https://www.aimspress.com/journal/mine

Mathematics in Engineering, 6(1): 155–172.
DOI:10.3934/mine.2024007
Received: 07 December 2023
Revised: 15 February 2024
Accepted: 18 February 2024
Published: 28 February 2024

Research article

Local well-posedness of 1D degenerate drift diffusion equation†

La-Su Mai* **and Suriguga**

School of Mathematics Sciences, Inner Mongolia University,Hohhot 010021, China

† **This contribution is part of the Special Issue:** Math aspects of classical and quantum fluid dynamics
Guest Editors: Paolo Antonelli; Luigi Forcella
Link: www.aimspress.com/mine/article/6556/special-articles

* **Correspondence:** Email: lasu_mai@163.com.

Abstract: This paper proves the well-posedness of locally smooth solutions to the free boundary value problem for the 1D degenerate drift diffusion equation. At the free boundary, the drift diffusion equation becomes a degenerate hyperbolic-Poisson coupled equation. We apply the Hardy's inequality and weighted Sobolev spaces to construct the appropriate a priori estimates, overcome the degeneracy of the system and successfully establish the existence of solutions in the Lagrangian coordinates.

Keywords: drift diffusion equation; free boundary; degeneracy; local well-posedness

1. Introduction

We consider the well-known drift diffusion equation which usually describes the motion of elections in the semiconductor device [21]:

$$\begin{cases} \rho_t - p_x(\rho) = (\rho\phi_x)_x, \\ \phi_{xx} = \rho - \mathcal{D}(x), \end{cases} \tag{1.1}$$

where ρ, ϕ represent the electron density and the electrostatic potential, respectively. The function $\mathcal{D}(x)$ is called the doping profile standing for the density of impurities in semiconductor device, which plays an important role for the existence of solution to the mathematical models of semiconductor [1, 11, 12, 14, 23, 24, 26, 27]. The pressure $p(\rho)$ is given by γ-law, namely,

$$p(\rho) = \rho^\gamma \text{ for } \gamma > 1. \tag{1.2}$$

From the point of view of mathematics, one of the main motivations for studying system (1.1) is to consider the relation between with the unipolar hydrodynamic semiconductor model [5, 6]. The main reason is that the unipolar hydrodynamic semiconductor model reduces to the system (1.1) as the relaxation time (parameter in the semiconductor model) goes to the zero, which calls the zero-relaxation limit. However, the system (1.1) is a coupled system of a parabolic equation and the Poisson equation, but the unipolar hydrodynamic semiconductor model is a coupled system of a hyperbolic equation and the Poisson equation. This property occurs the initial layer and makes the mathematical justification of the relaxation limit more complicated. There have been a lot of works to study the 1D zero-relaxation limit such as in [8, 13, 15, 20–23]. In particular, Marcati and Natalini [20] made a pioneering work in the field and provided new methods and perspectives for the relevant research field. To the best of our knowledge, beside we [19] proved the local well-posedness of smooth solution for the spherically symmetric drift diffusion equation, there is still a lack of research in this area. This paper aims to fill this gap by providing further analysis and investigation on this topic.

In this paper, we mainly discuss the case of $\gamma = 2$. Let us introduce the velocity of elections in the system (1.1) by:

$$v(x,t) = -\frac{\left(\rho^2\right)_x}{\rho} + \phi_x,$$

then the system (1.1) can be written as the hyperbolic-Poisson coupled form:

$$\begin{cases} \rho_t + (\rho v)_x = 0, \\ v(x,t) = -\dfrac{\left(\rho^2\right)_x}{\rho} + \phi_x, \\ \phi_{xx} = \rho - \mathcal{D}(x). \end{cases} \tag{1.3}$$

The aim of this paper is to study the well-posedness for the local smooth solutions to the free boundary value problem of the system (1.3) in $(x,t) \in (a(t), b(t)) \times (0,T)$ with the following free boundary and the initial conditions:

$$\begin{cases} \rho > 0, \text{ in } (a(t), b(t)), \\ \rho(a(t),t) = \rho(b(t),t) = \mathcal{D}(a(t)) = \mathcal{D}(b(t)) = 0, \\ \dfrac{da(t)}{dt} = v(a(t),t), \ \dfrac{db(t)}{dt} = v(b(t),t), \\ (\rho, v)(x,0) = (\rho_0(x), v_0(x)), \ x \in (a(0), b(0)) = (0,1), \\ 0 < |\dfrac{d}{dx}(\rho_0^2)| < \infty, \text{ at 0 and 1}, \end{cases} \tag{1.4}$$

The condition $(1.4)_2$ implies the electron density ρ occurs the vacuum on the free boundary which makes the system (1.3) being a degenerate system. The condition $(1.4)_4$ confirms that ρ_0 is equivalent to the distance function $d(x)$ of the boundary near $x = 0, 1$, which is called the *physical vacuum condition* for the compressible fluids (cf. [2–4, 9, 10, 17, 18]).

This paper is to investigate the well-posedenss of the local smooth solution for the free boundary value problem (1.3) and (1.4). Under the Lagrangian variable (2.1), the free boundary value problem (1.3) and (1.4) will be reduced to an equivalent system with the initial boundary value problem (2.7) and (2.11). The well-posedness of local smooth solutions to the problem (1.3) and (1.4) will be stated in Theorem 2.1.

Note that, due to the degeneracy of the system (1.3) on the moving boundary caused by the pressure term, the classical theory of Friedrich-Lax-Kato for quasilinear strictly hyperbolic system can not be directly applied to prove the existence of local smooth solutions. Due to the physical vacuum condition $(1.4)_5$, the initial data ρ_0 is equivalent to the distance function near the boundary. Thus, the initial data ρ_0 plays the role of weight in the weighted Sobolev embedding inequality (1.5), which is the connection between L^2-norm and the weighted Sobolev spaces. Especially, the initial data ρ_0 plays the basic weight in the coefficient of the Lagrangian form (2.7) of the system (1.3). This observation helps us to overcome the obstacle by using the Hardy's inequality in a certain weighted Sobolev space. Compared to the analysis in the previous studies [2–4, 9, 10, 16–18], the Lagrangian form of the compressible Euler equation is a degenerate quasilinear wave equation, but the Lagrangian form in (2.7) of the drift diffusion equation (1.3) is a degenerate quasilinear parabolic equation, which makes some essential difference between two systems.

