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Research article

Kolmogorov variation: KAM with knobs *(à la Kolmogorov)*†

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Abstract: In this paper we reconsider the original Kolmogorov normal form algorithm [26] with a variation on the handling of the frequencies. At difference with respect to the Kolmogorov approach, we do not keep the frequencies fixed along the normalization procedure. Besides, we select the frequencies of the final invariant torus and determine *a posteriori* the corresponding starting ones. In particular, we replace the classical *translation step* with a change of the frequencies. The algorithm is based on the original scheme of Kolmogorov, thus exploiting the fast convergence of the Newton-Kantorovich method.

Keywords: Kolmogorov normal form; perturbation theory; KAM theorem

Foreword

The present manuscript marks a first step in answering a question raised by Prof. Antonio Giorgilli in 2014 about our recent result [20] on the construction of lower dimensional elliptic tori in planetary systems. The question sounded pretty much like "can we fix the final frequencies and determine where we have to start from?" Indeed, from the *mathematical* point of view the result in [20] was satisfactory, we obtained a result that is valid in measure. However, Antonio has always pursued explicit algorithms that can be effectively implemented in order to study the behavior of a specific dynamical system, like the dynamics of the solar system or the FPU problem. Thus, the fact that given a specific value of the frequencies one does not know if the corresponding lower dimensional torus exists or not, left us with a bad taste in our mouth*.

We therefore believe that the manuscript has an appropriate place in this Volume in honor of Prof. Antonio Giorgilli. We will do our best to follow the line traced by Antonio and preserve his legacy, always looking for rigorous *constructive* results†.

1. Introduction

The aim of this paper is to reconsider the proof of the Kolmogorov theorem [26] with a variation on the handling of the frequencies.

1.1. About the genesis of this approach

The motivation behind the development of this approach has strong connections with the problem of the persistence of lower dimensional elliptic invariant tori under sufficiently small perturbations. Indeed, in [20] the authors gave an *almost* constructive proof of the existence of lower dimensional elliptic tori for planetary systems, adapting the classical Kolmogorov normalization algorithm (see also [48]) and a result of Pöschel [39], that allows to estimate the measure of a suitable set of non-resonant frequencies. The key point is that both the *internal* frequencies of the torus and the *transversal* ones vary at each normalization step, and cannot be kept fixed as in Kolmogorov algorithm. This makes the accumulation of small divisors much more tricky to control and, more important, the result is only valid *in measure* and therefore one cannot know *a priori* if a specific invariant torus exists or not.

A different approach, based on Lindstedt's series, that allows to control the frequencies has been proposed in [4, 5] in the context of FPU problem. However, the algorithm has been so far introduced and used, up to our knowledge, only in a formal way and the literature lacks of rigorous convergence estimates. Recently, a comparison of the Lindstedt's method and the Kolmogorov normal form has been studied in [33].

The idea is to overcome the issue of having a result that is valid only in measure, playing with the frequency like one does with a control *knob*, hence the title of the paper. The present work focuses on full dimensional invariant tori, thus representing a first step in this direction. We are well aware that, considering full dimensional invariant tori, the original Kolmogorov normalization algorithm allows to have a complete control of the frequencies, which are kept fixed along the whole normalization procedure. However, considering lower dimensional elliptic tori, as explained in detail by Pöschel [39], one cannot keep the frequencies fixed, but have to let them vary. Thus, as a first result, we decide to adapt the classical Kolmogorov normalization algorithm in order to avoid the translation that keeps the frequencies fixed by introducing a *detuning*‡ between the prescribed *final* frequencies and the corresponding initial ones, to be determined *a posteriori*. We remark that a similar approach has been adopted in [50], dealing with an application of the KAM theorem in dissipative dynamical systems.

Finally, let us stress that our approach (see also [7, 40]), in principle, also allows to start from a resonant torus that *by construction* falls into a strongly nonresonant one.

*As Antonio said, the result was satisfactory for an *analyst*, not for a *mathematical physicist*.

†Where *constructive* means that one must be able to *build*: (i) the proof of the theorem, (ii) the code that implement it; (iii) the computer that run the code; and of course (iv) the desk and the chair where we actually write down the proof.

‡The detuning can be figured as the action of turning a control *knob*.

1.2. KAM theory

In order to better illustrate our point of view, we briefly recall here some classical results on KAM theory. Consider the so-called fundamental problem of dynamics as stated by Poincaré, i.e., a canonical system of differential equations with Hamiltonian

$$H(p,q) = H_0(p) + \varepsilon H_1(p,q;\varepsilon) , \tag{1}$$

where $(p,q) \in \mathcal{G} \times \mathbb{T}^n$ are action-angle variables, being $\mathcal{G} \subseteq \mathbb{R}^n$ an open set and ε is a small parameter. The functions $H_0(p)$ and $H_1(p,q;\varepsilon)$ are assumed to be analytic in the variables and in the small parameter, and bounded. Kolmogorov [26], in his seminal paper, that together with the works of Moser [36] and Arnold [1] gave birth to the celebrated KAM theory, proved the existence of quasi periodic solutions for this Hamiltonian, with given strongly nonresonant frequencies.

The original idea of Kolmogorov is to select the actions $p^* \in \mathcal{G}$ such that the frequency vector $\omega = \nabla_p H_0(p^*)$ satisfies a Diophantine condition

$$|k \cdot \omega| > \gamma |k|^{-\tau} , \qquad \text{for all } k \in \mathbb{Z}^n, k \neq 0 , \tag{2}$$

for some positive γ and $\tau \geq n-1$. Hence the term H_0 in (1) can be expanded in a neighborhood of p^*, denoting again by p the translated actions, and (forgetting the unessential constant term) the Hamiltonian reads

$$H(p,q) = \omega \cdot p + O(p^2) + \varepsilon H_1(p,q;\varepsilon) . \tag{3}$$

The Kolmogorov theorem ensures the persistence of the torus $p = 0$ ($p = p^*$ in the original variables) carrying quasi-periodic solutions with frequencies ω, if ε is small enough and $H_0(p)$ is nondegenerate.

Let us stress here a technical point. The role of the nondegeneracy assumption on $H_0(p)$ is twofold: (i) it allows to select the desired frequencies, parameterized by the actions; (ii) it allows to perform the *translation step* that keeps the frequency fixed along the normalization procedure. However, if the Hamiltonian is already in the form (3) or satisfies the so-called *twistless* property, i.e., it consists of a sum of a kinetic term, quadratic in p, and of a potential energy, depending only on the angles, it turns out that the nondegeneracy assumption can be removed, see, e.g., [10–12, 16].

