoonup \mathbb{E}(\mathbf{z}), \quad w^* - L^\infty(\Omega;\mathbb{R}^{3\times3}). \tag{4.40}$$

By taking account of $\mathcal{L}(\mathbf{e}_\alpha) = 0$ for $\alpha = 1, 2$, (2.32) entails

$$\begin{aligned} &\mathcal{L}(\mathbf{y}_j - \mathbf{x}) = \mathcal{L}((y_{j,3} - x_3)\mathbf{e}_3) + \mathcal{L}((\mathbf{y}_j - \mathbf{R}_j\mathbf{x} - \mathbf{c}_j)_\alpha\mathbf{e}_\alpha) \\ &+\mathcal{L}((\mathbf{R}_j\mathbf{x} - \mathbf{x})_\alpha\mathbf{e}_\alpha) \leq h_j\mathcal{L}(\mathbf{R}_j\mathbf{u}_j) \end{aligned} \tag{4.41}$$

thus, since η is increasing, by (2.21)–(2.23), (2.25), (4.37)–(4.40) we get for large j

$$\begin{aligned} \mathcal{G}_j(\mathbf{y}_j) &\geq \frac{1}{h_j^2}\int_{B_j} \mathcal{V}(\mathbf{x}, h_j\mathbf{D}_j + h_j\mathbf{F}_j)\, d\mathbf{x} - \mathcal{L}(\mathbf{R}_j\mathbf{u}_j) \\ &\geq \int_{B_j} Q(\mathbf{x}, \mathbf{D}_j + \mathbf{F}_j)\, d\mathbf{x} - \int_{B_j} \eta(h_j\mathbf{D}_j + h_j\mathbf{F}_j)|\mathbf{D}_j + \mathbf{F}_j|^2\, d\mathbf{x} - \mathcal{L}(\mathbf{R}_j\mathbf{u}_j) \\ &\geq \int_{\Omega} Q(\mathbf{x}, \mathbf{1}_{B_j}(\mathbf{D}_j + \mathbf{F}_j))\, d\mathbf{x} - \eta(3\sqrt{h_j}) \int_{\Omega} |\mathbf{1}_{B_j}(\mathbf{D}_j + \mathbf{F}_j)|^2\, d\mathbf{x} - \mathcal{L}(\mathbf{R}_j\mathbf{u}_j). \end{aligned} \tag{4.42}$$

Since $h_j\nabla\mathbf{u}_j^T\nabla\mathbf{u}_j \to 0$ a.e. in Ω and $|\mathbf{1}_{B_j}h_j\nabla\mathbf{u}_j^T\nabla\mathbf{u}_j| \leq 1$, by taking account of $|\Omega \setminus B_j| \to 0$ as $j \to +\infty$ we get $\mathbf{1}_{B_j}h_j\nabla\mathbf{u}_j^T\nabla\mathbf{u}_j \rightharpoonup 0$ weakly in $L^2(\Omega, \mathbb{R}^{3\times3})$. By taking account of $\mathbf{1}_{B_j}\nabla\mathbf{u}_j \rightharpoonup \nabla\mathbf{u}$ weakly in $L^2(\Omega, \mathbb{R}^{3\times3})$ and (4.40), we then obtain

$$\mathbf{1}_{B_j}(\mathbf{D}_j + \mathbf{F}_j) \rightharpoonup \mathbb{E}(\mathbf{u}) + \mathbb{E}(\mathbf{z}) = \mathbb{E}(\mathbf{u}) + \frac{1}{2}b_\alpha(\mathbf{e}_\alpha \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_\alpha) \ \ (\alpha = 1, 2) \tag{4.43}$$

weakly in $L^2(\Omega, \mathbb{R}^{3\times3})$. Since $\mathbf{R}_j \to \mathbf{R} \in \mathcal{S}_{\mathcal{L},E}$, then

$$\lim_{j\to+\infty} \mathcal{L}(-\mathbf{R}_j\mathbf{u}_j) = -\mathcal{L}(\mathbf{R}\mathbf{u})$$

and by (2.21) and (4.42), the weak $L^2(\Omega, \mathbb{R}^{3\times3})$ lower semicontinuity of the map $\mathbf{B} \mapsto \int_\Omega Q(\mathbf{x}, \mathbf{B})d\mathbf{x}$ entails

$$\liminf_{j\to+\infty} \mathcal{G}_j(\mathbf{y}_j) \geq \int_\Omega Q(\mathbf{x}, \mathbb{E}(\mathbf{u}) + \mathbb{E}(\mathbf{z}))\, d\mathbf{x} - \mathcal{L}(\mathbf{R}\mathbf{u}) \geq \widetilde{\mathcal{G}}(\mathbf{u})$$

which, by recalling (4.30), ends the proof. □

Remark 4.5. If condition (2.32) is not satisfied then the thesis of Lemma 4.4 may fail. Indeed let $\mathbf{f} := -\mathbf{e}_3 + 6(x_3 - \frac{1}{2})\mathbf{e}_1,\ \mathbf{g} = \mathbf{0}$ and

$$E = \Omega := \{\mathbf{x} : x_1^2 + x_2^2 < 1,\ 0 < x_3 < 1\}.$$

It is straightforward checking that $\mathcal{L}(\mathbf{e}_3) < 0 = \mathcal{L}(\mathbf{e}_1) = \mathcal{L}(\mathbf{e}_2)$ and

$$\begin{aligned} &\mathcal{L}((\mathbf{R} - \mathbf{I})\mathbf{x}) - \min_{\mathbf{x}\in E_{ess}}((\mathbf{R}\mathbf{x})_3 - x_3)\}\, \mathcal{L}(\mathbf{e}_3) \\ &= \frac{\pi}{2}R_{13} - \frac{\pi}{2}(1 - R_{33}) - \pi\sqrt{1 - R_{33}^2} \\ &\leq \frac{\pi}{2}\sqrt{1 - R_{33}^2} - \frac{\pi}{2}(1 - R_{33}) - \pi\sqrt{1 - R_{33}^2} \\ &= -\frac{\pi}{2}\sqrt{1 - R_{33}}\left\{\sqrt{1 - R_{33}} + \sqrt{1 + R_{33}}\right\} \leq -\frac{\pi\sqrt{2}}{2}\sqrt{1 - R_{33}} \leq 0 \end{aligned} \tag{4.44}$$

for every $\mathbf{R} \in SO(3)$. On the other hand if $R_{13} > 0$ we have

$$\mathcal{L}((\mathbf{R}\mathbf{x} - \mathbf{x})_\alpha \mathbf{e}_\alpha) = \frac{\pi}{2}R_{13} > 0,$$

so (2.33) is satisfied while (2.32) is not. Choose now $h_j := j^{-1}$,

$$\mathbf{R}_j := \mathbf{e}_2 \otimes \mathbf{e}_2 + (1 - j^{-2})(\mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_3 \otimes \mathbf{e}_3) + j^{-1}\sqrt{2 - j^{-2}}\,(-\mathbf{e}_3 \otimes \mathbf{e}_1 + \mathbf{e}_1 \otimes \mathbf{e}_3)$$

and set $\mathbf{y}_j := \mathbf{R}_j\mathbf{x} + j^{-1}\sqrt{2 - j^{-2}}\mathbf{e}_3$. It is readily seen that

$$\begin{aligned} y_{j,3} &= -j^{-1}\sqrt{2 - j^{-2}}x_1 + (1 - j^{-2})x_3 + j^{-1}\sqrt{2 - j^{-2}} \\ &\geq (1 - j^{-2})x_3 \geq x_3 - j^{-1}x_3 \end{aligned} \tag{4.45}$$