The remaining sections of this paper are as follows. In Section 2, under the Lagrangian transformation, we transform the free boundary problem into an initial boundary value problem and provide the main theorems. In Section 3, we mainly establish energy estimates for higher-order temporal derivatives and higher-order spatial regularization elliptic derivative estimates. In Section 4, we prove the existence and uniqueness.

Notation and weighted Sobolev spaces

Let $H^k(0,1)$ denote the usual Sobolev spaces with the norm $\|\cdot\|_k$, especially $\|\cdot\|_0 = \|\cdot\|$. For real number l, the Sobolev spaces $H^l(0,1)$ and the norm $\|\cdot\|_l$ are defined by interpolation. The function space $L^\infty(0,1)$ is simplified by L^∞.

Let $d(x)$ be distance function to boundary $\Gamma = \{0,1\}$ as $d(x) = dist(x,\Gamma) = \min\{x, 1-x\}$ for $x \in \Gamma$. For any $a > 0$ and nonnegative b, the weighted Sobolev space $H^{a,b}$ is given by

$$H^{a,b} := \{d^{\frac{a}{2}} F \in L^2(0,1) : \int_0^1 d^a |\partial_x^k F|^2 dx \le \infty,\ 0 \le k \le b\}$$

with the norm

$$\|F\|^2_{H^{a,b}} := \sum_0^b \int_0^1 d^a |\partial_x^k F| dx.$$

Then, it holds the following embedding: $H^{a,b}(0,1) \hookrightarrow H^{b-a/2}(0,1)$, with the estimate $\|F\|_{b-a/2} \le C_0 \|F\|_{H^{a,b}}$. In particular, we have

$$\|F\|_0^2 \le C_0 \int_0^1 d(x)^2 \left(|F(x)|^2 + |F'(x)|^2\right) dx, \tag{1.5}$$

$$\|F\|_{1/2}^2 \le C_0 \int_0^1 d(x) \left(|F(x)|^2 + |F'(x)|^2\right) dx. \tag{1.6}$$

2. Initial boundary value problem

In order to transform the region $(0, R(t))$ into $(0,1)$, we define the Lagrangian variables $\eta(x,t)$ as:

$$\partial_t \eta(x,t) = \upsilon(\eta(x,t),t), \quad \eta(x,0) = x. \tag{2.1}$$

We also have:

$$\begin{aligned} u(x,t) &= \upsilon(\eta(x,t),t),\\ f(x,t) &= \rho(\eta(x,t),t),\\ \Phi(x,t) &= \phi(\eta(x,t),t). \end{aligned}$$

Then, the first equation of (1.3) is equivalent to

$$f = \frac{\rho_0(x)}{\eta_x}, \tag{2.2}$$

which in combination with (1.3) leads to

$$\rho_0 u + \left(\frac{{\rho_0}^2}{{\eta_x}^2}\right)_x = \rho_0\phi_\eta, \tag{2.3}$$

$$\phi_{\eta\eta} = \rho - \mathcal{D}(\eta). \tag{2.4}$$

We have form (2.4)

$$\phi_\eta(\eta) = \int_{a(t)}^{\eta} [\rho(y,t) - \mathcal{D}(y)]\, dy + M(t). \tag{2.5}$$

where $M(t)$ being function of t. Without loss of generality, we take $\phi_\eta(+\infty) = -\phi_\eta(+\infty)$ and obtain

$$\begin{aligned} \phi_\eta(+\infty) &= \frac{1}{2}\int_{-\infty}^{+\infty} [\rho(y,t) - \mathcal{D}(y)]\, dy,\\ \phi_\eta(-\infty) &= -\frac{1}{2}\int_{-\infty}^{+\infty} [\rho(y,t) - \mathcal{D}(y)]\, dy, \end{aligned} \tag{2.6}$$

and

$$M(t) = -\frac{1}{2}\int_{a(t)}^{+\infty} [\rho(y,t) - \mathcal{D}(y)]\, dy + \frac{1}{2}\int_{-\infty}^{a(t)} [\rho(y,t) - \mathcal{D}(y)]\, dy.$$

Due to $(1.4)_2$, it holds that $\rho(\eta,t) = \mathcal{D}(\eta) = 0$, when $\eta \le a(t)$ or $\eta \ge b(t)$, which implies

$$M(t) = -\frac{1}{2}\int_{a(t)}^{b(t)} [\rho(y,t) - \mathcal{D}(y)]\, dy.$$

By using (2.1), it follows that

$$\begin{aligned} \phi_\eta = \frac{\Phi_x}{\eta_x} &= \int_{a(t)}^{\eta} [\rho(y,t) - \mathcal{D}(y)]\, dy - \frac{1}{2}\int_{a(t)}^{b(t)} [\rho(y,t) - \mathcal{D}(y)]\, dy\\ &= \int_{\eta(0,t)}^{\eta(x,t)} [\rho(\eta,t) - \mathcal{D}(\eta)]\, dy - \frac{1}{2}\int_{\eta(0,t)}^{\eta(1,t)} [\rho(\eta,t) - \mathcal{D}(\eta)]\, dy\\ &= \int_0^x [f(y,t) - \mathcal{D}(\eta)]\, \eta_y dy - \frac{1}{2}\int_0^1 [f(y,t) - \mathcal{D}(\eta)]\, \eta_y dy\\ &= \int_0^x \rho_0 dy - \frac{1}{2}\int_0^1 \rho_0 dy - \frac{1}{2}\int_0^x \mathcal{D}(\eta)\eta_y dy + \frac{1}{2}\int_x^1 \mathcal{D}(\eta)\eta_y dy. \end{aligned}$$

Thus, we can rewrite (2.3) as

$$\rho_0 u + \left(\frac{{\rho_0}^2}{{\eta_x}^2}\right)_x = \rho_0 F, \tag{2.7}$$

where

$$F = \int_0^x \rho_0 dy - \frac{1}{2}\int_0^1 \rho_0 dy - \frac{1}{2}\int_0^x \mathcal{D}(\eta)\eta_y dy + \frac{1}{2}\int_x^1 \mathcal{D}(\eta)\eta_y dy. \tag{2.8}$$

Taking ∂_t over (2.7), we have

$$\rho_0 u_t - 2\left({\rho_0}^2 \eta_x^{-3} u_x\right)_x = \rho_0 G_t, \tag{2.9}$$

$$G = -\frac{1}{2}\int_0^x D(\eta)\eta_y dy + \frac{1}{2}\int_x^1 D(\eta)\eta_y dy. \tag{2.10}$$