Nowadays, the literature about KAM theory is so vast that an exhaustive list would fill several pages. Indeed, quoting Pöschel [40], *After all, KAM theory is not only a collection of specific theorems, but rather a methodology, a collection of ideas of how to approach certain problems in perturbation theory connected with "small divisors"*. Hence, as this is a paper in honor of Antonio Giorgilli, we have decided to just mention his main contributions in the field[§], i.e., [2, 3, 8, 13, 16–20, 22, 24, 28–30, 34, 35, 49].

A final remark is about the so-called *quadratic* (or superconvergent or Newton-like) method, originally adopted by Kolmogorov and considered crucial until Russmann [41, 42] pointed out that a careful analysis of the accumulation of the small divisors allows to sharpen some estimates and get rid of it. Eventually, a proof of Kolmogorov theorem via classical expansion in a small parameter has been obtained by Giorgilli and Locatelli in [18]. The approach based on classical expansions allows to unveil the mechanism of the accumulation of the small divisors and leads in a natural way to

[§]For an historical account on the role played by Antonio on the development of KAM theory in Milan, see [9] in this same Special Issue.

introduce a more relaxed nonresonant condition for the frequency vector ω, the so-called $\boldsymbol{\tau}$-condition introduced by Antonio in [23] and later adopted in [20, 21], precisely

$$-\sum_{r\geq 1} \frac{\log \alpha_r}{r(r+1)} = \Gamma < \infty\ , \qquad \text{with} \quad \min_{0<|k|\leq rK} |k\cdot\omega| \geq \alpha_r\ , \tag{4}$$

where K and Γ are two positive constants. Such a non-resonance condition is equivalent to the Bruno's one, which is the weakest one that can be assumed to prove the persistence of invariant tori (see [6, 16, 52, 53]). Furthermore, the classical approach is the only way to directly implement KAM theory in practical applications via computer algebra (see, e.g., [25]) and it proved advantageous in different contexts, e.g., the construction of lower dimensional elliptic tori in planetary systems in [48, 49], the study of the long term dynamics of exoplanets in [27, 47, 51], the investigation of the effective stability in the spin-orbit problem in [45, 46], the design of an a priori control for symplectic maps related to betatronic motion in [44] and the continuation of periodic orbits on resonant tori in [37, 38, 43].

In the present paper, we adopt the original quadratic approach by Kolmogorov, which turns out to be better suited in order to devise a normal form algorithm that introduces a detuning of the initial frequencies that will be determined along the normalization procedure and complement it with rigorous convergence estimates.

1.3. Statement of the main result

Consider a $2n$-dimensional phase space with canonical action-angle variables $(p, q) \in \mathcal{G}\times\mathbb{T}^n$, where $\mathcal{G} \subseteq \mathbb{R}^n$ is an open set containing the origin.

The Hamiltonian (1) is assumed to be a bounded real analytic function for sufficiently small values of ε and real bounded holomorphic function of the (p, q) variables in the complex domain $\mathcal{D}_{\rho_0,\sigma_0} = \mathcal{G}_{\rho_0}\times\mathbb{T}^n_{\sigma_0}$ where ρ_0 and σ_0 are positive parameters, $\mathcal{G}_{\rho_0} = \bigcup_{p\in\mathcal{G}} \Delta_{\rho_0}(p)$, with $\Delta_{\rho_0}(p) = \{z \in \mathbb{C}^n : |p_j - z_j| < \rho_0\}$ and $\mathbb{T}^n_{\sigma_0} = \{q \in \mathbb{C}^n : |\mathrm{Im}(q_j)| < \sigma_0\}$ that are the usual complex extensions of the real domains.

Given a point $p_0 \in \mathcal{G}$, denote by $\omega_0(p_0) \in \mathbb{R}^n$ the corresponding frequency vector and expand the Hamiltonian H_0 in a neighborhood of p_0, denoting again by p the translated actions $p - p_0$, precisely

$$H(p, q) = \omega_0 \cdot p + O(p^2) + \varepsilon H_1(p, q; \varepsilon)\ . \tag{5}$$

As remarked in the previous subsection, one can assume a nondegeneracy condition on $H_0(p)$ so as to ensure that the frequency vector is parameterized by the actions. However, if the Hamiltonian is already in this form, no nondegeneracy assumption is required.

We can now state our main theorem

Theorem 1.1. *Consider the Hamiltonian* (5) *and pick a strongly nonresonant frequency vector* $\omega \in \mathbb{R}^n$ *satisfying the Diophantine condition* (2) *with some* $\gamma > 0$ *and* $\tau \geq n - 1$*. Then there exists a positive* ε^* *such that the following statement holds true: for* $|\varepsilon| < \varepsilon^*$ *there exist a frequency vector* ω_0 *and a real analytic near to the identity canonical transformation* $(p, q) = C^{(\infty)}(p^{(\infty)}, q^{(\infty)})$ *leading the Hamiltonian* (5) *in normal form, i.e.,*

$$H^{(\infty)} = \omega \cdot p^{(\infty)} + O({p^{(\infty)}}^2)\ . \tag{6}$$

A more quantitative statement, including a detailed definition of the threshold on the smallness of the perturbation, is given in Proposition 4.1.

A few comments are in order. At difference with respect to the original Kolmogorov theorem, we do not keep the frequencies fixed along the normalization procedure. The idea, that will be fully detailed in the next section, is to replace the classical *translation step* with an *unknown* detuning $\delta\omega$ of the frequencies. Thus, once selected the *final* KAM torus, the theorem ensures the existence of a *starting* one which is invariant in the integrable approximation with $\varepsilon = 0$ and, by construction, falls into the wanted invariant torus. Let us remark that in order to apply the Kolmogorov theorem, e.g., for constructing an invariant torus for a planetary system, it is somehow natural to determine the final angular velocity vector ω by using some numerical techniques like, e.g., Frequency Analysis (see [31, 32]).

