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Research article

A limiting case in partial regularity for quasiconvex functionals†

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Abstract: Local minimizers of nonhomogeneous quasiconvex variational integrals with standard p-growth of the type

$$w \mapsto \int [F(Dw) - f \cdot w] \, \mathrm{d}x$$

feature almost everywhere BMO-regular gradient provided that f belongs to the borderline Marcinkiewicz space $L(n, \infty)$.

Keywords: regularity; quasiconvex functionals; degenerate variational integrals

Dedicated to Giuseppe Mingione on the occasion of his 50^{th} *birthday, with admiration.*

1. Introduction

In this paper we provide a limiting partial regularity criterion for vector-valued minimizers $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$, $n \geq 2$, $N > 1$, of nonhomogeneous, quasiconvex variational integrals as:

$$W^{1,p}(\Omega; \mathbb{R}^N) \ni w \mapsto \mathcal{F}(w; \Omega) := \int_\Omega [F(Dw) - f \cdot w] \, \mathrm{d}x, \tag{1.1}$$

with standard p-growth. More precisely, we infer the optimal [31, Section 9] ε-regularity condition

$$\sup_{B_\varrho \Subset \Omega} \varrho^m \fint_{B_\varrho} |f|^m \, \mathrm{d}x \lesssim \varepsilon \implies Du \text{ has a.e. bounded mean oscillation,}$$

and the related borderline function space criterion

$$f \in L(n,\infty) \implies \sup_{B_\varrho \Subset \Omega} \varrho^m \fint_{B_\varrho} |f|^m \, \mathrm{d}x \lesssim \varepsilon.$$

This is the content of our main theorem.

Theorem 1.1. *Under assumptions* $(1.6)_{1,2,3}$, (1.7) *and* (1.10)*, let* $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ *be a local minimizer of functional* (1.1)*. Then, there exists a number* $\varepsilon_* \equiv \varepsilon_*(\mathtt{data}) > 0$ *such that if*

$$\|f\|_{L^{n,\infty}(\Omega)} \le \left(\frac{|B_1|}{4^{n/m}}\right)^{1/n} \varepsilon_*, \tag{1.2}$$

then there exists an open set $\Omega_u \subset \Omega$ *with* $|\Omega \setminus \Omega_u| = 0$ *such that*

$$Du \in BMO_{\mathrm{loc}}(\Omega_u; \mathbb{R}^{N\times n}). \tag{1.3}$$

Moreover, the set Ω_u *can be characterized as follows*

$$\Omega_u := \left\{ x_0 \in \Omega : \exists \varepsilon_{x_0}, \varrho_{x_0} > 0 \ \ \textit{such that } \mathscr{E}(u; B_\varrho(x_0)) \le \varepsilon_{x_0} \textit{ for some } \varrho \le \varrho_{x_0} \right\},$$

where $\mathscr{E}(\cdot)$ *is the usual excess functional defined as*

$$\mathscr{E}(w, z_0; B_\varrho(x_0)) := \left(\fint_{B_\varrho(x_0)} |z_0|^{p-2}|Dw - z_0|^2 + |Dw - z_0|^p \, \mathrm{d}x \right)^{\frac{1}{p}}. \tag{1.4}$$

We immediately refer to Section 1.2 below for a description of the structural assumptions in force in Theorem 1.1. Let us put our result in the context of the available literature. The notion of quasiconvexity was introduced by Morrey [38] in relation to the delicate issue of semicontinuity of multiple integrals in Sobolev spaces: an integrand $F(\cdot)$ is a *quasiconvex* whenever

$$\fint_{B_1(0)} F(z + D\varphi) \, \mathrm{d}x \ge F(z) \quad \text{holds for all } \ z \in \mathbb{R}^{N\times n}, \ \ \varphi \in C^\infty_{\mathrm{c}}(B_1(0), \mathbb{R}^N). \tag{1.5}$$

Under power growth conditions, (1.5) is proven to be necessary and sufficient for the sequential weak lower semicontinuity on $W^{1,p}(\Omega; \mathbb{R}^N)$; see [1, 4, 35, 36, 38]. It is worth stressing that quasiconvexity is a strict generalization of convexity: the two concepts coincide in the scalar setting ($N = 1$), or for 1-d problems ($n = 1$), but sharply differ in the multidimensional case: every convex function is quasiconvex thanks to Jensen's inequality, while the determinant is quasiconvex (actually polyconvex), but not convex, cf. [24, Section 5.1]. Another distinctive trait is the nonlocal nature of quasiconvexity: Morrey [38] conjectured that there is no condition involving only $F(\cdot)$ and a finite number of its derivatives that is both necessary and sufficient for quasiconvexity, fact later on confirmed by Kristensen [29]. A peculiarity of quasiconvex functionals is that minima and critical points (i.e., solutions to the associated Euler-Lagrange system) might have very different behavior under the (partial) regularity viewpoint. In fact, a classical result of Evans [22] states that the gradient of minima is locally Hölder continuous outside a negligible, "singular" set, while a celebrated counterexample due to Müller and Šverák [39] shows that the gradient of critical points may be everywhere discontinuous.

After Evans seminal contribution [22], the partial regularity theory was extended by Acerbi and Fusco [2] to possibly degenerate quasiconvex functionals with superquadratic growth, and by Carozza, Fusco and Mingione [8] to subquadratic, nonsingular variational integrals. A unified approach that allows simultaneously handling degenerate/nondegenerate, and singular/nonsingular problems, based on the combination of $\mathscr{A}$-harmonic approximation [21], and p-harmonic approximation [20], was eventually proposed by Duzaar and Mingione [19]. Moreover, Kristensen and Mingione [30] proved that the Hausdorff dimension of the singular set of Lipschitz continuous minimizers of quasiconvex multiple integrals is strictly less than the ambient space dimension n, see also [5] for further developments in this direction. We refer to [3, 15, 16, 25–28, 37, 41, 42] for an (incomplete) account of classical, and more recent advances in the field. In all the aforementioned papers are considered homogeneous functionals, i.e., $f \equiv 0$ in (1.1). The first sharp ε-regularity criteria for nonhomogeneous quasiconvex variational integrals guaranteeing almost everywhere gradient continuity under optimal assumptions on f were obtained by De Filippis [12], and De Filippis and Stroffolini [14], by connecting the classical partial regularity theory for quasiconvex functionals with nonlinear potential theory for degenerate/singular elliptic equations, first applied in the context of partial regularity for strongly elliptic systems by Kuusi and Mingione [33]. Potential theory for nonlinear PDE originates from the classical problem of determining the best condition on f implying gradient continuity in the Poisson equation $-\Delta u = f$, that turns out to be formulated in terms of the uniform decay to zero of the Riesz potential, in turn implied by the membership of f to the Lorentz space $L(n, 1)$, [9, 31]. In this respect, a breakthrough result due to Kuusi and Mingione [32, 34] states that the same is true for the nonhomogeous, degenerate p-Laplace equation–in other words, the regularity theory for the nonhomogeneous p-Laplace PDE coincides with that of the Poisson equation up to the C^1-level. This important result also holds in the case of singular equations [18, 40], for general, uniformly elliptic equations [6], up to the boundary [10, 11, 13], and at the level of partial regularity for p-Laplacian type systems without Uhlenbeck structure, [7, 33]. We conclude by highlighting that our Theorem 1.1 fits this line of research as, it determines for the first time in the literature optimal conditions on the inhomogeneity f assuring partial BMO-regularity for minima of quasiconvex functionals expressed in terms of the limiting function space $L(n, \infty)$.

1.1. Outline of the paper

In Section 2 we recall some well-known results from the study of nonlinear problems also establishing some Caccioppoli and Gehring type lemmas. In Section 3 we prove the excess decay estimates; considering separately the nondegenerate and the degenerate case. Section 4 is devoted to the proof of Theorem 1.1.

1.2. Structural assumptions

In (1.1), the integrand $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ satisfies

$$\begin{cases} F \in C^2_{\mathrm{loc}}(\mathbb{R}^{N\times n}) \\ \Lambda^{-1}|z|^p \le F(z) \le \Lambda|z|^p \\ |\partial^2 F(z)| \le \Lambda|z|^{p-2} \\ |\partial^2 F(z_1) - \partial^2 F(z_2)| \le \mu\left(\dfrac{|z_2 - z_1|}{|z_2| + |z_1|}\right)\left(|z_1|^2 + |z_2|^2\right)^{\frac{p-2}{2}} \end{cases} \tag{1.6}$$

for all $z \in \mathbb{R}^{N\times n}$, $\Lambda \geq 1$ being a positive absolute constant and $\mu\colon [0,\infty) \to [0,1]$ being a concave nondecreasing function with $\mu(0) = 0$. In the rest of the paper we will always assume $p \geq 2$. In order to derive meaningful regularity results, we need to update (1.5) to the stronger strict quasiconvexity condition

$$\int_B [F(z + D\varphi) - F(z)] \;\mathrm{d}x \geq \lambda \int_B (|z|^2 + |D\varphi|^2)^{\frac{p-2}{2}} |D\varphi|^2 \,\mathrm{d}x, \tag{1.7}$$

holding for all $z \in \mathbb{R}^{N\times n}$ and $\varphi \in W^{1,p}_0(B, \mathbb{R}^N)$, with λ being a positive, absolute constant. Furthermore, we allow the integrand $F(\cdot)$ to be degenerate elliptic in the origin. More specifically, we assume that $F(\cdot)$ features degeneracy of p-Laplacian type at the origin, i.e.,

$$\left| \frac{\partial F(z) - \partial F(0) - |z|^{p-2}z}{|z|^{p-1}} \right| \to 0 \qquad \text{as } |z| \to 0, \tag{1.8}$$

which means that we can find a function $\omega\colon (0,\infty) \to (0,\infty)$ such that

$$|z| \leq \omega(s) \implies |\partial F(z) - \partial F(0) - |z|^{p-2}z| \leq s|z|^{p-1}, \tag{1.9}$$

for every $z \in \mathbb{R}^{N\times n}$ and all $s \in (0,\infty)$. Moreover, the right-hand side term $f\colon \Omega \to \mathbb{R}^N$ in (1.1) verifies as minimal integrability condition the following

$$f \in L^m(\Omega, \mathbb{R}^N) \quad \text{with } \; 2 > m > \begin{cases} 2n/(n+2) & \text{if } \; n > 2, \\ 3/2 & \text{if } \; n = 2, \end{cases} \tag{1.10}$$

which, being $p \geq 2$, in turn implies that

$$f \in W^{1,p}(\Omega, \mathbb{R}^N)^* \qquad \text{and} \qquad m' < 2^* \leq p^*. \tag{1.11}$$

Here it is intended that, when $p \geq n$, the Sobolev conjugate exponent p^* can be chosen as large as needed - in particular it will always be larger than p. By (1.5) and $(1.6)_2$ we have

$$|\partial F(z)| \leq c|z|^{p-1}, \tag{1.12}$$

with $c \equiv c(n, N, \Lambda, p)$; see for example [35, proof of Theorem 2.1]. Finally, (1.7) yields that for all $z \in \mathbb{R}^{N\times n}$, $\xi \in \mathbb{R}^N$, $\zeta \in \mathbb{R}^n$ it is

$$\partial^2 F(z)\langle \xi \otimes \zeta, \xi \otimes \zeta \rangle \geq 2\lambda |z|^{p-2}|\xi|^2|\zeta|^2, \tag{1.13}$$

see [24, Chapter 5].

2. Preliminaries

In this section we display our notation and collect some basic results that will be helpful later on.