The initial and boundary conditions (1.4) can be transformed to

$$\begin{cases} \rho_0 > 0, \ \text{in } (0,1), \\ \rho_0 = 0, \ \text{at } x = 0, 1, \\ (\eta, u)(x, 0) = (x, \upsilon_0(x)), \ x \in (0,1), \\ 0 < |\rho_0'(x)| < +\infty, \ \text{at } x = 0, 1. \end{cases} \tag{2.11}$$

Define the energy functional $E(t)$ by:

$$\begin{aligned} E(t) :=& \|u(t)\|^2_{H^2(0,1)} + \|\rho_0 u_{xx}\|^2_{H^1(0,1)} + \|\rho_0 u_{xxx}\|\|^2_0 \\ &+ \|u_t\|^2_{H^1(0,1)} + \| u_{xt}\|^2_0 + \|\rho_0 u_{xt}\|^2_{H^1(0,1)} + \|\rho_0 u_{xxt}\|^2_0 \\ &+ \| u_{tt}\|^2_0 + \|\rho_0 u_{xtt}\|^2_0, \end{aligned} \tag{2.12}$$

with the following compatibility conditions for $k = 1, 2$:

$$\partial_t^k u(x, 0) := \partial_t^{k-1}\left[F - \frac{1}{\rho_0}(\frac{\rho_0^2}{\eta_x^2})\right]|_{t=0}. \tag{2.13}$$

where F is given by (2.8). Throughout the whole paper, we denote P as a generic polynomial function of its argument and $P_0 = P(E(0))$.

We describe the main result of this paper as follows.

Theorem 2.1. *Let the initial data $\rho_0 \in C^2[0,1]$, the doping profile $\mathcal{D} \in C^3[0,1]$ satisfying* (2.11) *and* (2.13)*, and*

$$E(0) < +\infty \text{ and } \|\rho_0\|_{C^2[0,1]} + \|\mathcal{D}\|_{C^3[0,1]} \le M_0,$$

with M_0 being a positive constant. In addition, let the following degeneracy condition of the doping profile $\mathcal{D}$ satisfying

$$0 < \mathcal{D}(x) < d(x) \text{ near } 0 \text{ and } 1, \tag{2.14}$$

where $d(x)$ is a distance function. Then, there exists a positive constant $\overline{T}$ such that the problem (2.7) *and* (2.11) *has a unique smooth solution (η, u) in $[0,1] \times [0, \overline{T}]$ satisfying*

$$\sup_{t\in[0,\overline{T}]} E(t) \le 2P_0. \tag{2.15}$$

3. A priori estimates

In this section, we mainly establish the prior estimates. More preciously, Section 3.1 develops high-order time derivative estimates, while Section 3.2 establishes high-order spatial derivative estimates. For this purpose, we begin by assuming that there exist a smooth solution (η, u) to the problem (2.7) and (2.11) on $[0,1]\times[0,T]$ satisfying

$$\sup_{t\in[0,T]} \|u_x\|_{L^\infty} \le M_0, \tag{3.1}$$

for some constant $M_0 > 0$ determined later, which implies there is a small enough time $0 < \overline{T} < T < 1$ such that for any $(x,t) \in (0,t)\times(0,\overline{T}]$,

$$\frac{1}{2} \le \eta_x(x,t) \le \frac{3}{2}. \tag{3.2}$$

3.1. Energy estimates

This subsection mainly proves the high-order time derivative estimates of local smooth solutions for the initial boundary value problem (2.7) and (2.11).

Lemma 3.1. *Assume that* (3.1) *holds on* $[0,1]\times[0,T]$. *Then it holds that for* $t\in(0,T]$,

$$\int_0^t\int_0^1 \rho_0 u_{ttt}^2 dxds + \|\rho_0 u_{xtt}\|_0^2 \le E(0) + CtP\left(\sup_{0\le\tau\le t} E(\tau)\right). \tag{3.3}$$

Proof. From (2.9), we have

$$\rho_0 u_{ttt} - 2\left({\rho_0}^2\eta_x^{-3}\partial_t^2 u_x\right)_x = 2\left(\rho_0^2 \sum_{l=1}^{2} C_2^l \partial_t^l \eta_x^{-3}\partial_t^{2-l} u_x\right)_x + \rho_0 G_{ttt}. \tag{3.4}$$

Multiplying (3.4) by $\partial_t^3 u$ and integrating over $(0,t)\times(0,1)$ shows

$$\begin{aligned}
&\int_0^t\int_0^1 \rho_0 u_{ttt}^2 dxds + \int_0^1 \rho_0^2\eta_x^{-3}(\partial_t^2 u_x)^2 dx \\
&= 2\int_0^t\int_0^1 \left(\rho_0^2 \sum_{l=1}^{2} C_2^l \partial_t^l \eta_x^{-3}\partial_t^{2-l} u_x\right)_x \partial_t^3 u dxds + \int_0^t\int_0^1 \rho_0 G_{ttt}\partial_t^3 u dxds \\
&+ \int_0^1 {\rho_0}^2\eta_x^{-3}(\partial_t^2 u_x)^2(0) dx + \int_0^t\int_0^1 \rho_0^2\partial_t\eta_x^{-3}(\partial_t^2 u_x)^2 dxds.
\end{aligned} \tag{3.5}$$

The fourth term on the right side of (3.5) is estimated as follows:

$$\begin{aligned}
\left|\int_0^t\int_0^1 \rho_0^2\partial_t\eta_x^{-3}(\partial_t^2 u_x)^2 dxds\right| &\le C\int_0^t \|u_x\|_{L^\infty}\|\rho_0\partial_t^2 u_x\|_0^2 ds \\
&\le CtP\left(\sup_{0\le\tau\le t} E(\tau)\right).
\end{aligned}$$

By using the integrating by parts, the first term on the right side of (3.5) is

$$
\begin{aligned}
&2\int_0^t\int_0^1\left(\rho_0^2\sum_{l=1}^2 C_2^l\partial_t^l\eta_x^{-3}\partial_t^{2-l}u_x\right)_x\partial_t^3 udxds\\
=&-2\int_0^t\int_0^1\left(\rho_0^2\sum_{l=1}^2 C_2^l\partial_t^l\eta_x^{-3}\partial_t^{2-l}u_x\right)\partial_t^3 u_xdxds\\
=&-2\int_0^1\rho_0^2\sum_{l=1}^2 C_2^l\partial_t^l\eta_x^{-3}\partial_t^{2-l}u_x\partial_t^2u_xdx+2\int_0^1\rho_0^2\sum_{l=1}^2 C_2^l\partial_t^l\eta_x^{-3}\partial_t^{2-l}u_x\partial_t^2u_x(0)dx\\
&+2\int_0^t\int_0^1\left(\rho_0^2\sum_{l=1}^2 C_2^l\partial_t^l\eta_x^{-3}\partial_t^{2-l}u_x\right)_t\partial_t^2u_xdxds.
\end{aligned}\tag{3.6}
$$