2. Analytic setting and expansion of the Hamiltonian

We now define the norms we are going to use. For real vectors $x \in \mathbb{R}^n$, we use

$$|x| = \sum_{j=1}^{n} |x_j| \,.$$

For an analytic function $f(p, q)$ with $q \in \mathbb{T}^n$, we use the weighted Fourier norm

$$\|f\|_{\rho,\sigma} = \sum_{k\in\mathbb{Z}^n} |f_k|_\rho e^{|k|\sigma} \,,$$

with

$$|f_k|_\rho = \sup_p |f_k(p)| \,.$$

We introduce the classes of functions $\mathcal{P}_l$, with integers $l \geq 0$, such that $g \in \mathcal{P}_l$ can be written as

$$g(p, q) = \sum_{|m|=l} \sum_{k} c_{m,k} p^m e^{\mathrm{i}k\cdot q} \,,$$

with $c_{m,k} \in \mathbb{C}$. For consistency reasons, we also set $\mathcal{P}_{-1} = \{0\}$. Finally, we will also omit the dependence of the functions from the variables, unless it has some special meaning.

The Hamiltonian (5), expanded in power series of the actions p, reads

$$H(p, q) = \omega_0 \cdot p + \sum_{l\geq 0} h_l \,, \tag{7}$$

where $h_l \in \mathcal{P}_l$ are bounded as

$$\|h_0\|_{\rho,\sigma} \leq \varepsilon E \,, \quad \|h_1\|_{\rho,\sigma} \leq \frac{\varepsilon E}{2} \qquad \text{and} \qquad \|h_l\|_{\rho,\sigma} \leq \frac{E}{2^l} \quad \text{for } l \geq 2 \,. \tag{8}$$

provided $\rho_0 \leq 1/4$, with $E = 2^{n-1} E_0$ where

$$E_0 = \max\left(\sup_{p\in\Delta_{\rho_0}} |H_0(p)| \,, \sup_{(p,q)\in\mathcal{D}_{\rho_0,\sigma_0}} |H_1(p, q; \varepsilon)| \right) . \tag{9}$$

3. Formal algorithm

We present in this section the algorithm leading the Hamiltonian (7) in normal form. The procedure is described here from a purely formal point of view, while the study of the convergence is postponed to the next section.

First we introduce the unknown detuning $\delta\omega$ and rewrite the Hamiltonian as

$$H(p,q) = \omega \cdot p + \delta\omega \cdot p + \sum_{l\geq 0} h_l(p,q) \,, \tag{10}$$

with $h_l \in \mathcal{P}_l$. Let us stress again that the quantity $\delta\omega$ is unknown and will be determined at the end of the normalization procedure.

As in the original Kolmogorov proof scheme, the algorithm consists in iterating infinitely many times a single normalization step: starting from H, we apply two near to the identity canonical transformations with generating functions $\chi_0(q)$ and $\chi_1(p,q)$, i.e.,

$$H' = \exp(L_{\chi_1}) \circ \exp(L_{\chi_0}) H \,.$$

The generating functions are determined in order to kill the unwanted terms $h_0(q)$ and $h_1(p,q)$. At difference with respect to the original approach designed by Kolmogorov we do not introduce a translation of the actions p, since we do not keep fixed the *initial* frequency ω_0. Indeed, in our algorithm the role of the translation step is played by the detuning of the frequency $\delta\omega$.

The functions $\chi_0(q)$ and $\chi_1(p,q)$ are determined by solving

$$L_{\chi_0}\omega \cdot p + h_0 = 0 \,, \tag{11}$$

$$L_{\chi_1}\omega \cdot p + \sum_{s\geq 0} \frac{1}{s!} L^s_{\chi_0} h_{s+1} = \sum_{s\geq 0} \frac{1}{s!} \langle L^s_{\chi_0} h_{s+1} \rangle_q \,, \tag{12}$$

where $\langle\cdot\rangle_q$ denotes the average with respect to the angles q.

First, considering the Fourier expansion of h_0, and neglecting the constant term, one has

$$h_0(q) = \sum_{k\neq 0} c_k e^{\mathbf{i}k\cdot q} \,,$$

and can easily check that the solution of (11) is given by

$$\chi_0(q) = \sum_{k\neq 0} \frac{c_k}{\mathbf{i}k\cdot\omega} e^{\mathbf{i}k\cdot q} \,.$$

The intermediate Hamiltonian $\hat{H} = \exp(L_{\chi_0})H$ reads

$$\hat{H}(p,q) = \omega \cdot p + \delta\omega' \cdot p + \sum_{l\geq 0} \hat{h}_l(p,q) \,, \tag{13}$$

with

$$
\begin{aligned}
\delta\omega' \cdot p &= \delta\omega \cdot p + \sum_{s=0}^{\infty} \frac{1}{s!} \, \langle L_{\chi_0}^{s} h_{s+1} \rangle_q \, , \\
\hat{h}_0 &= L_{\chi_0}\Big(\delta\omega' \cdot p - \sum_{s=1}^{\infty} \frac{1}{s!} \, \langle L_{\chi_0}^{s} h_{s+1} \rangle_q \Big) \\
&\quad + L_{\chi_0}\Big(h_1 - \langle h_1 \rangle_q \Big) + \sum_{s=2}^{\infty} \frac{1}{s!} \, L_{\chi_0}^{s} h_s \, , \\
\hat{h}_1 &= \sum_{s=0}^{\infty} \frac{1}{s!} \, L_{\chi_0}^{s} h_{s+1} + (\delta\omega - \delta\omega') \cdot p \, , \\
\hat{h}_l &= \sum_{s=0}^{\infty} \frac{1}{s!} \, L_{\chi_0}^{s} h_{s+l} \, , \qquad\qquad \text{for } l \geq 2 \, .
\end{aligned}
\tag{14}
$$

where the unessential constant term $\langle h_0 \rangle_q$ has been neglected in the expression above.

Second, considering the Fourier expansion

$$
\hat{h}_1(p, q) = \sum_{k \neq 0} \hat{c}_k(p) e^{\mathbf{i} k \cdot q} \, ,
$$

one can easily check that the solution of (12) is given by

$$
\chi_1(p, q) = \sum_{k \neq 0} \frac{\hat{c}_k(p)}{\mathbf{i} k \cdot \omega} e^{\mathbf{i} k \cdot q} \, .
$$

We complete the normalization step by computing the Hamiltonian $H' = \exp(L_{\chi_1})\hat{H}$ that takes the form (10) with $\delta\omega'$ as in (14) and

$$
\begin{aligned}
h'_0 &= \sum_{s=0}^{\infty} \frac{1}{s!} \, L_{\chi_1}^{s} \hat{h}_0 \, , \\
h'_1 &= \sum_{s=1}^{\infty} \frac{s}{(s+1)!} \, L_{\chi_1}^{s} \hat{h}_1 + \sum_{s=1}^{\infty} \frac{1}{s!} \, L_{\chi_1}^{s} \delta\omega' \cdot p \, , \\
h'_l &= \sum_{s=0}^{\infty} \frac{1}{s!} L_{\chi_1}^{s} \hat{h}_l \qquad\qquad \text{for } l \geq 2 \, .
\end{aligned}
\tag{15}
$$

The justification of the formulæ (14) and (15) is just a matter of straightforward computations, exploiting (11) and (12).

