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Research article

Nonlinear nonlocal equations involving subcritical or power nonlinearities and measure data

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Abstract: Let $s \in (0,1)$, $1 < p < \frac{N}{s}$ and $\Omega \subset \mathbb{R}^N$ be an open bounded set. In this work we study the existence of solutions to problems $(E_\pm)$ $Lu \pm g(u) = \mu$ and $u = 0$ a.e. in $\mathbb{R}^N \setminus \Omega$, where $g \in C(\mathbb{R})$ is a nondecreasing function, μ is a bounded Radon measure on Ω and L is an integro-differential operator with order of differentiability $s \in (0,1)$ and summability $p \in (1, \frac{N}{s})$. More precisely, L is a fractional p-Laplace type operator. We establish sufficient conditions for the solvability of problems $(E_\pm)$. In the particular case $g(t) = |t|^{\kappa-1}t$; $\kappa > p-1$, these conditions are expressed in terms of Bessel capacities.

Keywords: fractional p-Laplace operator; critical exponents; Bessel capacities; Wolff potentials

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be an open bounded domain, $s \in (0,1)$ and $1 < p < \frac{N}{s}$. In this article we are concerned with the existence of very weak solutions to the quasilinear nonlocal problems

$$\begin{cases} Lu \pm g(u) = \mu, & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{$P_\pm$}$$

where μ is a bounded Radon measure on Ω and $g \in C(\mathbb{R})$ is a nondecreasing function such that $g(0) = 0$. Here, the nonlocal operator L is defined by

$$Lu(x) := P.V. \int_{\mathbb{R}^N} |u(x) - u(y)|^{p-2}(u(x) - u(y))K(x,y)\mathrm{d}y, \quad \forall x \in \Omega,$$

where the symbol P.V. stands for the principle value integral and $K : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ is a measurable and symmetric (i.e., $K(x,y) = K(y,x)$) function. Note that if $K(x,y) \equiv |x-y|^{-N-sp}$ then L coincides with the standard fractional p-Laplace operator $(-\Delta)^s_p$.

Throughout this work, we assume that there exists a positive constant $\Lambda_K \geq 1$ such that the following ellipticity condition holds

$$\Lambda_K^{-1}|x-y|^{-N-sp} \leq K(x,y) \leq \Lambda_K|x-y|^{-N-sp}, \quad \forall (x,y) \in \mathbb{R}^N \times \mathbb{R}^N \text{ and } x \neq y.$$

In addition, we denote by $\mathfrak{M}_b(\Omega)$ the space of Radon measures on $\mathbb{R}^N$ such that $\mu(\mathbb{R}^N \setminus \Omega) = 0$, as well as by $\mathfrak{M}_b^+(\Omega)$ its positive cone.

Let

$$C_{N,s} := 2^{2s}\pi^{-\frac{N}{2}} s \frac{\Gamma(\frac{N+2s}{2})}{\Gamma(1-s)} > 0.$$

For $p = 2$ and $K(x,y) = C_{N,s}|x-y|^{-N-2s}$, operator L reduces to the well-known fractional Laplace operator $(-\Delta)^s$ and the problem P_+ becomes

$$\begin{cases} (-\Delta)^s u + g(u) = \mu, & \text{in } \Omega \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{1.1}$$

When g satisfies the subcritical integral condition

$$\int_1^\infty (g(s) - g(-s)) s^{-\frac{N}{N-2s}-1} \mathrm{d}s < \infty,$$

Chen and Véron [9] showed that problem (1.1) admits a unique very weak solution for any $\mu \in \mathfrak{M}_b(\Omega)$. In addition they showed that problem (1.1) with $g(u) = |u|^{\kappa-1}u$ ($\kappa > 1$) possesses a very weak solution if and only if μ is absolutely continuous with respect to Bessel capacity $C_{L_{2s,\kappa'}}$, i.e., μ vanishes on compact set E of Ω satisfying $\mathrm{Cap}_{2s,\kappa'}(E) = 0$ (see (3.21) for the definition of the Bessel capacities). Their approach is based on the properties of the Green Kernel associated with fractional Laplace operator $(-\Delta)^s$ in Ω.

In the local theory and more precisely when $Lu = -\Delta_p u = -\mathrm{div}(|\nabla u|^{p-2}\nabla u)$, related problems have been studied in [4–6, 15, 30–32]. In particular, in the power case, i.e.,

$$\begin{cases} -\Delta_p u + |u|^{\kappa-1}u = \mu, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

Bidaut-Véron, Nguyen and Véron [5] established that if $\mu \in \mathfrak{M}_b(\Omega)$ is absolutely continuous with respect to the Bessel capacity $\mathrm{Cap}_{p,\frac{\kappa}{\kappa-p+1}}$, then there exists a renormalized solution to problem (1.2) with $\kappa > p-1$. A main ingredient in the proof of this result is the pointwise estimates for p-superharmonic functions in Ω. These pointwise estimates are expressed in terms of the truncated Wolff potentials $W_{1,p}^R[\mu]$ (see, e.g., [17, 19, 20, 31]). We recall here that the truncated Wolff potential is given by

$$W_{\alpha,p}^R[\mu](x) := \int_0^R \left(\frac{|\mu|(B_r(x))}{r^{N-\alpha p}}\right)^{\frac{1}{p-1}} \frac{\mathrm{d}r}{r}, \tag{1.3}$$

for any $R > 0$ and $\alpha \in (0, N)$ such that $p \in (1, \frac{N}{\alpha})$. Conversely, Bidaut-Véron [4] showed that if problem (1.2) with $\kappa > p-1$ admits a renormalized solution, then μ is absolutely continuous with respect to the Bessel capacity $\mathrm{Cap}_{p,\frac{\kappa}{\kappa-p+1}+\varepsilon}$, for any $\varepsilon > 0$.

Phuc and Verbitsky [31] showed that if $\tau \in \mathfrak{M}_b^+(\Omega)$ has compact support in Ω, then the problem

$$\begin{cases} -\Delta_p u - |u|^\kappa = \rho\tau, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

admits a nonnegative renormalized solution for some $\rho > 0$, if and only if, there exists a positive constant C such that

$$\tau(K) \leq C\mathrm{Cap}_{p,\frac{\kappa}{\kappa-p+1}}(K), \tag{1.5}$$

for any compact $K \subset \Omega$. Moreover, they showed that (1.5) is equivalent to

$$W_{1,p}^{2\mathrm{diam}\,(\Omega)}[(W_{1,p}^{2\mathrm{diam}\,(\Omega)}[\tau])^\kappa] \leq C W_{1,p}^{2\mathrm{diam}\,(\Omega)}[\tau], \quad \text{a.e. in } \Omega,$$

for some positive constant $C > 0$.

Recently, a great attention has been drawn to the study of the fractional p-Laplacian or more general nonlocal operators (see for example [2, 11, 12, 18, 21–29]). More precisely, Kuusi, Mingione and Sire [26] dealt with the problem

$$\begin{cases} L_\Phi u = \mu, & \text{in } \Omega, \\ u = g, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.6}$$

where $g \in W^{s,p}(\mathbb{R}^N)$, L_Φ is a nonlocal operator defined by

$$< L_\Phi u, \zeta >:= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \Phi(u(x)-u(y))(\zeta(x)-\zeta(y))K(x,y)\mathrm{d}y\mathrm{d}x, \quad \forall \zeta \in C_0^\infty(\Omega).$$

Here $\Phi : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $\Phi(0) = 0$ and

$$\Lambda_\Phi^{-1}|t|^p \leq \Phi(t)t \leq \Lambda_\Phi |t|^p.$$

When $2 - \frac{s}{N} < p$, they show the existence of a very weak solution to (1.6), which they called SOLA (Solutions obtained as limits of approximations). They also showed local pointwise estimates for SOLA to (1.6) in terms of the truncated Wolff Potential $W_{s,p}^R[\mu]$. In the particular case $\Phi(t) = |t|^{p-2}t$ and $g = 0$, the existence of very weak solutions was established in [2] for any $1 < p < \frac{N}{s}$.

The objective of this work is to determine the subcritical integral conditions on g, which ensure the existence of very weak solutions to problems $(P_\pm)$. In addition, in the power case, i.e., $g(u) = |u|^{\kappa-1}u$; $\kappa > p-1$, we aim to find sufficient conditions, expressed in terms of Bessel capacities like above, for the solvability of $(P_\pm)$.

Let us mention here that our work is inspired by the article [5] for problem (P_+) and by the articles [30, 31] for problem (P_-) with $g(u) = |u|^{\kappa-1}u$; $\kappa > p-1$. However, due to the presence of the nonlocal operator, new essential difficulties arise which complicate drastically the study of problems $(P_\pm)$.

In order to state our main results, we need to introduce the notion of the very weak solutions.

Definition 1.1. Let $s \in (0,1)$, $1 < p < \frac{N}{s}$, $\tilde{g} \in C(\mathbb{R})$, $\Omega \subset \mathbb{R}^N$ be an open bounded domain and $\mu \in \mathfrak{M}(\Omega)$. We will say that $u : \mathbb{R}^N \to \mathbb{R}$ is a very weak solution to the problem

$$\begin{cases} Lu + \tilde{g}(u) = \mu, & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.7}$$

if $\tilde{g}(u) \in L^1_{loc}(\Omega)$ and if the following conditions are valid:

(i) $u = 0$ a.e. in $\mathbb{R}^N \setminus \Omega$ and $u \in W^{h,q}(\mathbb{R}^N)$ for any $0 < h < s$ and for any $0 < q < \frac{N(p-1)}{N-s}$.

(ii) $T_k(u) := \max(-k, \min(u,k)) \in W^{s,p}_0(\Omega)$ for any $k > 0$.

(iii)

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |u(x)-u(y)|^{p-2}(u(x)-u(y))(\phi(x)-\phi(y))K(x,y)\mathrm{d}x\mathrm{d}y + \int_\Omega \tilde{g}(u)\phi\mathrm{d}x = \int_\Omega \phi\mathrm{d}\mu$$

for any $\phi \in C^\infty_0(\Omega)$.

We note here that if $2 - \frac{s}{N} < p < \frac{N}{s}$, then the very weak solution u belongs to the fractional Sobolev space $W^{h,q}(\mathbb{R}^N)$ for any $q \in (1, \frac{N(p-1)}{N-s})$. If $p \le 2 - \frac{s}{N}$, the space $W^{h,q}(\mathbb{R}^N)$ in the above definition is no longer a fractional Sobolev space, however it is defined in the same way (see (2.1)).

In Section 2, we discuss the existence and main properties of the very weak solutions of problem (1.7) with $\tilde{g} \equiv 0$. Particularly, in the spirit of [26], we show the existence of a SOLA u satisfying statements (i)–(iii) of the above definition (see Proposition 2.8). The approximation sequence consists of solutions of (1.7) with $\tilde{g} \equiv 0$ and smooth data. In addition, we prove that these solutions satisfy a priori estimates (2.8) and (2.11). As a result, we establish that the very weak solution satisfies (2.11) and

$$\left\| |u|^{p-1} \right\|^*_{L_w^{\frac{N}{N-sp}}(\mathbb{R}^N)} \le C(N,p,s,\Lambda_K) \int_\Omega |\mu|\mathrm{d}x, \tag{1.8}$$

where $\|\cdot\|^*_{L_w^{\frac{N}{N-sp}}(\mathbb{R}^N)}$ has been defined in (2.4) and is related to the Marcinkiewicz spaces. Finally, when $\mu \in \mathfrak{M}^+_b(\Omega)$, we construct this solution (see Propositions 2.9 and 2.10) such that $u \ge 0$ and

$$C^{-1}(N,p,s,\Lambda_K) W^{\frac{d(x)}{8}}_{s,p}[\mu](x) \le u(x) \le C(N,p,s,\Lambda_K) W^{2\mathrm{diam}\,(\Omega)}_{s,p}[\mu](x), \quad \text{a.e. in } \Omega,$$

where $d(x) = \mathrm{dist}\,(x, \partial\Omega)$. The lower estimate in the above display can be obtained as a consequence of [26, estimate (1.25)]. The upper estimate in the above display is an application of [21, Theorem 5.3] and (1.8).

Using the above properties of the very weak solutions and the fact that if u, g satisfies (1.8) and (1.9) respectively then $g(u) \in L^1(\Omega)$, we obtain the following result.

Theorem 1.2. *Let $s \in (0,1)$, $1 < p < \frac{N}{s}$, $\mu \in \mathfrak{M}_b(\Omega)$. We assume that $g \in C(\mathbb{R})$ is a nondecreasing function satisfying $g(0) = 0$ and*

$$\int_1^\infty (g(s) - g(-s)) s^{-\frac{N(p-1)}{N-sp}-1}\mathrm{d}s < \infty. \tag{1.9}$$

Then there exist a very weak solution u to problem (P_+) *satisfying* (1.8) *and*

$$-C(N,p,s,\Lambda_K)W_{s,p}^{2\mathrm{diam}(\Omega)}[\mu^-] \le u \le C(N,p,s,\Lambda_K)W_{s,p}^{2\mathrm{diam}(\Omega)}[\mu^+], \quad \textit{a.e. in } \Omega. \tag{1.10}$$

In addition, for any $q \in (0, \frac{N(p-1)}{N-s})$ *and* $h \in (0,s)$, *there exists a positive constant* $c = c(N,p,s,\Lambda_K,q,h,|\Omega|)$ *such that*

$$\left(\int_\Omega |g(u)|\mathrm{d}x\right)^{\frac{1}{p-1}} + \left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^q}{|x-y|^{N+hq}}\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{q}} \le c(|\mu|(\Omega))^{\frac{1}{p-1}}. \tag{1.11}$$

We note here that the integral conditions (1.9) and (1) coincide for $p = 2$. In addition, in the corresponding local case, the integral condition (1.9) with $s = 1$ ensures the existence of the associated renormalized solutions (see [32, Theorem 5.1.2 and (5.1.40)]).

Let us consider problem (P_+) with a power absorption, i.e.,

$$\begin{cases} Lu + |u|^{\kappa-1}u = \mu, & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{1.12}$$

We first notice that the function $g(t) = |t|^{\kappa-1}t$ with $k > 0$ satisfies (1.9) if and only if $0 < \kappa < \frac{N(p-1)}{N-sp}$, hence problem (1.12) admits a very weak solution in this case. In the supercritical case $\kappa \ge \frac{N(p-1)}{N-sp}$, the sufficient condition for the solvability of problem (1.12) is expressed in terms of the Bessel capacity $\mathrm{Cap}_{sp,\frac{\kappa}{\kappa-p+1}}$ as follows.

Theorem 1.3. *Let* $s \in (0,1)$, $1 < p < \frac{N}{s}$, $\kappa > p-1$ *and* $\mu \in \mathfrak{M}_b(\Omega)$. *In addition we assume that* μ *is absolutely continuous with respect to the Bessel capacity* $\mathrm{Cap}_{sp,\frac{\kappa}{\kappa-p+1}}$. *Then there exists a very weak solution u to problem* (1.12) *such that*

$$-CW_{s,p}^{2\mathrm{diam}(\Omega)}[\mu^-] \le u \le CW_{s,p}^{2\mathrm{diam}(\Omega)}[\mu^+], \quad \textit{a.e. in } \Omega. \tag{1.13}$$

In addition, for any $q \in (0, \frac{N(p-1)}{N-s})$ *and* $h \in (0,s)$, *there exists a positive constant* $c = c(N,p,s,\Lambda_K,q,h,|\Omega|)$ *such that*

$$\left(\int_\Omega |u|^\kappa\mathrm{d}x\right)^{\frac{1}{p-1}} + \left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^q}{|x-y|^{N+hq}}\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{q}} \le c(|\mu|(\Omega))^{\frac{1}{p-1}}. \tag{1.14}$$

In view of the discussion on the existence of solutions to problem (1.4), we expect that the existence phenomenon occurs for (P_-) only for measures $\mu \in \mathfrak{M}_b(\Omega)$ with small enough total mass. Indeed, using the Schauder fixed point theorem and sharp weak Lebesgue estimates, we prove the following existence result for any $\mu \in \mathfrak{M}_b(\Omega)$ with small enough total mass.