The first term on the right side of (3.6) is

$$
\begin{aligned}
&-2\int_0^1\rho_0^2\sum_{l=1}^2 C_2^l\partial_t^l\eta_x^{-3}\partial_t^{2-l}u_x\partial_t^2u_xdx\\
&=-4\int_0^1\rho_0^2\partial_t\eta_x^{-3}\partial_tu_x\partial_t^2u_xdx-2\int_0^1\rho_0^2\partial_t^2\eta_x^{-3}u_x\partial_t^2u_xdx.
\end{aligned}\tag{3.7}
$$

The first term on the right side of (3.7) can be controlled as

$$
\begin{aligned}
&\left|-4\int_0^1\rho_0^2\partial_t\eta_x^{-3}\partial_tu_x\partial_t^2u_xdx\right|\\
=&\left|12\int_0^1\rho_0^2\eta_x^{-4}u_x\partial_tu_x\partial_t^2u_xdx\right|\\
\le&C(\varepsilon)\|u_x(0)\|_{L^\infty}^2\int_0^1\rho_0^2[u_{xt}(0)+\int_0^t\partial_t^2u_xd\tau]^2dx\\
&+C(\varepsilon)\|\rho_0u_{xt}\|_{L^\infty}^2\int_0^t\|u_{xt}\|_0^2d\tau+\varepsilon\int_0^1\rho_0^2(\partial_t^2u_x)^2dx\\
\le&P(0)+CtP\left(\sup_{0\le\tau\le t}E(\tau)\right),
\end{aligned}\tag{3.8}
$$

with ε being a positive constant. The second term on the right side of (3.7) is

$$
-2\int_0^1\rho_0^2\partial_t^2\eta_x^{-3}u_x\partial_t^2u_xdx=-24\int_0^1\rho_0^2\eta_x^{-5}u_x^3\partial_t^2u_xdx+6\int_0^1\rho_0^2\eta_x^{-4}u_{xt}u_x\partial_t^2u_xdx.\tag{3.9}
$$

Due to the Cauchy inequality, the fundamental theorem of calculus shows that the first term on the right side of (3.9) can be estimated for any positive constant ε,

$$
\begin{aligned}
&\left|-24\int_0^1\rho_0^2\eta_x^{-5}u_x^3\partial_t^2u_xdx\right|\\
\le&C(\varepsilon)\int_0^1\rho_0^2u_x^6(0)dx+C(\varepsilon)\int_0^1\rho_0^2u_x^4(0)\left(\int_0^t u_{xt}d\tau\right)^2dx
\end{aligned}
$$

$$
\begin{aligned}
&\le P(0) + C(\varepsilon)\|\rho_0 u_x^2(0)\|_{L^\infty}^2 \int_0^t \|u_{xt}\|_0^2 d\tau \\
&\quad + C(\varepsilon)\|\rho_0 u_x(0)\|_{L^\infty}^2 \|u_x\|_{L^\infty}^2 \int_0^t \|u_{xt}\|_0^2 d\tau + C(\varepsilon)\|u_x\|_{L^\infty}^4 \int_0^t \|u_{xt}\|_0^2 d\tau \\
&\quad + C(\varepsilon)\int_0^1 \rho_0^2 \left(\int_0^t u_{xt} d\tau\right)^2 {u_x}^4 dx + \varepsilon \int_0^1 \rho_0^2 (\partial_t^2 u_x)^2 dx \\
&\le P(0) + CtP\left(\sup_{0\le\tau\le t} E(\tau)\right). \qquad (3.10)
\end{aligned}
$$

Similarly, the second term on the right side of (3.9) is

$$
\begin{aligned}
&\left| 6\int_0^1 \rho_0^2 \eta_x^{-4} u_{xt} u_x \partial_t^2 u_x dx \right| \\
&\le C(\varepsilon)\int_0^1 \rho_0^2 {u_x}^2 (\partial_t u_x)^2 dx + \varepsilon \int_0^1 \rho_0^2 (\partial_t^2 u_x)^2 dx \\
&\le C(\varepsilon)\|u_x(0)\|_{L^\infty}^2 \int_0^1 \rho_0^2 [u_{xt}(0) + \int_0^t \partial_t^2 u_x d\tau]^2 dx \\
&\quad + C(\varepsilon)\|\rho_0 u_{xt}\|_{L^\infty}^2 \int_0^t \|u_{xt}\|_0^2 d\tau + \varepsilon \int_0^1 \rho_0^2 (\partial_t^2 u_x)^2 dx \\
&\le P(0) + CtP\left(\sup_{0\le\tau\le t} E(\tau)\right). \qquad (3.11)
\end{aligned}
$$

The last term on the right side of (3.6) is

$$
\begin{aligned}
&2\int_0^t \int_0^1 \left(\rho_0^2 \sum_1^2 C_2^l \partial_t^l \eta_x^{-3} \partial_t^{2-l} u_x\right)_t \partial_t^2 u_x dx ds \\
=&6\int_0^t \int_0^1 \rho_0^2 \partial_t^2 \eta_x^{-3} \partial_t u_x \partial_t^2 u_x dx ds \\
&+ 4\int_0^t \int_0^1 \rho_0^2 \partial_t \eta_x^{-3} \partial_t^2 u_x \partial_t^2 u_x dx ds + 2\int_0^t \int_0^1 \rho_0^2 \partial_t^3 \eta_x^{-3} u_x \partial_t^2 u_x dx ds. \qquad (3.12)
\end{aligned}
$$

We only estimate the last term on the right side of (3.12), while the other terms can be controlled similarly, as

$$
\begin{aligned}
&\left| 2\int_0^t \int_0^1 \rho_0^2 \partial_t^3 \eta_x^{-3} u_x \partial_t^2 u_x dx ds \right| \\
\le& C\int_0^t (\|u_x\|_{L^\infty}^8 + \|\rho_0 \partial_t^2 u_x\|_0^2) ds + C\int_0^t (\|u_x\|_{L^\infty}^4 \|\rho_0 u_{xt}\|_0^2 + \|\rho_0 \partial_t^2 u_x\|_0^2) ds \\
&+ C\|u_x\|_{L^\infty} \int_0^t \|\rho_0 \partial_t^2 u_x\|_0^2 ds \\
\le& CtP\left(\sup_{0\le\tau\le t} E(\tau)\right). \qquad (3.13)
\end{aligned}
$$