hence $\mathbf{y}_j \in \mathcal{A}_j$ and by taking (4.2) into account we get $(\mathbf{y}_j - \mathbf{R}_j\mathbf{x} - \mathbf{c}_j)_\alpha \equiv 0,\ \alpha = 1, 2$. Therefore bearing in mind that $\mathbf{R}_j^T \to \mathbf{I}$ we have

$$\mathbf{u}_j = j\mathbf{R}_j^T((y_{j,3} - x_3)\mathbf{e}_3) = \mathbf{R}_j^T\left\{(\sqrt{2 - j^{-2}}(1 - x_1) + j^{-1}x_3)\mathbf{e}_3\right\} \to \mathbf{u} := \sqrt{2}(1 - x_1)\mathbf{e}_3$$

and by Lemma 3.8 we get $\mathbf{Ru} = \mathbf{u}$ for every $\mathbf{R} \in \mathcal{S}_{\mathcal{L},E}$, hence

$$\widetilde{\mathcal{G}}(\mathbf{u}) \geq -\max_{\mathbf{R}\in\mathcal{S}_{\mathcal{L},E}} \mathcal{L}(\mathbf{Ru}) = -\mathcal{L}(\mathbf{u}) = \pi\sqrt{2}.$$

On the other hand by taking account of

$$y_{j,1} - x_1 = -j^{-2}x_1 + j^{-1}\sqrt{2 - j^{-2}}x_3,\ y_{j,2} - x_2 = 0$$

and

$$y_{j,3} - x_3 = j^{-1}\sqrt{2 - j^{-2}}(1 - x_1) + j^{-2}x_3$$

it is straightforward checking that

$$\mathcal{G}_j(\mathbf{y}_j) = -j\mathcal{L}(\mathbf{y}_j - \mathbf{x}) = \pi\sqrt{2 - j^{-2}} + \pi j^{-1} - \frac{\pi}{2}\sqrt{2 - j^{-2}} \to \frac{\pi\sqrt{2}}{2} < \widetilde{\mathcal{G}}(\mathbf{u})$$

thus proving that the claim of Lemma 4.4 fails in this case.

Lemma 4.6. (Upper bound) *Assume* (2.13)*,* (2.19)–(2.22)*,* (2.32)*,* (2.33)*,* $\mathcal{L}(\mathbf{e}_3) < 0$ *and let* $0 < h_j \to 0^+$ *as* $j \to +\infty$*. For every* $\mathbf{u} \in C^1(\overline{\Omega}, \mathbb{R}^3)$ *there exists* $\widetilde{\mathbf{y}}_j \in C^1(\overline{\Omega}, \mathbb{R}^3)$ *such that*

$$\limsup_{j\to+\infty} \mathcal{G}_j(\widetilde{\mathbf{y}}_j) \leq \widetilde{\mathcal{G}}(\mathbf{u}).$$

Proof. We assume without loss of generality that $\mathbf{u} \in \mathcal{A}$ and let

$$\mathbf{b}^* \in \operatorname{argmin}\left\{\int_\Omega \mathcal{Q}(\mathbf{x}, \mathbb{E}(\mathbf{u}) + \frac{1}{2}b_\alpha(\mathbf{e}_\alpha \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_\alpha))\, d\mathbf{x} : \mathbf{b} \in \mathbb{R}^2\right\},$$

$$\widetilde{\mathbf{u}}(\mathbf{x}) := \mathbf{u}(\mathbf{x}) + x_3(b_1^*\mathbf{e}_1 + b_2^*\mathbf{e}_2). \tag{4.46}$$

It is readily seen that $\widetilde{\mathbf{u}} \in \mathcal{A}$, that $\mathbb{E}(\widetilde{\mathbf{u}}) = \mathbb{E}(\mathbf{u}) + \frac{1}{2}b_\alpha^*(\mathbf{e}_\alpha \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_\alpha)$ hence

$$\mathcal{I}(\mathbf{u}) = \int_\Omega \mathcal{Q}(\mathbf{x}, \mathbb{E}(\widetilde{\mathbf{u}}))\, d\mathbf{x}. \tag{4.47}$$

Moreover, by (3.1) of Lemma 3.1 and Remark 3.2 we obtain

$$\mathcal{L}(\mathbf{R}\widetilde{\mathbf{u}}) = \mathcal{L}(\mathbf{R}\mathbf{u}) + \mathcal{L}(x_3(b_1^*\mathbf{R}\mathbf{e}_1 + b_2^*\mathbf{R}\mathbf{e}_2)) = \mathcal{L}(\mathbf{R}\mathbf{u}), \qquad \forall\, \mathbf{R} \in \mathcal{S}_{\mathcal{L},E}. \tag{4.48}$$

Therefore by choosing

$$\widetilde{\mathbf{R}} \in \operatorname{argmin}\{-\mathcal{L}(\mathbf{R}\widetilde{\mathbf{u}}) : \mathbf{R} \in \mathcal{S}_{\mathcal{L},E}\}$$

we get

$$\widetilde{\mathcal{G}}(\mathbf{u}) = \int_\Omega \mathcal{Q}(\mathbf{x}, \mathbb{E}(\widetilde{\mathbf{u}}))\, d\mathbf{x} - \mathcal{L}(\widetilde{\mathbf{R}}\widetilde{\mathbf{u}}).$$

By setting $\widetilde{\mathbf{y}}_j := \widetilde{\mathbf{R}}(\mathbf{x}+h_j\widetilde{\mathbf{u}})$, taking account $\mathcal{S}_{\mathcal{L},E} \subset \{\mathbf{R} : \mathbf{R}\mathbf{e}_3 = \mathbf{e}_3\}$ and Lemma 3.8, we get $\mathcal{L}(\widetilde{\mathbf{R}}\mathbf{x}-\mathbf{x}) = 0$ and for q.e. $\mathbf{x} \in E$

$$\widetilde{y}^*_{j,3} = x_3 + h_j\widetilde{u}_3 \geq (1 - h_j)x_3.$$

Therefore $\widetilde{\mathbf{y}}_j \in \mathcal{A}_j$ and by (2.24) we get

$$\limsup_{j\to+\infty} |\mathcal{G}_{h_j}(\mathbf{y}_j) - \widetilde{\mathcal{G}}(\mathbf{u})| \leq \limsup_{j\to+\infty} \int_\Omega \left| \frac{1}{h_j^2}\mathcal{W}(\mathbf{x}, \mathbf{I} + h_j\nabla\widetilde{\mathbf{u}}) - \mathcal{Q}(\mathbf{x}, \mathbb{E}(\widetilde{\mathbf{u}}))\right| d\mathbf{x} = 0$$

which proves the lemma. □

We are now in a position to prove our main theorem.