Theorem 1.4. *Let* $s \in (0,1)$, $1 < p < \frac{N}{s}$ *and* $\tau \in \mathfrak{M}_b(\Omega)$ *be such that* $|\tau|(\Omega) \le 1$. *Assume that* $g \in C(\mathbb{R})$ *is a nondecreasing function satisfying* (1.9) *and*

$$|g(s)| \le a|s|^d \quad \textit{for some } a > 0,\ d > 1 \quad \textit{and for any } |s| \le 1. \tag{1.15}$$

Then there exists a positive constant ρ_0 *depending on* $N, |\Omega|, \Lambda_g, \Lambda_K, a, s, p, d, |\Omega|$ *such that for every* $\rho \in (0, \rho_0)$ *the following problem*

$$\begin{cases} Lv = g(v) + \rho\tau, & in \ \ \Omega, \\ \ v = 0, & in \ \ \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.16}$$

admits a very weak solution v *satisfying*

$$\||v|^{p-1}\|^*_{L_w^{\frac{N}{N-sp}}(\mathbb{R}^N)} \leq t_0. \tag{1.17}$$

Here, $t_0 > 0$ *depends on* $N, |\Omega|, \Lambda_g, \Lambda_K, a, s, p, d, \rho_0$*. In addition, for any* $q \in (0, \frac{N(p-1)}{N-s})$ *and* $h \in (0, s)$*, there exists a positive constant* c *depending only on* $N, p, s, \Lambda_g, \Lambda_K, q, h, |\Omega|, a, d, \rho_0$ *and* t_0*, such that*

$$\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v(x)-v(y)|^q}{|x-y|^{N+hq}} \mathrm{d}x\mathrm{d}y\right)^{\frac{1}{q}} \leq c(1+\rho|\tau|(\Omega))^{\frac{1}{p-1}}. \tag{1.18}$$

In the linear case, i.e., $p = 2$, problem (P_-) with $L = (-\Delta)^s$ was thoroughly studied in [7]. More precisely, the authors in [7] showed that the same existence result occurs provided g satisfies (1) and (1.15).

Problem (P_-) with $g(t) = |t|^{\kappa-1}t$ and $\mu \in \mathfrak{M}_b^+(\Omega)$ becomes

$$\begin{cases} Lv = |v|^{\kappa-1}v + \rho\tau, & \text{in } \ \Omega, \\ \ v = 0, & \text{in } \ \mathbb{R}^N \setminus \Omega. \end{cases} \tag{1.19}$$

When $p = 2$, problem (P_-) with $L = (-\Delta)^s$ and $\tau = \delta_0$ was studied in [8]. Here δ_0 denotes the dirac measure concentrated at a point $x_0 \in \Omega$. In particular, the authors in [8] established that if $\kappa \geq \frac{N}{N-2s}$ and u is a nonnegative solution of (1.19) then $\rho = 0$. Concerning problem (1.19), conditions (1.9) and (1.15) are satisfied if κ belongs to the subcritical range, that is when $p-1 < \kappa < \frac{N(p-1)}{N-sp}$. In general, a sufficient condition for the solvability of (1.19) is the following.

Proposition 1.5. *Let* $s \in (0,1)$, $1 < p < \frac{N}{s}$, $\kappa > p-1$ *and* $\tau \in \mathfrak{M}_b^+(\Omega)$ *be such that*

$$W_{s,p}^{2\mathrm{diam}\,(\Omega)}[(W_{s,p}^{2\mathrm{diam}\,(\Omega)}[\tau])^\kappa] \leq M W_{s,p}^{2\mathrm{diam}\,(\Omega)}[\tau], \quad a.e. \ \ in \ \ \Omega, \tag{1.20}$$

for some positive constant M*. Then problem* (1.19) *admits a nonnegative very weak solution* u *for some* $\rho > 0$*. Furthermore, there holds*

$$M^{-1} W_{s,p}^{\frac{d(x)}{8}}[\mu](x) \leq u(x) \leq M W_{s,p}^{2\mathrm{diam}\,(\Omega)}[\rho\tau](x), \qquad for\ a.e.\ x \in \Omega, \tag{1.21}$$

where $\mathrm{d}\mu = u^\kappa \mathrm{d}x + \rho \mathrm{d}\tau$ *and the positive constant* M *depends only on* C, N, p, q, Λ_K*.*

Finally, inspired from Phuc and Verbitsky's ideas in [30, 31], we establish the following existence result in the whole range $\kappa > p-1$.

Theorem 1.6. *Let* $s \in (0,1)$, $1 < p < \frac{N}{s}$, $\kappa > p-1$ *and* $\tau \in \mathfrak{M}_b^+(\Omega)$ *with compact support in* Ω*. Then the following statements are equivalent.*

(i) Problem (1.19) *admits a nonnegative very weak solution* u_ρ *for some* $\rho > 0$ *such that*

$$C_1^{-1} W_{s,p}^{\frac{d(x)}{8}}[\mu](x) \le u_\rho(x) \le C_1 W_{s,p}^{2\mathrm{diam}\,(\Omega)}[\rho\tau](x), \quad \text{for a.e. } x \in \Omega, \tag{1.22}$$

where $\mathrm{d}\mu = u^\kappa \mathrm{d}x + \rho \mathrm{d}\tau$ *and for some constant* $C_1 > 0$.

(ii) There exists a positive constant C_2 *such that*

$$\tau(E) \le C_2 \mathrm{Cap}_{sp,\frac{\kappa}{\kappa-p+1}}(E) \tag{1.23}$$

for any Borel set $E \subset \mathbb{R}^N$.

(iii) There exists a positive constant C_3 *such that*

$$\int_B (W_{s,p}^{2\mathrm{diam}\,(\Omega)}[\tau_{\lfloor B}])^\kappa \mathrm{d}x \le C_3 \tau(B) \tag{1.24}$$

for any ball $B \subset \mathbb{R}^N$, *where* $\mathrm{d}\tau_{\lfloor B} = \chi_B \mathrm{d}\tau$.

(iv) There exists a positive constant C_4 *such that*

$$W_{s,p}^{2\mathrm{diam}\,(\Omega)}[(W_{s,p}^{2\mathrm{diam}\,(\Omega)}[\tau])^\kappa] \le C_4 W_{s,p}^{2\mathrm{diam}\,(\Omega)}[\tau], \quad \text{a.e. in } \Omega.$$

We note here that if $p - 1 < q < \frac{N(p-1)}{N-sp}$ then $\frac{spq}{q-p+1} > N$, this implies that $\mathrm{Cap}_{sp,\frac{q}{q-p+1}}(\{x\}) > 0$ for any $x \in \mathbb{R}^N$ (see [1, Section 2.6]). Hence, the statement (ii) in the above theorem is always satisfied in the subcritical range.

Section 2 is devoted to the study of the very weak solutions to problem (1.7) with $\tilde{g} \equiv 0$. In Section 3, we discuss problem (P_+) as well as Theorems 1.2 and 1.3 are proved in Subsections 3.2 and 3.3 respectively. In section 4, we deal with problem (P_-). More precisely, we prove Theorem 1.4 in Subsection 4.1 and demonstrate Proposition 1.5 and Theorem 1.6 in Subsection 4.2.

2. Very weak solutions

We start with the definition of the fractional spaces, which will be used frequently in this work. For any $s \in (0, 1)$ and $q > 0$, we denote by $W^{s,q}(\mathbb{R}^N)$ the fractional space

$$W^{s,q}(\mathbb{R}^N) := \left\{ \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^q}{|x - y|^{N+sq}} \mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^N} |u|^q \mathrm{d}x < \infty \right\}, \tag{2.1}$$

endowed with the quasinorm

$$\|u\|_{W^{s,q}(\mathbb{R}^N)} := \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^q}{|x - y|^{N+sq}} \mathrm{d}x\mathrm{d}y \right)^{\frac{1}{q}} + \left(\int_{\mathbb{R}^N} |u|^q \mathrm{d}x \right)^{\frac{1}{q}}.$$

When $q \ge 1$, $W^{s,q}(\mathbb{R}^N)$ is a Banach space and is called fractional Sobolev space. Finally, for any $p > 1$, we denote by $W_0^{s,p}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in the norm $\|\cdot\|_{W^{s,p}(\mathbb{R}^N)}$ and by $(W_0^{s,p}(\Omega))^*$ its dual space.

2.1. Weak solutions and a priori estimates

In this subsection, we introduce the notion of the weak solution of the following problem

$$\begin{cases} Lu = \mu, & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{2.2}$$

where $\mu \in (W_0^{s,p}(\Omega))^*$. In addition, when $\mu \in L^{p'}(\Omega)$, we establish a priori estimates, which will be used in the construction of the very weak solutions of the above problem with measure data.

Definition 2.1. Let $s \in (0,1)$, $p > 1$, and $\mu \in (W_0^{s,p}(\Omega))^*$. We will say that $u \in W_0^{s,p}(\Omega)$ is a weak solution of (2.2), if it satisfies

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |u(x)-u(y)|^{p-2}(u(x)-u(y))(\phi(x)-\phi(y))K(x,y)\mathrm{d}x\mathrm{d}y = <\mu,\phi>, \quad \forall \phi \in W_0^{s,p}(\Omega).$$

Let us now give the definition of weak supersolutions of L in Ω.

Definition 2.2. Let $s \in (0,1)$ and $p > 1$. We will say that $u \in W^{s,p}(\mathbb{R}^N)$ is a weak supersolution (resp. subsolution) of L in Ω, if and only if satisfies

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |u(x)-u(y)|^{p-2}(u(x)-u(y))(\phi(x)-\phi(y))K(x,y)\mathrm{d}x\mathrm{d}y \geq 0 \ (resp. \leq 0)$$

for any nonnegative $\phi \in W_0^{s,p}(\Omega)$.

Next we state the comparison principle.

Proposition 2.3 ([23, Lemma 6])**.** *Let $u \in W^{s,p}(\mathbb{R}^N)$ be a weak supersolution of L in Ω as well as let $v \in W^{s,p}(\mathbb{R}^N)$ be a weak subsolution of L in Ω such that $(v-u)_+ \in W_0^{s,p}(\Omega)$. Then, $u \geq v$ a.e. in $\mathbb{R}^N$.*

In view of the proof [11, Theorem 2.3], we may obtain the following existence result.

Proposition 2.4. *For any $\mu \in (W_0^{s,p}(\Omega))^*$ there exists a unique weak solution of* (2.2).

In order to state the first a priori estimate for the weak solution of (2.2), we need to give the definition and the main properties of Marcinkiewicz spaces. Let $D \subset \mathbb{R}^N$ be a domain. Denote $L^p_w(D)$, $1 \leq p < \infty$, the weak L^p space (or Marcinkiewicz space) defined as follows. A measurable function f in D belongs to this space if there exists a constant c such that

$$\lambda_f(a) := |\{x \in D : |f(x)| > a\}| \leq ca^{-p}, \quad \forall a > 0. \tag{2.3}$$

The function λ_f is called the distribution function of f. For $p \geq 1$, denote

$$L^p_w(D) = \{f \text{ Borel measurable} : \sup_{a>0} a^p \lambda_f(a) < \infty\},$$

$$\|f\|^*_{L^p_w(D)} = (\sup_{a>0} a^p \lambda_f(a))^{\frac{1}{p}}. \tag{2.4}$$

The $\|.\|^*_{L^p_w(D)}$ is not a norm, but for $p > 1$, it is equivalent to the norm

$$\|f\|_{L^p_w(D)} = \sup\left\{\frac{\int_\omega |f|\mathrm{d}x}{|\omega|^{1/p'}} : \omega \subset D, \omega \text{ measurable}, \ 0 < |\omega| < \infty\right\}. \tag{2.5}$$

More precisely,

$$\|f\|^*_{L^p_w(D)} \leq \|f\|_{L^p_w(D)} \leq \frac{p}{p-1}\, \|f\|^*_{L^p_w(D)}\,. \tag{2.6}$$

Notice that,

$$L^p_w(D) \subset L^r(D), \quad \forall r \in [1, p).$$

From (2.4) and (2.6), one can derive the following estimate which is useful in the sequel.

$$\int_{\{|u|\geq s\}} \mathrm{d}x \leq s^{-p}\, \|u\|^p_{L^p_w(D)}\,. \tag{2.7}$$

Proposition 2.5. *Let* $1 < p < \frac{N}{s}$, $\mu \in L^{p'}(\Omega)$ *and* $u \in W^{s,p}_0(\Omega)$ *be the unique weak solution of* (2.2). *Then there exists a positive constant* $C = C(p, s, N, \Lambda_K)$ *such that*

$$\left\||u|^{p-1}\right\|^*_{L_w^{\frac{N}{N-sp}}(\mathbb{R}^N)} \leq C \int_\Omega |\mu|\mathrm{d}x. \tag{2.8}$$

Proof. Let $k > 0$. Taking $T_k(u)$ as test function and using the fact that

$$|u(x) - u(y)|^{p-2}(u(x) - u(y))(T_k(u)(x) - T_k(u)(y)) \geq |T_k(u)(x) - T_k(u)(y)|^p, \quad \forall x, y \in \mathbb{R}^N,$$

we obtain

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|T_k(u)(x) - T_k(u)(y)|^p}{|x-y|^{N+sp}}\mathrm{d}x\mathrm{d}y \leq \Lambda_K k \int_\Omega |\mu|\mathrm{d}x. \tag{2.9}$$

Now, by the above inequality and the fractional Sobolev inequality we have

$$|\{|u(x)| \geq k\}| = |\{|T_k(u)(x)| \geq k\}| \leq k^{-\frac{Np}{N-sp}} \int_{\mathbb{R}^N} |T_k(u)(x)|^{\frac{Np}{N-sp}}\mathrm{d}x \leq Ck^{-\frac{N(p-1)}{N-sp}} \left(\int_\Omega |\mu|\mathrm{d}x\right)^{\frac{N}{N-sp}},$$

which implies the desired result. □

Proposition 2.6. *Let* $\mu \in L^{p'}(\Omega)$ *and* $u \in W^{s,p}_0(\mathbb{R}^N)$ *be the unique weak solution of* (2.2). *Then there exists a positive constant* $C = C(p, s, N, \Lambda_K)$ *such that*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{(d + |u(x)| + |u(y)|)^\xi}\frac{\mathrm{d}x\mathrm{d}y}{|x-y|^{N+sp}} \leq \frac{Cd^{1-\xi}}{(\xi - 1)}\int_\Omega |\mu|\mathrm{d}x \tag{2.10}$$

for any $\xi > 1$ *and* $d > 0$.

Proof. The proof is very similar to that of [26, Lemma 3.1] (see also [25, Lemma 8.4.1]). For the sake of convenience we give it below.

Set $\phi_\pm := \pm(d^{1-\xi} - (d + u_\pm)^{1-\xi})$. Using $\phi_\pm$ as test function we obtain

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |u(x) - u(y)|^{p-2}(u(x) - u(y))(\phi_\pm(x) - \phi_\pm(y))K(x, y)\mathrm{d}x\mathrm{d}y = \int_\Omega \phi_\pm \mu \mathrm{d}x.$$

Now,

$$(\phi_\pm(x) - \phi_\pm(y)) = \pm(\xi - 1)(u_\pm(x) - u_\pm(y)) \int_0^1 (d + tu_\pm(y) + (1 - t)u_\pm(x))^{-\xi} dt,$$

which implies

$$\begin{aligned}&|u(x) - u(y)|^{p-2}(u(x) - u(y))(\phi_\pm(x) - \phi_\pm(y))K(x, y)\\ &\qquad \geq (\xi - 1)|u(x) - u(y)|^{p-2}(u_\pm(x) - u_\pm(y))^2(d + |u(y)| + |u(x)|)^{-\xi}.\end{aligned}$$

Combining all above we can easily reach the desired result. □

We conclude this subsection by the following a priori estimate for the weak solutions of (2.2) in the whole range $p > 1$.