By (2.10), a divert computation shows

$$|G_{ttt}| \leq C(\|u\|_{L^\infty}^3 + \|u\|_{L^\infty}\|u_t\|_{L^\infty} + \|u\|_{L^\infty}^2 + \|u_{tt}\|_{L^2} \\ + \|u_x\|_{L^\infty}\|u_t\|_{L^\infty} + \|u\|_{L^\infty}\|u_{xt}\|_{L^2} + \|u_{xtt}\|_{L^2}), \tag{3.14}$$

where C is a positive constant depending on $\|\mathcal{D}\|_{C^3[0,1]}$. Then, we have the estimate of the second term on the right side of (3.5) for any positive constant ε,

$$\begin{aligned}&\left|\int_0^t \int_0^1 \rho_0 G_{ttt}\partial_t^3 u dxds\right| \\ \leq & C(\varepsilon)\int_0^t \int_0^1 \rho_0^2 G_{ttt}^2 dxds + \varepsilon\int_0^t \int_0^1 (\partial_t^3 u)^2 dxds \\ \leq & C(\varepsilon)\int_0^t \|\rho_0 G_{ttt}\|_0^2 ds + \varepsilon\int_0^t \int_0^1 (\partial_t^3 u)^2 dxds \\ \leq & CtP\left(\sup_{0\leq\tau\leq t} E(\tau)\right).\end{aligned} \tag{3.15}$$

Substituting (3.6)–(3.15) into (3.5) obtain (3.3). This is the end of proof. □

3.2. Elliptic type estimates

The primary focus of this subsection is to establish the high-order spatial derivative estimates in (3.16) for the local smooth solution of the problem (2.7) and (2.11) on the interval $[0,1]\times[0,T]$, assuming (3.1).

Lemma 3.2. *Assume that* (3.1) *holds on* $[0,1]\times[0,T]$. *Then it holds that for* $t\in(0,T]$,

$$\|(u_x, \rho_0 u_{xx}, u_{xt}, \rho_0 u_{xxt}, \rho_0 u_{xxx})\|_0^2 \\ \leq E(0) + CtP\left(\sup_{0\leq\tau\leq t} E(\tau)\right). \tag{3.16}$$

Proof. We divide our proof into the following three steps.

Step1. Estimate of $\|(\rho_0 u_{xx}, u_x)\|_0^2$.

We can rewrite (2.9) as

$$\rho_0 u_{xx} + 2\rho_{0x}u_x = \frac{1}{2}u_t - \rho_0(\eta_x^{-3}-1)u_{xx} - 2\rho_{0x}(\eta_x^{-3}-1)u_x \\ + 3\rho_0\eta_x^{-4}u_x\eta_{xx} - \frac{1}{2}G_t. \tag{3.17}$$

Taking L^2-norm, we have

$$\|\rho_0 u_{xx} + 2\rho_{0x}u_x\|_0^2 \leq \|\frac{1}{2}u_t\|_0^2 + \|\rho_0(\eta_x^{-3}-1)u_{xx}\|_0^2 + \|2\rho_{0x}(\eta_x^{-3}-1)u_x\|_0^2 \\ + \|3\rho_0\eta_x^{-4}u_x\eta_{xx}\|_0^2 + \|\frac{1}{2}G_t\|_0^2. \tag{3.18}$$

The left-hand side of (3.18) is estimated as follows

$$\begin{aligned}
&\|\rho_0 u_{xx} + 2\rho_{0x} u_x\|_0^2 \\
=&\|\rho_0 u_{xx}\|_0^2 + 2\|\rho_{0x} u_x\|_0^2 - 2\int_0^1 \rho_0\rho_{0xx} u_x^2 dx \\
\geq&\|\rho_0 u_{xx}\|_0^2 + 2\|\rho_{0x} u_x\|_0^2 - P(0) - CtP\left(\sup_{0\leq\tau\leq t} E(\tau)\right),
\end{aligned} \tag{3.19}$$

where we have used

$$\begin{aligned}
2\int_0^1 \rho_0\rho_{0xx} u_x^2 dx &= 2\int_0^1 \rho_0\rho_{0xx}(u_x(0) + \int_0^t u_{xt} d\tau)^2 dx \\
&\leq P(0) + \|\rho_0\rho_{0xx}\|_{L^\infty} \int_0^t \|u_{xt}\|_0^2 d\tau \\
&\leq P(0) + CtP\left(\sup_{0\leq\tau\leq t} E(\tau)\right).
\end{aligned} \tag{3.20}$$

The first term on the right side of (3.18) is estimated as follows:

$$\|\frac{1}{2} u_t\|_0^2 \leq C\|u_t\|_0^2 = C\|u_t(0) + \int_0^t u_{tt} d\tau\|_0^2 \leq P(0) + CtP\left(\sup_{0\leq\tau\leq t} E(\tau)\right). \tag{3.21}$$

Similarly, we have the estimates of the other terms on the ride side of (3.18) as

$$\|\rho_0(\eta_x^{-3} - 1)u_{xx}\|_0^2 \leq C\int_0^t \|u_x\|_{L^\infty}^2 d\tau \|\rho_0 u_{xx}\|_0^2 \leq CtP\left(\sup_{0\leq\tau\leq t} E(\tau)\right),$$

$$\|2\rho_{0x}(\eta_x^{-3} - 1)u_x\|_0^2 \leq C\int_0^t \|u_x\|_{L^\infty}^2 d\tau \|u_x\|_0^2 \leq CtP\left(\sup_{0\leq\tau\leq t} E(\tau)\right),$$

and

$$\|3\rho_0\eta_x^{-4} u_x \eta_{xx}\|_0^2 \leq C\int_0^t \|\rho_0 u_{xx}\|_0^2 d\tau \|u_x\|_{L^\infty}^2 \leq CtP\left(\sup_{0\leq\tau\leq t} E(\tau)\right).$$

Finally, we have for the last term on the ride side of (3.18)

$$\begin{aligned}
\|\frac{1}{2} G_t\|_0^2 &\leq C\|G_t\|_0^2 \leq C\|u\|_0^2 + C\|u_x\|_0^2 \\
&= C\|u(0) + \int_0^t u_t d\tau\|_0^2 + C\|u_x(0) + \int_0^t u_{xt} d\tau\|_0^2 \\
&\leq P(0) + CtP\left(\sup_{0\leq\tau\leq t} E(\tau)\right).
\end{aligned} \tag{3.22}$$

From (3.18)–(3.22), we have

$$\|\rho_0 u_{xx}\|_0^2 + \|u_x\|_0^2 \leq \|P(0) + CtP\left(\sup_{0\leq\tau\leq t} E(\tau)\right). \tag{3.23}$$

Step 2. Estimate of $\| (\rho_0 u_{xxt}, u_{xt}) \|_0^2$.