4. Quantitative estimates

In this section, we translate our formal algorithm into a recursive scheme of estimates on the norms of the functions. This essentially requires to bound the norm of the Lie series. In order to shorten the notation, we will replace $|\cdot|_{\alpha(\rho,\sigma)}$ by $|\cdot|_{\alpha}$ and $\|\cdot\|_{\alpha(\rho,\sigma)}$ by $\|\cdot\|_{\alpha}$, being α any positive real number. The useful estimates are collected in the following statements.

Lemma 4.1. *Let f and g be analytic respectively in $\mathcal{D}_1$ and $\mathcal{D}_{(1-d')}$ for some $0 \leq d' < 1$ with finite norms $\|f\|_1$ and $\|g\|_{1-d'}$. Therefore,*

i. for $0 < d < 1$ and for $1 \leq j \leq n$ we have

$$\left\|\frac{\partial f}{\partial p_j}\right\|_{(1-d)} \leq \frac{1}{d\rho}\|f\|_1 \, , \quad \left\|\frac{\partial f}{\partial q_j}\right\|_{(1-d)} \leq \frac{1}{ed\sigma}\|f\|_1 \, ; \tag{16}$$

ii. for $0 < d < 1 - d'$ we have

$$\|\{f, g\}\|_{(1-d'-d)} \leq \frac{2}{ed(d + d')\rho\sigma}\|f\|_1\|g\|_{(1-d')} \, . \tag{17}$$

Lemma 4.2. *Let d and d' be real numbers such that $d > 0$, $d' \geq 0$ and $d + d' < 1$; let X and g be two analytic functions on $\mathcal{D}_{(1-d')}$ having finite norms $\|X\|_{1-d'}$ and $\|g\|_{1-d'}$, respectively. Then, for $j \geq 1$, we have*

$$\frac{1}{j!}\left\|L_X^j g\right\|_{1-d-d'} \leq \frac{1}{e^2}\left(\frac{2e}{\rho\sigma}\right)^j \frac{1}{d^{2j}}\|X\|_{1-d'}^j\|g\|_{1-d'} \, . \tag{18}$$

The proofs of these lemmas are straightforward and can be found, e.g., in [14].

We are now ready to write the statement of Theorem 8.1 in a more detailed form.

Proposition 4.1. *Consider the Hamiltonian* (10) *and assume the following hypotheses:*

(i) h_l, for $l \geq 0$, satisfy (8)*;*

(ii) $\omega \in \mathbb{R}^n$ satisfy the Diophantine condition (2) *with some $\gamma > 0$ and $\tau \geq n - 1$.*

Then, there exists a positive ε^* depending on n, τ, γ, ρ and σ such that for $|\varepsilon| < \varepsilon^*$ and $\delta \leq 1/8$ there exists a real analytic near to the identity canonical transformation $(p, q) = C(p', q')$ satisfying

$$|p_j - p'_j| \leq \delta^{\tau+3}\rho \, , \quad |q_j - q'_j| \leq \delta^{\tau+3}\sigma \, , \qquad j = 1 \, , \, \ldots \, , \, n \, , \tag{19}$$

for all $(p', q') \in \mathcal{D}_{1-4\delta}$ which gives the Hamiltonian the Kolmogorov normal form (6)*. Moreover, the detuning is bounded as*

$$\|\delta\omega \cdot p\|_{\frac{1}{2}} \leq \frac{E}{2}\delta^{2\tau+4} \, .$$

The proof of this Proposition is given in the next two subsections. Indeed, it is divided in two parts: first the quantitative analytic estimates for a single step are obtained in the so-called Iterative Lemma, and finally the convergence of the infinite sequence of iterations is proved.

4.1. The iterative lemma

The aim of this subsection is to translate the algorithm of Section 3 into a scheme of estimates for the norms of all functions involved.

Lemma 4.3. *Let H be as in* (10) *and assume that the hypotheses (i)–(ii) of Proposition 4.1 hold true. Let $\delta \leq 1/8$ and ρ^*, σ^* be positive constants satisfying*

$$(1 - 4\delta)\rho \geq \rho^* \quad \textit{and} \quad (1 - 4\delta)\sigma \geq \sigma^* \, .$$

Then there exists a positive constant $\Lambda = \Lambda(n, \tau, \gamma, \rho^*, \sigma^*)$ such that the following holds true: if

$$\frac{\Lambda}{\delta^{3\tau+6}}\varepsilon \le 1 \ , \tag{20}$$

assuming that the following "a priori" bound on the detunings holds true,

$$\|\delta\omega' \cdot p\|_{1-\delta} \le \frac{\varepsilon}{2\delta^{\tau+2}} \ ,$$

then there exists a canonical transformation $(p, q) = C(p', q')$ satisfying

$$\begin{aligned} |p_j - p_j'| &\le \frac{\Lambda\varepsilon}{\delta^{3\tau+6}}\delta^{\tau+3}\rho \le \delta^{\tau+3}\rho \ , \\ |q_j - q_j'| &\le \frac{\Lambda\varepsilon}{\delta^{3\tau+6}}\delta^{\tau+3}\sigma \le \delta^{\tau+3}\sigma \ , \qquad j = 1 \ , \ \ldots \ , \ n \ , \end{aligned} \tag{21}$$

for all $(p', q') \in \mathcal{D}_{1-4\delta}$, which brings the Hamiltonian in the Kolmogorov normal form (6) *with the same ω and with new functions $\delta\omega' \cdot p$ and h_l', for $l \ge 1$, satisfying the hypotheses (i)–(ii) with new positive constants ε', ρ', σ' given by*

$$\varepsilon' = \frac{\Lambda}{\delta^{3\tau+6}}\varepsilon^2 \ , \quad \rho' = (1 - 4\delta)\rho \quad \textit{and} \quad \sigma' = (1 - 4\delta)\sigma \ .$$

Furthermore, the variation of the detuning frequency vector is bounded as follows

$$\|\delta\omega' \cdot p - \delta\omega \cdot p\|_{1-2\delta} \le \frac{\Lambda\varepsilon}{\delta^{3\tau+6}}\delta^{2\tau+4}\frac{E}{2} \le \delta^{2\tau+4}\frac{E}{2} \ .$$

A crucial role in the proof of the Iterative Lemma is played by the control of the accumulation of the small divisors. This topic has been deeply investigated by Antonio Giorgilli, see, e.g., [15, Section 8.2.4].