Proof of Theorem 2.4. If $(\overline{\mathbf{y}}_j)_{j\in\mathbb{N}} \subset H^1(\Omega, \mathbb{R}^3)$ is a minimizing sequence for $\mathcal{G}_j$ then $\mathcal{G}_j(\overline{\mathbf{y}}_j) \leq \mathcal{G}_j(\mathbf{x}) = 0$, moreover if $\mathbf{R}_j$ belong $\mathcal{A}(\overline{\mathbf{y}}_j)$ and $\overline{\mathbf{c}}_j$ is defined by (4.2) and (4.3), then Lemma 4.1 entails that the sequence

$$\overline{\mathbf{u}}_j(\mathbf{x}) := h_j^{-1}\mathbf{R}_j^T\left\{(\overline{\mathbf{y}}_j - \mathbf{R}_j\mathbf{x} - \overline{\mathbf{c}}_j)_\alpha\mathbf{e}_\alpha + (\overline{y}_{j,3} - x_3)\mathbf{e}_3\right\}$$

is bounded in $H^1(\Omega; \mathbb{R}^3)$. Therefore up to subsequences $\overline{\mathbf{u}}_j \to \overline{\mathbf{u}}$ weakly in $H^1(\Omega; \mathbb{R}^3)$, so, by Lemma 4.4, we have $\overline{\mathbf{u}} \in \mathcal{A}$ and

$$\liminf_{j\to+\infty} \mathcal{G}_j(\overline{\mathbf{y}}_j) \geq \widetilde{\mathcal{G}}(\overline{\mathbf{u}}).$$

On the other hand, by Lemma 4.6, for every $\mathbf{u} \in C^1(\overline{\Omega}, \mathbb{R}^3) \cap \mathcal{A}$ there exists a sequence $\mathbf{y}_j \in C^1(\overline{\Omega}, \mathbb{R}^3)$ such that

$$\limsup_{j\to+\infty} \mathcal{G}_j(\mathbf{y}_j) \leq \widetilde{\mathcal{G}}(\mathbf{u}).$$

Since

$$\mathcal{G}_j(\overline{\mathbf{y}}_j) + o(1) = \inf_{H^1(\Omega,\mathbb{R}^3)} \mathcal{G}_j \leq \mathcal{G}_j(\mathbf{y}_j), \qquad \text{as } j \to +\infty, \tag{4.49}$$

by passing to the limit as $j \to +\infty$, we get

$$\widetilde{\mathcal{G}}(\overline{\mathbf{u}}) \leq \widetilde{\mathcal{G}}(\mathbf{u}), \qquad \forall \mathbf{u} \in C^1(\overline{\Omega}, \mathbb{R}^3) \cap \mathcal{A}. \tag{4.50}$$

Now fix a generic $\mathbf{u} \in \mathcal{A}$ and denote again by $\mathbf{u}$ a Sobolev extension of $\mathbf{u}$ to the whole $\mathbb{R}^3$. We claim that there exists $\mathbf{u}_j \in C^1(\overline{\Omega}, \mathbb{R}^3) \cap \mathcal{A}$ such that $\mathbf{u}_j \to \mathbf{u}$ in $H^1(\Omega; \mathbb{R}^3)$: Indeed, since $\widetilde{u}_3(\mathbf{x}) + x_3 \geq 0$ for q.e. $\mathbf{x} \in E$, by Lemma A.4 it is enough to choose $u_{3,j} := \eta_j - x_3$ where $\eta_j \in C^1(\mathbb{R}^3)$, $\eta_j \geq$

0 q.e. in E, $\eta_j \to u_3 + x_3$ in $H^1(\mathbb{R}^3)$ (here $u_3 + x_3$ denotes also an extension to the whole $H^1(\mathbb{R}^3)$) and $u_{\alpha,j} := u_\alpha * \rho_j$, $\alpha = 1, 2$ where ρ_j is a sequence of smooth mollifiers. By (4.50) we have

$$\widetilde{\mathcal{G}}(\overline{\mathbf{u}}) \leq \widetilde{\mathcal{G}}(\mathbf{u}_j)$$

whence by Remark 2.2,

$$\widetilde{\mathcal{G}}(\overline{\mathbf{u}}) \leq \lim_{j\to+\infty} \widetilde{\mathcal{G}}(\mathbf{u}_j) = \widetilde{\mathcal{G}}(\mathbf{u}), \qquad \forall\, \mathbf{u} \in \mathcal{A},$$

that is $\overline{\mathbf{u}} \in \operatorname{argmin} \widetilde{\mathcal{G}}$.

We show that $\mathcal{G}_j(\mathbf{y}_j) \to \widetilde{\mathcal{G}}(\overline{\mathbf{u}})$: By Lemma A.4 in the Appendix, for every $\varepsilon > 0$ there is $\overline{\mathbf{u}}_\varepsilon \in C^1(\overline{\Omega}; \mathbb{R}^3)) \cap \mathcal{A}$ such that

$$\widetilde{\mathcal{G}}(\overline{\mathbf{u}}_\varepsilon) < \widetilde{\mathcal{G}}(\overline{\mathbf{u}}) + \varepsilon$$

and by Lemma 4.6 there exists $\mathbf{y}_{j,\varepsilon} \in C^1(\overline{\Omega}; \mathbb{R}^3)$ such that by taking account of (4.49) we have

$$\limsup_{j\to+\infty} \mathcal{G}_j(\overline{\mathbf{y}}_j) \leq \limsup_{j\to+\infty} \mathcal{G}_j(\mathbf{y}_{j,\varepsilon}) \leq \widetilde{\mathcal{G}}(\mathbf{u}_\varepsilon) < \widetilde{\mathcal{G}}(\overline{\mathbf{u}}) + \varepsilon$$

for every $\varepsilon > 0$. Since by Lemma 4.4,

$$\liminf_{j\to+\infty} \mathcal{G}_j(\overline{\mathbf{y}}_j) \geq \widetilde{\mathcal{G}}(\overline{\mathbf{u}}),$$

we get $\mathcal{G}_j(\mathbf{y}_j) \to \widetilde{\mathcal{G}}(\overline{\mathbf{u}})$ as claimed.

We are only left to show that $\min \widetilde{\mathcal{G}} = \min \mathcal{G}$. To this aim we show first that for every $\mathbf{u} \in \mathcal{A}$ there exists $\mathbf{u}_* \in \mathcal{A}$ such that $\mathcal{G}(\mathbf{u}_*) = \widetilde{\mathcal{G}}(\mathbf{u})$. Indeed if $\widetilde{\mathbf{u}}$ is defined as in (4.46) then by (4.47) and (4.48) we get

$$\widetilde{\mathcal{G}}(\mathbf{u}) = \mathcal{I}(\mathbf{u}) - \max_{\mathbf{R}\in\mathcal{S}_{\mathcal{L},E}} \mathcal{L}(\mathbf{R}\mathbf{u}) = \int_\Omega \mathcal{Q}(\mathbf{x}, \mathbb{E}(\widetilde{\mathbf{u}})\, d\mathbf{x} - \max_{\mathbf{R}\in\mathcal{S}_{\mathcal{L},E}} \mathcal{L}(\mathbf{R}\widetilde{\mathbf{u}}) = \mathcal{G}(\widetilde{\mathbf{u}}) \tag{4.51}$$

as claimed. By recalling that $\widetilde{\mathcal{G}}(\overline{\mathbf{u}}) = \min \widetilde{\mathcal{G}} \leq \inf \mathcal{G}$ let us assume that inequality is strict. Then by (4.51) there exists $\overline{\mathbf{u}}_* \in \mathcal{A}$ such that $\mathcal{G}(\overline{\mathbf{u}}_*) = \widetilde{\mathcal{G}}(\overline{\mathbf{u}}) < \inf \mathcal{G}$, a contradiction. Thus again by (4.51) $\mathcal{G}(\overline{\mathbf{u}}_*) = \widetilde{\mathcal{G}}(\overline{\mathbf{u}}) = \min \mathcal{G}$. □

5. The gap with Signorini problem

In this section we will exhibit a choice of energy density $\mathcal{W}$, open set Ω, dead loads $\mathbf{f}, \mathbf{g}$ and set $E \subset \overline{\Omega}$ fulfilling all the assumptions of Theorem 2.4 but such that the minimum of the limit functional $\mathcal{G}$ is strictly less than the minimum of the Signorini functional (see [45] for a counterexample exhibiting an analogous gap for unconstrained pure traction problem).