Proposition 2.7. *Let* $\overline{q} = \min\{\frac{N(p-1)}{N-s}, p\}$ $\mu \in L^{p'}(\Omega)$ *and* $u \in W^{s,p}_0(\mathbb{R}^N)$ *be the unique weak solution of* (2.2). *For any* $q \in (0, \overline{q})$ *and* $h \in (0, s)$, *there exists a positive constant c depending only on* N, s, p, Λ_K, q *and* $|\Omega|$ *such that*

$$\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^q}{|x - y|^{N+hq}}\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{q}} \leq c\left(\int_\Omega |\mu|\mathrm{d}x\right)^{\frac{1}{p-1}}. \tag{2.11}$$

Proof. The proof is an adaptation of the argument in [26, Lemma 3.2]. Let $R = \mathrm{diam}\,(\Omega)$ and $x_0 \in \Omega$. First, we note that

$$\begin{aligned}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^q}{|x - y|^{N+hq}}\mathrm{d}x\mathrm{d}y &= \int_{B_{2R}(x_0)}\int_{B_{2R}(x_0)} \frac{|u(x) - u(y)|^q}{|x - y|^{N+hq}}\mathrm{d}x\mathrm{d}y\\ &\quad + \int_{\mathbb{R}^N\setminus B_{2R}(x_0)}\int_{\mathbb{R}^N\setminus B_{2R}(x_0)} \frac{|u(x) - u(y)|^q}{|x - y|^{N+hq}}\mathrm{d}x\mathrm{d}y\\ &\quad + 2\int_{\mathbb{R}^N\setminus B_{2R}(x_0)}\int_{B_{2R}(x_0)} \frac{|u(x) - u(y)|^q}{|x - y|^{N+hq}}\mathrm{d}x\mathrm{d}y.\end{aligned}$$

Taking into account that $u = 0$ a.e. in $\mathbb{R}^N \setminus \Omega$, we can easily prove that

$$\int_{\mathbb{R}^N\setminus B_{2R}(x_0)}\int_{\mathbb{R}^N\setminus B_{2R}(x_0)} \frac{|u(x) - u(y)|^q}{|x - y|^{N+hq}}\mathrm{d}x\mathrm{d}y = 0$$

and

$$\begin{aligned}\int_{\mathbb{R}^N\setminus B_{2R}(x_0)}\int_{B_{2R}(x_0)} \frac{|u(x) - u(y)|^q}{|x - y|^{N+hq}}\mathrm{d}x\mathrm{d}y &= \int_{\mathbb{R}^N\setminus B_{2R}(x_0)}\int_{B_R(x_0)} \frac{|u(x)|^q}{|x - y|^{N+hq}}\mathrm{d}x\mathrm{d}y\\ &\approx \int_{B_R(x_0)} |u(x)|^q \mathrm{d}x \int_{\mathbb{R}^N\setminus B_{2R}(x_0)} \frac{1}{(1 + |y - x_0|)^{N+hq}}\mathrm{d}y\end{aligned}$$

$$\approx \int_{B_R(x_0)} |u(x)|^q \mathrm{d}x.$$

Here, we have also used the fact that $|x-y| \approx 1+|y-x_0|$ for any $(x,y) \in B_R(x_0)\times(\mathbb{R}^N \setminus B_{2R}(x_0))$, where the implicit constants in the last estimate depend only on R. Similarly, we have

$$\begin{aligned}\int_{B_{2R}(x_0)}\int_{B_{2R}(x_0)} \frac{|u(x)-u(y)|^q}{|x-y|^{N+hq}}\mathrm{d}x\mathrm{d}y &= \int_{B_{\frac{3R}{2}}(x_0)}\int_{B_{\frac{3R}{2}}(x_0)} \frac{|u(x)-u(y)|^q}{|x-y|^{N+hq}}\mathrm{d}x\mathrm{d}y \\ &+ \int_{B_{2R}(x_0)\setminus B_{\frac{3R}{2}}(x_0)}\int_{B_{2R}(x_0)\setminus B_{\frac{3R}{2}}(x_0)} \frac{|u(x)-u(y)|^q}{|x-y|^{N+hq}}\mathrm{d}x\mathrm{d}y \\ &+ 2\int_{B_{2R}(x_0)\setminus B_{\frac{3R}{2}}(x_0)}\int_{B_{\frac{3R}{2}}(x_0)} \frac{|u(x)-u(y)|^q}{|x-y|^{N+hq}}\mathrm{d}x\mathrm{d}y \\ &\approx \int_{B_{\frac{3R}{2}}(x_0)}\int_{B_{\frac{3R}{2}}(x_0)} \frac{|u(x)-u(y)|^q}{|x-y|^{N+hq}}\mathrm{d}x\mathrm{d}y + \int_{B_R(x_0)} |u|^q \mathrm{d}x.\end{aligned}$$

Combining all above, we have

$$\begin{aligned}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^q}{|x-y|^{N+hq}}\mathrm{d}x\mathrm{d}y &\approx \int_{B_{2R}(x_0)}\int_{B_{2R}(x_0)} \frac{|u(x)-u(y)|^q}{|x-y|^{N+hq}}\mathrm{d}x\mathrm{d}y + \int_\Omega |u|^q \mathrm{d}x \\ &\approx \int_{B_{2R}(x_0)}\int_{B_{2R}(x_0)} \frac{|u(x)-u(y)|^q}{|x-y|^{N+hq}}\mathrm{d}x\mathrm{d}y.\end{aligned} \tag{2.12}$$

Now, by Hölder inequality we obtain

$$\begin{aligned}&\int_{B_{2R}(x_0)}\int_{B_{2R}(x_0)} \frac{|u(x)-u(y)|^q}{|x-y|^{N+hq}}\mathrm{d}x\mathrm{d}y \\ &= \int_{B_{2R}(x_0)}\int_{B_{2R}(x_0)} \left(\frac{|u(x)-u(y)|^p}{(d+|u(x)|+|u(y)|)^\xi |x-y|^{ps}}(d+|u(x)|+|u(y)|)^\xi |x-y|^{p(s-h)}\right)^{\frac{q}{p}} \frac{\mathrm{d}x\mathrm{d}y}{|x-y|^N} \\ &\le \left(\int_{B_{2R}(x_0)}\int_{B_{2R}(x_0)} \frac{|u(x)-u(y)|^p}{(d+|u(x)|+|u(y)|)^\xi |x-y|^{N+sp}}\mathrm{d}x\mathrm{d}y\right)^{\frac{q}{p}} \\ &\times \left(\int_{B_{2R}(x_0)}\int_{B_{2R}(x_0)} \frac{(d+|u(x)|+|u(y)|)^{\frac{\xi q}{p-q}}}{|x-y|^{N-\frac{qp(s-h)}{p-q}}}\mathrm{d}x\mathrm{d}y\right)^{\frac{p-q}{p}}.\end{aligned} \tag{2.13}$$

Setting

$$d = \left(\int_\Omega |u(y)|^{\frac{\xi q}{p-q}} \mathrm{d}x\right)^{\frac{p-q}{\xi q}}$$

and combining (2.10) and (2.13), we conclude

$$\int_{B_{2R}(x_0)}\int_{B_{2R}(x_0)} \frac{|u(x)-u(y)|^q}{|x-y|^{N+hq}}\mathrm{d}x\mathrm{d}y \le c d^{\frac{q}{p}} \Big(\int_\Omega |\mu| \mathrm{d}x\Big)^{\frac{q}{p}}. \tag{2.14}$$

If $p > 2-\frac{s}{N}$, without loss of generality, we may assume that $q > 1$. Therefore, we may apply the fractional Sobolev inequality to d as in the proof of [26, Lemma 3.2] to obtain the desired result.

If $1 < p \le 2 - \frac{N}{s}$, we have that $0 < q \le 1$, therefore, we can not apply the fractional Sobolev inequality to d. To overcome this difficulty we use (2.8) instead of fractional Sobolev inequality. More precisely, let $1 < p < \frac{N}{s}$, then $0 < q < \frac{N(p-1)}{N-s} < p$. Hence, we may choose $\xi > 1$ such that $1 < \gamma := \frac{\xi q}{(p-1)(p-q)} < \frac{N}{N-sp}$. Thus, by (2.6) and (2.8), we deduce

$$\left(\int_\Omega |u|^{\gamma(p-1)}\right)^{\frac{1}{\gamma}} \le C(\gamma, N, p, s, |\Omega|, \Lambda_K) \int_\Omega |\mu| \mathrm{d}x,$$

which in turn implies

$$d \le C(\gamma, N, p, s, |\Omega|, \Lambda_K) \left(\int_\Omega |\mu| \mathrm{d}x\right)^{\frac{1}{p-1}}.$$

The desired result follows by (2.12), (2.14) and the above inequality. □

2.2. Existence and main properties

In this subsection, we construct a very weak solution to problem (2.2) which possesses several important properties, such as it satisfies pointwise estimates in terms of Wolff's potential. These estimates play an important role in the study of problems $(P_\pm)$.

We start with the following existence result.

Proposition 2.8. *Let $1 < p < \frac{N}{s}$ and $\mu \in \mathfrak{M}_b(\Omega)$. Then there exists a very weak solution to* (2.2) *satisfying*

$$\left\| |u|^{p-1} \right\|^*_{L_w^{\frac{N}{N-sp}}(\mathbb{R}^N)} \le C_1(N, p, s, \Lambda_K) \mu(\Omega) \tag{2.15}$$

and

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|T_k(u)(x) - T_k(u)(y)|^p}{|x-y|^{N+sp}} \mathrm{d}x\mathrm{d}y \le k\Lambda_K |\mu|(\Omega), \quad \forall k > 0. \tag{2.16}$$

In addition, for any $q \in (0, \frac{N(p-1)}{N-s})$ and $h \in (0, s)$, there exist a positive constant $C_2 = C_2(N, p, s, \Lambda_K, q, h, |\Omega|)$ such that

$$\left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^q}{|x-y|^{N+hq}} \mathrm{d}x\mathrm{d}y\right)^{\frac{1}{q}} \le C_2 |\mu|(\Omega)^{\frac{1}{p-1}}. \tag{2.17}$$

Proof. Let $\{\rho_n\}_n$ be a sequence of mollifiers and $\mu_n = \rho_n * \mu$. Then $\mu_n \in C_0^\infty(\mathbb{R}^N)$ and $\mu_n \rightharpoonup \mu$ weakly in $\mathbb{R}^N$. We denote by u_n the weak solution of (2.2) with $\mu = \mu_n$.

By (2.8), (2.9) and (2.11), there exist positive constants C_1 and C_2 such that

$$\left\| |u_n|^{p-1} \right\|^*_{L_w^{\frac{N}{N-sp}}(\mathbb{R}^N)} \le C_1(N, p, s, \Lambda_K) \mu(\Omega), \qquad \forall n \in \mathbb{N}, \tag{2.18}$$

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|T_k(u_n)(x) - T_k(u_n)(y)|^p}{|x-y|^{N+sp}} \mathrm{d}x\mathrm{d}y \le k\Lambda_K \mu(\Omega), \qquad \forall k > 0 \text{ and } n \in \mathbb{N}, \tag{2.19}$$

and

$$\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_n(x)-u_n(y)|^q}{|x-y|^{N+hq}}\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{q}}\le C_2(N,p,s,\Lambda_K,q,h)\mu(\Omega)^{\frac{1}{p-1}} \tag{2.20}$$

for any $n\in\mathbb{N}$, $q\in(0,\frac{N(p-1)}{N-s})$ and $h\in(0,s)$.

In the spirit of the proof of [10, Theorem 3.4], we will show that the existence of a subsequence (still denoted by $\{u_n\}$) and a function $u:\mathbb{R}^N\to\mathbb{R}$ satisfying the following properties:

(i) $u\in W^{h,q}(\mathbb{R}^N)$ for any $0<q<\frac{N(p-1)}{N-s}$ and $0<h<s$.

(ii) $u_n\to u$ a.e. in $\mathbb{R}^N$, $u=0$ a.e. in $\mathbb{R}^N\setminus\Omega$ and $\|u-u_n\|_{W^{h,q}(\mathbb{R}^N)}\to 0$ for any $q\in(0,\frac{N(p-1)}{N-s})$ and $h\in(0,s)$.

(iii) $T_k(u)\in W_0^{s,p}(\Omega)$ for any $k>0$.

Step 1. There exists a subsequence, still denoted by u_n, such that

$$\lim_{n\to\infty}\lim_{m\to\infty}|\{x\in\Omega:\ |u_n-u_m|>\eta\}|=0,\quad\forall\,\eta>0.$$

Let $n,m\in\mathbb{N}$ and $\eta,\rho>0$. Then

$$\{|u_n-u_m|>\eta\}\subset\{|T_k(u_n)|>k\}\cup\{|T_k(u_m)|>k\}\cup\{|T_k(u_n)-T_k(u_m)|>\eta\}.$$

By (2.18) and (2.7), there exists $k_0>0$ such that

$$|\{|T_k(u_n)|>k\}|+|\{|T_k(u_m)|>k\}|\le\frac{\rho}{2},\quad\forall k\ge k_0. \tag{2.21}$$

By (2.19), the fractional Sobolev embedding theorem (see e.g., [13, Corollary 7.2]) and the fact that $W_0^{s,p}(\Omega)$ is a reflexive Banach space, we may prove the existence of a subsequence $T_{k_0}(u_{n_j})$ of $T_{k_0}(u_n)$ such that $T_{k_0}(u_{n_j})\to v_{k_0}$ in $L^p(\mathbb{R}^N)$ and a.e. in $\mathbb{R}^N$ as well as $T_{k_0}(u_{n_j})\rightharpoonup v_{k_0}$ in $W_0^{s,p}(\Omega)$. Hence,

$$|\{|T_{k_0}(u_{n_j})-T_{k_0}(u_{m_{\tilde{j}}})|>\eta\}|\le\frac{\rho}{2},\quad\forall\,j,\tilde{j}\ge n_0. \tag{2.22}$$

The desired result follows by (2.21) and (2.22).

Step 2. Weak convergence of the truncates. Since $u_n\to u$ a.e. in $\mathbb{R}^N$, we have that $T_k(u_n)\to T_k(u)$ a.e. in $\mathbb{R}^N$. Furthermore, by (2.19) and the fractional Sobolev embedding theorem, we can find a subsequence $\{T_k(u_{n_j})\}_{j=1}^{\infty}$ such that $T_k(u_{n_j})\to v_k$ in $L^p(\mathbb{R}^N)$ and $T_k(u_{n_j})\rightharpoonup v_k$ in $W_0^{s,p}(\Omega)$. Since $v_k=T_k(u)$ a.e. in $\mathbb{R}^N$, we have that $T_k(u)\in W_0^{s,p}(\Omega)$. This implies that the limit does not depend on the subsequence. Hence, for the same subsequence u_n of the Step 1, we have that

$$T_k(u_n)\rightharpoonup T_k(u)\ \text{ in } W_0^{s,p}(\Omega),\ \forall k>0.$$

Furthermore, by (2.18)–(2.20) and Fatou's lemma, we have that

$$\left\||u|^{p-1}\right\|^*_{L_w^{\frac{N}{N-sp}}(\mathbb{R}^N)}\le C_1(N,p,s,\Lambda_K)\mu(\Omega), \tag{2.23}$$

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|T_k(u)(x)-T_k(u)(y)|^p}{|x-y|^{N+sp}}\mathrm{d}x\mathrm{d}y\le k\Lambda_K\mu(\Omega),\qquad\forall k>0, \tag{2.24}$$

and

$$\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^q}{|x-y|^{N+hq}}\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{q}} \le C_2(N,p,s,\Lambda_K,q,h)\mu(\Omega)^{\frac{1}{p-1}} \tag{2.25}$$

for any $q \in (0, \frac{N(p-1)}{N-s})$ and $h \in (0, s)$.

By (2.20), (2.25) and the fact that $u_n \to u$ a.e. in $\mathbb{R}^N$, We can easily show that $\|u-u_n\|_{W^{h,q}(\mathbb{R}^N)} \to 0$ for any $q \in (0, \frac{N(p-1)}{N-s})$ and $h \in (0, s)$. Let $\phi \in C_0^\infty(\Omega)$, $q \in (p-1, \frac{N(p-1)}{N-s})$ and $h \in (\max(\frac{sp-1}{p-1}, 0), s)$. For any bounded Borel set $E \subset \mathbb{R}^N$, we have that

$$\begin{aligned}
&\left|\int_E\int_E |u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))(\phi(x)-\phi(y))K(x,y)\mathrm{d}x\mathrm{d}y\right| \\
&\quad\le C(\phi,\Lambda_K)\int_E\int_E\frac{|u_n(x)-u_n(y)|^{p-1}}{|x-y|^{N+hp-1+sp-h(p-1)-1}} \\
&\quad\le C(\phi,\Lambda_K)\left(\int_E\int_E\frac{|u_n(x)-u_n(y)|^q}{|x-y|^{N+hq}}\mathrm{d}x\mathrm{d}y\right)^{\frac{p-1}{q}}\left(\int_E\int_E|x-y|^{-N-\frac{q(sp-h(p-1)-1)}{q-p+1}}\mathrm{d}x\mathrm{d}y\right)^{\frac{q-p+1}{q}}.
\end{aligned}$$

This, together with (i), (ii) and the fact that $\frac{q(sp-h(p-1)-1)}{q-p+1} < 0$, implies that

$$\begin{aligned}
\int_\Omega \phi\mathrm{d}\mu &= \lim_{n\to\infty}\int_\Omega \phi\mu_n\mathrm{d}x \\
&= \lim_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))(\phi(x)-\phi(y))K(x,y)\mathrm{d}x\mathrm{d}y \\
&= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|u(x)-u(y)|^{p-2}(u_n(x)-u_n(y))(\phi(x)-\phi(y))K(x,y)\mathrm{d}x\mathrm{d}y.
\end{aligned}$$

The proof is complete. □

In the next theorem, we establish a priori pointwise estimates for a certain nonnegative very weak solution of problem (2.2) with $\mu \in \mathfrak{M}_b^+(\Omega)$.