Taking ∂_t over (3.17), we have

$$\begin{aligned}\rho_0 u_{xxt} + 2\rho_{0x} u_{xt} =& \frac{1}{2}u_{tt} + 6\rho_0\eta_x^{-4}u_x u_{xx} - \rho_0(\eta_x^{-3}-1)u_{xxt}\\ &+ 6\rho_{0x}\eta_x^{-4}u_x^2 - 2\rho_{0x}(\eta_x^{-3}-1)u_{xt}\\ &- 12\rho_0\eta_x^{-5}u_x^2\eta_{xx} + 3\rho_0\eta_x^{-4}u_{xt}\eta_{xx} - \frac{1}{2}G_{tt}.\end{aligned} \tag{3.24}$$

Taking L^2-norm, we have

$$\begin{aligned}\|\rho_0 u_{xxt} + 2\rho_{0x} u_{xt}\|_0^2 \le& \|\frac{1}{2}u_{tt}\|_0^2 + \|6\rho_0\eta_x^{-4}u_x u_{xx}\|_0^2 + \|\rho_0(\eta_x^{-3}-1)u_{xxt}\|_0^2\\ &+ \|6\rho_{0x}\eta_x^{-4}u_x^2\|_0^2 + \|2\rho_{0x}(\eta_x^{-3}-1)u_{xt}\|_0^2\\ &+ \|12\rho_0\eta_x^{-5}u_x^2\eta_{xx}\|_0^2 + \|3\rho_0\eta_x^{-4}u_{xt}\eta_{xx}\|_0^2 + \|\frac{1}{2}G_{tt}\|_0^2.\end{aligned} \tag{3.25}$$

The first term on the right side of (3.25) is estimated as follows:

$$\begin{aligned}\|\frac{1}{2}u_{tt}\|_0^2 \le C\|u_{tt}\|_0^2 &\le C\int_0^1 \rho_0^2\left(u_{tt}^2 + u_{xtt}^2\right)dx\\ &\le P(0) + C\int_0^t \|\rho_0 u_{ttt}\|_0^2 d\tau + C\int_0^1 \rho_0^2 u_{xtt}^2 dx\\ &\le P(0) + CtP\left(\sup_{0\le\tau\le t} E(\tau)\right).\end{aligned} \tag{3.26}$$

The second term on the right side of (3.25) is

$$\begin{aligned}&\|6\rho_0\eta_x^{-4}u_x u_{xx}\|_0^2\\ \le& C\|\rho_0 u_x(0)\left(u_{xx}(0) + \int_0^t u_{xxt}d\tau\right)\|_0^2 + C\|\rho_0\int_0^t u_{xt}d\tau u_{xx}\|_0^2\\ \le& P(0) + CtP\left(\sup_{0\le\tau\le t} E(\tau)\right).\end{aligned} \tag{3.27}$$

Similarly, we have

$$\begin{aligned}\|\rho_0\left(\eta_x^{-3}-1\right)u_{xxt}\|_0^2 \le& C\|\rho_0\int_0^t u_x d\tau u_{xxt}\|_0^2\\ \le& C\|\rho_0 u_{xxt}\|_0^2\int_0^t \|u_x\|_{L^\infty}^2 d\tau\\ \le& CtP\left(\sup_{0\le\tau\le t} E(\tau)\right),\end{aligned}$$

and

$$
\begin{aligned}
&\|6\rho_{0x}\eta_x^{-4}u_x^2\|_0^2\\
\leq &C\|u_x(0)\|_{L^\infty}^4 + C\|u_x(0)\|_{L^\infty}^2\int_0^t \|u_{xt}\|_0^2 d\tau + C\|u_x\|_{L^\infty}^2\int_0^t \|u_{xt}\|_0^2 d\tau\\
\leq &P(0) + CtP\left(\sup_{0\leq\tau\leq t} E(\tau)\right).
\end{aligned}
$$

We turn to estimate the seventh term on the ride side of (3.25) as

$$
\|\frac{1}{2}G_{tt}\|_0^2 \leq C\left(\|u\|_{L^\infty}^2 + \|u_t\|_{L^2} + \|u\|_{L^\infty}\|u_x\|_{L^2} + \|u_{xt}\|_{L^2}\right) \leq P(0) + CtP\left(\sup_{0\leq\tau\leq t} E(\tau)\right).
$$

We can also estimate the other terms on the right side of (3.25) and obtain similar to (3.23)

$$
\|\rho_0 u_{xxt}\|_0^2 + \|u_{xt}\|_0^2 \leq P(0) + CtP\left(\sup_{0\leq\tau\leq t} E(\tau)\right). \tag{3.28}
$$

Step 3. Estimate of $\|(\rho_0 u_{xxx}, u_{xx})\|_0^2$.

Taking ∂_x over (3.17), we have

$$
\begin{aligned}
&\rho_0 u_{xxx} + 3\rho_{0x}u_{xx}\\
=&\frac{1}{2}u_{xt} - 3\rho_{0x}\left(\eta_x^{-3} - 1\right)u_{xx} + 3\rho_0\eta_x^{-4}\eta_{xx}u_{xx}\\
&-\rho_0\left(\eta_x^{-3} - 1\right)u_{xxx} - 2\rho_{0xx}\left(\eta_x^{-3} - 2\right)u_x + 9\rho_{0x}\eta_x^{-4}\eta_{xx}u_x\\
&-12\rho_0\eta_x^{-5}\eta_{xx}^2u_x + 3\rho_0\eta_x^{-4}\eta_{xxx}u_x - \frac{1}{2}G_{tx}.
\end{aligned} \tag{3.29}
$$