We shall consider the energy density already defined in (2.26) by setting

$$\mathcal{W}(\mathbf{F}) := \begin{cases} \mathcal{W}_{\text{iso}}\left(\dfrac{\mathbf{F}}{(\det \mathbf{F})^{1/3}}\right) + \mathcal{W}_{\text{vol}}(\mathbf{F}), & \text{if } \det \mathbf{F} > 0, \\ +\infty, & \text{if } \det \mathbf{F} \leq 0, \end{cases} \tag{5.1}$$

where $\mathcal{W}_{\text{iso}}$ is the energy density of Yeoh type defined in (2.27) with $2c_1 = \mu > 0$ and $\mathcal{W}_{\text{vol}}(\mathbf{F}) = g(\det \mathbf{F})$ where $g : \mathbb{R}_+ \to \mathbb{R}$ is the convex C^2 function (satisfying (2.28) with $r = 2$) defined by

$$g(t) = \frac{\mu}{6}(t^2 - 1 - 2\log t).$$

By recalling Example 2.1 it is readily seen that $\mathcal{W}$ satisfies (2.14)–(2.17) and by taking into account of

$$\det(\mathbf{I}+h\mathbf{B}) = 1 + h\,\mathrm{Tr}\mathbf{B} + (h^2/2)((\mathrm{Tr}\mathbf{B})^2 - \mathrm{Tr}\,\mathbf{B}^2) + h^3\det\mathbf{B}$$

and $\mathrm{Tr}(\mathbf{B}^T\mathbf{B}) = |\mathbf{B}|^2$ for every $\mathbf{B}\in\mathbb{R}^{3\times3}$, we obtain as $h\to 0$

$$\frac{|\mathbf{I}+h\mathbf{B}|^2}{\det(\mathbf{I}+h\mathbf{B})^{2/3}} - 3 \;=\; h^2\big(\,2\,|\mathbf{B}|^2 - \tfrac{2}{3}|\mathrm{Tr}\mathbf{B}|^2\,\big) + o(h^2)$$

for every $\mathbf{B}\in\mathbb{R}^{3\times3}_{\mathrm{sym}}$. Moreover by recalling (2.27) (with $2c_1=\mu$)

$$\mathcal{W}_{\mathrm{vol}}(\mathbf{I}+h\mathbf{B}) = g(\det(\mathbf{I}+h\mathbf{B})) = \frac{h^2}{2}|\mathrm{Tr}\mathbf{B}|^2 + o(h^2) = \frac{\mu}{3}\,|\mathrm{Tr}\mathbf{B}|^2\,h^2 \;+\; o(h^2),$$

$$\mathcal{W}_{\mathrm{iso}}(\mathbf{I}+h\mathbf{B}) \;=\; \frac{\mu}{2}\,h^2\left(2\,|\mathbf{B}|^2 - \frac{2}{3}(\mathrm{Tr}\mathbf{B})^2\right) \;+\; o(h^2),$$

so

$$\frac{1}{2}\mathbf{B}\,D^2\mathcal{W}(\mathbf{I})\,\mathbf{B} \;=\; \mu\,|\mathbf{B}|^2. \tag{5.2}$$

Let

$$\Omega := \{\mathbf{x}\in\mathbb{R}^3 : x_1^2+x_2^2<1,\ 0<x_3<1\},\quad E := \{\mathbf{x}\in\mathbb{R}^3 : x_1^2+x_2^2<1,\ x_3=0\} \tag{5.3}$$

and $\varphi\in C^2(\overline{E})$ such that

$$\Delta\varphi \not\equiv 0\,,\quad \varphi(x_1,x_2)=\phi(r)\,,\quad r := \sqrt{x_1^2+x_2^2}\,,\quad \phi(1)=\phi'(1)=\int_0^1 r^2\phi'(r)\,dr = 0 \tag{5.4}$$

(for instance $\phi(r) := 1-6r^2+9r^4-4r^6$ fulfills (5.4)). It is readily seen that condition (2.13) is fulfilled and that $E_{ess}=\overline{E}$. We define

$$\begin{aligned} \mathbf{R}_* \quad &:= -\mathbf{e}_1\otimes\mathbf{e}_2 + \mathbf{e}_2\otimes\mathbf{e}_1 + \mathbf{e}_3\otimes\mathbf{e}_3, \\ \mathcal{L}(\mathbf{u}) \;&:= \int_\Omega u_\alpha\,\nabla_\alpha\varphi\,d\mathbf{x} - \int_E u_3(x_1,x_2,0)\,dx_1\,dx_2. \end{aligned} \tag{5.5}$$

Condition (2.32) is satisfied since

$$\mathcal{L}((\mathbf{R}\mathbf{x}-\mathbf{x})_\alpha\mathbf{e}_\alpha) = \pi\,(R_{11}+R_{22}-2)\int_0^1 r^2\phi'(r)\,dr = 0,\qquad \forall\,\mathbf{R}\in SO(3).$$

Moreover $\mathcal{L}(\mathbf{e}_1)=\mathcal{L}(\mathbf{e}_2)=0,\ \mathcal{L}(\mathbf{e}_3)<0$ and

$$\begin{aligned} \Phi(\mathbf{R},E,\mathcal{L}) \;&= \pi\,(R_{11}+R_{22}-2)\int_0^1 r^2\phi'(r)\,dr + \pi\,\min_{E_{ess}}\{R_{31}x_1+R_{32}x_2+(R_{33}-1)x_3\} \\ &= -\pi\sqrt{R_{31}^2+R_{32}^2}\le 0. \end{aligned} \tag{5.6}$$

so (2.33) is fulfilled too. By taking account of $\mathbf{R}_*\mathbf{e}_3=\mathbf{e}_3$, we get

$$\Phi(\mathbf{R}_*,E,\mathcal{L}) = -2\pi\int_0^1 r^2\phi'(r)\,dr = 0$$

whence $\mathbf{R}_* \in \mathcal{S}_{\mathcal{L},E}$ and Lemma 3.8 entails $\mathcal{S}_{\mathcal{L},E} = \{\mathbf{R} : \mathbf{R}\mathbf{e}_3 = \mathbf{e}_3\}$. Since E fulfills (2.13), $\mathcal{W}$ satisfies (2.14)–(2.17) and $\mathcal{L}$ satisfies (2.27) together with $\mathcal{L}(\mathbf{e}_3) < 0$ then, by taking into account of (5.2), Theorem 2.4 entails

$$\inf \mathcal{G}_j \to \min_{\mathbf{u}\in\mathcal{A}} \mathcal{G} = \min\left\{\mu\int_\Omega |\mathbb{E}(\mathbf{u})|^2\,d\mathbf{x} - \max_{\mathbf{R}\in\mathcal{S}_{\mathcal{L},E}} \mathcal{L}(\mathbf{R}\mathbf{u}) : \mathbf{u}\in\mathcal{A}\right\}. \tag{5.7}$$