Proposition 2.9. *Let* $1 < p < \frac{N}{s}$ *and* $\mu \in \mathfrak{M}_b^+(\Omega)$. *Then there exist a nonnegative very weak solution* u *of* (2.2) *and a positive constant* C *depending only on* N, s, p, Λ_K *such that*

$$\begin{aligned}
C^{-1}W_{s,p}^{\frac{d(x)}{8}}[\mu](x) &\le u(x) \\
&\le C\left(\operatorname*{ess\,inf}_{B_{\frac{d(x)}{4}}(x)} u + W_{s,p}^{\frac{d(x)}{2}}[\mu](x) + \left(\left(\frac{d(x)}{4}\right)^{sp}\int_{\mathbb{R}^N\setminus B_{\frac{d(x)}{4}}(x)}\frac{u(y)^{p-1}}{|x-y|^{N+sp}}\mathrm{d}y\right)^{\frac{1}{p-1}}\right)
\end{aligned} \tag{2.26}$$

for a.e. $x \in \Omega$.

Proof. Let u be the solution constructed in Proposition 2.8 and $\{u_n\}$ be the sequence defined in Proposition 2.8 such that

(i) $u \in W^{h,q}(\mathbb{R}^N)$ for any $0 < q < \frac{N(p-1)}{N-s}$ and $0 < h < s$.

(ii) $u_n \to u$ a.e. in $\mathbb{R}^N$, $u = 0$ a.e. in $\mathbb{R}^N \setminus \Omega$ and $\|u-u_n\|_{W^{h,q}(\mathbb{R}^N)} \to 0$ for any $q \in (0, \frac{N(p-1)}{N-s})$ and $h \in (0, s)$.

Since $\mu_n \in C_0^\infty(\mathbb{R}^N)$ is a nonnegative function, by (2.3), we have that $u_n \geq 0$ a.e. in $\mathbb{R}^N$. Hence, by [23, Lemma 7], $u_{k,n} = \min(u_n, k)$ is a nonnegative weak supersolution. By properties (i) and (ii), we may show that $u_k = \min(u, k)$ is a nonnegative weak supersolution. Hence, there exists a nonnegative Radon measure $\mu_k \in \mathfrak{M}^+(\Omega)$ such that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |u_k(x) - u_k(y)|^{p-2}(u_k(x) - u_k(y))(\phi(x) - \phi(y))K(x,y)\mathrm{d}x\mathrm{d}y = \int_\Omega \phi(x)\mathrm{d}\mu_k \tag{2.27}$$

for any $\phi \in C_0^\infty(\Omega)$. Since $u_k \to u$ in $\mathbb{R}^N$, we have that $\|u - u_k\|_{W^{h,q}(\mathbb{R}^N)} \to 0$ for any $h \in (0, s)$ and $q \in (0, \frac{N(p-1)}{N-s})$. This, together with (2.27), implies

$$\int_\Omega \phi(x)\mathrm{d}\mu_k \to \int_\Omega \phi(x)\mathrm{d}\mu, \qquad \forall \phi \in C_0^\infty(\Omega). \tag{2.28}$$

Now, we remark that, in view of the proof of [26, Theorem 1.3], we may apply [26, estimate (1.25)] to u_k. Hence,

$$C^{-1} W_{s,p}^{\frac{d(x)}{8}}[\mu_k](x) \leq u_k(x), \qquad \text{for a.e. } x \in \Omega \text{ and } \forall k > 0.$$

Letting $k \to \infty$ in the above inequality and using some elementary manipulations, we may obtain the lower estimate in (2.26).

For the upper estimate in (2.26), by [23, Theorem 9], we have that

$$v_k(x) := \operatorname{ess}\liminf_{y\to x} u_k(y) = u_k(x), \qquad \text{for a.e. } x \in \mathbb{R}^N.$$

Hence, v_k is a lower semicontinuous functions in Ω and a nonnegative weak supersolution. By [23, Theorem 12], v_k is (s, p)-superharmonic function in Ω (see [23, Definition 1] for the definition of (s, p)-superharmonic function). This, together with [23, Lemma 12], implies that $v := \lim_{k\to\infty} v_k$ is (s, p)-superharmonic function in Ω and $v = u$ a.e. in $\mathbb{R}^N$. The desired result follows by applying [21, Theorem 5.3] to v and the fact that $v = u$ a.e. in $\mathbb{R}^N$. ◻

Proposition 2.10. *Let $\mu \in \mathfrak{M}_b(\Omega)$. Then there exists a very weak solution u of* (2.2) *and a positive constant C depending only on N, s, p and Λ_K such that*

$$-CW_{s,p}^{2\mathrm{diam}\,(\Omega)}[\mu^-] \leq u \leq CW_{s,p}^{2\mathrm{diam}\,(\Omega)}[\mu^+], \qquad \textit{a.e. in } \Omega. \tag{2.29}$$

Proof. Let u be the solution constructed in Proposition 2.8 and $x_0 \in \Omega$. Set $R = \mathrm{diam}\,(\Omega)$, $\mu_n = \rho_n * \mu$ and $\mu_n^\oplus = \rho_n * \mu^+$. We denote by $v_n^\oplus \in W_0^{s,p}(\Omega)$ the solution of

$$\begin{cases} Lv_n^\oplus = \mu_n^\oplus, & \text{in } B_{2R}(x_0), \\ v_n^\oplus = 0, & \text{in } \mathbb{R}^N \setminus B_{2R}(x_0). \end{cases}$$

By Proposition 2.3, we have that $v_n^\oplus \geq 0$ and $v_n^\oplus \geq u_n$, where $u_n \in W_0^{s,p}(\Omega)$ is the weak solution of (2.2) with $\mu = \mu_n$. By statements (i)–(iii) in the proof of Proposition 2.8, there exist subsequences $\{u_{n_k}, v_{n_k}^\oplus\}_{k=1}^\infty$ such that $u_{n_k} \to u$ and $v_{n_k}^\oplus \to v^\oplus$ a.e. in $\mathbb{R}^N$ and

$$\left\|u - u_{n_k}\right\|_{W^{h,q}(\mathbb{R}^N)} + \left\|v^\oplus - v_{n_k}^\oplus\right\|_{W^{h,q}(\mathbb{R}^N)} \to 0$$

for any $h \in (0,s)$ and $q \in (0, \frac{N(p-1)}{N-s})$. Combining all above, we may deduce that $u \le v^{\oplus}$ a.e. in $\mathbb{R}^N$ and $v^{\oplus}$ is a nonnegative very weak solution to

$$\begin{cases} Lv^{\oplus} = \mu^+, & \text{in } B_{2R}(x_0), \\ v^{\oplus} = 0, & \text{in } \mathbb{R}^N \setminus B_{2R}(x_0). \end{cases}$$

In addition, in view of the proof of Proposition 2.9, there exists a positive constant $C = C(p, s, \Lambda_K, N)$ such that

$$u(x) \le v^{\oplus}(x) \le C\Big(W^R_{s,p}[\mu^+](x) + \operatorname*{ess\,inf}_{B_{\frac{R}{2}}(x)} v^{\oplus} + \mathrm{Tail}(v^{\oplus}; x, \frac{R}{2})\Big), \quad \text{for a.e. } x \in \Omega, \tag{2.30}$$

where

$$\mathrm{Tail}(v^{\oplus}; x, \frac{R}{2}) = \left(\left(\frac{R}{2}\right)^{sp} \int_{\mathbb{R}^N \setminus B_{\frac{R}{2}}(x)} \frac{|v^{\oplus}(y)|^{p-1}}{|x-y|^{N+sp}} \mathrm{d}y\right)^{\frac{1}{p-1}}.$$

By (2.15) and (2.6), we derive that

$$\operatorname*{ess\,inf}_{B_{\frac{R}{2}}(x)} v^{\oplus} \lesssim \left(\fint_{B_{\frac{R}{2}}(x)} |v^{\oplus}|^{p-1} \mathrm{d}x\right)^{\frac{1}{p-1}} \lesssim R^{-\frac{N-sp}{p-1}} \mu^+(B_R(x))^{\frac{1}{p-1}} \lesssim W^{2R}_{s,p}[\mu^+](x), \tag{2.31}$$

and

$$\mathrm{Tail}(v^{\oplus}; x_0, \frac{R}{2}) \lesssim \left(\fint_{B_{2R}(x_0})|v^{\oplus}|^{p-1} \mathrm{d}x\right)^{\frac{1}{p-1}} \lesssim W^{2R}_{s,p}[\mu^+](x), \quad \forall x \in \Omega, \tag{2.32}$$

where the implicit constants in (2.31) and (2.32) depend only on p, s, Λ_K, N. The inequalities in (2.32) follow by the fact that $v^{\oplus} = 0$ in $\mathbb{R}^N \setminus B_{2R}(x_0)$ and $\mu(\mathbb{R}^N \setminus \Omega) = 0$.

Combining (2.30)–(2.32), we obtain the upper bound in (2.29).

The proof of the lower bound in (2.29) is similar and we omit it. □

3. Nonlocal equations with absorption nonlinearities

3.1. The variational problem

We assume that $g \in C(\mathbb{R})$ and $rg(r) \ge 0$. Let $\Omega \subset \mathbb{R}^N$ be an open bounded domain and $\mu \in (W^{s,p}_0(\Omega))^*$. Set $G(r) = \int_0^r g(s)\mathrm{d}s$,

$$J(v) = \frac{1}{p} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |v(x) - v(y)|^p K(x,y) \mathrm{d}x\mathrm{d}y + \int_{\Omega} G(v)\mathrm{d}x - < \mu, v >$$

and

$$\mathbf{X}_G(\Omega) = \{v \in W^{s,p}_0(\Omega) : \ G(v) \in L^1(\Omega)\}.$$

Theorem 3.1. *Let $s \in (0,1)$, $p > 1$ and $\mu \in (W^{s,p}_0(\Omega))^*$. Then, there exists a minimizer u_μ of J in $\mathbf{X}_G(\Omega)$. Furthermore, u_μ is a weak solution of J, in the sense of*

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |u_\mu(x) - u_\mu(y)|^{p-2}(u_\mu(x) - u_\mu(y))(\zeta(x) - \zeta(y))K(x,y)\mathrm{d}x\mathrm{d}y + \int_{\Omega} g(u_\mu)\zeta \mathrm{d}x =< \mu, \zeta > \tag{3.1}$$

for any $\zeta \in W^{s,p}_0(\Omega) \cap L^\infty(\Omega)$.

If g is nondecreasing the solution u_μ is unique and the mapping $\mu \mapsto u_\mu$ is nondecreasing.

Proof. We adapt the argument used in the proof of [15, Theorem 5.1]. Let $\{v_n\}$ be a minimizing sequence. Taking in to account that $G(t) \geq 0$ for any $t \in \mathbb{R}$ and the fractional Sobolev inequality, we can easily show the existence of a positive constant $C = C(p, \Omega, \Lambda_K)$ such that

$$\|v_n\|^p_{W^{s,p}_0(\Omega)} \leq C(J(v_n) + \|\mu\|^{p'}_{(W^{s,p}_0(\Omega))^*}), \qquad \forall n \in \mathbb{N}. \tag{3.2}$$

This implies that v_n is uniformly bounded in $W^{s,p}_0(\Omega)$. Thus, by the fractional Sobolev embedding theorem (see e.g., [13, Corollary 7.2]) and the fact that $W^{s,p}_0(\Omega)$ is a reflexive Banach space, we may prove the existence of a subsequence, still denoted by $\{v_n\}$ and a function $v \in W^{s,p}_0(\Omega)$ such that there hold:

(i) $v_n \to v$ a.e. in $\mathbb{R}^N$.

(ii) $v_n \rightharpoonup v$ in $W^{s,p}_0(\Omega)$ and $v_n \to v$ in $W^{h,q}(\mathbb{R}^N)$ for any $h \in (0, s)$ and $q \in (1, p)$.

By Fatou's lemma, we obtain

$$\begin{aligned}\frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |v(x) - v(y)|^p K(x,y)\mathrm{d}x\mathrm{d}y + \int_\Omega G(v)\mathrm{d}x \\ \leq \liminf_{k\to\infty} \frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |v_k(x) - v_k(y)|^p K(x,y)\mathrm{d}x\mathrm{d}y + \int_\Omega G(v_k)\mathrm{d}x.\end{aligned}$$

Hence v is a minimizer. If g is nondecreasing, the uniqueness of the minimizer follows by the fact that J is strictly convex.

We next show (3.1). Let v_k be the minimizer of J associated with $g_k = \max(-k, \min(g, k))$. Then, in view of the proof of [11, Theorem 2.3], v_k satisfies

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |v_k(x) - v_k(y)|^{p-2}(v_k(x) - v_k(y))(\zeta(x) - \zeta(y))K(x,y)\mathrm{d}x\mathrm{d}y + \int_\Omega g_k(v_k)\zeta\mathrm{d}x =< \mu, \zeta > \tag{3.3}$$

for any $\zeta \in W^{s,p}_0(\Omega)$. Taking v_k as test function, we have

$$\begin{aligned}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |v_k(x) - v_k(y)|^p K(x,y)\mathrm{d}x\mathrm{d}y + \int_\Omega g(v_k)v_k\mathrm{d}x =< \mu, v_k > \\ \leq \frac{1}{p}\|v_k\|^p_{W^{s,p}_0(\Omega)} + \frac{1}{p'}\|\mu\|^{p'}_{(W^{s,p}_0(\Omega))^*},\end{aligned}$$

which implies

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v_k(x) - v_k(y)|^p}{|x-y|^{N+sp}}\mathrm{d}x\mathrm{d}y + \int_\Omega g_k(v_k)v_k\mathrm{d}x \leq C(\Lambda_K, p)\,\|\mu\|^{p'}_{(W^{s,p}_0(\Omega))^*} =: M. \tag{3.4}$$

By the above inequality, we may deduce that there exists a subsequence, still denoted by $\{v_k\}$ and a function $v \in W^{s,p}_0(\Omega)$ such that they satisfy statements (i) and (ii).

Let $\zeta \in L^\infty(\Omega)$ with $\|\zeta\|_{L^\infty(\Omega)} = N$ and $E \subset \Omega$ be a Borel set. Then, for any $\lambda > 0$, we have

$$\int_{E\cap\{|v_k|>\lambda\}} |\zeta g_k(v_k)|\mathrm{d}x \leq \frac{1}{\lambda}\int_{E\cap\{v_k>\lambda\}} |\zeta||v_k g_k(v_k)|\mathrm{d}x \leq \frac{N}{\lambda}\int_\Omega v_k g_k(v_k)\mathrm{d}x \leq \frac{MN}{\lambda}.$$

Also,

$$\int_{E\cap\{|u_k|\leq\lambda\}} |\zeta g_k(v_k)|\mathrm{d}x \leq |E| N \sup\{|g(t)| : |t| \leq \lambda\}.$$

Let $\varepsilon > 0$, $\lambda = \frac{2MN}{\varepsilon}$ and $\delta = \frac{\varepsilon}{2N\sup\{|g(t)|:|t|\leq\frac{2MN}{\varepsilon}\}+1}$. Then for any Borel set $E \subset \Omega$ with $|E| < \delta$, we have

$$\int_E |\zeta g_k(v_k)|\mathrm{d}x < \varepsilon.$$

Thus, by Vitali's theorem, we conclude

$$\int_\Omega g_k(v_k)\zeta\mathrm{d}x \to \int_\Omega g(v)\zeta\mathrm{d}x. \tag{3.5}$$

Combining all above, we obtain that v satisfies (3.1).