2.1. Notation

In this paper, $\Omega \subset \mathbb{R}^n$ is an open, bounded domain with Lipschitz boundary, and $n \geq 2$. By c we will always denote a general constant larger than one, possibly depending on the data of the problem. Special occurrences will be denoted by $c_*, \tilde{c}$ or likewise. Noteworthy dependencies on parameters will be highlighted by putting them in parentheses. Moreover, to simplify the notation, we shall array the main parameters governing functional (1.1) in the shorthand $\texttt{data} := (n, N, \lambda, \Lambda, p, \mu(\cdot), \omega(\cdot))$. By $B_r(x_0) := \{x \in \mathbb{R}^n : |x - x_0| < r\}$, we denote the open ball with radius r, centred at x_0; when not necessary or clear from the context, we shall omit denoting the center, i.e., $B_r(x_0) \equiv B_r$ - this will happen, for instance, when dealing with concentric balls. For $x_0 \in \Omega$, we abbreviate $d_{x_0} := \min\{1, \operatorname{dist}(x_0, \partial\Omega)\}$. Moreover, with $B \subset \mathbb{R}^n$ being a measurable set with bounded positive Lebesgue measure $0 < |B| < \infty$, and $a \colon B \to \mathbb{R}^k$, $k \geq 1$, being a measurable map, we denote

$$(a)_B \equiv \fint_B a(x) \, \mathrm{d}x := \frac{1}{|B|} \int_B a(x) \, \mathrm{d}x.$$

We will often employ the almost minimality property of the average, i.e.,

$$\left(\fint_B |a - (a)_B|^t \, \mathrm{d}x \right)^{1/t} \leq 2 \left(\fint_B |a - z|^t \, \mathrm{d}x \right)^{1/t} \tag{2.1}$$

for all $z \in \mathbb{R}^{N \times n}$ and any $t \geq 1$. Finally, if $t > 1$ we will indicate its conjugate by $t' := t/(t-1)$ and its Sobolev exponents as $t^* := nt/(n-t)$ if $t < n$ or any number larger than one for $t \geq n$ and $t_* := \max\{nt/(n+t), 1\}$.

2.2. Tools for nonlinear problems

When dealing with p-Laplacian type problems, we shall often use the auxiliary vector field $V_s \colon \mathbb{R}^{N \times n} \to \mathbb{R}^{N \times n}$, defined by

$$V_s(z) := (s^2 + |z|^2)^{(p-2)/4} z \qquad \text{with } p \in (1, \infty), \quad s \geq 0, \quad z \in \mathbb{R}^{N \times n},$$

incorporating the scaling features of the p-Laplacian. If $s = 0$ we simply write $V_s(\cdot) \equiv V(\cdot)$. A couple of useful related inequalities are

$$\begin{cases} |V_s(z_1) - V_s(z_2)| \approx (s^2 + |z_1|^2 + |z_2|^2)^{(p-2)/4} |z_1 - z_2|, \\ |V_s(z_1 + z_2)| \lesssim |V_s(z_1)| + |V_s(z_2)|, \\ |V_{s_1}(z)| \approx |V_{s_2}(z)|, \text{ if } \frac{1}{2} s_2 \leq s_1 \leq 2 s_2, \\ |V(z_1) - V(z_2)|^2 \approx |V_{|z_1|}(z_1 - z_2)|^2, \text{ if } \frac{1}{2}|z_2| \leq |z_1| \leq 2|z_2|, \end{cases} \tag{2.2}$$

and

$$|V_s(z)|^2 \approx s^{p-2}|z|^2 + |z|^p \qquad \text{with } p \geq 2, \tag{2.3}$$

where the constants implicit in "$\lesssim$", "$\approx$" depend on n, N, p. A relevant property which is relevant for the nonlinear setting is recorded in the following lemma.

Lemma 2.1. *Let $t > -1$, $s \in [0,1]$ and $z_1, z_2 \in \mathbb{R}^{N\times n}$ be such that $s + |z_1| + |z_2| > 0$. Then*

$$\int_0^1 \left[s^2 + |z_1 + y(z_2 - z_1)|^2\right]^{\frac{t}{2}} \, dy \approx (s^2 + |z_1|^2 + |z_2|^2)^{\frac{t}{2}},$$

with constants implicit in "≈" depending only on n, N, t.

The following iteration lemma will be helpful throughout the rest of the paper; for a proof we refer the reader to [24, Lemma 6.1].

Lemma 2.2. *Let $h\colon [\varrho_0, \varrho_1] \to \mathbb{R}$ be a non-negative and bounded function, and let $\theta \in (0,1)$, $A, B, \gamma_1, \gamma_2 \ge 0$ be numbers. Assume that $h(t) \le \theta h(s) + A(s-t)^{-\gamma_1} + B(s-t)^{-\gamma_2}$ holds for all $\varrho_0 \le t < s \le \varrho_1$. Then the following inequality holds $h(\varrho_0) \le c(\theta, \gamma_1, \gamma_2)[A(\varrho_1 - \varrho_0)^{-\gamma_1} + B(\varrho_1 - \varrho_0)^{-\gamma_2}]$.*

We will often consider the "quadratic" version of the excess functional defined in (1.4), i.e.,

$$\widetilde{\mathscr{E}}(w, z_0; B_\varrho(x_0)) := \left(\fint_{B_\varrho(x_0)} |V(Dw) - z_0|^2 \, dx \right)^{\frac{1}{2}}. \tag{2.4}$$

In the particular case $z_0 = (Dw)_{B_\varrho(x_0)}$ ($z_0 = (V(Dw))_{B_\varrho(x_0)}$, resp.) we shall simply write $\mathscr{E}(w, (Dw)_{B_\varrho(x_0)}; B_\varrho(x_0)) \equiv \mathscr{E}(w; B_\varrho(x_0))$ ($\widetilde{\mathscr{E}}(w, (V(Dw))_{B_\varrho(x_0)}; B_\varrho(x_0)) \equiv \widetilde{\mathscr{E}}(w; B_\varrho(x_0))$, resp.). A simple computation shows that

$$\mathscr{E}(w; B_\varrho(x_0))^{p/2} \approx \widetilde{\mathscr{E}}(w; B_\varrho(x_0)). \tag{2.5}$$

Moreover, from (2.1) and from [23, Formula (2.6)] we have that

$$\widetilde{\mathscr{E}}(w; B_\varrho(x_0)) \approx \widetilde{\mathscr{E}}(w, V((Dw)_{B_\varrho(x_0)}); B_\varrho(x_0)). \tag{2.6}$$

2.3. Basic regularity results

In this section we collect some basic estimates for local minimizers of nonhomogeneous quasiconvex functionals. We start with a variation of the classical Caccioppoli inequality accounting for the presence of a nontrivial right-hand side term, coupled with an higher integrability result of Gehring-type.

Lemma 2.3. *Under assumptions* (1.6)$_{1,2,3}$, (1.7) *and* (1.10)*, let $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ be a local minimizer of functional* (1.1).

- *For every ball $B_\varrho(x_0) \Subset \Omega$ and any $u_0 \in \mathbb{R}^N$, $z_0 \in \mathbb{R}^{N\times n} \setminus \{0\}$ it holds that*

$$\begin{aligned} \mathscr{E}(u, z_0; B_{\varrho/2}(x_0))^p \ \le\ & c \fint_{B_\varrho(x_0)} |z_0|^{p-2} \left|\frac{u-\ell}{\varrho}\right|^2 + \left|\frac{u-\ell}{\varrho}\right|^p dx \\ & + \frac{c}{|z_0|^{p-2}} \left(\varrho^m \fint_{B_\varrho(x_0)} |f|^m \, dx \right)^{\frac{2}{m}}, \end{aligned} \tag{2.7}$$

where $\mathscr{E}(\cdot)$ is defined in (1.4)*, $\ell(x) := u_0 + \langle z_0, x - x_0\rangle$ and $c \equiv c(n, N, \lambda, \Lambda, p)$.*

- *There exists an higher integrability exponent $p_2 \equiv p_2(n,N,\lambda,\Lambda,p) > p$ such that $Du \in L^{p_2}_{\rm loc}(\Omega,\mathbb{R}^{N\times n})$ and the reverse Hölder inequality*

$$
\begin{aligned}
&\left(\fint_{B_{\varrho/2}(x_0)} |Du-(Du)_{B_\varrho(x_0)}|^{p_2}\,{\rm d}x\right)^{\frac{1}{p_2}}\\
&\qquad \le c\left(\fint_{B_\varrho(x_0)} |Du|^p\,{\rm d}x\right)^{\frac{1}{p}} + c\left(\varrho^m\fint_{B_\varrho(x_0)} |f|^m\,{\rm d}x\right)^{\frac{1}{m(p-1)}},
\end{aligned}
\tag{2.8}
$$

is verified for all balls $B_\varrho(x_0) \Subset \Omega$ with $c \equiv c(n,N,\lambda,\Lambda,p)$.

Proof. For the ease of exposition, we split the proof in two steps, each of them corresponding to the proof of (2.7) and (2.8) respectively.

Step 1: proof of (2.7)**.**

We choose parameters $\varrho/2 \le \tau_1 < \tau_2 \le \varrho$, a cut-off function $\eta \in C^1_c(B_{\tau_2}(x_0))$ such that $\mathbb{1}_{B_{\tau_1}(x_0)} \le \eta \le \mathbb{1}_{B_{\tau_2}(x_0)}$ and $|D\eta| \lesssim (\tau_2-\tau_1)^{-1}$. Set $\varphi_1 := \eta(u-\ell)$, $\varphi_2 := (1-\eta)(u-\ell)$ and use (1.7) and the equivalence in $(2.2)_1$ to estimate

$$
\begin{aligned}
c\int_{B_{\tau_2}(x_0)} |V_{|z_0|}(D\varphi_1)|^2\,{\rm d}x &\le \int_{B_{\tau_2}(x_0)} [F(z_0+D\varphi_1)-F(z_0)]\,{\rm d}x\\
&= \int_{B_{\tau_2}(x_0)} [F(Du-D\varphi_2)-F(Du)]\,{\rm d}x\\
&\quad + \int_{B_{\tau_2}(x_0)} [F(Du)-F(Du-D\varphi_1)]\,{\rm d}x\\
&\quad + \int_{B_{\tau_2}(x_0)} [F(z_0+D\varphi_2)-F(z_0)]\,{\rm d}x =: {\rm I}_1+{\rm I}_2+{\rm I}_3,
\end{aligned}
\tag{2.9}
$$

where we have used the simple relation $D\varphi_1 + D\varphi_2 = Du - z_0$. Terms ${\rm I}_1$ and ${\rm I}_3$ can be controlled as done in [19, Proposition 2]; indeed we have

$$
\begin{aligned}
{\rm I}_1+{\rm I}_3 &\le c\int_{B_{\tau_2}(x_0)\setminus B_{\tau_1}(x_0)} |V_{|z_0|}(D\varphi_2)|^2\,{\rm d}x + c\int_{B_{\tau_2}(x_0)\setminus B_{\tau_1}(x_0)} |V_{|z_0|}(Du-z_0)|^2\,{\rm d}x\\
&\overset{(2.2)_2}{\le} c\int_{B_{\tau_2}(x_0)\setminus B_{\tau_1}(x_0)} |V_{|z_0|}(Du-z_0)|^2 + \left|V_{|z_0|}\left(\frac{u-\ell}{\tau_2-\tau_1}\right)\right|^2\,{\rm d}x,
\end{aligned}
\tag{2.10}
$$

for $c \equiv c(n,N,\lambda,\Lambda,p)$. Concerning term ${\rm I}_2$, we exploit (1.10), the fact that $\varphi_1 \in W^{1,p}_0(B_{\tau_2}(x_0),\mathbb{R}^N)$ and apply Sobolev-Poincaré inequality to get