We set

$$\mathcal{E}(\mathbf{u}) := \mu\int_\Omega |\mathbb{E}(\mathbf{u})|^2\,d\mathbf{x} - \mathcal{L}(\mathbf{u}) \tag{5.8}$$

and, for every $\mathbf{R}\in\mathcal{S}_{\mathcal{L},E}$,

$$\mathcal{E}_{\mathbf{R}}(\mathbf{u}) := \mu\int_\Omega |\mathbb{E}(\mathbf{u})|^2\,d\mathbf{x} - \mathcal{L}(\mathbf{R}\mathbf{u}). \tag{5.9}$$

We aim to show

$$\min\{\mathcal{E}_{\mathbf{R}_*}(\mathbf{u}) : \mathbf{u}\in\mathcal{A}\} < \min\{\mathcal{E}(\mathbf{u}) : \mathbf{u}\in\mathcal{A}\} \tag{5.10}$$

so that, once (5.10) is proved, we deduce

$$\min_{\mathbf{u}\in\mathcal{A}} \mathcal{G} < \min_{\mathbf{u}\in\mathcal{A}} \mathcal{E}. \tag{5.11}$$

In order to show inequality (5.10) we need some properties of minimizers of $\mathcal{E}$ which have been essentially proven in [45]. In the following $\overline{\mathbb{E}}(\cdot)$ will denote the upper-left 2×2 submatrix of $\mathbb{E}(\cdot)$ and $\overline{\mathbf{R}}\in SO(2)$ the upper-left 2×2 submatrix of any $\mathbf{R}\in\mathcal{S}_{\mathcal{L},E}$.

Lemma 5.1. *Let* $\mathbf{u}\in\mathcal{A}$ *and let*

$$\mathbf{v}(\mathbf{x}) := v_\alpha(x_1,x_2)\mathbf{e}_\alpha + v_3(x_3)\mathbf{e}_3,\ \ \alpha = 1,2 \tag{5.12}$$

where

$$v_\alpha(x_1,x_2) := \int_0^1 u_\alpha(\mathbf{x})\,dx_3,\ \ \alpha=1,2, \quad v_3(x_3) := \pi^{-1}\int_E u_3(\mathbf{x})dx_1dx_2.$$

Then $\mathbf{v}\in\mathcal{A}$ *and*

$$\mathcal{E}_{\mathbf{R}}(\mathbf{u}) \ge \mathcal{J}_{\overline{\mathbf{R}}}(\overline{\mathbf{v}}), \qquad \forall\,\mathbf{R}\in .\mathcal{S}_{\mathcal{L},E}, \tag{5.13}$$

where $\overline{\mathbf{v}} := v_\alpha\mathbf{e}_\alpha$, *and*

$$\mathcal{J}_{\overline{\mathbf{R}}}(\overline{\mathbf{v}}) := \mu\int_E |\overline{\mathbb{E}}(\overline{\mathbf{v}})|^2\,dx_1dx_2 - \int_E \overline{\mathbf{R}}^T\nabla\varphi\cdot\overline{\mathbf{v}}\ dx_1dx_2. \tag{5.14}$$

In particular if $\mathbf{R}=\mathbf{I}$, *then* (5.13) *reduces to* $\mathcal{E}(\mathbf{u}) \ge \mathcal{J}(\overline{\mathbf{v}})$ *having set* $\mathcal{J} := \mathcal{J}_{\overline{\mathbf{I}}}$.

Proof. Since $u_3^* \ge 0$ q.e. on $E = \partial\Omega\cap\{x_3=0\}$ then by Remark 2.3 we get $u_3\ge 0$ $\mathcal{H}^2$- q.e. in E that is $v_3(0)\ge 0$ hence, again by Remark 2.3, $\mathbf{v}\in\mathcal{A}$. Moreover, by using the notation $u_{3,3} := \partial_3 u_3$, Jensen inequality entails

$$\begin{aligned}\mathcal{E}_{\mathbf{R}}(\mathbf{u}) \ &\ge\ \mu\int_E\left|\int_0^1 \overline{\mathbb{E}}(\mathbf{u})\,dx_3\right|^2 dx_1\,dx_2 \ +\ \mu\pi\int_0^1\left|\frac{1}{\pi}\int_E u_{3,3}\,dx_1\,dx_2\right|^2 dx_3\\ &\quad - \int_E \overline{R}_{\beta\alpha}\nabla_\beta\varphi\left(\int_0^1 u_\alpha\,dz\right)dx_1\,dx_2 \ +\ \int_E u_3(x_1,x_2,0)\,dx_1\,dx_2\\ &\ge\ \mathcal{J}_{\overline{\mathbf{R}}}(\overline{v}) + \mu\pi\int_0^1 |\dot{v}_3|^2\,dx_3 + \pi v_3(0) \ge \mathcal{J}_{\overline{\mathbf{R}}}(\overline{v}),\end{aligned} \tag{5.15}$$

thus proving the lemma. □

We need now the following characterization of minimizers of $\mathcal{J}$ which has been given in [45].

Lemma 5.2. *There exists* $\overline{\Phi} \in H^2(E)$ *such that*

$$\min_{\mathbf{u}\in H^1(E)} \mathcal{J}(\mathbf{u}) = \mathcal{J}(\nabla\overline{\Phi}) \geq \min_{\Phi\in H^2(E)} \int_E (2\,\mu\,\Phi^2_{,12} + \mu\,\Phi^2_{,11} + \mu\,\Phi^2_{,22} + \Phi\Delta\varphi)\,dx_1dx_2, \tag{5.16}$$

where we have used the notation $\Phi_{,\alpha\beta} := \partial^2_{\alpha\beta}\Phi$.

A straightforward application of Lemma 5.1 (with $\mathbf{R} = \mathbf{I}$) and Lemma 5.2 yields the following precise calculation of the energy level of $\mathbf{u} \in \operatorname{argmin}_{\mathcal{A}} \mathcal{E}$.

Lemma 5.3. *There holds*

$$\min_{\mathbf{u}\in\mathcal{A}} \mathcal{E}(\mathbf{u}) = \min_{\Phi\in H^2(E)} \int_E (2\mu\Phi^2_{,12} + \mu\Phi^2_{,11} + \mu\Phi^2_{,22} + \Phi\Delta\varphi)\,dx_1dx_2. \tag{5.17}$$

Proof. It is readily seen that any displacement of the kind $(\nabla\Phi(x_1,x_2), v_3(x_3)) \in \mathcal{A}$ if and only if $\Phi \in H^2(E)$, $v_3 \in H^1(0,1)$ and $v_3(0) \geq 0$. Therefore, by Lemmas 5.1 and 5.2, we get

$$\begin{aligned}
\min_{\mathbf{u}\in\mathcal{A}} \mathcal{E}(\mathbf{u}) \;&\geq \min_{\Phi\in H^2(E)} \int_E (2\mu\Phi^2_{,12} + \mu\Phi^2_{,11} + \mu\Phi^2_{,22} + \Phi\Delta\varphi)\,dx_1dx_2 \\
&\quad + \inf\left\{\mu\,\pi\int_0^1 |\dot v_3|^2\,dx_3 + \pi\,v_3(0)\;:\; v_3 \in H^1(0,1),\; v_3(0)\geq 0\right\} \\
&= \min_{\Phi\in H^2(E)} \int_E (2\mu\Phi^2_{,12} + \mu\Phi^2_{,11} + \mu\Phi^2_{,22} + \Phi\Delta\varphi)\,dx_1dx_2.
\end{aligned}$$

The opposite inequality follows by choosing $\mathbf{v} := (\nabla\Phi, 0)$ with $\Phi \in H^2(B)$ and by taking into account of

$$\min_{\mathbf{u}\in\mathcal{A}} \mathcal{E}(\mathbf{u}) \leq \mathcal{E}(\mathbf{v})$$

for every choice of $\Phi \in H^2(E)$. □

Let now $\Phi \in H^2(E)$. Then $\mathbf{v} := \Phi_{,2}\mathbf{e}_1 - \Phi_{,1}\mathbf{e}_2 \in \mathcal{A}$ and a direct computation shows that

$$\min_{\mathcal{A}} \mathcal{E}_{\mathbf{R}_*} \leq \min_{\Phi\in H^2(E)} \int_E (2\mu{\Phi_{,12}}^2 + \frac{\mu}{2}(\Phi_{,22} - \Phi_{,11})^2 + \Phi\Delta\varphi)\,dx_1dx_2. \tag{5.18}$$

Therefore inequality (5.10) is an immediate consequence of the next proposition.