Now for any $u \in \mathbf{X}_G(\Omega)$, we have that $u \in \mathbf{X}_{G_k}(\Omega)$, $G_k(u) \leq G(u)$ and

$$\begin{aligned}
&\frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|v_k(x)-v_k(y)|^pK(x,y)\mathrm{d}x\mathrm{d}y + \int_\Omega G_k(v_k)\mathrm{d}x - <\mu, v_k> \\
&\quad\leq \frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|u(x)-u(y)|^pK(x,y)\mathrm{d}x\mathrm{d}y + \int_\Omega G_k(u)\mathrm{d}\sigma - <\mu, u>,
\end{aligned}$$

where $G_k(r) = \int_0^r g_k(s)\mathrm{d}s$. By the above inequality and Fatou's Lemma, we deduce that v is a minimizer of J in $\mathbf{X}_G(\Omega)$.

Let g be nondecreasing and u_ν be the minimizer of J associated with $\nu \in (W_0^{s,p}(\Omega))^*$, such that $\nu \leq \mu$. Then, using $v_k = \min\{(u_\nu - u_\mu)_+, k\}$ as test function, we have that

$$\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|u_\nu(x)-u_\nu(y)|^{p-2}(u_\nu(x)-u_\nu(y))(v_k(x)-v_k(y))K(x,y)\mathrm{d}x\mathrm{d}y \\
&\quad - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|u_\mu(x)-u_\mu(y)|^{p-2}(u_\mu(x)-u_\mu(y))(v_k(x)-v_k(y))K(x,y)\mathrm{d}x\mathrm{d}y \\
&\qquad = -\int_\Omega(g(u_\nu)-g(u_\mu))v_k\mathrm{d}x + <\nu-\mu, v_k> \leq 0.
\end{aligned}$$

Letting $k \to \infty$ in the above inequality and then proceeding as in the proof of [23, Lemma 6], we obtain that $u_\nu \leq u_\mu$ a.e. in $\mathbb{R}^N$. □

When $\mu \in L^{p'}(\Omega)$, we derive the following result which will be useful in the next subsection.

Lemma 3.2. *Let $\mu \in L^{p'}(\Omega)$, $g \in C(\mathbb{R}^N)$ be a nondecreasing function with $g(0) = 0$ and $u \in W_0^{s,p}(\Omega)$ satisfy* (3.1). *Then there holds,*

$$\int_\Omega |g(u)|\mathrm{d}x \leq \int_\Omega |\mu|\mathrm{d}x. \tag{3.6}$$

In addition, if we assume that $\mu \geq 0$, then $u \geq 0$ a.e. in $\mathbb{R}^N$.

Proof. Let $k > 0$. Using $\phi_k = \tanh(ku)$ as test function in (3.1), we obtain

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|u(x)-u(y)|^{p-2}(u(x)-u(y))(\phi_k(x)-\phi_k(y))K(x,y)\mathrm{d}x\mathrm{d}y + \int_\Omega g(u)\phi_k\mathrm{d}x = \int_\Omega \mu\phi_k\mathrm{d}x.$$

If $\infty > u(x) > u(y) > -\infty$, then there exists $\xi \in (u(y), u(x))$ such that

$$\phi_k(x) - \phi_k(y) = (1-\tanh^2(k\xi))(u(x)-u(y)) \geq c(\xi,k)(u(x)-u(y)).$$

Combining the last two displays, we can easily obtain that

$$\int_\Omega g(u)\phi_k \mathrm{d}x \le \int_\Omega |\mu| \mathrm{d}x.$$

Since $g(u)\phi_k \ge 0$ a.e. in Ω, by Fatou's lemma and the above inequality, we can easily deduce (3.6). □

3.2. Subcritical nonlinearities

In this subsection, we always assume that $s \in (0,1)$, $1 < p < \frac{N}{s}$ and $g \in C(\mathbb{R})$ is nondecreasing such that $g(0) = 0$.

Lemma 3.3. *Let $g \in L^\infty(\mathbb{R})$ and $\lambda_i \in \mathfrak{M}_b^+(\Omega)$ $(i = 1, 2)$. Then there exist very weak solutions u, u_i $(i = 1, 2)$ to problems*

$$\begin{cases} Lu + g(u) = \lambda_1 - \lambda_2, & \text{in } \Omega, \\ \qquad u = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{3.7}$$

$$\begin{cases} Lu_1 + g(u_1) = \lambda_1, & \text{in } \Omega, \\ \qquad u = 0, & \text{in } \mathbb{R}^N \setminus \Omega \end{cases} \tag{3.8}$$

and

$$\begin{cases} Lu_2 - g(-u_2) = \lambda_2, & \text{in } \Omega, \\ \qquad u = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{3.9}$$

such that there hold

$$u_1, u_2 \ge 0 \quad \text{and} \quad -u_2 \le u \le u_1, \quad \text{a.e. in } \mathbb{R}^N. \tag{3.10}$$

In addition, for any $q \in (0, \frac{N(p-1)}{N-s})$ and $h \in (0, s)$, there exists a positive constant $c = c(N, p, s, \Lambda_K, q, h, |\Omega|)$ such that

$$\left(\int_\Omega |g(u)| \mathrm{d}x\right)^{\frac{1}{p-1}} + \left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^q}{|x-y|^{N+hq}} \mathrm{d}x\mathrm{d}y\right)^{\frac{1}{q}} \le c(\lambda_1(\Omega) + \lambda_2(\Omega))^{\frac{1}{p-1}} \tag{3.11}$$

and

$$\left(\int_\Omega |g((-1)^{i+1}u_i)| \mathrm{d}x\right)^{\frac{1}{p-1}} + \left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_i(x)-u_i(y)|^q}{|x-y|^{N+hq}} \mathrm{d}x\mathrm{d}y\right)^{\frac{1}{q}} \le c\lambda_i(\Omega)^{\frac{1}{p-1}}. \tag{3.12}$$

Finally, there exist very weak solutions v_i to (2.2) with $\mu = \lambda_i$ (i=1,2) such that

$$0 \le u_i \le v_i \le C_i W_{s,p}^{2\mathrm{diam}(\Omega)}[\lambda_i], \quad \text{a.e. in } \Omega, \tag{3.13}$$

where C_i is a positive constant depending only on p, s, Λ_K and N.

Proof. Let $\{\rho_n\}_1^\infty$ be a sequence of mollifiers and $\lambda_{n,i} = \rho_n * \lambda_i$. Then $\lambda_{n,i} \in C_0^\infty(\mathbb{R}^N)$. By Proposition 3.1, there exist unique solutions $u_n, u_{n,i}, v_{n,i} \in W_0^{s,p}(\Omega)$ to the following problems

$$\begin{cases} Lu_n + g(u_n) = \lambda_{n,1} - \lambda_{n,2}, & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

$$\begin{cases} Lu_{n,1} + g(u_{n,1}) = \lambda_{n,1}, & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

$$\begin{cases} Lu_{n,2} - g(-u_{n,2}) = \lambda_{n,2}, & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

and

$$\begin{cases} Lv_{n,i} = \lambda_{n,i} & \text{in } \Omega \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

such that there holds

$$-v_{n,2} \le -u_{n,2} \le u_n \le u_{n,1} \le v_{n,1}, \quad \text{a.e. in } \mathbb{R}^N. \tag{3.14}$$

By Lemma 3.2 and Proposition 2.7, for any $q \in (0, \frac{N(p-1)}{N-s})$ and $h \in (0, s)$, there exists a positive constant $c = c(N, p, s, \Lambda_K, q, h, |\Omega|)$ such that

$$\left(\int_\Omega |g(u_n)|\mathrm{d}x\right)^{\frac{1}{p-1}} + \left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^q}{|x-y|^{N+hq}}\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{q}} \le c\left(\int_\Omega \lambda_{n,1} + \lambda_{n,2}\mathrm{d}x\right)^{\frac{1}{p-1}}, \tag{3.15}$$

$$\left(\int_\Omega |g(u_{n,1})|\mathrm{d}x\right)^{\frac{1}{p-1}} + \left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_{n,1}(x) - u_{n,1}(y)|^q}{|x-y|^{N+hq}}\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{q}} \le c\left(\int_\Omega \lambda_{n,1}\mathrm{d}x\right)^{\frac{1}{p-1}}, \tag{3.16}$$

$$\left(\int_\Omega |g(-u_{n,2})|\mathrm{d}x\right)^{\frac{1}{p-1}} + \left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_{n,2}(x) - u_{n,2}(y)|^q}{|x-y|^{N+hq}}\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{q}} \le c\left(\int_\Omega \lambda_{n,2}\mathrm{d}x\right)^{\frac{1}{p-1}} \tag{3.17}$$

and

$$\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v_{n,i}(x) - v_{n,i}(y)|^q}{|x-y|^{N+hq}}\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{q}} \le c\left(\int_\Omega \lambda_{n,i}\mathrm{d}x\right)^{\frac{1}{p-1}}. \tag{3.18}$$

Furthermore, in view of the proof of (2.16), we have that $T_k(u_n), T_k(u_{n,i}), T_k(v_{n,i}) \in W_0^{s,p}(\Omega)$ and satisfy (2.19) with $\mu = \lambda_1 + \lambda_2$.

Since the sequences $\{\lambda_{n,i}\}_n$ are uniformly bounded in $\mathfrak{M}_b(\Omega)$, as in the proof of Proposition 2.8, we may show that there exist subsequences, still denoted by the same index, such that $u_n \to u$, $u_{n,i} \to u_i$

$v_{n,i} \to v_i$ in $W^{h,q}(\mathbb{R}^N)$ and a.e. in $\mathbb{R}^N$. In addition, we may prove that $T_k(u), T_k(u_i), T_k(v_i) \in W_0^{s,p}(\Omega)$ for any $k > 0$. Finally, by dominated convergence theorem, we deduce that $g(u_n) \to g(u)$, $g(u_{n,1}) \to g(u_1)$, $g(-u_{n,2}) \to g(-u_2)$ in $L^1(\Omega)$. Hence, combining all above, we can easily show that u, u_i are very weak solutions of problems (3.7)–(3.9) respectively and v_i are very weak solutions of problem (2.2) with $\mu = \lambda_i$ $(i = 1, 2)$.

By proceeding as in the proof of Proposition 2.10 and using (3.14), we derive (3.13).

Estimates (3.11) and (3.12) follow by (3.15), (3.16) and Fatou's lemma. □

Lemma 3.4. *Let $\lambda_i \in \mathfrak{M}_b^+(\Omega)$ for $i = 1, 2$. We also assume that $g((-1)^{1+i} C W_{s,p}^{2R}[\lambda_i]) \in L^1(\Omega)$, where C is the constant in Proposition 2.10. Then the conclusion of Lemma 3.3 holds true.*

Proof. Let $T_n(t) = \max(-n, \min(t, n))$ for any $n \in \mathbb{N}$. By Lemma 3.3, there exist very weak solutions $u_n, u_{n,i}, v_{n,i} \in W_0^{s,p}(\Omega)$ of the following problems

$$\begin{cases} Lu_n + T_n o g(u_n) = \lambda_1 - \lambda_2 & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

$$\begin{cases} Lu_{n,1} + T_n o g(u_{n,1}) = \lambda_1, & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

$$\begin{cases} Lu_{n,2} - T_n o g(-u_{n,2}) = \lambda_2, & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega \end{cases}$$

and

$$\begin{cases} Lv_i = \lambda_i, & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

such that there holds

$$-C W_{1,p}^{2\mathrm{diam}\,(\Omega)}[\lambda_2] \le -v_2 \le -u_{n,2} \le u_n \le u_{n,1} \le v_1 \le C W_{1,p}^{2\mathrm{diam}\,(\Omega)}[\lambda_1], \quad \text{a.e. in } \mathbb{R}^N$$

and for any $n \in \mathbb{N}$. The rest of the proof can proceed similarly to the proof of Lemma 3.3 and we omit it. □

Proposition 3.5. *Assume*

$$\Lambda_g := \int_1^\infty s^{-\tilde{q}-1}(g(s) - g(-s))\mathrm{d}s < \infty \tag{3.19}$$

for $\tilde{q} > 0$. Let v be a measurable function defined in Ω. For $s > 0$, set

$$E_s(v) := \{x \in \Omega : |v(x)| > s\} \quad \textit{and} \quad e(s) := |E_s(v)|.$$

Assume that there exists a positive constant C_0 such that

$$e(s) \le C_0 s^{-\tilde{q}}, \quad \forall s \ge 1. \tag{3.20}$$

Then for any $s_0 \geq 1$, *there hold*

$$\|g(|v|)\|_{L^1(\Omega)} \leq \int_{\Omega\setminus E_{s_0}(v)} g(|v|)dx + \tilde{q}C_0 \int_{s_0}^{\infty} s^{-\tilde{q}-1} g(s)\mathrm{d}s,$$

$$\|g(-|v|)\|_{L^1(\Omega)} \leq -\int_{\Omega\setminus E_{s_0}(v)} g(-|v|)\mathrm{d}x - \tilde{q}C_0 \int_{s_0}^{\infty} s^{-\tilde{q}-1} g(-s)\mathrm{d}s.$$

Proof. The proof is very similar to the one of [16, Lemma 5.1] and we omit it. □

Proof of Theorem 1.2. Let $\lambda_1 = \mu^+$ and $\lambda_2 = \mu^-$. By Lemma 3.3, there exist very weak solutions u_n, v_i of the following problems

$$\begin{cases} Lu_n + T_n o g(u_n) = \lambda_1 - \lambda_2, & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega \end{cases}$$

and

$$\begin{cases} Lv_i = \lambda_i, & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

such that there holds

$$-v_2 \leq u_n \leq v_1, \quad \text{a.e. in } \mathbb{R}^N \text{ and } \forall n \in \mathbb{N}.$$

Also, taking into consideration that g in nondecreasing with $g(0) = 0$, we may show that $T_k(u_n), T_k(v_i)$ satisfy (2.19) with $\mu = \lambda_1 + \lambda_2$. In addition, by (2.15), there holds

$$\left\|v_1^{p-1}\right\|^*_{L_w^{\frac{N}{N-sp}}(\mathbb{R}^N)} + \left\|v_2^{p-1}\right\|^*_{L_w^{\frac{N}{N-sp}}(\mathbb{R}^N)} \leq C_1(N, p, s, \Lambda_K)(\lambda_1(\Omega) + \lambda_2(\Omega)).$$

By (2.7) and Proposition 3.5, we have that $|T_n o g(u_n)| \leq g(v_1) - g(-v_2)$ and

$$\begin{aligned}\|T_n o g(u_n)\|_{L^1(\Omega)} &\leq \|g(v_1)\|_{L^1(\Omega)} + \|g(-v_2)\|_{L^1(\Omega)} \\ &\leq (g(s_0) - g(s_0))|\Omega| \\ &+ \tilde{q}C_1(N, p, s, \Lambda_K, \Lambda_g)(\lambda_1(\Omega) + \lambda_2(\Omega))^{\frac{N(p-1)}{N-sp}} \int_{s_0}^{\infty} s^{-\tilde{q}-1}(g(s) - g(-s)\mathrm{d}s, \quad \forall n \in \mathbb{N},\end{aligned}$$

where $\tilde{q} = \frac{N(p-1)}{N-sp}$. The desired result follows by proceeding as in the proof of Lemma 3.3. □

3.3. Power nonlinearities: proof of Theorem 1.3

In order to prove Theorem 1.3, we need to introduce some notations concerning the Bessel capacities, we refer the reader to [1] for more detail. For $\alpha \in \mathbb{R}$ we define the Bessel kernel of order α by $G_\alpha(\xi) = \mathcal{F}^{-1}(1 + |.|^2)^{-\frac{\alpha}{2}}(\xi)$, where $\mathcal{F}$ is the Fourier transform of moderate distributions in $\mathbb{R}^N$. For any $\beta > 1$, the Bessel space $L_{\alpha,\beta}(\mathbb{R}^N)$ is given by

$$L_{\alpha,\beta}(\mathbb{R}^N) := \{f = G_\alpha * g : g \in L^\beta(\mathbb{R}^N)\},$$

with norm

$$\|f\|_{L_{\alpha,\beta}(\mathbb{R}^N)} := \|g\|_{L^\beta(\mathbb{R}^N)} = \|G_{-\alpha} * f\|_{L^\beta(\mathbb{R}^N)}.$$

The Bessel capacity is defined as follows.