$$
\begin{aligned}
\mathrm{I}_2 &\le |B_{\tau_2}(x_0)|\left(\tau_2^m \fint_{B_{\tau_2}(x_0)} |f|^m \, \mathrm{d}x\right)^{1/m}\left(\tau_2^{-m'} \fint_{B_{\tau_2}(x_0)} |\varphi_1|^{m'} \, \mathrm{d}x\right)^{\frac{1}{m'}} \\
&\le |B_{\tau_2}(x_0)|\left(\tau_2^m \fint_{B_{\tau_2}(x_0)} |f|^m \, \mathrm{d}x\right)^{1/m}\left(\fint_{B_{\tau_2}(x_0)} \left|\frac{\varphi_1}{\tau_2}\right|^{2^*} \, \mathrm{d}x\right)^{\frac{1}{2^*}} \\
&\le |B_{\tau_2}(x_0)|\left(\tau_2^m \fint_{B_{\tau_2}(x_0)} |f|^m \, \mathrm{d}x\right)^{1/m}\left(\fint_{B_{\tau_2}(x_0)} |D\varphi_1|^2 \, \mathrm{d}x\right)^{\frac{1}{2}} \\
&\le \varepsilon \int_{B_{\tau_2}(x_0)} |V_{|z_0|}(D\varphi_1)|^2 \, \mathrm{d}x + \frac{c|B_\varrho(x_0)|}{\varepsilon |z_0|^{p-2}}\left(\varrho^m \fint_{B_\varrho(x_0)} |f|^m \, \mathrm{d}x\right)^{\frac{2}{m}}, \qquad (2.11)
\end{aligned}
$$

where $c \equiv c(n, N, m)$ and we also used that $\varrho/2 \le \tau_2 \le \varrho$. Merging the content of the two above displays, recalling that $\eta \equiv 1$ on $B_{\tau_1}(x_0)$ and choosing $\varepsilon > 0$ sufficiently small, we obtain

$$
\begin{aligned}
\int_{B_{\tau_1}(x_0)} |V_{|z_0|}(Du - z_0)|^2 \, \mathrm{d}x &\le c \int_{B_{\tau_2}(x_0)\setminus B_{\tau_1}(x_0)} |V_{|z_0|}(Du - z_0)|^2 + \left| V_{|z_0|}\left(\frac{u - \ell}{\tau_2 - \tau_1}\right)\right|^2 \, \mathrm{d}x \\
&\quad + \frac{c|B_\varrho(x_0)|}{|z_0|^{p-2}}\left(\varrho^m \fint_{B_\varrho(x_0)} |f|^m \, \mathrm{d}x\right)^{\frac{2}{m}},
\end{aligned}
$$

with $c \equiv c(n, N, \lambda, \Lambda, p)$. At this stage, the classical hole-filling technique, Lemma 2.2 and (2.3) yield (2.7) and the first bound in the statement is proven.

Step 2: proof of (2.8)**.**

To show the validity of (2.8), we follow [33, proof of Proposition 3.2] and first observe that if u is a local minimizer of functional $\mathcal{F}(\cdot)$ on $B_\varrho(x_0)$, setting $f_\varrho(x) := \varrho f(x_0 + \varrho x)$, the map $u_\varrho(x) := \varrho^{-1}u(x_0 + \varrho x)$ is a local minimizer on $B_1(0)$ of an integral with the same integrand appearing in (1.1) satisfying $(1.6)_{1,2,3}$ and f_ϱ replacing f. This means that (2.10) still holds for all balls $B_{\sigma/2}(\tilde{x}) \subseteq B_{\tau_1}(\tilde{x}) \subset B_{\tau_2}(\tilde{x}) \subseteq B_\sigma(\tilde{x}) \Subset B_1(0)$, with $\tilde{x} \in B_1(0)$ being any point, in particular it remains true if $|z_0| = 0$, while condition $|z_0| \neq 0$ was needed only in the estimate of term I_2 in (2.11), that now requires some change. So, in the definition of the affine map ℓ we choose $z_0 = 0$, $u_0 = (u_\varrho)_{B_\sigma(\tilde{x})}$ and rearrange estimates (2.10) and (2.11) as:

$$
\mathrm{I}_1 + \mathrm{I}_3 \overset{(2.3)}{\le} c \int_{B_{\tau_2}(\tilde{x})\setminus B_{\tau_1}(\tilde{x})} |Du_\varrho|^p + \left| \frac{u_\varrho - (u_\varrho)_{B_\sigma(\tilde{x})}}{\tau_2 - \tau_1}\right|^p \, \mathrm{d}x,
$$

and, recalling that $\varphi_1 \in W^{1,p}_0(B_{\tau_2}(\tilde{x}), \mathbb{R}^N)$, via Sobolev Poincaré, Hölder and Young inequalities and $(1.11)_2$, we estimate

$$
\begin{aligned}
\mathrm{I}_2 &\le \left|B_{\tau_2}(\tilde{x})\right|\left(\tau_2^{(p^*)'}\fint_{B_{\tau_2}(\tilde{x})}\left|f_\varrho\right|^{(p^*)'}\,\mathrm{d}x\right)^{\frac{1}{(p^*)'}}\left(\tau_2^{-p^*}\fint_{B_{\tau_2}(\tilde{x})}\left|\varphi_1\right|^{p^*}\,\mathrm{d}x\right)^{\frac{1}{p^*}}\\
&\le c\left|B_{\tau_2}(\tilde{x})\right|\left(\tau_2^{(p^*)'}\fint_{B_{\tau_2}(\tilde{x})}\left|f_\varrho\right|^{(p^*)'}\mathrm{d}x\right)^{\frac{1}{(p^*)'}}\left(\fint_{B_{\tau_2}(\tilde{x})}\left|D\varphi_1\right|^{p}\mathrm{d}x\right)^{\frac{1}{p}}\\
&\le \frac{c|B_\sigma(\tilde{x})|}{\varepsilon^{1/(p-1)}}\left(\sigma^{(p^*)'}\fint_{B_\sigma(\tilde{x})}\left|f_\varrho\right|^{(p^*)'}\mathrm{d}x\right)^{\frac{p}{(p^*)'(p-1)}}+\varepsilon\int_{B_{\tau_2}(\tilde{x})}\left|D\varphi_1\right|^{p}\mathrm{d}x,
\end{aligned}
$$

with $c \equiv c(n, N, p)$. Plugging the content of the two previous displays in (2.9), reabsorbing terms and applying Lemma 2.2, we obtain

$$
\fint_{B_{\sigma/2}(\tilde{x})}|Du_\varrho|^p\,\mathrm{d}x \le c\fint_{B_\sigma(\tilde{x})}\left|\frac{u_\varrho-(u_\varrho)_{B_\sigma(\tilde{x})}}{\sigma}\right|^p\,\mathrm{d}x + c\left(\sigma^{(p^*)'}\fint_{B_\sigma(\tilde{x})}\left|f_\varrho\right|^{(p^*)'}\mathrm{d}x\right)^{\frac{p}{(p^*)'(p-1)}}, \tag{2.12}
$$

for $c \equiv c(n, N, \Lambda, \lambda, p)$. Notice that

$$
n\left(\frac{p}{(p^*)'(p-1)}-1\right)\le\frac{p}{p-1}, \tag{2.13}
$$

with equality holding when $p < n$, while for $p \ge n$ any value of $p^* > 1$ will do. We then manipulate the second term on the right-hand side of (2.12) as

$$
\begin{aligned}
&\left(\sigma^{(p^*)'}\fint_{B_\sigma(\tilde{x})}\left|f_\varrho\right|^{(p^*)'}\mathrm{d}x\right)^{\frac{p}{(p^*)'(p-1)}}\\
&\le \sigma^{\frac{p}{p-1}-n\left(\frac{p}{(p^*)'(p-1)}-1\right)}\left(\fint_{B_1(0)}|f_\varrho|^{(p^*)'}\,\mathrm{d}x\right)^{\frac{p}{(p^*)'(p-1)}-1}\fint_{B_\sigma(\tilde{x})}|f_\varrho|^{(p^*)'}\mathrm{d}x\\
&\overset{(2.13)}{\le}\left(\fint_{B_1(0)}|f_\varrho|^{(p^*)'}\,\mathrm{d}x\right)^{\frac{p}{(p^*)'(p-1)}-1}\fint_{B_\sigma(\tilde{x})}|f_\varrho|^{(p^*)'}\,\mathrm{d}x\\
&=:\fint_{B_\sigma(\tilde{x})}|\mathfrak{K}_\varrho f_\varrho|^{(p^*)'}\,\mathrm{d}x,
\end{aligned}
$$

where we set

$$
\mathfrak{K}_\varrho^{(p^*)'} := |B_1(0)|^{1-\frac{p}{(p^*)'(p-1)}}\|f_\varrho\|_{L^{(p^*)'}(B_1(0))}^{\frac{p}{p-1}-(p^*)'}.
$$

Plugging the content of the previous display in (2.12) and applying Sobolev-Poincaré inequality we get

$$
\fint_{B_{\sigma/2}(\tilde{x})}|Du_\varrho|^p\,\mathrm{d}x \le c\left(\fint_{B_\sigma(\tilde{x})}|Du_\varrho|^{p_*}\,\mathrm{d}x\right)^{\frac{p}{p_*}} + c\fint_{B_\sigma(\tilde{x})}|\mathfrak{K}_\varrho f_\varrho|^{(p^*)'}\,\mathrm{d}x,
$$

with $c \equiv c(n, N, \Lambda, \lambda, p)$. Now we can apply a variant of Gehring lemma [24, Corollary 6.1] to determine a higher integrability exponent $\mathfrak{s} \equiv \mathfrak{s}(n, N, \Lambda, \lambda, p)$ such that $1 < \mathfrak{s} \le m/(p^*)'$ and

$$
\left(\fint_{B_{\sigma/2}(\tilde{x})}|Du_\varrho|^{\mathfrak{s}p}\,\mathrm{d}x\right)^{\frac{1}{\mathfrak{s}p}} \le c\left(\fint_{B_\sigma(\tilde{x})}|Du_\varrho|^{p}\,\mathrm{d}x\right)^{\frac{1}{p}} + c\mathfrak{K}_\varrho^{(p^*)'/p}\left(\fint_{B_\sigma(\tilde{x})}|f_\varrho|^{\mathfrak{s}(p^*)'}\,\mathrm{d}x\right)^{\frac{1}{\mathfrak{s}p}}
$$

for $c \equiv c(n, N, \Lambda, \lambda, p)$. Next, notice that

$$\mathfrak{K}_{\varrho}^{(p^*)'/p} = \left(\fint_{B_1(0)} |f_{\varrho}|^{(p^*)'}\,\mathrm{d}x\right)^{\frac{1}{(p^*)'(p-1)}-\frac{1}{p}} \le \left(\fint_{B_1(0)} |f_{\varrho}|^{\mathfrak{s}(p^*)'}\,\mathrm{d}x\right)^{\frac{1}{\mathfrak{s}(p^*)'(p-1)}-\frac{1}{\mathfrak{s}p}},$$

so plugging this last inequality in (2.14) and recalling that $\mathfrak{s}(p^*)' \le m$, we obtain

$$\left(\fint_{B_{\sigma/2}(\tilde{x})} |Du_{\varrho}|^{\mathfrak{s}p}\,\mathrm{d}x\right)^{\frac{1}{\mathfrak{s}p}} \le c\left(\fint_{B_{\sigma}(\tilde{x})} |Du_{\varrho}|^{p}\,\mathrm{d}x\right)^{\frac{1}{p}} + c\left(\fint_{B_{\sigma}(\tilde{x})} |f_{\varrho}|^{m}\,\mathrm{d}x\right)^{\frac{1}{m(p-1)}}.$$

Setting $p_2 := \mathfrak{s}p > p$ above and recalling that $\tilde{x} \in B_1(0)$ is arbitrary, we can fix $\tilde{x} = 0$, scale back to $B_{\varrho}(x_0)$ and apply (2.1) to get (2.8) and the proof is complete. □

3. Excess decay estimate

In this section we prove some excess decay estimates considering separately two cases: when a smallness condition on the excess functional of our local minimizer u is satisfied and when such an estimate does not hold true.