Proposition 5.4. *There holds*

$$\begin{aligned}
&\min_{\Phi\in H^2(E)} \int_E (2\mu{\Phi_{,12}}^2 + \frac{\mu}{2}(\Phi_{,22} - \Phi_{,11})^2 + \Phi\Delta\varphi)\,dx_1dx_2 \\
&< \min_{\Phi\in H^2(E)} \int_E (2\mu\Phi^2_{,12} + \mu\Phi^2_{,11} + \mu\Phi^2_{,22} + \Phi\Delta\varphi)\,dx_1dx_2.
\end{aligned} \tag{5.19}$$

Proof. The proof is the same of formula (5.14) of [45]. □

The previous explicit example shows that a gap phenomenon may actually develop. Nevertheless one can prove that whenever $\mathbf{f}$, $\mathbf{g}$ satisfy (2.33) then they can be suitable rotated in order to avoid the gap. In order to state such result, we introduce suitable notation: Set

$$\mathcal{L}_{\mathbf{R}}(\mathbf{v}) := \mathcal{L}(\mathbf{R}\mathbf{v}) = \int_\Omega \mathbf{R}^T\mathbf{f}\cdot\mathbf{v}\,d\mathbf{x} + \int_{\partial\Omega} \mathbf{R}^T\mathbf{g}\cdot\mathbf{v}\,d\mathcal{H}^2, \qquad \forall\,\mathbf{R}\in SO(3), \tag{5.20}$$

say $\mathcal{L}_{\mathbf{R}}$ is the load functional associated to the external forces $\mathbf{R}^T\mathbf{f}, \mathbf{R}^T\mathbf{g}$ and let $\mathcal{E}_{\mathbf{R}}$ be the functional defined by replacing $\mathcal{L}$ with $\mathcal{L}_{\mathbf{R}}$ in the definition of $\mathcal{E}$.

Theorem 5.5. *Assume* (2.13)*,* (2.33)*,* $\mathcal{L}(\mathbf{e}_3) < 0$ *and* $\mathbf{R}\in\mathcal{S}_{\mathcal{L},E}$*. Then the functional* $\mathcal{L}_{\mathbf{R}}$ *fulfills* (2.33) *and* $\mathcal{S}_{\mathcal{L},E} \equiv \mathcal{S}_{\mathcal{L}_{\mathbf{R}},E}$*. Moreover, if* $\mathbf{u}$ *minimizes* $\mathcal{G}$ *over* $H^1\Omega,\mathbb{R}^3)$, $\mathbf{R}\in\mathcal{S}_{\mathcal{L},E}$ *attains the maximum in definition* (2.42) *of* $\mathcal{G}(\mathbf{u})$ *then* $\mathbf{u}\in\operatorname{argmin}\mathcal{E}_{\mathbf{R}}$ *and*

$$\min_{H^1(\Omega,\mathbb{R}^3)} \mathcal{G} \;=\; \min_{H^1(\Omega,\mathbb{R}^3)} \mathcal{E}_{\mathbf{R}}. \tag{5.21}$$

Proof. By Lemma 3.8 we have either $\mathcal{S}_{\mathcal{L},E}\equiv\{\mathbf{I}\}$ or $\mathcal{S}_{\mathcal{L},E} = \{\mathbf{R} : \mathbf{R}\mathbf{e}_3 = \mathbf{e}_3\}$: In the first case there is nothing to prove, in the second one by (2.33) we get

$$0 = \Phi(\mathbf{R}, E, \mathcal{L}) = \mathcal{L}((\mathbf{R}-\mathbf{I})\mathbf{x}). \tag{5.22}$$

Therefore for any other $\mathbf{S}\in SO(3)$ by taking account of $\mathbf{R}\mathbf{e}_3 = \mathbf{e}_3$ and (5.22) we have

$$\Phi(\mathbf{S}, E, \mathcal{L}_{\mathbf{R}}) = \Phi(\mathbf{R}\mathbf{S}, E, \mathcal{L}) \le 0 \tag{5.23}$$

that is $\mathcal{L}_{\mathbf{R}}$ satisfies (2.33). By Remark 3.5 conditions (4.9)–(4.11) of Theorem 4.5 in [12] are fulfilled hence $\mathcal{E}_{\mathbf{R}}$ achieves a finite minimum. Moreover since $\mathbf{R}\mathbf{e}_3 = \mathbf{e}_3$ implies $\mathbf{R}^2\mathbf{e}_3 = \mathbf{e}_3$, (5.23) together with Lemma 3.8 entails

$$\Phi(\mathbf{R}, E, \mathcal{L}_{\mathbf{R}}) = \Phi(\mathbf{R}^2, E, \mathcal{L}) = 0, \tag{5.24}$$

whence $\mathbf{R}\in\mathcal{S}_{\mathcal{L}_{\mathbf{R}},E}$ whenever $\mathbf{R}\in\mathcal{S}_{\mathcal{L},E}$. Then since $\mathcal{L}_{\mathbf{R}}(\mathbf{e}_3) = \mathcal{L}(\mathbf{R}\mathbf{e}_3) = \mathcal{L}(\mathbf{e}_3) < 0$ we get, again by Lemma 3.8, $\mathcal{S}_{\mathcal{L}_{\mathbf{R}},E}\not\equiv\{\mathbf{I}\}$ hence $\mathcal{S}_{\mathcal{L}_{\mathbf{R}},E} = \{\mathbf{R} : \mathbf{R}\mathbf{e}_3 = \mathbf{e}_3\} = \mathcal{S}_{\mathcal{L},E}$ as claimed.