Definition 3.6. Let $\alpha > 0$, $1 < \beta < \infty$ and $E \subset \mathbb{R}^N$. Set

$$\mathcal{S}_E := \{g \in L^\beta(\mathbb{R}^N) : \; g \geq 0, \; G_\alpha * g(x) \geq 1 \; \text{ for any } x \in E\}.$$

Then

$$\mathrm{Cap}_{\alpha,\beta}(E) := \inf\{\|g\|^\beta_{L^\beta(\mathbb{R}^N)}; g \in \mathcal{S}_E\}. \tag{3.21}$$

If $\mathcal{S}_E = \emptyset$, we set $\mathrm{Cap}_{\alpha,\beta}(E) = \infty$.

In the sequel, we denote by $L_{-\alpha,\beta'}(\mathbb{R}^N)$ the dual of $L_{\alpha,\beta}(\mathbb{R}^N)$ and we set

$$\mathbb{G}_\alpha[\mu](x) = \int_{\mathbb{R}^N} G_\alpha(x,y)\mathrm{d}\mu(y), \quad \forall \mu \in \mathfrak{M}(\mathbb{R}^N).$$

Proof of Theorem 1.3. Since μ is absolutely continuous with respect to the capacity $\mathrm{Cap}_{sp,\frac{\kappa}{\kappa-p+1}}$, the measures μ^+, μ^- have the same property. Thus, by [5, Theorem 2.5] (see also [3]), there are nondecreasing sequences $\{\mu_n^\pm\}_n \subset L^{-sp,\frac{\kappa}{p-1}}(\mathbb{R}^N) \cap \mathfrak{M}_b^+(\mathbb{R}^N)$ with compact support in Ω, such that they converge to $\mu^\pm$ in the narrow topology. Furthermore, by [5, Theorem 2.3] (see also [1, Corollary 3.6.3]),

$$\left\|W^{2\mathrm{diam}\,(\Omega)}_{\alpha,p}[\mu_n^\pm]\right\|^\kappa_{L^\kappa(\mathbb{R}^N)} \approx \left\|\mathbb{G}_{sp}[\mu_n^\pm]\right\|^{\frac{\kappa}{p-1}}_{L^{\frac{\kappa}{p-1}}(\mathbb{R}^N)} < \infty.$$

By Lemma 3.4, there exist solutions $u_n, u_{n,i}, v_i$ to the problems

$$\begin{cases} Lu_n + |u_n|^{\kappa-1}u_n = \lambda_{n,1} - \lambda_{n,2}, & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{3.22}$$

$$\begin{cases} Lu_{n,1} + |u_{n,1}|^{\kappa-1}u_{n,1} = \lambda_{n,1}, & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{3.23}$$

$$\begin{cases} Lu_{n,2} + |u_{n,2}|^{\kappa-1}u_{n,2} = \lambda_{n,2}, & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{3.24}$$

and

$$\begin{cases} Lv_{n,i} = \lambda_{n,i}, & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

such that there holds

$$-v_{n,2} \leq -u_{n,2} \leq u_n \leq u_{n,1} \leq v_{n,1}, \quad \text{a.e. in } \mathbb{R}^N. \tag{3.25}$$

Furthermore, in view of the proof of Lemmas 3.3 and 3.4, the sequences $\{u_{n,i}\}, \{v_{n,i}\}$ satisfy (3.15)–(3.18) with $g(t) = |t|^\kappa \mathrm{sign}(t)$, $\lambda_{n,1} = \mu_n^+$ and $\lambda_{n,2} = \mu_n^-$, as well as they can be constructed such that

$$u_{n,i} \leq u_{n+1,i} \quad \text{and} \quad v_{n,i} \leq v_{n+1,i}, \quad \text{a.e. in } \mathbb{R}^N, \forall n \in \mathbb{N} \text{ and } i = 1,2. \tag{3.26}$$

By (3.15) and (3.16) with $g(t) = |t|^\kappa \mathrm{sign}(t)$, $\lambda_{n,1} = \mu_n^+$ and $\lambda_{n,2} = \mu_n^-$, we have

$$\int_\Omega |u_{n,1}|^\kappa \mathrm{d} \le \mu^+(\Omega) \quad \text{and} \quad \int_\Omega |u_{n,2}|^\kappa \mathrm{d} \le \mu^-(\Omega), \quad \forall n \in \mathbb{N}.$$

By (3.15)–(3.18) with $g(t) = |t|^\kappa \mathrm{sign}(t)$, $\lambda_{n,1} = \mu_n^+$ and $\lambda_{n,2} = \mu_n^-$, there are subsequences, still denoted by the same index, such that $u_n \to u$, $u_{n,i} \to u_i$ $v_{n,i} \to v$ in $W^{h,q}(\mathbb{R}^N)$ and a.e. in $\mathbb{R}^N$. In addition, $T_k(u), T_k(u_i), T_k(v_i) \in W_0^{s,p}(\mathbb{R}^N)$ and

$$\int_\Omega |u_1|^k \mathrm{d}x \le \mu^+(\Omega), \quad \text{and} \quad \int_\Omega |u_2|^k \mathrm{d}x \le \mu^-(\Omega).$$

Therefore, by dominated convergence theorem, we obtain that $|u_n|^\kappa \to |u|^\kappa$, $|u_{n,1}|^\kappa \to |u_1|^\kappa$, $|u_{n,2}|^\kappa \to |u_2|^\kappa$ in $L^1(\Omega)$. This, implies that u, u_i are very weak solutions of problems (3.7)–(3.9) respectively and v_i are very weak solution of problem (2.2) with $\mu = \lambda_i$, where $\lambda_1 = \mu^+$ and $\lambda_2 = \mu^-$.

Estimate (1.13) follows by (3.25) and (3.13). Estimate (1.14) follows by (3.15) with $g(t) = |t|^\kappa \mathrm{sign}(t)$, $\lambda_{n,1} = \mu_n^+$, $\lambda_{n,2} = \mu_n^-$ and Fatou's lemma. □

4. Nonlocal equations with source nonlinearities

4.1. Subcritical nonlinearities

In this subsection, we investigate the existence of solutions to the following problem

$$\begin{cases} Lv = g(v) + \rho\tau, & \text{in } \Omega, \\ v = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{4.1}$$

where $\rho > 0$, $g \in C(\mathbb{R})$ is a nondecreasing function and

$$|g(t)| \le a|t|^d \quad \text{for some } a > 0,\ d > p-1 \text{ and for any } |t| \le 1. \tag{4.2}$$

Let us state the first existence result.

Lemma 4.1. *Let* $1 < p < \frac{N}{s}$ *and* $\tau \in C_0^\infty(\mathbb{R}^N)$ *be such that* $\|\tau\|_{L^1(\mathbb{R}^N)} \le 1$. *Assume that* $g \in L^\infty(\Omega) \cap C(\mathbb{R})$ *satisfies* (3.19) *for*

$$\tilde{q} = \frac{N(p-1)}{N-sp}.$$

In addition, we assume that g is nondecreasing and satisfies (4.2).

Then there exists a positive constant ρ_0 *depending on* $N, \Omega, \Lambda_g, \Lambda_K, a, d, p, s$ *such that for every* $\rho \in (0, \rho_0)$, *problem* (4.1) *admits a weak solution* $v \in W_0^{s,p}(\Omega)$ *satisfying*

$$\||v|^{p-1}\|_{L_w^{\frac{N}{N-sp}}(\Omega)} \le t_0, \tag{4.3}$$

where $t_0 > 0$ *depends on* $N, \Omega, \Lambda_g, \Lambda_K, a, d, p, s$.

Proof. We shall use Schauder fixed point theorem to show the existence of a positive weak solution of (4.1).

Let $1 < \kappa < \min\{\frac{N}{N-sp}, \frac{d}{p-1}\}$ and $v \in L^1(\Omega)$. Since $g \in L^\infty(\Omega)$, we can easily show that the following problem

$$\begin{cases} Lu = g(|v|^{\frac{1}{p-1}}\operatorname{sign}(v)) + \rho\tau, & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{4.4}$$

admits a unique weak solution $\mathbb{T}(v) \in W_0^{s,p}(\Omega)$.

We define the operator $\mathbb{S}$ by

$$\mathbb{S}(v) := |\mathbb{T}(v)|^{p-1}\operatorname{sign}(\mathbb{T}(v)), \quad \forall v \in L^1(\Omega). \tag{4.5}$$

By (2.8), we obtain

$$\begin{aligned} \|\mathbb{S}(v)\|_{L_w^{\frac{N}{N-sp}}(\Omega)} &\le C(s,p,N,\Lambda_K)\left(\rho\int_\Omega |\tau|\mathrm{d}x + \int_\Omega |g(|v|^{\frac{1}{p-1}}\operatorname{sign}(v))|\mathrm{d}x\right) \\ &\le C(s,p,N,\Lambda_K)\left(\rho + \int_\Omega g(|v|^{\frac{1}{p-1}}) - g(-|v|^{\frac{1}{p-1}})\mathrm{d}x\right). \end{aligned} \tag{4.6}$$

Let $v \in L_w^{\frac{N}{N-sp}}(\Omega)$. For any $\lambda > 0$, we set $E_\lambda := \{x \in \Omega : |v(x)|^{\frac{1}{p-1}} > \lambda\}$ and $e(\lambda) = \int_{E_\lambda} \mathrm{d}x$. By (2.4) and (2.6), we can easily show that

$$e(\lambda) \le C(N,s,p)\, \|v\|^{\frac{N}{N-sp}}_{L_w^{\frac{N}{N-sp}}(\Omega)} \lambda^{-\frac{N(p-1)}{N-sp}}.$$

By the above inequality and Lemma 3.5 with $\lambda_0 = 1$ and $\tilde{q} = \frac{N(p-1)}{N-sp}$, we deduce

$$\int_\Omega g(|v|^{\frac{1}{p-1}}) - g(-|v|^{\frac{1}{p-1}})\mathrm{d}x \le 2a\int_\Omega |v|^\kappa \mathrm{d}x + C(p,s,N)\, \|v\|^{\frac{N}{N-sp}}_{L_w^{\frac{N}{N-sp}}(\Omega)} \Lambda_g.$$

Let $\lambda = \|v\|_{L_w^{\frac{N}{N-sp}}(\Omega)}$. By (2.6), we have that

$$\begin{aligned} \int_\Omega |v|^\kappa \mathrm{d} &= \int_0^\infty |\{x \in \Omega : \ |v| \ge t\}|\mathrm{d}t^\kappa \\ &= \int_0^\lambda |\{x \in \Omega : \ |v| \ge t\}|\mathrm{d}t^\kappa + \int_\lambda^\infty |\{x \in \Omega : \ |v| \ge t\}|\mathrm{d}t^\kappa \\ &\le |\Omega|\lambda^\kappa + \kappa\lambda^{\frac{N}{N-sp}} \int_\lambda^\infty t^{\kappa - \frac{N}{N-sp} - 1}\mathrm{d}t \le C(\Omega,\kappa,s,p,N)\lambda^k. \end{aligned}$$

Combining all above, we may prove that

$$\|\mathbb{S}(v)\|_{L_w^{\frac{N}{N-sp}}(\Omega)} \le C(p,N,\kappa,|\Omega|,\Lambda_g,\Lambda_K,a)\left(\rho + \|v\|^{\frac{N}{N-sp}}_{L_w^{\frac{N}{N-sp}}(\Omega)} + \|v\|^\kappa_{L_w^{\frac{N}{N-sp}}(\Omega)}\right).$$

Therefore, if $\|v\|_{L_w^{\frac{N}{N-sp}}(\Omega)} \le t$ then

$$\|\mathbb{S}(v)\|_{L_w^{\frac{N}{N-sp}}(\Omega)} \le C\left(t^{\frac{N}{N-sp}} + t^{\kappa} + \rho\right). \tag{4.7}$$

Since $1 < \kappa < \frac{N}{N-sp}$, there exist $t_0 > 0$ and $\rho_0 > 0$ depending on $|\Omega|, \Lambda_g, p, \kappa, N, a$ such that for any $t \in (0, t_0]$ and $\rho \in (0, \rho_0)$, the following inequality holds

$$C\left(t^{\frac{N}{N-sp}} + t^{\kappa} + \rho\right) \le t_0,$$

where C is the constant in (4.7). Hence,

$$\|v\|_{L_w^{\frac{N}{N-sp}}(\Omega)} \le t_0 \Longrightarrow \|\mathbb{S}(v)\|_{L_w^{\frac{N}{N-sp}}(\Omega)} \le t_0. \tag{4.8}$$

Next, we apply Schauder fixed point theorem to our setting.

We claim that $\mathbb{S}$ is continuous. First we assume that $v_n \to v$ in $L^1(\Omega)$ and $\mathbb{T}(v_n) \to \mathbb{T}(v)$ in $W_0^{1,p}(\Omega)$, then by fractional Sobolev inequality, we have

$$\begin{aligned}\int_\Omega |\mathbb{T}(v_n) - \mathbb{T}(v)| dx &\le |\Omega|^{\frac{pN-N+sp}{Np}} \|\mathbb{T}(v_n) - \mathbb{T}(v)\|_{L^{\frac{Np}{N-sp}}(\Omega)} \\ &\le C|\Omega|^{\frac{pN-N+sp}{Np}} \|\mathbb{T}(v_n) - \mathbb{T}(v)\|_{W_0^{1,p}(\Omega)} \to 0.\end{aligned} \tag{4.9}$$

Let $k > 0$ and $\varepsilon > 0$, then

$$\begin{aligned}\int_\Omega |\mathbb{S}(v_n) - \mathbb{S}(v)| dx &= \int_{\{x\in\Omega:\ |\mathbb{S}(v_n)(x)|\le k\}\cap\{x\in\Omega:\ |\mathbb{S}(v)(x)|\le k\}} |\mathbb{S}(v_n) - \mathbb{S}(v)|\, dx \\ &+ \int_{\Omega\setminus(\{x\in\Omega:\ |\mathbb{S}(v_n)(x)|\le k\}\cap\{x\in\Omega:\ |\mathbb{S}(v)(x)|\le k\})} |\mathbb{S}(v_n)(x) - \mathbb{S}(v)(x)|\, dx.\end{aligned} \tag{4.10}$$

By (4.6) and the fact that $g \in L^\infty(\mathbb{R})$, we have that $\mathbb{S}(v_n) \in L^\beta(\Omega)$ and $\{\mathbb{S}(v_n)\}$ is uniformly bounded in $L^\beta(\Omega)$ for any $\beta \in (1, \frac{N}{N-sp})$. Hence, there exists $k_0 \in \mathbb{N}$, such that

$$\int_{\Omega\setminus(\{x\in\Omega:\ |\mathbb{S}(v_n)(x)|\le k\}\cap\{x\in\Omega:\ |\mathbb{S}(v)(x)|\le k\})} |\mathbb{S}(v_n) - \mathbb{S}(v)|\, dx \le \frac{\varepsilon}{3} \quad \forall k \ge k_0, \quad \text{and} \quad n \in \mathbb{N}. \tag{4.11}$$

Now, we set

$$A_{k_0,n} = \{x \in \Omega :\ |\mathbb{T}(v_n)(x)| \le k_0^{\frac{1}{p-1}}\} \cap \{x \in \Omega :\ |\mathbb{T}(v)(x)| \le k_0^{\frac{1}{p-1}}\}$$

and $B_{\delta,n} = \{x \in \Omega :\ |\mathbb{T}(v)(x) - \mathbb{T}(v_n)(x)| \le \delta\}$. Then, we have that

$$\begin{aligned}&\int_{\Omega\cap\{x\in\Omega:\ |\mathbb{S}(v_n)|\le k_0\}\cap\{x\in\Omega:\ |\mathbb{S}(v)|\le k_0\}} |\mathbb{S}(v_n) - \mathbb{S}(v)|\, dx \\ &= \int_{A_{k_0,n}\cap B_{\delta,n}} \left||\mathbb{T}(v_n)|^{p-1}\text{sign}(\mathbb{T}(v_n)) - |\mathbb{T}(v)|^{p-1}\text{sign}(\mathbb{T}(v))\right| dx \\ &+ \int_{A_{k_0,n}\setminus B_{\delta,n}} \left||\mathbb{T}(v_n)|^{p-1}\text{sign}(\mathbb{T}(v_n)) - |\mathbb{T}(v)|^{p-1}\text{sign}(\mathbb{T}(v))\right| dx.\end{aligned} \tag{4.12}$$

Since $h(t) = t^{p-1}\text{sign}(t)$ is uniformly continuous in $[-k_0, k_0]$, there exists $\delta_0 > 0$ independent of n such that

$$\int_{A_{k_0,n}\cap B_{\delta_0,n}} \left| |\mathbb{T}(v_n)|^{p-1}\text{sign}(\mathbb{T}(v_n)) - |\mathbb{T}(v)|^{p-1}\text{sign}(\mathbb{T}(v)) \right| \mathrm{d}x \le \frac{\varepsilon}{3}. \tag{4.13}$$

Moreover, by (4.9), there exists $n_0 = n_0(\delta_0, k_0, p) \in \mathbb{N}$ such that

$$\int_{A_{k_0,n_0}\setminus B_{\delta_0,n_0}} \left| |\mathbb{T}(v_{n_0})|^{p-1}\text{sign}(\mathbb{T}(v_{n_0})) - |\mathbb{T}(v)|^{p-1}\text{sign}(\mathbb{T}(v)) \right| \mathrm{d}x \le \frac{\varepsilon}{3}. \tag{4.14}$$

Hence, combining (4.9)–(4.14), we obtain that $\mathbb{S}(v_n) \to \mathbb{S}(v)$ in $L^1(\Omega)$.