3.1. The nondegenerate scenario

We start working assuming that a suitable smallness condition on the excess functional $\mathscr{E}(u; B_{\varrho}(x_0))$ is fulfilled. In particular, we prove the following proposition.

Proposition 3.1. *Under assumptions* $(1.6)_{1,2,3}$, (1.7) *and* (1.10)*, let* $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ *be a local minimizer of functional* (1.1)*. Then, for* $\tau_0 \in (0, 2^{-10})$*, there exists* $\varepsilon_0 \equiv \varepsilon_0(\mathtt{data}, \tau_0) \in (0,1)$ *and* $\varepsilon_1 \equiv \varepsilon_1(\mathtt{data}, \tau_0) \in (0,1)$ *such that the following implications hold true.*

- *If the conditions*

$$\mathscr{E}(u; B_{\varrho}(x_0)) \le \varepsilon_0 |(Du)_{B_{\varrho}(x_0)}|, \tag{3.1}$$

and

$$\left(\varrho^m \fint_{B_{\varrho}(x_0)} |f|^m\,\mathrm{d}x\right)^{\frac{1}{m}} \le \varepsilon_1 |(Du)_{B_{\varrho}(x_0)}|^{\frac{p-2}{2}} \mathscr{E}(u; B_{\varrho}(x_0))^{\frac{p}{2}}, \tag{3.2}$$

are verified on $B_{\varrho}(x_0)$*, then it holds that*

$$\mathscr{E}(u; B_{\tau_0\varrho}(x_0)) \le c_0 \tau_0^{\beta_0} \mathscr{E}(u; B_{\varrho}(x_0)), \tag{3.3}$$

for all $\beta_0 \in (0, 2/p)$*, with* $c_0 \equiv c_0(\mathtt{data}) > 0$.

- *If condition* (3.1) *holds true and*

$$\left(\varrho^m \fint_{B_{\varrho}(x_0)} |f|^m\,\mathrm{d}x\right)^{\frac{1}{m}} > \varepsilon_1 |(Du)_{B_{\varrho}(x_0)}|^{\frac{p-2}{2}} \mathscr{E}(u; B_{\varrho}(x_0))^{\frac{p}{2}}, \tag{3.4}$$

is satisfied on $B_{\varrho}(x_0)$*, then*

$$\mathscr{E}(u; B_{\tau_0\varrho}(x_0)) \le c_0 \left(\varrho^m \fint_{B_{\varrho}(x_0)} |f|^m\,\mathrm{d}x\right)^{\frac{1}{m(p-1)}}, \tag{3.5}$$

for $c_0 \equiv c_0(\mathtt{data}) > 0$.

Proof of Proposition 3.1. For the sake of readability, since all balls considered here are concentric to $B_\varrho(x_0)$, we will omit denoting the center. Moreover, we will adopt the following notation $(Du)_{B_\varsigma(x_0)} \equiv (Du)_\varsigma$ and, for all $\varphi \in C_c^\infty(B_\varrho;\mathbb{R}^N)$, we will denote $\|D\varphi\|_{L^\infty(B_\varrho)} \equiv \|D\varphi\|_\infty$. We spilt the proof in two steps.

Step 1: proof of (3.3)**.**

With no loss of generality we can assume that $\mathscr{E}(u;B_\varrho) > 0$, which clearly implies, thanks to (3.1), that $|(Du)_\varrho| > 0$.

We begin proving that condition (3.1) implies that

$$\fint_{B_\varrho} |Du|^p \,\mathrm{d}x \le c|(Du)_\varrho|^p, \tag{3.6}$$

for a constant $c \equiv c(p,\varepsilon_0) > 0$. Indeed,

$$\begin{aligned} \fint_{B_\varrho} |Du|^p \,\mathrm{d}x &\le c\fint_{B_\varrho} |Du-(Du)_\varrho|^p \,\mathrm{d}x + c|(Du)_\varrho|^p \\ &\overset{(1.4)}{\le} c\,\mathscr{E}(u;B_\varrho)^p + c|(Du)_\varrho|^p \\ &\overset{(3.1)}{\le} c(\varepsilon_0^p+1)|(Du)_\varrho|^p, \end{aligned}$$

and (3.6) follows.

Consider now

$$B_\varrho \ni x \mapsto u_0(x) := \frac{|(Du)_\varrho|^{\frac{p-2}{2}}\Big(u(x)-(u)_\varrho - \langle (Du)_\varrho, x-x_0\rangle\Big)}{\mathscr{E}(u;B_\varrho)^{p/2}}, \tag{3.7}$$

and

$$d := \left(\frac{\mathscr{E}(u;B_\varrho)}{|(Du)_\varrho|}\right)^{\frac{p}{2}}.$$

Let us note that we have

$$\begin{aligned} &\fint_{B_\varrho} |Du_0|^2 \,\mathrm{d}x + d^{p-2}\fint_{B_\varrho} |Du_0|^p \,\mathrm{d}x \\ &\quad\le \frac{|(Du)_\varrho|^{p-2}}{\mathscr{E}(u;B_\varrho)^p}\fint_{B_\varrho} |Du-(Du)_\varrho|^2 \,\mathrm{d}x \\ &\qquad + \left(\frac{\mathscr{E}(u;B_\varrho)}{|(Du)_\varrho|}\right)^{\frac{p(p-2)}{2}} \frac{|(Du)_\varrho|^{\frac{p(p-2)}{2}}}{\mathscr{E}(u;B_\varrho)^{\frac{p^2}{2}}}\fint_{B_\varrho} |Du-(Du)_\varrho|^p \,\mathrm{d}x \\ &\quad\le \frac{1}{\mathscr{E}(u;B_\varrho)^p}\fint_{B_\varrho} |(Du)_\varrho|^{p-2}|Du-(Du)_\varrho|^2 \,\mathrm{d}x \\ &\qquad + \frac{1}{\mathscr{E}(u;B_\varrho)^p}\fint_{B_\varrho} |Du-(Du)_\varrho|^p \,\mathrm{d}x \le 1. \end{aligned}$$

Since $|(Du)_\varrho| > 0$ we have that the hypothesis of [12, Lemma 3.2] are satisfied with

$$\mathscr{A} := \partial^2 F((Du)_\varrho)|(Du)_\varrho|^{2-p}. \tag{3.8}$$

Then,

$$\begin{aligned}\left|\fint_{B_\varrho} \mathscr{A}\langle Du_0, D\varphi\rangle \,\mathrm{d}x\right| &\leq \frac{c\|D\varphi\|_\infty |(Du)_\varrho|^{\frac{2-p}{2}}}{\mathscr{E}(u;B_\varrho)^{\frac{p}{2}}}\left(\varrho^m \fint_{B_\varrho} |f|^m \,\mathrm{d}x\right)^{\frac{1}{m}} \\ &\quad + c\|D\varphi\|_\infty \mu\left(\frac{\mathscr{E}(u;B_\varrho)}{|(Du)_\varrho|}\right)^{\frac{1}{p}}\left[1+\left(\frac{\mathscr{E}(u;B_\varrho)}{|(Du)_\varrho|}\right)^{\frac{p-2}{2}}\right] \\ &\overset{(3.1),(3.2)}{\leq} c\varepsilon_1\|D\varphi\|_\infty + c\|D\varphi\|_\infty \mu(\varepsilon_0)^{\frac{1}{p}}\left[1+\varepsilon_0^{\frac{p-2}{2}}\right].\end{aligned}$$

Fix $\varepsilon > 0$ and let $\delta \equiv \delta(\mathtt{data}, \varepsilon) > 0$ be the one given by [33, Lemma 2.4] and choose ε_0 and ε_1 sufficiently small such that

$$c\,\varepsilon_1 + c\mu(\varepsilon_0)^{\frac{1}{p}}\left[1+\varepsilon_0^{\frac{p-2}{2}}\right] \leq \delta. \tag{3.9}$$

With this choice of ε_0 and ε_1 it follows that u_0 is almost $\mathscr{A}$-harmonic on B_ϱ, in the sense that

$$\left|\fint_{B_\varrho} \mathscr{A}\langle Du_0, D\varphi\rangle \,\mathrm{d}x\right| \leq \delta\|D\varphi\|_\infty,$$

with $\mathscr{A}$ as in (3.8). Hence, by [33, Lemma 2.4] we obtain that there exists $h_0 \in W^{1,2}(B_\varrho;\mathbb{R}^N)$ which is $\mathscr{A}$-harmonic, i.e.,

$$\int_{B_\varrho} \mathscr{A}\langle Dh_0, D\varphi\rangle \,\mathrm{d}x = 0 \quad \text{for all } \varphi \in C_c^\infty(B_\varrho;\mathbb{R}^N),$$

such that

$$\fint_{B_{3\varrho/4}} |Dh_0|^2 \,\mathrm{d}x + d^{p-2}\fint_{B_{3\varrho/4}} |Dh_0|^p \,\mathrm{d}x \leq 8^{2np}, \tag{3.10}$$

and

$$\fint_{B_{3\varrho/4}} \left|\frac{u_0 - h_0}{\varrho}\right|^2 + d^{p-2}\left|\frac{u_0-h_0}{\varrho}\right|^p \mathrm{d}x \leq \varepsilon. \tag{3.11}$$

We choose now $\tau_0 \in (0, 2^{-10})$, which will be fixed later on, and estimate

$$\begin{aligned}&\fint_{B_{2\tau_0\varrho}} \left|\frac{u_0(x) - h_0(x_0) - \langle Dh_0(x_0), x - x_0\rangle}{\tau_0\varrho}\right|^2 \mathrm{d}x \\ &\leq c\fint_{B_{2\tau_0\varrho}} \left|\frac{h_0(x) - h_0(x_0) - \langle Dh_0(x_0), x-x_0\rangle}{\tau_0\varrho}\right|^2 \mathrm{d}x + c\fint_{B_{2\tau_0\varrho}}\left|\frac{u_0 - h_0}{\tau_0\varrho}\right|^2 \mathrm{d}x \\ &\overset{(3.11)}{\leq} c(\tau_0\varrho)^2 \sup_{B_{\varrho/2}} |D^2 h_0|^2 + \frac{c\varepsilon}{\tau_0^{n+2}} \\ &\leq c\,\tau_0^2 \fint_{B_{3\varrho/4}} |Dh_0|^2\,\mathrm{d}x + \frac{c\varepsilon}{\tau_0^{n+2}} \\ &\overset{(3.10)}{\leq} c\,\tau_0^2 + \frac{c\varepsilon}{\tau_0^{n+2}},\end{aligned} \tag{3.12}$$

where $c \equiv c(\texttt{data}) > 0$ and where we have used the following property of $\mathscr{A}$-harmonic functions

$$\varrho^{\gamma} \sup_{B_{\varrho/2}} |D^2 h_0|^{\gamma} \le c \fint_{B_{3\varrho/4}} |Dh_0|^{\gamma} \, \mathrm{d}x, \tag{3.13}$$

with $\gamma > 1$ and c depending on n, N, and on the ellipticity constants of $\mathscr{A}$.