Taking L^2-norm, we have

$$
\begin{aligned}
&\|\rho_0 u_{xxx} + 3\rho_{0x}u_{xx}\|_0^2\\
\leq &\|\frac{1}{2}u_{xt}\|_0^2 + \|3\rho_{0x}\left(\eta_x^{-3} - 1\right)u_{xx}\|_0^2 + \|3\rho_0\eta_x^{-4}\eta_{xx}u_{xx}\|_0^2\\
&+\|\rho_0\left(\eta_x^{-3} - 1\right)u_{xx}\|_0^2 + \|2\rho_{0xx}\left(\eta_x^{-3} - 2\right)u_x\|_0^2 + \|3\rho_{0x}\eta_x^{-4}\eta_{xx}u_x\|_0^2\\
&+\|12\rho_0\eta_x^{-5}\eta_{xx}^2u_x\|_0^2 + \|3\rho_0\eta_x^{-4}\eta_{xxx}u_x\|_0^2 + \|\frac{1}{2}G_{tx}\|_0^2.
\end{aligned} \tag{3.30}
$$

The estimate for the third term on the right-hand side of (3.30) is given by

$$
\begin{aligned}
&\|3\rho_0\eta_x^{-4}\eta_{xx}u_{xx}\|_0^2\\
\leq &C\|\rho_0\int_0^t u_{xx}d\tau u_{xx}\|_0^2\\
\leq &C\|u_{xx}\|_0^2\int_0^t \|\rho_0 u_{xx}\|_{L^\infty}^2 d\tau\\
\leq &CtP\left(\sup_{0\leq\tau\leq t} E(\tau)\right).
\end{aligned}
$$

We make the following procedure for the seventh term on the right-hand side of (3.30)

$$\begin{aligned}\|12\rho_0\eta_x^{-5}\eta_{xx}^2u_x\|_0^2 &\le C\|\rho_0\eta_{xx}^2u_x\|_0^2 \le C\|\rho_0\left(\int_0^t u_{xx}d\tau\right)^2 u_x\|_0^2\\&\le C\|u_x\|_{L^\infty}^2\int_0^t\|\rho_0u_{xx}\|_{L^\infty}^2d\tau\int_0^t\|u_{xx}\|_0^2d\tau\\&\le CtP\left(\sup_{0\le\tau\le t}E(\tau)\right).\end{aligned}$$

Considering the eighth term on the right-hand side of (3.30), we can obtain the following estimate

$$\begin{aligned}&\|3\rho_0\eta_x^{-4}\eta_{xxx}u_x\|_0^2\\\le&C\|\rho_0\int_0^t u_{xxx}d\tau u_x\|_0^2\\\le&C\|u_x\|_{L^\infty}^2\int_0^t\|\rho_0u_{xxx}\|_0^2d\tau\\\le&CtP\left(\sup_{0\le\tau\le t}E(\tau)\right).\end{aligned}$$

We can control the right-hand side of (3.30) by a similar estimate to (3.28), and obtain

$$\|\rho_0u_{xxx}\|_0^2+\|u_{xx}\|_0^2\le P(0)+CtP\left(\sup_{0\le\tau\le t}E(\tau)\right). \tag{3.31}$$

Finally, we have (3.16) from (3.23), (3.28) and (3.31). □

4. Well-posedness of local smooth solutions

By (1.5), (1.6), (3.3), (3.16) and the fundamental theorem of calculous, we can get

$$E(t)\le P_0+CtP\left(\sup_{0\le\tau\le t}E(\tau)\right), \tag{4.1}$$

which implies (2.15), where we have used a polynomial-type inequality introduced in [2]. Based on the a priori estimate in (3.1), this subsection is contributed to prove the existence of local smooth solutions for the problem (2.7) and (2.11) on $[0,1]\times[0,T]$ by the similar method in [7] by using the fixed point theorem. We omit the detailed proof here.

We describe the uniqueness of smooth solutions in the following Lemma 4.1.

Lemma 4.1. *Assume that* (η,u) *is a solution to the problem* (2.7) *and* (2.11) *corresponding to the initial data* (ρ_0,u_0) *satisfying* (2.15) *and*

$$\eta=x_0+\int_0^t ud\tau. \tag{4.2}$$

Then, there exists a positive time $0<\tilde{T}<T$ *such that for any* $[0,1]\times[0,\widetilde{T}]$, *the solution* (η,u) *is unique.*

Proof. Set

$$\eta_1 = x + \int_0^t u_1 d\tau,\ \eta_2 = x + \int_0^t u_2 d\tau,$$
$$R = \eta_1 - \eta_2,\ R_t = U = u_1 - u_2. \tag{4.3}$$

Substituting (4.3) into (2.7) and subtracting the resulting equations, we write the resulting equation as

$$\begin{aligned}&\rho_0(u_1 - u_2) + \left(\frac{\rho_0^2}{\eta_{1x}^2} - \frac{\rho_0^2}{\eta_{2x}^2}\right)_x\\ =&\frac{1}{2}\rho_0\int_x^1 [\mathcal{D}(\eta_1)\eta_{1y} - \mathcal{D}(\eta_2)\eta_{2y}]dy\\ &-\frac{1}{2}\rho_0\int_0^x [\mathcal{D}(\eta_1)\eta_{1y} - \mathcal{D}(\eta_2)\eta_{2y}]dy.\end{aligned} \tag{4.4}$$

By a straightforward calculation, we can obtain

$$\begin{aligned}\rho_0 U - \left(\rho_0^2 R_x G_1\right)_x =&\frac{1}{2}\rho_0\int_x^1 [\mathcal{D}(\eta_1)R_y + G_2 R]dy\\ &-\frac{1}{2}\rho_0\int_0^x [\mathcal{D}(\eta_2)R_y + G_3 R]dy,\end{aligned} \tag{4.5}$$

where

$$\begin{aligned}G_1 &= \frac{\eta_{1x} + \eta_{2x}}{\eta_{1x}^2\eta_{2x}^2},\\ G_2 &= \eta_{2x}\int_0^1 \mathcal{D}_\eta[\eta_2 + \mu(\eta_1 - \eta_2)]d\mu,\\ G_3 &= \eta_{1x}\int_0^1 \mathcal{D}_\eta[\eta_2 + \mu(\eta_1 - \eta_2)]d\mu.\end{aligned}$$

Due to (2.15), there exists a positive constant K_0 such that

$$\begin{aligned}&\|\eta_x\|_{L^\infty} + \|u_x\|_{L^\infty} + \|\mathcal{D}_\eta\|_{L^\infty} \le K_0,\quad \mathcal{D}(\eta) \le C\rho_0,\\ &\sum_{i=1}^3 \|G_i\|_{L^\infty} \le C(K_0),\quad \|\partial_t G_1\|_{L^\infty} \le C(K_0).\end{aligned} \tag{4.6}$$