Therefore, it is enough to show that $\mathbb{T}(v_n) \to \mathbb{T}(v)$ in $W_0^{s,p}(\Omega)$. In order to prove this, we will consider two cases.

Case 1. $1 < p < 2$. Let $M := \sup_{t\in\mathbb{R}} |g(t)|$. We will show that $\mathbb{T}(v_n) \to \mathbb{T}(v)$ in $W_0^{s,p}(\Omega)$. Since $\mathbb{T}(v_n), \mathbb{T}(v) \in W_0^{s,p}(\Omega)$ are weak solutions of (4.4) with v_n and v respectively, we have

$$\begin{aligned}
\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} &|\mathbb{T}(v_n)(x) - \mathbb{T}(v_n)(y)|^p K(x,y)\mathrm{d}x\mathrm{d}y = \int_\Omega \mathbb{T}(v_n)(g(|v_n|^{\frac{1}{p-1}}\text{sign}(v_n))\mathrm{d}x + \int_\Omega \mathbb{T}(v_n)\tau \mathrm{d}x \\
&\le M|\Omega|^{\frac{p-1}{p}} \left(\int_\Omega |\mathbb{T}(v_n)|^p \mathrm{d}x\right)^{\frac{1}{p}} + \left(\int_\Omega |\mathbb{T}(v_n)|^p \mathrm{d}x\right)^{\frac{1}{p}} \left(\int_\Omega |\tau|^{\frac{p}{p-1}} \mathrm{d}x\right)^{\frac{p-1}{p}} \\
&\le C_1(M,\Omega,p,N,\tau,s)\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\mathbb{T}(v_n)(x) - \mathbb{T}(v_n)(y)|^p}{|x-y|^{N+sp}}\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{p}}.
\end{aligned} \tag{4.15}$$

Therefore,

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\mathbb{T}(v_n)(x) - \mathbb{T}(v_n)(y)|^p}{|x-y|^{N+sp}}\mathrm{d}x\mathrm{d}y \le C_1^{\frac{p-1}{p}}(M,\Omega,p,N,\tau,s,\Lambda_K). \tag{4.16}$$

Using $\phi = \mathbb{T}(v_n) - \mathbb{T}(v)$ as test function, we have

$$\begin{aligned}
I &:= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |\mathbb{T}(v_n)(x) - \mathbb{T}(v_n)(y)|^{p-2}(\mathbb{T}(v_n)(x) - \mathbb{T}(v_n)(y))\,(\phi(x) - \phi(y))\,K(x,y)\mathrm{d}x\mathrm{d}y \\
&- \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |\mathbb{T}(v)(x) - \mathbb{T}(v)(y)|^{p-2}(\mathbb{T}(v)(x) - \mathbb{T}(v)(y))\,(\phi(x) - \phi(y))\,K(x,y)\mathrm{d}x\mathrm{d}y \\
&= \int_\Omega \phi(g(|v_n|^{\frac{1}{p-1}}\text{sign}(v_n)) - g(|v|^{\frac{1}{p-1}}\text{sign}(v)))\mathrm{d}x =: II.
\end{aligned} \tag{4.17}$$

We first treat I. On one hand, since

$$(|a|^{p-2}a - |b|^{p-2}b)(a-b) \ge C(p)\frac{|a-b|^2}{(|a|+|b|)^{2-p}}$$

for any $(a,b) \in \mathbb{R}^{2N} \setminus \{(0,0)\}$ and $p \in (1,2)$, we have

$$I \ge C(p)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |\phi(x) - \phi(y)|^2\,(|\mathbb{T}(v_n)(x) - \mathbb{T}(v_n)(y)| + |\mathbb{T}(v)(x) - \mathbb{T}(v)(y)|)^{p-2}\,K(x,y)\mathrm{d}x\mathrm{d}y. \tag{4.18}$$

On the other hand, by Hölder inequality, we obtain

$$\begin{aligned}&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\phi(x)-\phi(y)|^p}{|x-y|^{N+sp}}\mathrm{d}x\mathrm{d}y\le\Lambda_K\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|\phi(x)-\phi(y)|^p K(x,y)\mathrm{d}x\mathrm{d}y\\&\le C(p,\Lambda_K)\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}(|\mathbb{T}(v_n)(x)-\mathbb{T}(v_n)(y)|+|\mathbb{T}(v)(x)-\mathbb{T}(v)(y)|)^p K(x,y)\mathrm{d}x\mathrm{d}y\right)^{\frac{2-p}{2}}I^{\frac{p}{2}}\\&\le C(p,C_1,\Omega,\Lambda_K)I^{\frac{p}{2}},\end{aligned}\tag{4.19}$$

where C_1 is the constant in (4.16). Hence, by (4.18) and (4.19), we obtain

$$C\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\phi(x)-\phi(y)|^p}{|x-y|^{N+sp}}\mathrm{d}x\mathrm{d}y\right)^{\frac{2}{p}}\le I.\tag{4.20}$$

Next we treat II. Let $r=\frac{Np}{N-sp}$, proceeding as in the proof of (4.15), we have

$$\begin{aligned}II&\le\left(\int_\Omega|\phi|^r\mathrm{d}\right)^{\frac{1}{r}}\left(\int_\Omega|g(|v_n|^{\frac{1}{p-1}}\operatorname{sign}(v_n))-g(|v|^{\frac{1}{p-1}}\operatorname{sign}(v))|^{r'}\mathrm{d}\right)^{\frac{1}{r'}}\\&\le C(N,p,s)\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\phi(x)-\phi(y)|^p}{|x-y|^{N+sp}}\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{p}}\\&\times\left(\int_\Omega|g(|v_n|^{\frac{1}{p-1}}\operatorname{sign}(v_n))-g(|v|^{\frac{1}{p-1}}\operatorname{sign}(v))|^{r'}\mathrm{d}\right)^{\frac{1}{r'}},\end{aligned}\tag{4.21}$$

where in the last inequality we used the fractional Sobolev inequality.

Combining (4.17), (4.20) and (4.21), we obtain

$$\begin{aligned}&\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\phi(x)-\phi(y)|^p}{|x-y|^{N+sp}}\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{p}}\\&\qquad\le C(p,C_1,\Omega,s,\Lambda_K)\left(\int_\Omega|g(|v_n|^{\frac{1}{p-1}}\operatorname{sign}(v_n))-g(|v|^{\frac{1}{p-1}}\operatorname{sign}(v))|^{r'}\mathrm{d}\right)^{\frac{1}{r'}}.\end{aligned}\tag{4.22}$$

Since $g\circ(|\cdot|^{p-1}\operatorname{sign}(\cdot))$ is uniformly continuous in $\mathbb{R}$, bounded and $v_n\to v$ in $L^1(\Omega)$, we obtain

$$\lim_{n\to\infty}\int_\Omega|g(|v|^{\frac{1}{p-1}}\operatorname{sign}(v))-g(|v_n|^{\frac{1}{p-1}}\operatorname{sign}(v_n))|^{r'}\mathrm{d}x=0,$$

which, together with (4.22), implies the desired result.

Case 2. $p\ge2$. We note here that

$$(|a|^{p-2}a-|b|^{p-2}b)(a-b)\ge C(p)|a-b|^p$$

for any $(a,b)\in\mathbb{R}^{2N}$ and $p\ge2$. Thus,

$$I\ge C(N,p,\Lambda_K)\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\phi(x)-\phi(y)|^p}{|x-y|^{N+sp}}\mathrm{d}x\mathrm{d}y\right).$$

By using a similar argument to the one in Case 1, we may show that $\mathbb{T}(v_n) \to \mathbb{T}(v)$ in $W_0^{s,p}(\Omega)$.

Next we claim that $\mathbb{S}$ is compact. Indeed, let $\{v_n\}$ be a sequence in $L^1(\Omega)$ then by (4.16), we obtain that $\mathbb{T}(v_n)$ is uniformly bounded in $W_0^{s,p}(\Omega)$. Hence there exists a subsequence still denoted by $\{\mathbb{T}(v_n)\}$ such that $\mathbb{T}(v_n) \rightharpoonup \psi$ in $W_0^{s,p}(\Omega)$ and $\mathbb{T}(v_n) \to \psi$ a.e. in $\mathbb{R}^N$. Furthermore, in view of (4.6), we can easily show that $\mathbb{S}(v_n) = |\mathbb{T}(v_n)|^{p-1}\operatorname{sign}(\mathbb{T}(v_n)) \to |\psi|^{p-1}\operatorname{sign}(\psi)$ in $L^1(\Omega)$.

Now set

$$\mathcal{O} := \{v \in L^1(\Omega) : \ \|v\|_{L_w^{\frac{N}{N-sp}}(\Omega)} \le t_0\}. \tag{4.23}$$

Then $\mathcal{O}$ is a closed, convex subset of $L^1(\Omega)$ and by (4.8), $\mathbb{S}(\mathcal{O}) \subset \mathcal{O}$. Thus we can apply Schauder fixed point theorem to obtain the existence of a function $v \in \mathcal{O}$ such that $\mathbb{S}(v) = v$. This means that $u = v^{\frac{1}{p-1}}\operatorname{sign}(v)$ is a solution of (4.1) satisfying (4.3). □

Proof of Theorem 1.4. Let $\{\rho_n\}_{n=1}^{\infty}$ be a sequence of mollifiers. Set $\tau_n = \rho_n * \tau$ and $g_n = \max(-n, \min(g, n))$. Then g_n satisfies (1.9) with the same constant Λ_g. Thus, there exists a weak solution $u_n \in W_0^{s,p}(\Omega)$ of

$$\begin{cases} Lv = g_n(v) + \rho\tau_n, & \text{in } \Omega, \\ v = 0, & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

In addition, it satisfies

$$\||u_n|^{p-1}\|_{L_w^{\frac{N}{N-sp}}(\Omega)} \le t_0, \tag{4.24}$$

where $t_0 > 0$ depends on $N, \Omega, \Lambda_g, \Lambda_K, a, s, p, d$.

By (4.24), we have that

$$|\{x \in \Omega : \ |u_n| > s\}| \le t_0^{\frac{N}{N-sp}} s^{-\frac{N(p-1)}{N-sp}}.$$

Hence by Proposition 3.5,

$$\int_\Omega |g_n(u_n)|\mathrm{d}x \le C, \quad \forall n \in \mathbb{N},$$

where C depends only on $N, \Omega, \Lambda_g, \Lambda_K, a, s, p, d$ and t_0. This, together with Proposition 2.7, implies that for any $q \in (p-1, \frac{N(p-1)}{N-s})$ and $h \in (0, s)$, there exists a positive constant $c = c(N, s, p, \Lambda_K, s, h, q, |\Omega|)$ such that

$$\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^q}{|x-y|^{N+hq}}\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{q}} \le c(C + \rho\int_\Omega |\tau_n|\mathrm{d}x)^{\frac{1}{p-1}}. \tag{4.25}$$

Therefore, in view of the proof of Proposition 2.8, we may show that there exists a subsequence, still denoted by the same notation, such that $u_n \to u$ in $W^{h,q}(\mathbb{R}^N)$ and a.e. in $\mathbb{R}^N$.

Now, we will show that $g_n(u_n) \to g(u)$ in $L^1(\Omega)$. We will prove it by using Vitali's convergence theorem. Let $E \subset \Omega$ be a Borel set. Then, by Lemma 3.5 and (4.24), we have

$$\begin{aligned} \int_E |g_n(u_n)|\mathrm{d}x &\le \int_\Omega |g(u_n)|\mathrm{d}x \\ &\le (g(s_0) - g(-s_0))|E| + C(t_0, p, \Lambda_g, N)\int_{s_0}^{\infty} (g(s) - g(-s))s^{-1-\frac{(p-1)N}{N-sp}}\mathrm{d}s, \quad \forall s_0 \ge 1. \end{aligned}$$

Let $\varepsilon > 0$, then there exists s_0 such that

$$C(t_0, p, \Lambda_g, N) \int_{s_0}^{\infty} (g(s) - g(-s)) s^{-1-\frac{(p-1)N}{N-sp}} \mathrm{d}s \leq \frac{\varepsilon}{2}.$$

Set $\delta = \frac{\varepsilon}{2(1+g(s_0)-g(-s_0))} > 0$. Then for any Borel set E with $|E| \leq \delta$, we have

$$g(s_0)|E| \leq \frac{\varepsilon}{2}.$$

Hence, by the last three inequalities, we may invoke Vitali's convergence theorem in order to prove that $g_n(u_n) \to g(u)$ in $L^1(\Omega)$.

In view of the proof of Proposition 2.8, we may deduce that u is a very weak solution of (1.16). Furthermore, by Fatou's lemma, we can easily show that u satisfies (1.17) and (1.18). □

4.2. Power nonlinearities: proof of Proposition 1.5 and Theorem 1.6

Proof of Proposition 1.5. Let $w = ACW_{s,p}^{2\mathrm{diam}\,(\Omega)}[\rho\tau]$, where C is the constant in (2.29) and $A > 1$ is a constant that will be determined later. Set $\mathrm{d}\nu = w^{\kappa}\mathrm{d}x + \rho\mathrm{d}\tau$, then by (1.20), we obtain

$$\begin{aligned} CW_{s,p}^{2(\mathrm{diam}\,(\Omega))}[\nu] &\leq 2^{\frac{1}{p-1}} C(W_{s,p}^{2\mathrm{diam}\,(\Omega)}[w^{\kappa}] + W_{s,p}^{2\mathrm{diam}\,(\Omega)}[\rho\mathrm{d}\tau]) \\ &\leq 2^{\frac{1}{p-1}} C((AC)^{\frac{\kappa}{p-1}} \rho^{\frac{\kappa}{(p-1)^2}} M W_{s,p}^{2\mathrm{diam}\,(\Omega)}[\tau] + W_{s,p}^{2\mathrm{diam}\,(\Omega)}[\rho\mathrm{d}\tau]) \\ &\leq 2^{\frac{1}{p-1}} C((AC)^{\frac{\kappa}{p-1}} M \rho^{\frac{\kappa-p+1}{(p-1)^2}} + 1) W_{s,p}^{2\mathrm{diam}\,(\Omega)}[\rho\mathrm{d}\tau]. \end{aligned}$$

If we choose $A = 2^{\frac{1}{p-1}+1}$ and ρ small enough such that $(AC)^{\frac{\kappa}{p-1}} M \rho^{\frac{\kappa-p+1}{(p-1)^2}} + 1 < 2$, we deduce that

$$CW_{s,p}^{2\mathrm{diam}\,(\Omega)}[\nu] \leq w. \tag{4.26}$$

Now, let $x_0 \in \Omega$ be such that $W_{s,p}^{2\mathrm{diam}\,(\Omega)}[\tau](x_0) < \infty$. If $0 \leq v \leq c_0 W_{s,p}^{2\mathrm{diam}\,(\Omega)}[\tau]$ a.e. in $\mathbb{R}^N$, for some constant $c_0 > 0$, then we have

$$\left(\int_{\Omega} |v|^{\kappa} \mathrm{d}x\right)^{\frac{1}{p-1}} \leq \left(\int_{B_{\mathrm{diam}\,(\Omega)}(x_0)} |v|^{\kappa} \mathrm{d}x\right)^{\frac{1}{p-1}} \leq C(\Omega, N, s, p, M, K, c_0) W_{s,p}^{2\mathrm{diam}\,(\Omega)}[\tau](x_0) < \infty.$$

Thus $v \in L^{\kappa}(\Omega)$.