We now collect all the estimates that allow to prove Lemma 4.3. Recalling the Diophantine condition (2), the elementary inequality $|k|^\tau e^{-|k|\delta\sigma} \le \left(\frac{\tau}{e\delta\sigma}\right)^\tau$ allows us to easily bound the generating function χ_0 as

$$\|\chi_0\|_{1-\delta} \le \frac{1}{\gamma}\left(\frac{\tau}{e\delta\sigma}\right)^\tau \varepsilon E \le \frac{K_1}{\delta^\tau}\varepsilon \ , \qquad K_1 = \frac{1}{\gamma}\left(\frac{\tau}{e\sigma}\right)^\tau E \ .$$

It is now convenient to provide some useful estimates to bound the terms appearing in $\hat{H}$. Assuming the smallness condition on ε

$$\frac{2eK_1\varepsilon}{\delta^{\tau+2}\rho\sigma} \le \frac{1}{2} \ ,$$

we easily get

$$
\begin{aligned}
\|L_{\chi_0} h_1\|_{1-2\delta} &\le \frac{1}{e^2}\left(\frac{2e}{\delta^2\rho\sigma}\frac{K_1}{\delta^\tau}\varepsilon\right)\frac{\varepsilon E}{2} \le \frac{K_2}{\delta^{\tau+2}}\frac{1}{2}\varepsilon^2\,, && K_2 = \frac{2K_1E}{e\rho\sigma}\,,\\
\|L_{\chi_0} h_{l+1}\|_{1-2\delta} &\le \frac{1}{e^2}\left(\frac{2e}{\delta^2\rho\sigma}\frac{K_1}{\delta^\tau}\varepsilon\right)\frac{E}{2^{l+1}} \le \frac{K_2}{\delta^{\tau+2}}\frac{1}{2^{l+1}}\varepsilon\,,\\
\sum_{s\ge2}\frac{1}{s!}\|L^s_{\chi_0} h_{l+s}\|_{1-2\delta} &\le \sum_{s\ge2}\frac{1}{e^2}\left(\frac{2e}{\delta^2\rho\sigma}\frac{K_1}{\delta^\tau}\varepsilon\right)^s\frac{E}{2^{l+s}}\\
&\le \frac{1}{e^2}\left(\frac{2e}{\delta^2\rho\sigma}\frac{K_1}{\delta^\tau}\varepsilon\right)^2\frac{E}{2^{l+2}}\sum_{s\ge0}\left(\frac{eK_1\varepsilon}{\delta^{\tau+2}\rho\sigma}\right)^s\\
&\le \frac{K_3}{\delta^{2\tau+4}}\frac{1}{2^{l+2}}\varepsilon^2\,, && K_3 = \frac{2^3K_1^2E}{\rho^2\sigma^2}\,,\\
\sum_{s\ge2}\frac{s}{s!}\|L^s_{\chi_0} h_{s}\|_{1-2\delta} &\le \sum_{s\ge2}\frac{s}{e^2}\left(\frac{2e}{\delta^2\rho\sigma}\frac{K_1}{\delta^\tau}\varepsilon\right)^s\frac{E}{2^{s}}\\
&\le \frac{1}{e^2}\left(\frac{2e}{\delta^2\rho\sigma}\frac{K_1}{\delta^\tau}\varepsilon\right)^2\frac{E}{2^{2}}\sum_{s\ge0}(s+2)\left(\frac{eK_1\varepsilon}{\delta^{\tau+2}\rho\sigma}\right)^s\\
&\le \frac{K_4}{\delta^{2\tau+4}}\frac{1}{2^{2}}\varepsilon^2\,, && K_4 = \frac{24K_1^2E}{\rho^2\sigma^2}\,,
\end{aligned}
\tag{22}
$$

where, in the last two inequalities, we used the well known sums

$$
\sum_{s\ge0}x^s = \frac{1}{1-x} \le 2 \quad \text{and} \quad \sum_{s\ge1}sx^s = \frac{x}{(1-x)^2} \le 2\,, \quad \text{for } |x| \le \frac{1}{2}\,.
$$

We now estimate the difference between the detunings, precisely,

$$
\begin{aligned}
\|\delta\omega'\cdot p - \delta\omega\cdot p\|_{1-2\delta} &\le \sum_{s\ge0}\frac{1}{s!}\|L^s_{\chi_0} h_{l+1}\|_{1-2\delta}\\
&\le \frac{\varepsilon E}{2} + \frac{K_2}{\delta^{\tau+2}}\frac{1}{2^2}\varepsilon + \frac{2^3K_1^2E}{\delta^{2\tau+4}\rho^2\sigma^2}\frac{1}{2^3}\varepsilon^2\\
&\le \frac{\varepsilon E}{2} + \frac{K_2}{\delta^{\tau+2}}\frac{1}{2^2}\varepsilon + \left(\frac{2eK_1\varepsilon}{\delta^{\tau+2}\rho\sigma}\right)\frac{K_2}{\delta^{\tau+2}}\frac{1}{2^2}\varepsilon\\
&\le \frac{K_5}{\delta^{\tau+2}}\frac{\varepsilon}{2}\,, && K_5 = E + K_2\,.
\end{aligned}
\tag{23}
$$