Multiplying (4.5) by R, integrating the resultant equation over $(0,t)\times(0,1)$, then the integration by parts implies

$$\begin{aligned}&\frac{1}{2}\int_0^1 \rho_0 R^2 dx + \int_0^t\int_0^1 \rho_0^2 R_x^2 G_1 dxds\\ =&\frac{1}{2}\int_0^t\int_0^1 \rho_0\int_x^1 \left[\mathcal{D}(\eta_1)R_y + G_2 R\right] dyRdxds\\ &-\frac{1}{2}\int_0^t\int_0^1 \rho_0\int_0^x \left[\mathcal{D}(\eta_2)R_y + G_3 R\right] dyRdxds.\end{aligned} \tag{4.7}$$

From (2.14), we have

$$
\begin{aligned}
&\left|\frac{1}{2}\int_0^t\int_0^1\rho_0\int_x^1[\mathcal{D}(\eta_1)R_y+G_2R]dyRdxds\right|\\
\le& C\int_0^t\int_0^1\rho_0R^2dxds+C\int_0^t\int_0^1\rho_0\left\{\int_x^1[\mathcal{D}(\eta_1)R_y+G_2R]dy\right\}^2dxds\\
\le& C\int_0^t\|\rho_0^{\frac{1}{2}}R\|_0^2ds+C\int_0^t\left(\|\rho_0R_x\|_0^2+\|\rho_0^{\frac{1}{2}}R\|_0^2\right)ds\\
\le& C\int_0^t\left(\|\rho_0R_x\|_0^2+\|\rho_0^{\frac{1}{2}}R\|_0^2\right)ds. \qquad (4.8)
\end{aligned}
$$

Similarly, the second term on the ride side of (4.7) can be controlled by

$$
C\int_0^t\left(\|\rho_0R_x\|_0^2+\|\rho_0^{\frac{1}{2}}R\|_0^2\right)ds.
$$

Thus,

$$
\frac{1}{2}\int_0^1\rho_0R^2dxds+\int_0^t\int_0^1\rho_0^2R_x^2G_1dxds\le C\int_0^t\left(\|\rho_0R_x\|_0^2+\|\rho_0^{\frac{1}{2}}R\|_0^2\right)ds. \qquad (4.9)
$$

Multiplying (4.5) by $U$and integration over $(0,t)\times(0,1)$, we have similar to (4.7)

$$
\begin{aligned}
&\int_0^t\int_0^1\rho_0U^2dxds+\frac{1}{2}\int_0^1\rho_0^2R_x^2G_1dx\\
=&\frac{1}{2}\int_0^t\int_0^1\rho_0\int_x^1[\mathcal{D}(\eta_1)R_y+G_2R]dyUdxds\\
&-\frac{1}{2}\int_0^t\int_0^1\rho_0\int_0^x[\mathcal{D}(\eta_2)R_y+G_3R]dyUdxds+\frac{1}{2}\int_0^t\int_0^1\rho_0^2R_x^2G_1dxds. \qquad (4.10)
\end{aligned}
$$

Similar to (4.8), it follows that

$$
\begin{aligned}
&\left|\frac{1}{2}\int_0^t\int_0^1\rho_0\int_x^1[\mathcal{D}(\eta_1)R_y+G_2R]dyUdxds\right|\\
\le&\varepsilon\int_0^t\int_0^1\rho_0U^2dxds+C\int_0^t\int_0^1\rho_0\left\{\int_x^1[\mathcal{D}(\eta_1)R_y+G_2R]dy\right\}^2dxds\\
\le&\varepsilon\int_0^t\int_0^1\rho_0U^2dxds+C\int_0^t\|\rho_0R_x\|_0^2+\left(\|\rho_0^{\frac{1}{2}}R\|_0^2\right)ds.
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
&\int_0^t\int_0^1\rho_0U^2dxds+\frac{1}{2}\int_0^1\rho_0^2R_x^2G_1dx\\
\le&\varepsilon\int_0^t\int_0^1\rho_0U^2dxds+C\int_0^t\|\rho_0R_x\|_0^2ds+\|\rho_0^{\frac{1}{2}}R\|_0^2. \qquad (4.11)
\end{aligned}
$$

From (4.9) and (4.11), we obtain

$$\begin{aligned}&\int_0^t\int_0^1 \rho_0 U^2 dxds + \int_0^t\int_0^1 \rho_0^2 R_x^2 G_1 dxds\\&+\frac{1}{2}\int_0^1 \rho_0 R^2 dx + \frac{1}{2}\int_0^1 \rho_0^2 R_x^2 G_1 dx\\ \le& C(K_0)\int_0^t \left(\|\rho_0 R_x\|_0^2 + \|\rho_0^{\frac{1}{2}} R\|_0^2\right) ds.\end{aligned}$$

By applying the Gronwall inequality, it holds that

$$\int_0^1 [\rho_0(\eta_1-\eta_2)^2 + \rho_0^2(\eta_{1x}-\eta_{2x})^2]dx \le 0,$$

which gives

$$\eta_1 = \eta_2 \text{ and } u_1 = u_2.$$

□

5. Conclusions

In this paper, we have obtained the well-posedness of local smooth solutions to the free boundary value problem in a one-dimensional degenerate drift-diffusion model, which becomes a degenerate hyperbolic-Poisson coupled equation at the free boundary. We have applied the Hardy's inequality and the the weighted Sobolev spaces to construct the appropriate a priori estimates, and establish the existence of solutions in the Lagrangian coordinates. Our result and the methods are new for the drift diffusion equation. In future research, we will continue to improve the method and study the related topics on the free boundary value problems to the drift diffusion equations, mainly including the well-posedness and the large time behaviors to the local and global smooth solutions for the one-dimensional, spherically symmetric, cylindrical symmetric and the three dimensional cases.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

We would like to express our sincere thanks to the referee for the valuable and helpful suggestions and comments, which made a significant change of the paper. The research was partially supported by the National Natural Science Foundation of China (No. 11601246, and No. 11971014), Young science and technology talents cultivation project of Inner Mongolia University (No. 21221505), Research and Educational Reform Project for Graduate Education of Inner Mongolia Autonomous Region in 2023 (No. JGCG2023007) and Outstanding Youth fund of Inner Mongolia Natural Science Foundation (No. 2023JQ13).

Conflict of interest

The authors declare no conflicts of interest.

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