We conclude by checking that if $\mathbf{u}$ minimizes $\mathcal{G}$ then it is also a minimizer of $\mathcal{E}_{\mathbf{R}}$ over $H^1(\Omega,\mathbb{R}^3)$. If $\mathbf{u}\in H^1(\Omega,\mathbb{R}^3)$ minimizes $\mathcal{G}$ and $\mathbf{R}$ attains the maximum then

$$\min_{H^1(\Omega,\mathbb{R}^3)} \mathcal{G} = \mathcal{G}(\mathbf{u}) = \int_\Omega \mathcal{Q}(\mathbf{x}, \mathbb{E}(\mathbf{u}))\,d\mathbf{x} - \mathcal{L}(\mathbf{R}\mathbf{u}) = \mathcal{E}_{\mathbf{R}}(\mathbf{u}).$$

Thus since $\mathcal{G}\le\mathcal{E}_{\mathbf{R}}$ then (5.21) is proven. □

Remark 5.6. By choosing $\mathcal{W}$ as in (5.1), Ω, E as in (5.3) we provide an example where the inclusion $\operatorname{argmin}\mathcal{G}\subset\operatorname{argmin}\widetilde{\mathcal{G}}$ is strict. Indeed Lemma 5.1 shows that for every $\mathbf{R}\in\mathcal{S}_{\mathcal{L},E}$ there exists $\mathbf{w}\in\operatorname{argmin}\mathcal{E}_{\mathbf{R}}$ such that $\mathbf{w}(\mathbf{x}) = w_\alpha(x_1,x_2)\mathbf{e}_\alpha$. By Theorem 5.5 there exists $\mathbf{u}\in\operatorname{argmin}\mathcal{G}\subset\operatorname{argmin}\widetilde{\mathcal{G}}$ such that $\mathbf{u}(\mathbf{x}) = u_\alpha(x_1,x_2)\mathbf{e}_\alpha$. Then we can set $\mathbf{u}_*(\mathbf{x}) := \mathbf{u}(\mathbf{x}) + x_3\mathbf{e}_1$. By taking account of Lemma 3.1 and Remark 3.2, we get $\mathcal{L}(\mathbf{R}\mathbf{u}) = \mathcal{L}(\mathbf{R}\mathbf{u}_*)$ for every $\mathbf{R}\in\mathcal{S}_{\mathcal{L},E}$ hence $\widetilde{\mathcal{G}}(\mathbf{u}) = \widetilde{\mathcal{G}}(\mathbf{u}_*)$ and $\mathbf{u}_*\in\operatorname{argmin}\widetilde{\mathcal{G}}$. Moreover, again by taking account of Lemma 3.1 and Remark 3.2, we have

$$\mathcal{G}(\mathbf{u}_*) - \mathcal{G}(\mathbf{u}) = \mu\int_\Omega |\mathbb{E}(\mathbf{u}_*)|^2\,d\mathbf{x} - \mu\int_\Omega |\mathbb{E}(\mathbf{u})|^2\,d\mathbf{x} = \frac{\mu}{2}|\Omega| > 0,$$

thus $\mathbf{u}_*\notin\operatorname{argmin}\mathcal{G}$ and the inclusion is strict in this case.

6. Conclusions

We have showed a rigorous variational linearization for a classical obstacle problem in nonlinear elasticity, namely an elastic body subject to pure traction load, supported on a unilateral rigid plane. Under suitable geometric admissibility conditions on the loads we obtain coincidence of minima with the classical Signorini problem in linear elasticity. On the other hand, we have shown the existence of loads violating such admissibility condition and entailing a gap between the minimum of the two problems.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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Appendix

For reader's convenience and aiming to the precise formulation of unilateral constraint, in this section we encompass some results about capacity theory which are essential to achieve the results of present paper and somehow present in the literature, though they are spread in several different contexts and not easy to find as stated in this form: In particular Proposition A.1 and Eq (A.2) can be proven as like as Propositions 5.8.3 and 5.8.4 in [7] although the results seem slightly different.

Proposition A.1. *Let G an open bounded subset of $\mathbb{R}^N$. Then*

$$\operatorname{cap} G = \inf\left\{\|w\|^2_{H^1(\mathbb{R}^N)} : w \in C_0^\infty(\mathbb{R}^N),\ w \geq 1 \text{ on } G\right\}. \tag{A.1}$$

The above property can be generalized to every bounded subset of $\mathbb{R}^N$ by the following:

Proposition A.2. *Let E a bounded subset of $\mathbb{R}^N$. Then*

$$\begin{aligned}\operatorname{cap} E &= \inf\left\{\|w\|^2_{H^1(\mathbb{R}^N)} : w \in C_0^\infty(\mathbb{R}^N),\ w \geq 1 \text{ on a neighborhood of } E\right\}\\ &= \inf\left\{\|w\|^2_{H^1(\mathbb{R}^N)} : w \in C_0^\infty(\mathbb{R}^N;[0,1]),\ w \equiv 1 \text{ on a neighborhood of } E\right\}.\end{aligned} \tag{A.2}$$

We state and prove some results which play a crucial role in the proof of our main theorem.

In the sequel Ω will denote an open bounded subset of $\mathbb{R}^N$ with Lipschitz boundary and E will denote a subset of $\overline{\Omega}$ such that $\operatorname{cap} E > 0$.

Lemma A.3. *Let $u \in H^1(\Omega)$, $u \geq 0$ a.e. in Ω such that $u^*(\mathbf{x}) = 0$ for q.e. $\mathbf{x} \in D$, where D is a closed subset of $\overline{\Omega}$. Then there is an extension $v \in H^1(\mathbb{R}^N)$ of u such that* $\operatorname{spt} v$ *is compact, $v \geq 0$ a.e. in $\mathbb{R}^N$ and*

$$\lim_{r\to 0^+} \frac{1}{|B_r(\mathbf{x})|}\int_{B_r(\mathbf{x})} v(\boldsymbol{\xi})\,d\boldsymbol{\xi} = 0$$

for q.e. $\mathbf{x} \in D$.

Proof. We recall that u^* is defined as

$$u^*(\mathbf{x}) = \lim_{r\to 0^+} \frac{1}{|B_r(\mathbf{x})|}\int_{B_r(\mathbf{x})} w(\boldsymbol{\xi})\,d\boldsymbol{\xi} \tag{A.3}$$

for q.e. $\mathbf{x} \in \overline{\Omega}$, where w is any Sobolev extension of u. Therefore the claim follows easily by choosing a cut off function $\varphi \in C_0^\infty(\mathbb{R}^N)$, $\varphi \equiv 1$ on Ω and by setting $v := w^+\varphi$ which is a Sobolev extension of u with compact support since $v = u$ a.e. in Ω, and $\operatorname{spt} v \subset \operatorname{spt}\varphi$. □

Lemma A.4. *Let $u \in H^1(\mathbb{R}^N)$ with compact support such that $u \geq 0$ a.e. in $\mathbb{R}^N$ and $u^*(\mathbf{x}) = 0$ for q.e. $\mathbf{x} \in E$. Then there exists a sequence $u_j \in C^1(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$, $u_j \geq 0$ in $\mathbb{R}^N$ such that $u_j(\mathbf{x}) = 0$ for q.e. $\mathbf{x} \in E$ and $u_j \to u$ in $H^1(\mathbb{R}^3)$.*

Proof. For every $j \in \mathbb{N}$, $j \geq 1$ let $\overline{u}_j := \min\{u, j^{\frac{1}{4}}\} \in H^1(\mathbb{R}^N)$. By Theorem 3.11.6 and Remark 3.11.7 of [57], there exists $v_j \in C^1(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$ with $\operatorname{spt} v_j \subset \{\mathbf{x} : d(\mathbf{x}, \operatorname{spt}\overline{u}_j) \leq j^{-1}\}$ such that if $F_j := \{\mathbf{x} : v_j(\mathbf{x}) \neq \overline{u}_j(\mathbf{x})\}$ then

$$\operatorname{Cap} F_j < \frac{1}{j}, \quad \|v_j - \overline{u}_j\|_{H^1(\mathbb{R}^N)} < \frac{1}{j}, \tag{A.4}$$

so (2.8) entails

$$\operatorname{cap} F_j < \beta j^{-1}. \tag{A.5}$$

By recalling that $\operatorname{spt} \overline{u}_j$ is compact we get that F_j is bounded so, by taking account of (A.2), there exists $w_j \in C_0^\infty(\mathbb{R}^N; [0,1])$, $w_j \equiv 1$ in a neighbourhood U_j of F_j such that