We now bound the term appearing in (13). The norm of the function $\hat{h}_0$ is bounded as

$$
\begin{aligned}
\|\hat{h}_0\|_{1-2\delta} \leq & \left\| L_{\chi_0}\Big(\delta\omega'\cdot p - \sum_{s=1}^{\infty}\frac{1}{s!}\,\langle L^s_{\chi_0} h_{s+1}\rangle_q\Big) \right.\\
& \qquad \left. + L_{\chi_0}\Big(h_1 - \langle h_1\rangle_q\Big) + \sum_{s=2}^{\infty}\frac{1}{s!}\,L^s_{\chi_0} h_s \right\|_{1-2\delta}\\
& \leq \left\| L_{\chi_0}\delta\omega'\cdot p\right\|_{1-2\delta} + \left\|\sum_{s\geq 2}\frac{s}{s!}L^s_{\chi_0}h_s\right\|_{1-2\delta}\\
& \qquad + \|L_{\chi_0}h_1\|_{1-2\delta} + \left\|\sum_{s\geq 2}\frac{1}{s!}L^s_{\chi_0}h_s\right\|_{1-2\delta}\\
& \leq \frac{1}{e^2}\left(\frac{2e}{\delta^2\rho\sigma}\frac{K_1}{\delta^\tau}\varepsilon\right)\frac{\varepsilon}{2\delta^{\tau+2}} + \frac{K_4}{\delta^{2\tau+4}}\frac{1}{2^2}\varepsilon^2\\
& \qquad + \frac{K_2}{\delta^{\tau+2}}\frac{1}{2}\varepsilon^2 + \frac{K_3}{\delta^{2\tau+4}}\frac{1}{2^2}\varepsilon^2\\
& \leq \frac{K_6}{\delta^{2\tau+4}}\varepsilon^2\,, \qquad\qquad K_6 = \frac{K_2}{2E} + \frac{K_4}{2^2} + \frac{K_2}{2} + \frac{K_3}{2^2}\,,
\end{aligned}
$$

while the norm of $\hat{h}_1$ satisfies

$$
\begin{aligned}
\|\hat{h}_1\|_{1-2\delta} & \leq \sum_{s\geq 0}\frac{1}{s!}\|L^s_{\chi_0}h_{s+1}\|_{1-2\delta}\\
& \leq \frac{\varepsilon E}{2} + \frac{K_2}{\delta^{\tau+2}}\frac{1}{2^2}\varepsilon + \frac{K_3}{\delta^{2\tau+4}}\frac{1}{2^3}\varepsilon^2\\
& \leq \frac{\varepsilon E}{2} + \frac{K_2}{\delta^{\tau+2}}\frac{1}{2^2}\varepsilon + \left(\frac{2eK_1\varepsilon}{\delta^{\tau+2}\rho\sigma}\right)\frac{K_2}{\delta^{\tau+2}}\frac{1}{2^2}\varepsilon\\
& \leq \frac{K_5}{\delta^{\tau+2}}\frac{\varepsilon}{2}\,.
\end{aligned}
$$

Finally, $\hat{h}_l$, for $l \geq 2$, one has

$$
\begin{aligned}
\|\hat{h}_l\|_{1-2\delta} & \leq \sum_{s\geq 0}\frac{1}{s!}\|L^s_{\chi_0}h_{s+l}\|_{1-2\delta}\\
& \leq \frac{E}{2^l} + \frac{K_2}{\delta^{\tau+2}}\frac{1}{2^{l+1}}\varepsilon + \frac{K_3}{\delta^{2\tau+4}}\frac{1}{2^{l+2}}\varepsilon^2\\
& \leq \frac{E}{2^l} + \left(\frac{2eK_1\varepsilon}{\delta^{\tau+2}\rho\sigma}\right)\frac{E}{2e^2}\frac{1}{2^l} + \left(\frac{2eK_1\varepsilon}{\delta^{\tau+2}\rho\sigma}\right)^2\frac{E}{2e^2}\frac{1}{2^l}\\
& \leq \frac{K_7}{2^l}\,, \qquad\qquad K_7 = E + \frac{E}{e^2}\,.
\end{aligned}
$$

This concludes the estimates for the first half of the normalization step.

Exploiting again the Diophantine condition (2) we easily bound the generating function χ_1 as

$$
\|\chi_1\|_{1-3\delta} \leq \frac{1}{\gamma}\left(\frac{\tau}{e\delta\sigma}\right)^\tau\frac{K_5}{\delta^{\tau+2}}\frac{1}{2}\varepsilon \leq \frac{K_8}{\delta^{2\tau+2}}\frac{1}{2}\varepsilon\,, \qquad K_8 = \frac{1}{\gamma}\left(\frac{\tau}{e\sigma}\right)^\tau K_5\,.
$$

Assuming the smallness condition

$$\frac{2e\|\chi_1\|_{1-3\delta}}{\delta^2\rho\sigma} \leq \frac{1}{2} \, ,$$

that can be written as

$$\frac{2eK_8\varepsilon}{\delta^{2\tau+4}\rho\sigma} \leq 1 \, ,$$

we now bound the terms appearing in (15). The norm of the function h'_0 is bounded as

$$\begin{aligned}
\|h'_0\|_{1-4\delta} &\leq \sum_{s\geq 0} \frac{1}{e^2}\left(\frac{2e}{\delta^2\rho\sigma}\frac{K_8}{\delta^{2\tau+2}}\frac{1}{2}\varepsilon\right)^s \frac{K_6}{\delta^{2\tau+4}}\varepsilon^2 \\
&\leq \frac{K_6}{e^2\delta^{2\tau+4}}\varepsilon^2 \sum_{s\geq 0}\left(\frac{eK_8\varepsilon}{\delta^{2\tau+4}\rho\sigma}\right)^s \\
&\leq \frac{K_9}{\delta^{2\tau+4}}\varepsilon^2 \, , \qquad\qquad K_9 = \frac{2K_6}{e^2} \, .
\end{aligned}$$

Similarly we get

$$\begin{aligned}
\|h'_1\|_{1-4\delta} &\leq \sum_{s\geq 1}\frac{1}{s!}\left\|L^s_{\chi_1}\hat{h}_1\right\|_{1-4\delta} + \sum_{s\geq 1}\frac{1}{s!}\left\|L^s_{\chi_1}\delta\omega'\cdot p\right\|_{1-4\delta} \\
&\leq \sum_{s\geq 1}\frac{1}{e^2}\left(\frac{2e}{\delta^2\rho\sigma}\frac{K_8}{\delta^{2\tau+2}}\frac{1}{2}\varepsilon\right)^s\left(\frac{K_5}{\delta^{\tau+2}}\frac{1}{2}\varepsilon + \frac{1}{\delta^{\tau+2}}\frac{1}{2}\varepsilon\right) \\
&\leq \frac{1}{e^2}\left(\frac{2e}{\delta^2\rho\sigma}\frac{K_8}{\delta^{2\tau+2}}\frac{1}{2}\varepsilon\right)\left(\frac{K_5}{\delta^{\tau+2}}\frac{1}{2}\varepsilon + \frac{1}{\delta^{\tau+2}}\frac{1}{2}\varepsilon\right) \\
&\qquad \sum_{s\geq 0}\left(\frac{2e}{\delta^2\rho\sigma}\frac{K_8}{\delta^{2\tau+2}}\frac{1}{2}\varepsilon\right)^s \\
&\leq \frac{K_8}{e\delta^{2\tau+4}\rho\sigma}\left(\frac{K_5+1}{\delta^{\tau+2}}\right)2\frac{\varepsilon^2}{2} \\
&\leq \frac{K_{10}}{\delta^{3\tau+6}}\frac{\varepsilon^2}{2} \, , \qquad\qquad K_{10} = \frac{2(K_5+1)K_8}{e\rho\sigma} \, .
\end{aligned}$$