Now, choosing

$$\varepsilon := \tau_0^{n+2p},$$

we have that this together with (3.9) gives that $\varepsilon_0 \equiv \varepsilon_0(\texttt{data}, \tau_0)$ and $\varepsilon_1 \equiv \varepsilon_1(\texttt{data}, \tau_0)$. Recalling the definition of u_0 in (3.7) and (3.12) we eventually arrive at

$$\begin{aligned} \fint_{B_{2\tau_0\varrho}} & \frac{|u - (u)_{\varrho} - \langle (Du)_{\varrho}, x - x_0\rangle - |(Du)_{\varrho}|^{\frac{2-p}{2}} \mathscr{E}(u; B_{\varrho})^{p/2} \left(h_0(x_0) - \langle Dh_0(x_0), x - x_0\rangle\right)|^2}{(\tau_0\varrho)^2} \, \mathrm{d}x \\ & \le c\, |(Du)_{\varrho}|^{2-p} \mathscr{E}(u; B_{\varrho})^{p} \tau_0^2, \end{aligned} \tag{3.14}$$

for $c \equiv c(\texttt{data}) > 0$. By a similar computation, always using (3.13), (3.10) and (3.11), we obtain that

$$d^{p-2} \fint_{B_{2\tau_0\varrho}} \left| \frac{u_0 - h_0(x_0) - \langle Dh_0(x_0), x - x_0\rangle}{\tau_0\varrho} \right|^p \mathrm{d}x \le c d^{p-2} (\tau_0\varrho)^p \sup_{B_{\varrho/2}} |D^2 h_0|^p + \frac{c\,\varepsilon}{\tau_0^{n+p}} \le c\, \tau_0^p.$$

In this way, as for (3.14), by the definition of u_0 in (3.7), we eventually arrive at

$$\begin{aligned} \fint_{B_{2\tau_0\varrho}} & \frac{|u - (u)_{\varrho} - \langle (Du)_{\varrho}, x - x_0\rangle - |(Du)_{\varrho}|^{\frac{2-p}{2}} \mathscr{E}(u; B_{\varrho})^{p/2} \left(h_0(x_0) - \langle Dh_0(x_0), x - x_0\rangle\right)|^p}{(\tau_0\varrho)^p} \, \mathrm{d}x \\ & \le c\, d^{2-p} |(Du)_{\varrho}|^{\frac{p(2-p)}{2}} \mathscr{E}(u; B_{\varrho})^{\frac{p^2}{2}} \tau_0^p \\ & \le c\, \mathscr{E}(u; B_{\varrho})^{p} \tau_0^2, \end{aligned} \tag{3.15}$$

with $c \equiv c(\texttt{data})$.

Denote now with $\ell_{2\tau_0\varrho}$ the unique affine function such that

$$\ell_{2\tau_0\varrho} \mapsto \min_{\ell \text{ affine}} \fint_{B_{2\tau_0\varrho}} |u - \ell|^2 \, \mathrm{d}x.$$

Hence, by (3.14) and (3.15), we conclude that

$$\fint_{B_{2\tau_0\varrho}} |(Du)_{\varrho}|^{p-2} \left| \frac{u - \ell_{2\tau_0\varrho}}{2\tau_0\varrho} \right|^2 + \left| \frac{u - \ell_{2\tau_0\varrho}}{2\tau_0\varrho} \right|^p \mathrm{d}x \le c\, \tau^2 \mathscr{E}(u; B_{\varrho})^p. \tag{3.16}$$

Notice that we have also used the property that

$$\fint_{B_{\varrho}} |u - \ell_{\varrho}|^p \, \mathrm{d}x \le c \fint_{B_{\varrho}} |u - \ell|^p \, \mathrm{d}x,$$

for $p \ge 2$, $c \equiv c(n, N, p) > 0$ and for any affine function ℓ; see [33, Lemma 2.3].

Recalling the definition of the excess functional $\mathscr{E}(\cdot)$, in (1.4), we can estimate the following quantity as follows

$$\begin{aligned}|D\ell_{2\tau_0\varrho}-(Du)_\varrho| &\le |D\ell_{2\tau_0\varrho}-(Du)_{2\tau_0\varrho}|+|(Du)_{2\tau_0\varrho}-(Du)_\varrho|\\ &\le c\left(\fint_{B_{2\tau_0\varrho}}|Du-(Du)_{2\tau_0\varrho}|^2\,\mathrm{d}x\right)^{\frac12}+\left(\fint_{B_{2\tau_0\varrho}}|Du-(Du)_\varrho|^2\,\mathrm{d}x\right)^{\frac12}\\ &\overset{(2.1)}{\le} \frac{c}{\tau_0^{n/2}}\left(\fint_{B_\varrho}|Du-(Du)_\varrho|^2\,\mathrm{d}x\right)^{\frac12}\\ &= \frac{c|(Du)_\varrho|^{\frac{2-p}{2}}}{\tau_0^{n/2}}\left(\fint_{B_\varrho}|(Du)_\varrho|^{p-2}|Du-(Du)_\varrho|^2\,\mathrm{d}x\right)^{\frac12}\\ &\le \frac{c(n)}{\tau_0^{n/2}}\left(\frac{\mathscr{E}(u,B_\varrho)}{|(Du)_\varrho|}\right)^{\frac p2}|(Du)_\varrho|, \end{aligned} \tag{3.17}$$

where we have used the following property of the affine function $\ell_{2\tau_0\varrho}$

$$|D\ell_{2\tau_0\varrho}-(Du)_{2\tau_0\varrho}|^p\le c\fint_{B_{2\tau_0\varrho}}|Du-(Du)_{2\tau_0\varrho}|^p\,\mathrm{d}x,$$

for a constant $c\equiv c(n,p)>0$; see for example [33, Lemma 2.2].

Now, starting from (3.1) and (3.9), we further reduce the size of ε_0 such that

$$\left(\frac{\mathscr{E}(u,B_\varrho)}{|(Du)_\varrho|}\right)^{\frac p2}\overset{(3.1)}{\le}\varepsilon_0^{\frac p2}\le\frac{\tau_0^{n/2}}{8c(n)}, \tag{3.18}$$

where $c\equiv c(n)$ is the same constant appearing in (3.17). Thus, combining (3.17) and (3.18), we get

$$|D\ell_{2\tau_0\varrho}-(Du)_\varrho|\le\frac{|(Du)_\varrho|}{8}. \tag{3.19}$$

The information provided by (3.18) combined with (3.16) allow us to conclude that

$$\fint_{B_{2\tau_0\varrho}}|D\ell_{2\tau_0\varrho}|^{p-2}\left|\frac{u-\ell_{2\tau_0\varrho}}{2\tau_0\varrho}\right|^2+\left|\frac{u-\ell_{2\tau_0\varrho}}{2\tau_0\varrho}\right|^p\mathrm{d}x\le c\,\tau^2\mathscr{E}(u;B_\varrho)^p. \tag{3.20}$$

By triangular inequality and (3.19) we also get

$$|D\ell_{2\tau_0\varrho}|\ge|(Du)_\varrho|-|D\ell_{2\tau_0\varrho}-(Du)_\varrho|\overset{(3.19)}{\ge}\frac{7|(Du)_\varrho|}{8}$$

which, therefore, implies that

$$\begin{aligned}
&\fint_{B_{\tau_0\varrho}} |D\ell_{2\tau_0\varrho}|^{p-2}|Du - D\ell_{2\tau_0\varrho}|^2 \,\mathrm{d}x + \inf_{z\in\mathbb{R}^{N\times n}} \fint_{B_{\tau_0\varrho}} |Du - z|^p \,\mathrm{d}x \\
&\overset{(2.7)}{\le} c \fint_{B_{2\tau_0\varrho}} |D\ell_{2\tau_0\varrho}|^{p-2} \left|\frac{u-\ell_{2\tau_0\varrho}}{2\tau_0\varrho}\right|^2 + \left|\frac{u-\ell_{2\tau_0\varrho}}{2\tau_0\varrho}\right|^p \,\mathrm{d}x \\
&\quad + \frac{c}{|D\ell_{2\tau_0\varrho}|^{p-2}} \left((2\tau_0\varrho)^m \fint_{B_{2\tau_0\varrho}} |f|^m \,\mathrm{d}x \right)^{\frac{2}{m}} \\
&\overset{(3.20)}{\le} c\,\tau_0^2 \mathscr{E}(u, B_\varrho)^p + \frac{c\tau_0^{2-2n/m}}{|(Du)_\varrho|^{p-2}} \left(\varrho^m \fint_{B_\varrho} |f|^m \,\mathrm{d}x \right)^{\frac{2}{m}},
\end{aligned} \tag{3.21}$$

where $c \equiv c(\mathtt{data}) > 0$. By triangular inequality, we can further estimate

$$\begin{aligned}
&\fint_{B_{\tau_0\varrho}} |(Du)_{\tau_0\varrho}|^{p-2}|Du - (Du)_{\tau_0\varrho}|^2 \,\mathrm{d}x \\
&\le c \fint_{B_{\tau_0\varrho}} |D\ell_{\tau_0\varrho} - (Du)_{\tau_0\varrho}|^{p-2}|Du - (Du)_{\tau_0\varrho}|^2 \,\mathrm{d}x \\
&\quad + c \fint_{B_{\tau_0\varrho}} |D\ell_{2\tau_0\varrho} - D\ell_{\tau_0\varrho}|^{p-2}|Du - (Du)_{\tau_0\varrho}|^2 \,\mathrm{d}x \\
&\quad + c \fint_{B_{\tau_0\varrho}} |D\ell_{2\tau_0\varrho}|^{p-2}|Du - (Du)_{\tau_0\varrho}|^2 \,\mathrm{d}x \\
&= \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3,
\end{aligned}$$

where $c \equiv c(p) > 0$. We now separately estimate the previous integrals. We begin considering I_1. By Young and triangular inequalities we get

$$\begin{aligned}
\mathrm{I}_1 &\le c|D\ell_{\tau_0\varrho} - (Du)_{\tau_0\varrho}|^p + c \fint_{B_{\tau_0\varrho}} |Du - (Du)_{\tau_0\varrho}|^p \,\mathrm{d}x \\
&\le c \fint_{B_{\tau_0\varrho}} |Du - (Du)_{\tau_0\varrho}|^p \,\mathrm{d}x \\
&\overset{(2.1)}{\le} c \inf_{z\in\mathbb{R}^N} \fint_{B_{\tau_0\varrho}} |Du - z|^p \,\mathrm{d}x \\
&\overset{(3.21)}{\le} c\,\tau_0^2 \mathscr{E}(u, B_\varrho)^p + \frac{c\tau_0^{2-2n/m}}{|(Du)_\varrho|^{p-2}} \left(\varrho^m \fint_{B_\varrho} |f|^m \,\mathrm{d}x \right)^{\frac{2}{m}},
\end{aligned}$$

with $c \equiv c(\texttt{data}) > 0$. In a similar fashion, we can treat the integral I_2

$$\begin{aligned}
\mathrm{I}_2 &\le c\,|D\ell_{2\tau_0\varrho} - D\ell_{\tau_0\varrho}|^p + c \fint_{B_{\tau_0\varrho}} |Du - (Du)_{\tau_0\varrho}|^p \,\mathrm{d}x \\
&\overset{(2.1)}{\le} c \fint_{B_{2\tau_0\varrho}} \left| \frac{u - \ell_{2\tau_0\varrho}}{2\tau_0\varrho} \right|^p \mathrm{d}x + c \inf_{z \in \mathbb{R}^{N\times n}} \fint_{B_{\tau_0\varrho}} |Du - z|^p \,\mathrm{d}x \\
&\overset{(3.20),(3.21)}{\le} c\,\tau_0^2 \mathscr{E}(u, B_\varrho)^p + \frac{c\tau_0^{2-2n/m}}{|(Du)_\varrho|^{p-2}} \left(\varrho^m \fint_{B_\varrho} |f|^m \,\mathrm{d}x \right)^{\frac{2}{m}},
\end{aligned}$$

where we have used the following property of the affine function $\ell_{2\tau_0\varrho}$

$$|D\ell_{2\tau_0\varrho} - D\ell_{\tau_0\varrho}|^p \le c \fint_{B_{2\tau_0\varrho}} \left| \frac{u - \ell_{2\tau_0\varrho}}{2\tau_0\varrho} \right|^p \mathrm{d}x,$$

for a given constant $c \equiv c(n, p) > 0$; see [33, Lemma 2.2]. Finally, the last integral I_3 can be treated recalling (3.21) and (2.1), i.e.,

$$\mathrm{I}_3 \le c\,\tau_0^2 \mathscr{E}(u, B_\varrho)^p + \frac{c\tau_0^{2-2n/m}}{|(Du)_\varrho|^{p-2}} \left(\varrho^m \fint_{B_\varrho} |f|^m \,\mathrm{d}x \right)^{\frac{2}{m}}.$$