Let $u_0 \geq 0$ be a very weak solution of

$$\begin{cases} Lu_0 = \rho\tau, & \text{in } \Omega, \\ v = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

satisfying $C^{-1} W_{s,p}^{\frac{d(x)}{8}}[\mu_{n-1}](x) \leq u_0(x) \leq CW_{s,p}^{2\mathrm{diam}\,(\Omega)}[\rho\tau](x)$ a.e. in Ω. We may construct a nondecreasing sequence $\{u_n\}_{n\geq 0}$, such that u_n is a very weak solution to problem

$$\begin{cases} Lu_n = u_{n-1}^{\kappa} + \rho\tau, & \text{in } \Omega, \\ v = 0, & \text{in } \mathbb{R}^N \setminus \Omega \end{cases}$$

and satisfies

$$C^{-1}W_{s,p}^{\frac{d(x)}{8}}[\mu_{n-1}](x) \le u_n(x) \le CW_{s,p}^{2\mathrm{diam}\,(\Omega)}[\mu_{n-1}](x), \qquad \text{for a.e. } x \in \Omega,$$

for any $n \in \mathbb{N}$, where $\mathrm{d}\mu_{n-1} = u_{n-1}^{\kappa}\mathrm{d}x + \rho \mathrm{d}\tau$. In addition, by (4.26) and the above inequality, there holds

$$C^{-1}W_{s,p}^{\frac{d(x)}{8}}[\mu_{n-1}](x) \le u_n(x) \le w(x), \qquad \text{for a.e. } x \in \Omega, \tag{4.27}$$

where the positive constant C^{-1} depends only on N, p, s, q. Finally, u_n satisfies (2.15)–(2.17) with $\mathrm{d}\mu = w^{\kappa}\mathrm{d}x + \rho\mathrm{d}\tau$.

Proceeding as in the proof of Proposition 2.8, we may show that there exists a subsequence, still denoted by $\{u_n\}$, such that $u_n \to u$ a.e. in $\mathbb{R}^N$ and u is a very weak solution of problem (1.19). By (4.27) and Fatou's Lemma, we obtain estimate (1.21). The proof is complete. □

Proof of Theorem 1.6. We will first prove that (*i*) implies (*ii*) by using some ideas from [30]. Without loss of generality we assume that $\rho = 1$. Extend μ to whole $\mathbb{R}^N$ by setting $\mu(\mathbb{R}^N \setminus \Omega) = 0$.

Let $0 \le g \in L^{\frac{\kappa}{p-1}}(\mathbb{R}^N;\mu)$. We set

$$M_\mu g(x) := \sup_{r>0,\, \mu(B(x,r))\neq 0} \mu(B(x,r))^{-1} \int_{B(x,r)} g(y)\mathrm{d}\mu.$$

It is well known that there exists a positive constant c_1 depending only on N, p, κ such that

$$\int_{\mathbb{R}^N} (M_\mu g(x))^{\frac{\kappa}{p-1}}\mathrm{d}\mu \le c_1 \int_{\mathbb{R}^N} |g(x)|^{\frac{\kappa}{p-1}}\mathrm{d}\mu \tag{4.28}$$

(see, e.g., [14]). Also,

$$\begin{aligned}\int_\Omega \left(W_{s,p}^{\frac{d(x)}{8}}[g\mu](x)\right)^{\kappa}\mathrm{d}x &\le \int_\Omega \left(W_{s,p}^{\frac{d(x)}{8}}[\mu](x)\right)^{\kappa}(M_\mu g(x))^{\frac{\kappa}{p-1}}\mathrm{d}x\\ &\le C^{\kappa}\int_\Omega u^{\kappa}(x)(M_\mu g(x))^{\frac{\kappa}{p-1}}\mathrm{d}x\\ &\le C^{\kappa}\int_\Omega (M_\mu g(x))^{\frac{\kappa}{p-1}}\mathrm{d}\mu\\ &\le c_2\int_{\mathbb{R}^N}|g(x)|^{\frac{\kappa}{p-1}}\mathrm{d}\mu.\end{aligned} \tag{4.29}$$

Let $K = \mathrm{supp}\,\tau$. By the assumption, we have that $r_0 := \mathrm{dist}\,(K, \partial\Omega) > 0$. Set $g = \mathbf{1}_K\tilde{g}$, for any nonnegative $\tilde{g} \in L^{\frac{\kappa}{p-1}}(\mathbb{R}^N;\mu_{\lfloor K})$. We first note that $B_{\frac{r_0}{8}}(x) \cap K = \emptyset$ if $x \in \Omega$ with $d(x) < \frac{r_0}{8}$ or if $x \in \mathbb{R}^N \setminus \Omega$, which implies

$$W_{s,p}^{\frac{d(x)}{24}}[\tilde{g}\mu_{\lfloor K}](x) = 0,$$

if $x \in \Omega$ with $d(x) < \frac{r_0}{24}$ or if $x \in \mathbb{R}^N \setminus \Omega$. Therefore, by the above equality and (4.29), we have

$$\int_{\mathbb{R}^N}\left(W_{s,p}^{\frac{r_0}{24}}[\tilde{g}\mu_{\lfloor K}](x)\right)^{\kappa}\mathrm{d}x \le \int_\Omega \left(W_{s,p}^{\frac{d(x)}{8}}[g\mu](x)\right)^{\kappa}\mathrm{d}x \le c_2\int_{\mathbb{R}^N}|\tilde{g}(x)|^{\frac{\kappa}{p-1}}\mathrm{d}\mu_{\lfloor K}.$$

Also, by [5, Theorem 2.3] (see also [1, Corollary 3.6.3]), we have

$$\int_{\mathbb{R}^N}\left(W_{s,p}^{\frac{r_0}{24}}[\tilde{g}\mu_{\lfloor K}](x)\right)^{\kappa}\mathrm{d}x \approx \int_{\mathbb{R}^N}(\mathbb{G}_{sp}[\tilde{g}\mu_{\lfloor K}])^{\frac{\kappa}{p-1}}\mathrm{d}x, \tag{4.30}$$

where the implicit constant depends only on s, p, N, κ and r_0.

Hence, combining the last two displays, we may show that there exists a positive constant $c_3 = c_3(N, p, s, \kappa, r_0)$ such that

$$\int_{\mathbb{R}^N} (\mathbb{G}_{sp}[\tilde{g}\mu_{\lfloor K}])^{\frac{\kappa}{p-1}} \mathrm{d}x \le c_3 \int_{\mathbb{R}^N} |\tilde{g}(x)|^{\frac{\kappa}{p-1}} \mathrm{d}\mu_{\lfloor K}. \tag{4.31}$$

Let $f \in L^{\frac{\kappa}{\kappa-p+1}}(\mathbb{R}^N)$. Then, for any $\tilde{g} \in L^{\frac{\kappa}{p-1}}(\mathbb{R}^N; \mu_{\lfloor K})$, there holds

$$\left|\int_{\mathbb{R}^N} f(x) G_{sp} * (\tilde{g}\mu_{\lfloor K})(x) \mathrm{d}x\right| = \left|\int_{\mathbb{R}^N} \tilde{g}(y) G_{sp} * f(y) \mathrm{d}\mu_{\lfloor K}\right| \le C_1 \|f\|_{L^{\frac{\kappa}{\kappa-p+1}}(\mathbb{R}^N)} \|\tilde{g}\|_{L^{\frac{\kappa}{p-1}}(\mathbb{R}^N; \mu_{\lfloor K})}.$$

The last inequality implies,

$$\int_{\mathbb{R}^N} |G_{sp} * f(x)|^{\frac{\kappa}{\kappa-p+1}} \mathrm{d}\mu_{\lfloor K} \le c_4 \int_{\mathbb{R}^N} |f|^{\frac{\kappa}{\kappa-p+1}} \mathrm{d}x \quad \forall f \in L^{\frac{\kappa}{\kappa-p+1}}(\mathbb{R}^N).$$

By [1, Theorem 7.2.1], the above inequality is equivalent to

$$\mu_{\lfloor K}(F) \le c_5 \mathrm{Cap}_{sp, \frac{\kappa}{\kappa-p+1}}(F), \tag{4.32}$$

for any compact $F \subset \mathbb{R}^N$. (1.23) follows by the above inequality and the fact that $\tau \le \mu_{\lfloor K}$.

Next, we prove that (ii) implies (iii). We note that proceeding as above, in the opposite direction, we may prove that (1.23) implies

$$\int_{\mathbb{R}^N} (\mathbb{G}_{sp}[\tilde{g}\tau])^{\frac{\kappa}{p-1}} \mathrm{d}x \le c_3 \int_{\mathbb{R}^N} |\tilde{g}(x)|^{\frac{\kappa}{p-1}} \mathrm{d}\tau, \quad \forall \tilde{g} \in L^{\frac{\kappa}{p-1}}(\mathbb{R}^N; \tau).$$

By (4.30) and taking $\tilde{g} = \mathbf{1}_B$, we can easily show that there exists a positive constant C depending only on N, s, p, Ω such that

$$\int_{\mathbb{R}^N} (W_{s,p}^{2\mathrm{diam}\,(\Omega)}[\tau_{\lfloor B}])^{\kappa} \mathrm{d}x \le C\tau(B).$$

We will show that (iii) implies (iv). Let $R = 2\mathrm{diam}\,(\Omega)$ and C_3 be the constant in (1.24). In the spirit of the proof of [31, Theorem 2.10], we need to prove that there exists a positive constant $c_0 = c_0(N, p, \kappa, s, C_3, R, \tau(\Omega)) > 0$ such that

$$\tau(B_t(x)) \le c_0 t^{\frac{\kappa(N-sp)-N(p-1)}{\kappa-p+1}} \tag{4.33}$$

for any $t \le R$ and $\forall x \in \Omega$.

Concerning the proof of the above inequality, we first note that for any $y \in B_t(x)$ and $t \le \frac{R}{4}$, there holds

$$\begin{aligned} W_{s,p}^{R}[\tau_{\lfloor B_t(x)}](y) &= \int_0^R \left(\frac{\tau(B_r(y) \cap B_t(x))}{r^{N-sp}}\right)^{\frac{1}{p-1}} \frac{\mathrm{d}r}{r} \ge \int_{2t}^{4t} \left(\frac{\tau(B_r(y) \cap B_t(x))}{r^{N-sp}}\right)^{\frac{1}{p-1}} \frac{\mathrm{d}r}{r} \\ &\ge C(N, p, s) \left(\frac{\tau(B_t(x))}{t^{N-sp}}\right)^{\frac{1}{p-1}}. \end{aligned}$$

By the above inequality, we deduce

$$t^N C^\kappa(N,p,s)\left(\frac{\tau(B_t(x))}{t^{N-sp}}\right)^{\frac{\kappa}{p-1}} \leq \int_{B_t(x)} (W^R_{s,p}[\tau_{\lfloor B_t(x)}](y))^\kappa \mathrm{d}y \leq C_3\tau(B_t(x)), \quad \forall t\in(0\frac{R}{4}], \tag{4.34}$$

where in the last inequality we used (1.24). This implies (4.33).

For any $x \in \Omega$ and $t < R$, we set

$$\nu_t(x) := \int_{B_t(x)} \Big(\int_0^t \left(\frac{\tau(B_r(y))}{r^{N-sp}}\right)^{\frac{1}{p-1}} \frac{\mathrm{d}r}{r}\Big)^\kappa \mathrm{d}y$$

and

$$\mu_t(x) := \int_{B_t(x)} \Big(\int_t^R \left(\frac{\tau(B_r(y))}{r^{N-sp}}\right)^{\frac{1}{p-1}} \frac{\mathrm{d}r}{r}\Big)^\kappa \mathrm{d}y.$$

Then we can easily prove that

$$W^R_{s,p}[(W^R_{s,p}[\tau])^\kappa] \leq C(q,p)\Big(\int_0^R \left(\frac{\nu_t(x)}{t^{N-sp}}\right)^{\frac{1}{p-1}} \frac{\mathrm{d}t}{t} + \int_0^R \left(\frac{\mu_t(x)}{t^{N-sp}}\right)^{\frac{1}{p-1}} \frac{\mathrm{d}t}{t}\Big). \tag{4.35}$$

Now, we treat the first term on the right hand in (4.35). By (1.24), we have

$$\nu_t(x) = \int_{B_t(x)} \Big(\int_0^t \left(\frac{\tau(B_r(y)\cap B_{2t}(x))}{r^{N-sp}}\right)^{\frac{1}{p-1}} \frac{\mathrm{d}r}{r}\Big)^\kappa \mathrm{d}y \leq C\tau(B_{2t}(x)), \tag{4.36}$$

which implies

$$\int_0^R \left(\frac{\nu_t(x)}{t^{N-sp}}\right)^{\frac{1}{p-1}} \frac{\mathrm{d}t}{t} \leq CW^{2R}_{s,p}[\tau](x). \tag{4.37}$$

Next, we treat the second term on the right hand in (4.35). First we note that

$$\begin{aligned}\mu_t(x) &\leq \int_{B_t(x)} \Big(\int_t^R \left(\frac{\tau(B_{2r}(x))}{r^{N-sp}}\right)^{\frac{1}{p-1}} \frac{\mathrm{d}r}{r}\Big)^\kappa \mathrm{d}y \\ &\leq C(N)t^N\Big(\int_t^{2R} \left(\frac{\tau(B_r(x))}{r^{N-sp}}\right)^{\frac{1}{p-1}} \frac{\mathrm{d}r}{r}\Big)^\kappa =: C(N)t^N\mu^\kappa_{1,t}(x),\end{aligned}$$

which implies

$$\begin{aligned}\int_0^R \left(\frac{\mu_t(x)}{t^{N-sp}}\right)^{\frac{1}{p-1}} \frac{\mathrm{d}t}{t} &\leq C(N,p)\int_0^R \mu_{1,t}^{\frac{\kappa}{p-1}}(x)t^{\frac{sp}{p-1}-1}\mathrm{d}t \\ &= C(N,p,s,q)\Big(\mu_{1,R}^{\frac{\kappa}{p-1}}(x)R^{\frac{sp}{p-1}} + \int_0^R (\mu_{1,t}(x))^{\frac{\kappa}{p-1}-1}\, t^{\frac{sp}{p-1}} \left(\frac{\tau(B_t(x))}{t^{N-sp}}\right)^{\frac{1}{p-1}} \frac{\mathrm{d}t}{t}\Big),\end{aligned}$$

where we have used integration by parts in the last equality. By (4.33), we have

$$\mu_{1,t}(x) = \int_t^{2R} \left(\frac{\tau(B_r(x))}{r^{N-sp}} \right)^{\frac{1}{p-1}} \frac{\mathrm{d}r}{r} \le C t^{-\frac{sp}{\kappa-p+1}}.$$

Combining the last two displays, we obtain

$$\int_0^R \left(\frac{\mu_t(x)}{t^{N-sp}} \right)^{\frac{1}{p-1}} \frac{\mathrm{d}t}{t} \le C(N,p,s,\kappa,R)\Big(\tau(B_{2R}(x))^{\frac{\kappa}{(p-1)^2}} + W^R_{s,p}[\tau](x) \Big). \tag{4.38}$$

The desired result follows by (4.35), (4.37), (4.38) and the fact that

$$\tau(B_{2R}(x))^{\frac{\kappa}{(p-1)^2}} \le \tau(\Omega)^{\frac{\kappa-p+1}{(p-1)^2}} \tau(B_{\frac{R}{2}}(x))^{\frac{1}{p-1}} \le C(R,N,p,\tau,s,\kappa) W^R_{s,p}[\tau](x), \quad \forall x \in \Omega.$$

□

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares no conflict of interest in this paper.

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