For $l \geq 2$, we bound the norm of h'_l as

$$\begin{aligned}
\|h'_l\|_{1-4\delta} &\leq \sum_{s\geq 0}\frac{1}{s!}\left\|L^s_{\chi_1}\hat{h}_l\right\|_{1-4\delta} \\
&\leq \sum_{s\geq 0}\frac{1}{e^2}\left(\frac{2e}{\delta^2\rho\sigma}\frac{K_8}{\delta^{2\tau+2}}\frac{1}{2}\varepsilon\right)^s\frac{K_7}{2^l} \\
&\leq \frac{K_{11}}{2^l} \, , \qquad\qquad K_{11} = \frac{2K_7}{e^2} \, .
\end{aligned}$$

To finish, we need to provide the convergence of the near to the identity change of coordinates. The first change of coordinates is bounded as follows

$$\begin{aligned}
\exp(L_{\chi_0})\hat{p}_j &= \hat{p}_j + \left.\frac{\partial\chi_0}{\partial q_j}\right|_{(\hat{q},\hat{p})} \, , \\
\exp(L_{\chi_0})\hat{q}_j &= \hat{q}_j \, ,
\end{aligned}$$

The second change of coordinates is bounded as

$$\begin{aligned}\|\exp(L_{\chi_1})p' - p'\|_{1-4\delta} &\le \sum_{s\ge1} \frac{1}{s!}\|L^s_{\chi_1} p'\|_{1-4\delta} \\ &\le \sum_{s\ge1} \frac{1}{s!}\frac{(s-1)!}{e^2}\left(\frac{2e}{\delta^2\rho\sigma}\|\chi_1\|_{1-4\delta}\right)^{s-1} \|L_{\chi_1} p'\|_{1-4\delta} \\ &\le \frac{\|\chi_1\|}{e^3\delta\sigma}\sum_{s\ge1}\frac{1}{s}\left(\frac{2e}{\delta^2\rho\sigma}\|\chi_1\|_{1-4\delta}\right)^{s-1} \\ &\le \frac{\delta\rho}{2}\left(\frac{1}{e^4}\frac{2eK_8\varepsilon}{\delta^{2\tau+4}\rho\sigma}\right) ,\end{aligned}$$

and similar computations give

$$\|\exp(L_{\chi_1})q' - q'\|_{1-4\delta} \le \frac{\delta\sigma}{2}\left(\frac{1}{e^3}\frac{2eK_8\varepsilon}{\delta^{2\tau+4}\rho\sigma}\right) . \tag{24}$$

Combining these bounds we eventually get

$$\begin{aligned}\|p' - p\|_{1-4\delta} &\le \frac{\delta\rho}{2}\frac{1}{e^4}\frac{2eK_8\varepsilon}{\delta^{2\tau+4}\rho\sigma} + \frac{1}{e\delta\sigma}\frac{K_1}{\delta^\tau}\varepsilon \\ &\le \frac{\delta\rho}{2}\left(\frac{1}{e^4}\frac{2eK_8}{\delta^{2\tau+4}\rho\sigma} + \frac{2K_1}{e\delta^{\tau+2}\rho\sigma}\right)\varepsilon .\end{aligned} \tag{25}$$

In order to conclude the proof, we now collect all the estimates. We define Λ as

$$\Lambda = \max\left(1,\ K_j \text{ for } j = 1,\dots,11\ ,\ \frac{2eK_1}{\rho\sigma}\ ,\ \frac{2eK_8}{\rho\sigma}\right) .$$

Let us stress that Λ depends only on τ, γ, ρ^*, σ^* and n (implicitly via τ). Thus all the convergence conditions are summarized by

$$\frac{\Lambda}{\delta^{3\tau+6}}\varepsilon \le 1 ,$$

and trivial computations conclude the proof of Lemma 4.3.

4.2. Conclusion of the proof

By repeated application of the iterative lemma, we construct an infinite sequence $\{\hat{C}^{(k)}\}_{k\ge1}$ of near the identity canonical transformations

$$(p^{(k-1)}, q^{(k-1)}) = \hat{C}^{(k)}(p^{(k)}, q^{(k)}) ,$$

where the upper index labels the coordinates at the k-th step. This introduces a sequence $\{H^{(k)}\}_{k\ge1}$ of Hamiltonians, where $H^{(0)} = H$ is the original one, satisfying

$$\varepsilon_k = \frac{\Lambda}{\delta_k^{3\tau+6}}\varepsilon_{k-1}^2 , \tag{26}$$

$$\rho_k = (1 - 4\delta_k)\rho_{k-1} \ , \tag{27}$$

$$\sigma_k = (1 - 4\delta_k)\sigma_{k-1} \ . \tag{28}$$

These sequences depend on the arbitrary sequence $\{\delta_k\}_{k\geq 1}$, that must be chosen so that for every k one has $\delta_k \leq 1/8$ and

$$\frac{\Lambda}{\delta_k^{3\tau+6}}\varepsilon_{k-1} \leq 1 \ , \tag{29}$$

$$(1 - 4\delta_k)\rho_{k-1} \geq \rho^* > 0 \ , \tag{30}$$

$$(1 - 4\delta_k)\sigma_{k-1} \geq \sigma^* > 0 \ . \tag{31}$$

Let us now make a choice of the parameters[¶], precisely

$$\varepsilon_k = \varepsilon_0^{k+1} \qquad \text{and} \qquad \delta_k = \frac{1}{\alpha^k} \ ,$$

where α is real positive constant to be determined.