All in all, combining the previous estimate

$$\begin{aligned}
\mathscr{E}(u; B_{\tau_0\varrho}) &\le c\,\tau_0^{2/p} \mathscr{E}(u, B_\varrho) + \frac{c\tau_0^{2/p-2n/(mp)}}{|(Du)_\varrho|^{\frac{p-2}{p}}} \left(\varrho^m \fint_{B_\varrho} |f|^m \,\mathrm{d}x \right)^{\frac{2}{mp}} \\
&\overset{(3.2)}{\le} c\,\tau_0^{2/p} \mathscr{E}(u, B_\varrho) + c\tau_0^{2/p-2n/(mp)} \varepsilon_1^{2/p} \mathscr{E}(u; B_{\tau_0\varrho}) \\
&\le c_0 \tau_0^{2/p} \mathscr{E}(u; B_{\tau_0\varrho}),
\end{aligned}$$

up to choosing ε_1 such that

$$\varepsilon_1 \le \tau_0^{n/m}.$$

Step 2: proof of (3.5)**.**

The proof follows by [12, Lemma 2.4] which yields

$$\begin{aligned}
\mathscr{E}(u; B_{\tau_0\varrho}(x_0))^{\frac{p}{2}} &\le \frac{2^{3p}}{\tau_0^{n/2}} \mathscr{E}(u; B_\varrho(x_0))^{\frac{p}{2}} \\
&\overset{(3.4)}{\le} \frac{2^{3p}}{\tau_0^{n/2}} \varepsilon_1^{-1} |(Du)_{B_\varrho(x_0)}|^{\frac{2-p}{2}} \left(\varrho^m \fint_{B_\varrho(x_0)} |f|^m \,\mathrm{d}x \right)^{\frac{1}{m}} \\
&\overset{(3.1)}{\le} \frac{2^{6(p-1)}}{\tau_0^{n(p-1)/p}} \varepsilon_0^{\frac{p-2}{2}} \varepsilon_1^{-1} \mathscr{E}(u; B_{\tau_0\varrho})^{\frac{2-p}{2}} \left(\varrho^m \fint_{B_\varrho(x_0)} |f|^m \,\mathrm{d}x \right)^{\frac{1}{m}}.
\end{aligned}$$

Multiplying both sides by $\mathscr{E}(u; B_{\tau_0\varrho})^{\frac{p-2}{2}}$ we get the desired estimate. □

3.2. The degenerate scenario

It remains to considering the case when condition (3.1) does not hold true. We start with two technical lemmas. The first one is an analogous of the Caccioppoli inequality (2.7), where we take in consideration the eventuality $z_0 = 0$.

Lemma 3.1. *Under assumptions* $(1.6)_{1,2,3}$, (1.7) *and* (1.10)*, let* $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ *be a local minimizer of functional* (1.1)*. For every ball* $B_\varrho(x_0) \Subset \Omega$ *and any* $u_0 \in \mathbb{R}^N$*,* $z_0 \in \mathbb{R}^{N\times n}$ *it holds that*

$$\begin{aligned}\mathscr{E}(u, z_0; B_{\varrho/2}(x_0))^p \quad &\le \quad c \fint_{B_\varrho(x_0)} |z_0|^{p-2}\left|\frac{u-\ell}{\varrho}\right|^2 + \left|\frac{u-\ell}{\varrho}\right|^p \,\mathrm{d}x \\ &\quad + c\left(\varrho^m \fint_{B_\varrho(x_0)} |f|^m \,\mathrm{d}x\right)^{\frac{p}{m(p-1)}},\end{aligned} \tag{3.22}$$

where $\mathscr{E}(\cdot)$ *is defined in* (1.4)*,* $\ell(x) := u_0 + \langle z_0, x - x_0\rangle$ *and* $c \equiv c(n, N, \lambda, \Lambda, p)$*.*

Proof. The proof is analogous to estimate (2.7), up to treating in a different way the term I_2 in (2.9), taking in consideration the eventuality $z_0 = 0$. Exploiting (1.10) and fact that $\varphi_1 \in W^{1,p}_0(B_{\tau_2}(x_0), \mathbb{R}^N)$, an application of the Sobolev-Poincaré inequality yields

$$\begin{aligned}\mathrm{I}_2 \quad &\le \quad |B_{\tau_2}(x_0)|\left(\tau_2^m \fint_{B_{\tau_2}(x_0)} |f|^m \,\mathrm{d}x\right)^{1/m}\left(\tau_2^{-m'} \fint_{B_{\tau_2}(x_0)} |\varphi_1|^{m'} \,\mathrm{d}x\right)^{\frac{1}{m'}} \\ &\le \quad |B_{\tau_2}(x_0)|\left(\tau_2^m \fint_{B_{\tau_2}(x_0)} |f|^m \,\mathrm{d}x\right)^{1/m}\left(\fint_{B_{\tau_2}(x_0)} \left|\frac{\varphi_1}{\tau_2}\right|^{p^*} \,\mathrm{d}x\right)^{\frac{1}{p^*}} \\ &\le \quad |B_{\tau_2}(x_0)|\left(\tau_2^m \fint_{B_{\tau_2}(x_0)} |f|^m \,\mathrm{d}x\right)^{1/m}\left(\fint_{B_{\tau_2}(x_0)} |D\varphi_1|^p \,\mathrm{d}x\right)^{\frac{1}{p}} \\ &\le \quad \varepsilon \int_{B_{\tau_2}(x_0)} |V_{|z_0|}(D\varphi_1)|^2 \,\mathrm{d}x + \frac{c|B_\varrho(x_0)|}{\varepsilon^{1/(p-1)}}\left(\varrho^m \fint_{B_\varrho(x_0)} |f|^m \,\mathrm{d}x\right)^{\frac{p}{m(p-1)}},\end{aligned} \tag{3.23}$$

where $c \equiv c(n, N, m)$ and we also used that $\varrho/2 \le \tau_2 \le \varrho$. Hence, proceeding as in the proof of (2.7), we obtain that

$$\begin{aligned}&\int_{B_{\tau_1}(x_0)} |V_{|z_0|}(Du - z_0)|^2 \,\mathrm{d}x \\ &\quad \le c \int_{B_{\tau_2}(x_0)\setminus B_{\tau_1}(x_0)} |V_{|z_0|}(Du - z_0)|^2 + \left|V_{|z_0|}\left(\frac{u-\ell}{\tau_2 - \tau_1}\right)\right|^2 \,\mathrm{d}x \\ &\quad\quad + \frac{c|B_\varrho(x_0)|}{\varepsilon^{1/(p-1)}}\left(\varrho^m \fint_{B_\varrho(x_0)} |f|^m \,\mathrm{d}x\right)^{\frac{p}{m(p-1)}},\end{aligned}$$

with $c \equiv c(n, N, \lambda, \Lambda, p)$. Concluding as in the proof of (2.7), we eventually arrive at (3.22). □

We will also need the following result.

Lemma 3.2. *Under assumptions* $(1.6)_{1,2,3}$, (1.7) *and* (1.10)*, let* $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ *be a local minimizer of functional* (1.1)*. For any* $B_\varrho(x_0) \Subset \Omega$ *and any* $s \in (0, \infty)$ *it holds that*

$$\begin{aligned} \left| \fint_{B_\varrho(x_0)} \langle |Du|^{p-2}Du, D\varphi \rangle \, \mathrm{d}x \right| \leq \; & s \left\| D\varphi \right\|_{L^\infty(B_\varrho(x_0))} \left(\fint_{B_\varrho(x_0)} |Du|^p \, \mathrm{d}x \right)^{\frac{p-1}{p}} \\ & + c\, \omega(s)^{-1} \left\| D\varphi \right\|_{L^\infty(B_\varrho(x_0))} \fint_{B_\varrho(x_0)} |Du|^p \, \mathrm{d}x \\ & + c \left\| D\varphi \right\|_{L^\infty(B_\varrho(x_0))} \left(\varrho^m \fint_{B_\varrho(x_0)} |f|^m \, \mathrm{d}x \right)^{1/m}, \end{aligned} \tag{3.24}$$

for any $\varphi \in C_0^\infty(B_\varrho(x_0), \mathbb{R}^N)$*, with* $c \equiv c(n, N, \Lambda, \lambda, p)$*.*

Proof. Given the regularity properties of the integrand F, we have that a local minimizer u of (1.1) solves weakly the following integral identity (see [42, Lemma 7.3])

$$\int_\Omega \left[\langle \partial F(Du), D\varphi \rangle - f \cdot \varphi \right] \mathrm{d}x = 0 \quad \text{for all } \varphi \in C_0^\infty(\Omega, \mathbb{R}^N). \tag{3.25}$$

Now, fix $\varphi \in C_0^\infty(B_\varrho(x_0), \mathbb{R}^N)$ and split

$$\begin{aligned} & \left| \fint_{B_\varrho(x_0)} \langle |Du|^{p-2}Du, D\varphi \rangle \, \mathrm{d}x \right| \\ & \quad \overset{(3.25)}{\leq} \left| \fint_{B_\varrho(x_0)} \langle \partial F(Du) - \partial F(0) - |Du|^{p-2}Du, D\varphi \rangle \, \mathrm{d}x \right| + \left| \fint_{B_\varrho(x_0)} f \cdot \varphi \, \mathrm{d}x \right| \\ & \quad =: \mathrm{I}_1 + \mathrm{I}_2. \end{aligned}$$

We begin estimating the first integral I_1. For $s \in (0, \infty)$ we get

$$\begin{aligned} \mathrm{I}_1 \leq \; & \frac{\left\| D\varphi \right\|_{L^\infty(B_\varrho(x_0))}}{|B_\varrho(x_0)|} \int_{B_\varrho(x_0) \cap \{|Du| \leq \omega(s)\}} |\partial F(Du) - \partial F(0) - |Du|^{p-2}Du| \, \mathrm{d}x \\ & + \frac{\left\| D\varphi \right\|_{L^\infty(B_\varrho(x_0))}}{|B_\varrho(x_0)|} \int_{B_\varrho(x_0) \cap \{|Du| > \omega(s)\}} |\partial F(Du) - \partial F(0) - |Du|^{p-2}Du| \, \mathrm{d}x \\ \leq \; & s \left\| D\varphi \right\|_{L^\infty(B_\varrho(x_0))} \left(\fint_{B_\varrho(x_0)} |Du|^p \, \mathrm{d}x \right)^{\frac{p-1}{p}} \\ & + c\, \omega(s)^{-1} \left\| D\varphi \right\|_{L^\infty(B_\varrho(x_0))} \fint_{B_\varrho(x_0)} |Du|^p \, \mathrm{d}x. \end{aligned} \tag{3.26}$$

On the other hand, the integral I_2 can be estimated as follows

$$\begin{aligned}
\mathrm{I}_2 &\leq \left(\varrho^m \fint_{B_\varrho(x_0)} |f|^m \,\mathrm{d}x\right)^{1/m} \left(\fint_{B_\varrho(x_0)} \left|\frac{\varphi}{\varrho}\right|^{m'} \,\mathrm{d}x\right)^{\frac{1}{m'}} \\
&\leq \left(\varrho^m \fint_{B_\varrho(x_0)} |f|^m \,\mathrm{d}x\right)^{1/m} \left(\fint_{B_\varrho(x_0)} \left|\frac{\varphi}{\varrho}\right|^{p^*} \,\mathrm{d}x\right)^{\frac{1}{p^*}} \\
&\leq \left(\varrho^m \fint_{B_\varrho(x_0)} |f|^m \,\mathrm{d}x\right)^{1/m} \left(\fint_{B_\varrho(x_0)} |D\varphi|^p \,\mathrm{d}x\right)^{\frac{1}{p}} \\
&\leq \left\|D\varphi\right\|_{L^\infty(B_\varrho(x_0))} \left(\varrho^m \fint_{B_\varrho(x_0)} |f|^m \,\mathrm{d}x\right)^{1/m}.
\end{aligned}$$

Combining the inequalities above we obtain (3.24). □

In this setting the analogous result of Proposition 3.1 is the following one.