$$\|w_j\|^2_{H^1(\mathbb{R}^N)} < \beta j^{-1}. \tag{A.6}$$

We define $u_j := (1 - w_j)v_j$: it is readily seen that $u_j \in C^1(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$, $u_j \geq 0$ in $\mathbb{R}^N$, that $u_j(\mathbf{x}) = 0$ for q.e. $\mathbf{x} \in E \cap U_j$ and that $u_j \equiv \overline{u}_j(1 - w_j)$ outside U_j, hence, by recalling that $\overline{u}_j{}^*(\mathbf{x}) = u^*(\mathbf{x}) = 0$ for q.e. $\mathbf{x} \in E$, we get $u_j(\mathbf{x}) = 0$ for q.e. $\mathbf{x} \in E \setminus U_j$ that is $u_j(\mathbf{x}) = 0$ for q.e. $\mathbf{x} \in E$. We claim that $u_j \to u$ in $H^1(\mathbb{R}^N)$: to this aim, by noticing that $u_j - u = v_j - \overline{u}_j + \overline{u}_j - u - v_j w_j$ and that $\overline{u}_j \to u$ in $H^1(\mathbb{R}^N)$, thanks to (A.4) we have only to show that $v_j w_j \to 0$ in $H^1(\mathbb{R}^N)$. We first notice that by setting

$$B_j := \{\mathbf{x} : |w_j(\mathbf{x})| \geq j^{-\frac{1}{4}}\},$$

then by (A.6) and Tchebichev inequality we get $|B_j| \leq \beta j^{-\frac{1}{2}}$, therefore

$$\begin{aligned}
\int_{\mathbb{R}^N} |w_j|^2 |v_j|^2 \, d\mathbf{x} &= \int_{B_j} |w_j|^2 |v_j|^2 \, d\mathbf{x} + \int_{\mathbb{R}^3 \setminus B_j} |w_j|^2 |v_j|^2 \, d\mathbf{x} \\
&\leq \int_{B_j} |v_j|^2 \, d\mathbf{x} + j^{-\frac{1}{2}} \int_{\mathbb{R}^N \setminus B_j} |v_j|^2 \, d\mathbf{x} \\
&\leq 2 \int_{B_j} |v_j - \overline{u}_j|^2 \, d\mathbf{x} + 2 \int_{B_j} |\overline{u}_j|^2 \, d\mathbf{x} \\
&\quad + 2 j^{-\frac{1}{2}} \int_{\mathbb{R}^N} |v_j - \overline{u}_j|^2 \, d\mathbf{x} + 2 j^{-\frac{1}{2}} \int_{\mathbb{R}^N} |\overline{u}_j|^2 \, d\mathbf{x} \to 0,
\end{aligned} \tag{A.7}$$

since

$$\|v_j - \overline{u}_j\|_{H^1(\mathbb{R}^N)} < \frac{1}{j}, \quad \overline{u}_j \to u \text{ in } H^1(\mathbb{R}^N), \quad |B_j| \to 0.$$

Analogously by recalling (A.6), that $w_j \equiv 1$ on U_j, that $v_j \equiv \overline{u}_j$ outside U_j and $\|\overline{u}_j\|^2_\infty \leq \sqrt{j}$, we get

$$\begin{aligned}
\int_{\mathbb{R}^N} |\nabla(w_j v_j)|^2 \, d\mathbf{x} &\leq 2 \int_{\mathbb{R}^N} |w_j|^2 |\nabla v_j|^2 \, d\mathbf{x} + 2 \int_{\mathbb{R}^N \setminus U_j} |v_j|^2 |\nabla w_j|^2 \, d\mathbf{x} \\
&\leq 2 \int_{\mathbb{R}^N} |w_j|^2 |\nabla v_j|^2 \, d\mathbf{x} + 2 \|\overline{u}_j\|^2_\infty \int_{\mathbb{R}^N \setminus U_j} |\nabla w_j|^2 \, d\mathbf{x} \\
&\leq 2 \int_{\mathbb{R}^N \setminus B_j} |w_j|^2 |\nabla v_j|^2 \, d\mathbf{x} + 2 \int_{B_j} |w_j|^2 |\nabla v_j|^2 \, d\mathbf{x} + 2 j^{-\frac{1}{2}} \to 0,
\end{aligned} \tag{A.8}$$

as in (A.7) thus proving the lemma. □

Lemma A.5. *Let $u \in H^1(\Omega)$ such that $u^*(\mathbf{x}) \geq 0$ for q.e. $\mathbf{x} \in E$. Then there exists a sequence $u_j \in C^1(\overline{\Omega})$ such that $u_j(\mathbf{x}) \geq 0$ for q.e. $\mathbf{x} \in E$ and $u_j \to u$ in $H^1(\Omega)$.*

Proof. We recall that by Remark 2.3 $u^*(\mathbf{x}) \geq 0$ for q.e. $\mathbf{x} \in E$ if and only if $(u^-)^*(\mathbf{x}) = 0$ for q.e. $\mathbf{x} \in E_{ess}$ and that E_{ess} is a closed subset of $\overline{\Omega}$. By Lemma A.3 there exists a Sobolev extension v of u^- such that spt v is compact, $v \geq 0$ a.e. in $\mathbb{R}^3$ and

$$\lim_{r\to 0^+} \frac{1}{|B_r(\mathbf{x})|} \int_{B_r(\mathbf{x})} v(\boldsymbol{\xi})\, d\boldsymbol{\xi} = 0$$

for q.e. $\mathbf{x} \in E_{ess}$, so by Lemma A.4, there exists a sequence $v_j \in C^1(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$, $v_j \geq 0$ in $\mathbb{R}^N$ such that $v_j(\mathbf{x}) = 0$ for q.e. $\mathbf{x} \in E_{ess}$ and $v_j \to v$ in $H^1(\mathbb{R}^N)$. Let now w be a Sobolev extension of u^+. We may assume without loss of generality that $w \geq 0$ a.e. in $\mathbb{R}^N$ and if ρ_j is a sequence of smooth mollifiers then $w_j := w * \rho_j \geq 0$ and $w_j \to w$ in $H^1(\mathbb{R}^N)$. Therefore by setting $u_j := w_j - v_j$, we have $u_j \in C^1(\overline{\Omega})$, $u_j(\mathbf{x}) \geq 0$ for q.e. $\mathbf{x} \in E$ and $u_j \to u$ in $H^1(\Omega)$ thus proving the lemma. □

Remark A.6. If E is a non empty subset of $\overline{\Omega}$ and $u \in H^1(\Omega)$ we say that $u \geq 0$ on E in the sense of $H^1(\Omega)$ if there exists a sequence $u_j \in C^1(\overline{\Omega})$ such that $u_j \geq 0$ on E and $u_j \to u$ in $H^1(\Omega)$ (according to [31, Definition 5.1]). We claim that $(u^-)^* = 0$ q.e. in E (or equivalently $u^* \geq 0$ q.e. in E) if and only if $u \geq 0$ on E in the sense of $H^1(\Omega)$: Indeed if $(u^-)_* = 0$ q.e. in E then Lemma A.5 provides a sequence $u_j \in C^1(\overline{\Omega})$ such that $u_j \to u$ in $H^1(\Omega)$, $u_j \geq 0$ on E, while the converse follows easily by recalling that if $u_j \to u$ in $H^1(\Omega)$ then, up to subsequences, $u_j(\mathbf{x}) \to u^*(\mathbf{x})$ for q.e. $\mathbf{x} \in E$.