Let's start with (29), that reads

$$\Lambda(\alpha^{3\tau+6})^k \varepsilon_0^k \leq 1 \ , \tag{32}$$

and holds true provided

$$\varepsilon_0 \leq \frac{1}{\Lambda\alpha^{3\tau+6}} \ . \tag{33}$$

Consider now the restrictions δ_k. We immediately get

$$\sum_{k\geq 1}\delta_k = \sum_{k\geq 1}\alpha^{-k} \leq \frac{1}{8} \ , \quad \text{for } \alpha \geq 9 \ . \tag{34}$$

We now prove that (31) and (30) hold true. Starting with

$$\ln\prod_{k\geq 1}(1 - 4\delta_k) = \sum_{k\geq 1}\ln(1 - 4\delta_k) \ ,$$

we easily get

$$0 \geq \sum \ln(1 - 4\delta_k) \geq -8\ln 2\sum_{k\geq 1}\delta_k > -\ln 2 \ ,$$

from which we have $\rho^* = \rho/2$ and $\sigma^* = \sigma/2$.

Let us now focus on the sequence of the detuning frequency vectors $\{\delta\omega^{(k)}\}_{k\geq 0}$, which requires some additional care. Indeed, Lemma 4.3 holds true provided $\|\delta\omega^{(k)} \cdot p\|_{1-\delta_k} \leq \frac{\varepsilon_{k-1}}{2\delta_k^{\tau+2}}$ and the sequence ε_k, by definition, satisfy $\lim_{k\to\infty}\delta\omega^{(k)} = 0$. The recursive definition in (14) allows us to compute $\delta\omega^{(k)} \cdot p$ as

$$\delta\omega^{(k)} \cdot p = \sum_{j\geq k+1}\left(\delta\omega^{(j-1)} \cdot p - \delta\omega^{(j)} \cdot p\right) \ , \quad \text{for } k \geq 0 \ , \tag{35}$$

and by using the inequality (23), we get

$$\|\delta\omega^{(k)} \cdot p\|_{1-\delta_k} \leq \sum_{j=k+1}^{\infty}\frac{K_5}{\delta_j^{\tau+2}}\varepsilon_{j-1} \ .$$

[¶]The choice is rather arbitrary, see [15], footnote 6, chapter 8.

Thus, the applicability of the Iterative Lemma 4.3 is then verified *a posteriori*, if the inequality

$$\sum_{j\geq k+1} \frac{K_5}{\delta_j^{\tau+2}} \varepsilon_{j-1} \leq \frac{\varepsilon_{k-1}}{\delta_k^{\tau+2}} \tag{36}$$

holds true for every positive integer k. We can rewrite this condition as

$$K_5 \sum_{j\geq k+1} (\alpha^{\tau+2}\varepsilon_0)^j = K_5 \frac{(\alpha^{\tau+2}\varepsilon_0)^{k+1}}{1-\alpha^{\tau+2}\varepsilon_0} \leq (\alpha^{\tau+2}\varepsilon_0)^k$$

from which we get

$$\varepsilon_0 \leq \frac{1}{\alpha^{\tau+2}(K_5+1)} \ . \tag{37}$$

Hence, once the choice of α is made so as to satisfy (34), $\alpha \geq 9$, one has two additional smallness conditions on ε_0, (33) and (37), that affects the threshold on the small parameter ε^*.

It remains to prove that the canonical transformation is well defined on some domain. To this end, consider the sequence of domains $\{\Delta_{\rho_k,\sigma_k}\}_{k\geq 0}$ with ρ_k and σ_k as in (27) and (28).

Then the canonical transformation $\hat{C}^{(k)} : \Delta_{\rho_k,\sigma_k} \to \Delta_{\rho_{k-1},\sigma_{k-1}}$ is analytic. Therefore, by composition, the transformation

$$C^{(k)} = \hat{C}^{(k)} \circ \cdots \circ \hat{C}^{(1)}$$

is canonical and analytic. Moreover, in view of (25) and (24) we have

$$|p^{(k)} - p^{(k-1)}| \leq \sigma \sum_{j=1}^{k} \delta_j^{\tau+3} \quad \text{and} \quad |q^{(k)} - q^{(k-1)}| \leq \rho \sum_{j=1}^{k} \delta_j^{\tau+3} \ ,$$

thus, since $\sum_{j\geq 1} \delta_j$ is convergent, the sequence $\{C^{(k)}\}_{k\geq 1}$ converges absolutely to

$$C^{(\infty)} : \Delta_{\rho_*,\sigma_*} \to \Delta_{\rho_0,\sigma_0} \ ,$$

with $\rho_* = \rho_0/2$ and $\sigma_* = \sigma_0/2$. The absolute convergence implies the uniform convergence in any compact subset of Δ_{ρ_*,σ_*}, hence $C^{(\infty)}$ is analytic. Finally, denoting by $(p^{(\infty)}, q^{(\infty)})$ the canonical coordinates in Δ_{ρ_*,σ_*}, and we immediately get

$$|p_j^{(\infty)} - p_j^{(0)}| \leq \frac{\sigma}{8^{\tau+3}} \quad \text{and} \quad |q_j^{(\infty)} - q_j^{(0)}| \leq \frac{\rho}{8^{\tau+3}} \ .$$

Lastly, we now focus on the sequence of detunings. We can bound the norm of $\delta\omega^{(0)}$ exploiting the recursive definition

$$\delta\omega^{(0)} \cdot p = \sum_{j\geq 1} \left(\delta\omega^{(j-1)} \cdot p - \delta\omega^{(j)} \cdot p\right) \ .$$

Indeed, one easily gets

$$\|\delta\omega^{(0)} \cdot p\|_{\frac{1}{2}} \leq \sum_{j\geq 1} \frac{\Lambda}{\delta_j^{\tau+2}} \frac{\varepsilon_{j-1}E}{2} \leq \frac{E}{2} \sum_{j\geq 1} \frac{\Lambda\varepsilon_{j-1}}{\delta_j^{\tau+2}} \leq \frac{E}{2} \sum_{j\geq 1} \delta_j^{2\tau+4} \leq \frac{E}{2} \frac{1}{8^{2\tau+4}} \ .$$

By the properties of the Lie series transformation, one also has that the sequence $\{H^{(k)}\}_{k\geq 0}$ converges to an analytic function $H^{(\infty)}$ which by construction is in normal form. This concludes the proof of Proposition 4.1.

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Conflict of interest

The authors declare no conflict of interest.

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