Proposition 3.2. *Under assumptions* $(1.6)_{1,2,3}$, (1.7) *and* (1.10)*, let* $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ *be a local minimizer of functional* (1.1)*. Then, for any* $\chi \in (0, 1]$ *and any* $\tau_1 \in (0, 2^{-10})$*, there exists* $\varepsilon_2 \equiv \varepsilon_2(\mathtt{data}, \chi, \tau_1) \in (0, 1)$ *such that if the smallness conditions*

$$\chi|(Du)_{B_\varrho(x_0)}| \leq \mathscr{E}(u; B_\varrho(x_0)), \quad \textit{and} \quad \mathscr{E}(u; B_\varrho(x_0)) \leq \varepsilon_2, \tag{3.27}$$

are satisfied on a ball $B_\varrho(x_0) \subset \mathbb{R}^n$*, then*

$$\mathscr{E}(u; B_{\tau_1\varrho}(x_0)) \leq c_1\tau_1^{\beta_1}\mathscr{E}(u; B_\varrho(x_0)) + c_1\left(\varrho^m \fint_{B_\varrho(x_0)} |f|^m \,\mathrm{d}x\right)^{\frac{1}{m(p-1)}}, \tag{3.28}$$

for any $\beta_1 \in (0, 2\alpha/p)$*, with* $\alpha \equiv \alpha(n, N, p) \in (0, 1)$ *is the exponent in* (3.34)*, and* $c_1 \equiv c_1(\mathtt{data}, \chi)$*.*

Proof. We adopt the same notations used in the proof of Proposition 3.1. Let us begin noticing that condition $(3.27)_1$ implies the following estimate

$$\fint_{B_\varrho} |Du|^p \,\mathrm{d}x \leq c_\chi \mathscr{E}(u; B_\varrho)^p \qquad \text{with } c_\chi := 2^p(1 + \chi^{-p}). \tag{3.29}$$

Indeed, by (1.4) and (3.27), we have

$$\begin{aligned}
\fint_{B_\varrho} |Du|^p \,\mathrm{d}x &\leq 2^p \fint_{B_\varrho} |Du - (Du)_{B_\varrho}|^p \,\mathrm{d}x + 2^p|(Du)_{B_\varrho}|^p \\
&\leq 2^p\mathscr{E}(u; B_\varrho)^p + \frac{2^p}{\chi^p}\mathscr{E}(u; B_\varrho)^p.
\end{aligned}$$

Consider now

$$\kappa := c_\chi\mathscr{E}(u; B_\varrho) + \left(\left(\frac{\varrho}{\varepsilon_3}\right)^m \fint_{B_\varrho} |f|^m \,\mathrm{d}x\right)^{\frac{1}{m(p-1)}} \quad \text{and} \quad v_0 := \frac{u}{\kappa},$$

for $\varepsilon_3 \in (0,1]$, which will be fixed later on. Applying (3.24) to the function v_0 yields

$$\left| \fint_{B_{\varrho/2}(x_0)} \langle |Dv_0|^{p-2} Dv_0, D\varphi \rangle \, \mathrm{d}x \right| \overset{(3.27)_2,(3.29)}{\leq} c \big\| D\varphi \big\|_\infty \left(s + \omega(s)^{-1} \varepsilon_2 + \varepsilon_3 \right).$$

For any $\varepsilon > 0$ and $\vartheta \in (0,1)$ and let δ be the one given by [17, Lemma 1.1]. Then, up to choosing s, ε_2 and ε_3 sufficiently small, we arrive at

$$c \left(s + \omega(s)^{-1} \varepsilon_2 + \varepsilon_3 \right) \leq \delta \|D\varphi\|_\infty^{p-1}.$$

Then, Lemma 1.1 in [17] implies

$$\left(\fint_{B_{\varrho/2}} |V(Dv_0) - V(Dh)|^{2\vartheta} \, \mathrm{d}x \right)^{\frac{1}{\vartheta}} \leq c\varepsilon \fint_{B_{\varrho/2}} |Du|^p \, \mathrm{d}x \overset{(3.29),(3.27)_2}{\leq} c\varepsilon\varepsilon_2^p,$$

up to taking ε as small as needed. Now, denoting with $\mathfrak{h}_0 := h\kappa$, we have that

$$\left(\fint_{B_{\varrho/2}} |V(Du) - V(D\mathfrak{h}_0)|^{2\vartheta} \, \mathrm{d}x \right)^{\frac{1}{\vartheta}} \leq \varepsilon\varepsilon_2^p \kappa^p.$$

Now, we choose $\vartheta := (\mathfrak{s})'/2$, with $\mathfrak{s}$ being the exponent given by (2.8). Note that by the proof of (2.8) it actually follows that $\vartheta < 1$. Thus, choosing $\varepsilon\varepsilon_2^p \kappa^p \leq \tau_1^{2n+4\alpha}$ (where $\alpha \in (0,1)$ is given by (3.34)) we arrive at

$$\left(\fint_{B_{\varrho/2}} |V(Du) - V(D\mathfrak{h}_0)|^{(\mathfrak{s})'} \, \mathrm{d}x \right)^{\frac{1}{(\mathfrak{s})'}} \leq c\, \tau_1^{n+2\alpha}.$$

By Hölder's Inequality, we have that

$$\begin{aligned} &\fint_{B_{\varrho/2}} |V(Du) - V(D\mathfrak{h}_0)|^2 \, \mathrm{d}x \\ &\quad \leq \left(\fint_{B_{\varrho/2}} |V(Du) - V(D\mathfrak{h}_0)|^{(\mathfrak{s})'} \, \mathrm{d}x \right)^{\frac{1}{(\mathfrak{s})'}} \left(\fint_{B_{\varrho/2}} |V(Du) - V(D\mathfrak{h}_0)|^{\mathfrak{s}} \, \mathrm{d}x \right)^{\frac{1}{\mathfrak{s}}}. \end{aligned} \tag{3.30}$$

Hence, since by (2.3) $V(z) \approx |z|^p$, an application of estimates (2.8) and (3.29) now yields

$$\begin{aligned} \left(\fint_{B_{\varrho/2}} |V(Du)|^{\mathfrak{s}} \, \mathrm{d}x \right)^{\frac{1}{\mathfrak{s}}} &\leq c \left(\fint_{B_{\varrho/2}} |Du - (Du)_\varrho|^{p_2} \, \mathrm{d}x \right)^{\frac{p}{p_2}} + c|(Du)_\varrho|^p \\ &\leq c \fint_{B_\varrho} |Du|^p \, \mathrm{d}x + c \left(\varrho^m \fint_{B_\varrho} |f|^m \, \mathrm{d}x \right)^{\frac{p}{m(p-1)}} + c|(Du)_\varrho|^p \\ &\leq c\, \mathscr{E}(u; B_\varrho)^p + c \left(\varrho^m \fint_{B_\varrho} |f|^m \, \mathrm{d}x \right)^{\frac{p}{m(p-1)}}, \end{aligned} \tag{3.31}$$

with $c \equiv c(\mathtt{data}, \chi)$.

On the other hand, by classical properties of p-harmonic functions, we have that

$$\left(\fint_{B_{\varrho/2}}|V(D\mathfrak{h}_0)|^{\mathfrak{s}}\,\mathrm{d}x\right)^{\frac{1}{\mathfrak{s}}}\le c\fint_{B_\varrho}|D\mathfrak{h}_0|^p\,\mathrm{d}x\le c\fint_{B_\varrho}|Du|^p\,\mathrm{d}x\le c\,\mathscr{E}(u;B_\varrho)^p. \tag{3.32}$$

Hence, combining (3.30)–(3.32), we get that

$$\fint_{B_{\varrho/2}}|V(Du)-V(D\mathfrak{h}_0)|^2\,\mathrm{d}x\le c\,\tau_1^{n+2\alpha}\mathscr{E}(u;B_\varrho)^p+c\,\tau_1^{n+2\alpha}\left(\varrho^m\fint_{B_\varrho}|f|^m\,\mathrm{d}x\right)^{\frac{p}{m(p-1)}}. \tag{3.33}$$

Let us recall that, for any $\tau_1\in(0,2^{-10})$, given the p-harmonic function $\mathfrak{h}_0$ we have

$$\widetilde{\mathscr{E}}(\mathfrak{h}_0;B_{\tau_1\varrho})^2\le c\tau_1^{2\alpha}\kappa^p,\qquad \alpha\equiv\alpha(n,N,p)\in(0,1). \tag{3.34}$$

Moreover, using Jensen's Inequality we can estimate the following difference as follows

$$\begin{aligned}|(Du)_{\tau_1\varrho}-(Du)_\varrho| &\le \left(\fint_{B_{\tau_1\varrho}}|Du-(Du)_\varrho|^p\,\mathrm{d}x\right)^{\frac{1}{p}}\\ &\le \tau_1^{-\frac{n}{p}}\left(\fint_{B_\varrho}|Du-(Du)_\varrho|^p\,\mathrm{d}x\right)^{\frac{1}{p}}\\ &\overset{(1.4),(3.27)_2}{\le} \tau_1^{-\frac{n}{p}}\varepsilon_2.\end{aligned}$$

Thus, up to taking ε_2 sufficiently small, by the triangular inequality, we obtain that

$$\frac{1}{2}|(Du)_{\tau_1\varrho}|\le|(Du)_\varrho|\le 2|(Du)_{\tau_1\varrho}|.$$

Hence, (2.2) yield

$$|V_{|(Du)_{\tau_1\varrho}|}(\cdot)|^2\approx|V_{|(Du)_\varrho|}(\cdot)|^2,$$

and

$$|V((Du)_{\tau_1\varrho})-V((Du)_\varrho)|^2\approx\left|V_{|(Du)_\varrho|}\Big((Du)_\varrho-(Du)_{\tau_1\varrho}\Big)\right|^2.$$

Then,

$$\begin{aligned}\mathscr{E}(u;B_{\tau_1\varrho})^p &\overset{(2.5)}{\le} c\,\widetilde{\mathscr{E}}(u;B_{\tau_1\varrho})^2\\ &\overset{(2.6)}{\le} c\fint_{B_{\tau_1\varrho}}|V(Du)-V((Du)_{\tau_1\varrho})|^2\,\mathrm{d}x\\ &\le c\,\tau_1^{-n}\fint_{B_{\varrho/2}}|V(Du)-V(D\mathfrak{h}_0)|^2\,\mathrm{d}x\\ &\quad +c\fint_{B_{\tau_1\varrho}}|V(D\mathfrak{h}_0)-V((D\mathfrak{h}_0)_{\tau_1\varrho})|^2\,\mathrm{d}x\\ &\overset{(2.6)}{\le} c\,\tau_1^{-n}\fint_{B_{\varrho/2}}|V(Du)-V(D\mathfrak{h}_0)|^2\,\mathrm{d}x+c\,\widetilde{\mathscr{E}}(\mathfrak{h}_0,B_{\tau_1\varrho})\\ &\overset{(3.33),(3.34)}{\le} c\,\tau_1^{2\alpha}\mathscr{E}(u;B_\varrho)^p+c\left(\varrho^m\fint_{B_\varrho}|f|^m\,\mathrm{d}x\right)^{\frac{p}{m(p-1)}},\end{aligned}$$

and the desired estimate (3.28) follows. □

4. Proof of the main result

This section is devoted to the proof of Theorem 1.1. First, we prove the following proposition.

Proposition 4.1. *Under assumptions* $(1.6)_{1,2,3}$, (1.7) *and* (1.10), *let* $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ *be a local minimizer of functional* (1.1). *Then, there exists* $\varepsilon_* \equiv \varepsilon_*(\mathtt{data}) > 0$ *such that if the following condition*

$$\mathscr{E}(Du; B_r) + \sup_{\varrho \le r} \left(\varrho^m \fint_{B_\varrho} |f|^m \, \mathrm{d}x \right)^{\frac{1}{m(p-1)}} < \varepsilon, \tag{4.1}$$

is satisfied on $B_r \subset \Omega$, *for some* $\varepsilon \in (0, \varepsilon_*]$, *then*

$$\sup_{\varrho \le r} \mathscr{E}(Du; B_\varrho) < c_3 \, \varepsilon, \tag{4.2}$$

for $c_3 \equiv c_3(\mathtt{data}) > 0$.

Proof. For the sake of readability, since all balls considered in the proof are concentric to $B_r(x_0)$, we will omit denoting the center.

Let us start fixing an exponent $\beta \equiv \beta(\alpha, p)$ such that

$$0 < \beta < \min\{\beta_0, \beta_1\} =: \beta_m, \tag{4.3}$$

where β_0 and β_1 are the exponents appearing in Propositions 3.1 and 3.2. Moreover, given the constant c_0 and c_1 from Propositions 3.1 and 3.2, choose $\tau \equiv \tau(\mathtt{data}, \beta)$ such that

$$(c_0 + c_1)\tau^{\beta_m - \beta} \le \frac{1}{4}. \tag{4.4}$$

With the choice of τ_0 as in (4.4) above, we can determine the constant ε_0 and ε_1 of Proposition 3.1. Now, we proceed applying Proposition 3.2 taking $\chi \equiv \varepsilon_0$ and τ_1 as in (4.4) there. This determines the constant ε_2 and c_2. We consider a ball $B_r \subset \Omega$ such that

$$\mathscr{E}(Du; B_r) < \varepsilon_2, \tag{4.5}$$

and

$$\sup_{\varrho \le r} c_2 \left(\varrho^m \fint_{B_\varrho} |f|^m \, \mathrm{d}x \right)^{\frac{1}{m(p-1)}} \le \frac{\varepsilon_2}{4}, \tag{4.6}$$

where the constant $c_2 := c_1 + c_0$, with c_0 appearing in (3.5) and c_1 in (3.28). In particular, see that by (4.5) and (4.6) we are in the case when (4.1) does hold true.

Now, we recall Proposition 3.2. Seeing that $(3.27)_2$ is satisfied (being (4.5)) we only check whether $(3.27)_1$ is verified too. If $\varepsilon_0 |(Du)_{B_r}| \le \mathscr{E}(Du; B_r)$ is satisfied then we obtain from (3.28), with $\tau_1 \equiv \tau$ in (4.4) that

$$\begin{aligned}\mathscr{E}(u;B_{\tau r}) &\leq \frac{\tau^\beta}{4}\mathscr{E}(u;B_r)+c_2\left(r^m\fint_{B_r}|f|^m\,\mathrm{d}x\right)^{\frac{1}{m(p-1)}}\\ &\leq \frac{\tau^\beta}{4}\mathscr{E}(u;B_r)+\sup_{\varrho\leq r}c_2\left(\varrho^m\fint_{B_\varrho}|f|^m\,\mathrm{d}x\right)^{\frac{1}{m(p-1)}}\\ &\leq \frac{\tau^\beta}{4}\mathscr{E}(u;B_r)+\frac{\varepsilon_2}{4}\leq\varepsilon_2,\end{aligned}\tag{4.7}$$

where the last inequality follows from (4.5) and (4.6). If on the other hand it holds $\varepsilon_0|(Du)_{B_r}| \geq \mathscr{E}(Du;B_r)$, by Proposition 3.1, then by (3.3) or (3.5) we eventually arrive at the same estimate (4.7).

Iterating now the seam argument we arrive at

$$\mathscr{E}(Du;B_{\tau^j r})<\varepsilon_2 \quad \text{for any } j\geq 0,$$

and the estimate

$$\mathscr{E}(u;B_{\tau^{j+1}r})\leq\frac{\tau^\beta}{4}\mathscr{E}(u;B_{\tau^j r})+c_2\left((\tau^j r)^m\fint_{B_{\tau^j r}}|f|^m\,\mathrm{d}x\right)^{\frac{1}{m(p-1)}},$$

holds true. By the inequality above we have that for any $k\geq 0$

$$\begin{aligned}\mathscr{E}(u;B_{\tau^{k+1}r}) &\leq \frac{\tau^{\beta(k+1)}}{4}\mathscr{E}(u;B_r)+c_2\sum_{j=0}^{k}(\tau^\beta)^{j-k}\left((\tau^j r)^m\fint_{B_{\tau^j r}}|f|^m\,\mathrm{d}x\right)^{\frac{1}{m(p-1)}}\\ &\leq \tau^{\beta(k+1)}\mathscr{E}(u;B_r)+c_2\sup_{\varrho\leq r}\left(\varrho^m\fint_{B_r r}|f|^m\,\mathrm{d}x\right)^{\frac{1}{m(p-1)}}.\end{aligned}$$

Applying a standard interpolation argument we conclude that, for any $t\leq r$, it holds

$$\mathscr{E}(Du,B_s)\leq c_3\left(\frac{s}{r}\right)^\beta\mathscr{E}(Du,B_r)+c_3\sup_{\varrho\leq r}\left(\varrho^m\fint_{B_r r}|f|^m\,\mathrm{d}x\right)^{\frac{1}{m(p-1)}},\tag{4.8}$$

where $c_3\equiv c_3(\mathtt{data})$. The desired estimate (4.2) now follows. □

Proof of Theorem 1.1. We proceed following the same argument used in [33, Theorem 1.5]. We star proving that, for any $1\leq m<n$ and any $\mathscr{O}\subset\Omega$, with positive measure, we have that

$$\|f\|_{L^m(\mathscr{O})}\leq\left(\frac{n}{n-m}\right)^{1/m}|\mathscr{O}|^{1/m-1/n}\|f\|_{L^{n,\infty}(\mathscr{O})}.\tag{4.9}$$

Indeed, fix $\bar\lambda$ which will be chosen later on. Then, we have that

$$\|f\|^m_{L^m(\mathscr{O})}=m\int_0^{\bar\lambda}\lambda^m|\{x\in\mathscr{O}:|f|>\lambda\}|\frac{\mathrm{d}\lambda}{\lambda}+m\int_{\bar\lambda}^\infty\lambda^m|\{x\in\mathscr{O}:|f|>\lambda\}|\frac{\mathrm{d}\lambda}{\lambda}.\tag{4.10}$$

The first integral on the righthand side of (4.10) can be estimated in the following way

$$\int_0^{\bar{\lambda}} \lambda^m |\{x \in \mathscr{O} : |f| > \lambda\}| \frac{\mathrm{d}\lambda}{\lambda} \le \frac{\bar{\lambda}^m |\mathscr{O}|}{m}.$$

On the other hand, the second integral can be estimated recalling the definition of the $L^{n,\infty}(\mathscr{O})$-norm. Indeed,

$$\int_{\bar{\lambda}}^{\infty} \lambda^m |\{x \in \mathscr{O} : |f| > \lambda\}| \frac{\mathrm{d}\lambda}{\lambda} \le \|f\|^n_{L^{n,\infty}(\mathscr{O})} \int_{\bar{\lambda}}^{\infty} \frac{\mathrm{d}\lambda}{\lambda^{1+n-m}} \le \frac{\|f\|^n_{L^{n,\infty}(\mathscr{O})}}{(n-m)\bar{\lambda}^{n-m}}.$$

Hence, putting all the estimates above in (4.10), choosing $\bar{\lambda} := \|f\|_{L^{n,\infty}(\mathscr{O})}/|\mathscr{O}|^{1/n}$, we obtain (4.9).

Now, recalling condition (1.2) we have that

$$\begin{aligned} \left(\varrho^m \fint_{B_\varrho} |f|^m \, \mathrm{d}x\right)^{1/m} &\le \left(\frac{n}{n-m}\right)^{1/m} |B_1|^{-1/n} \|f\|_{L^{n,\infty}(\Omega)} \\ &\overset{(1.10)}{\le} \left(\frac{4^{n/m}}{|B_1|}\right)^{1/n} \|f\|_{L^{n,\infty}(\Omega)} \overset{(1.2)}{\le} \varepsilon_*, \end{aligned}$$

where ε_* is the one obtained in the proof of Proposition 4.1. From this it follows that, we can choose a radius ϱ_1 such that

$$\sup_{\varrho \le \varrho_1} c_2 \left(\varrho^m \fint_{B_\varrho(x)} |f|^m \, \mathrm{d}x\right)^{1/m(p-1)} \le \frac{\varepsilon_*}{4c_3}. \tag{4.11}$$

We want to show that the set Ω_u appearing in (1.3) can be characterized by

$$\Omega_u := \left\{x_0 \in \Omega : \ \exists B_\varrho(x_0) \Subset \Omega \, \text{with} \, \varrho \le \varrho_1 \ : \mathscr{E}(Du, B_\varrho(x_0)) < \varepsilon_*/(4c_3)\right\},$$

thus fixing $\varrho_{x_0} := \varrho_1$ and $\varepsilon_{x_0} := \varepsilon_*/(4c_3)$. We first star noting that the the set Ω_u defined in (1.4) is such that $|\Omega \setminus \Omega_u| = 0$. Indeed, let us consider the set

$$\mathscr{L}_u := \left\{x_0 \in \Omega : \liminf_{\varrho \to 0} \widetilde{\mathscr{E}}(u; B_\varrho(x_0))^2 = 0\right\}, \tag{4.12}$$

which is such that $|\Omega \setminus \mathscr{L}_u| = 0$ by standard Lebesgue's Theory. Moreover, by (2.5) it follows that

$$\mathscr{L}_u := \left\{x_0 \in \Omega : \liminf_{\varrho \to 0} \mathscr{E}(u; B_\varrho(x_0)) = 0\right\},$$

so that, $\mathscr{L}_u \subset \Omega_u$ and we eventually obtained that $|\Omega \setminus \Omega_u| = 0$. Now we show that Ω_u is open. Let us fix $x_0 \in \Omega_u$ and find a radius $\varrho_{x_0} \le \varrho_1$ such that

$$\mathscr{E}(Du, B_{\varrho_{x_0}}(x_0)) < \frac{\varepsilon_*}{4c_3}. \tag{4.13}$$

By absolute continuity of the functional $\mathscr{E}(\cdot)$ we have that there exists an open neighbourhood $\mathscr{O}(x_0)$ such that, for any $x \in \mathscr{O}(x_0)$ it holds

$$\mathscr{E}(Du, B_{\varrho_{x_0}}(x)) < \frac{\varepsilon_*}{4c_3} \quad \text{and } B_{\varrho_{x_0}}(x) \Subset \Omega. \tag{4.14}$$

This prove that Ω_u is open. Now let us start noting that (4.11) and (4.14) yield that condition (4.1) is satisfied with $B_r \equiv B_{\varrho_{x_0}}(x)$. Hence, an application of Proposition 4.1 yields

$$\sup_{t\leq\varrho_{x_0}} \mathscr{E}(Du, B_t(x)) < \varepsilon_*,$$

for any $x \in \mathscr{O}(x_0)$. Thus concluding the proof. □

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares no conflict of interest.

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