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Research article

A mixed operator approach to peridynamics†

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Abstract: In the present paper we propose a model describing the nonlocal behavior of an elastic body using a peridynamical approach. Indeed, peridynamics is a suitable framework for problems where discontinuities appear naturally, such as fractures, dislocations, or, in general, multiscale materials. In particular, the regional fractional Laplacian is used as the nonlocal operator. Moreover, a combination of the fractional and classical Laplacian operators is used to obtain a better description of the phenomenological response in elasticity. We consider models with linear and nonlinear perturbations. In the linear case, we prove the existence and uniqueness of the solution, while in the nonlinear case the existence of at least two nontrivial solutions of opposite sign is proved. The linear and nonlinear problems are also solved by a numerical approach which estimates the regional fractional Laplacian by means of its singular integral representation. In both cases, a numerical estimation of the solutions is obtained, using in the nonlinear case an approach involving a random variation of an initial guess of the solution. Moreover, in the linear case a parametric analysis is made in order to study the effects of the parameters involved in the model, such as the order of the fractional Laplacian and the mixture law between local and nonlocal behavior.

Keywords: peridynamics; fractional Laplacian; mixed local and nonlocal operators; variational methods; numerical methods

Dedicated to the 50th birthday of Professor Giuseppe Mingione, with high feelings of admiration for his notable contributions in Mathematics and great affection.

1. Introduction

The last decades have seen an evolution in scientific understanding and human capabilities concerning new isolated nanoparticles set up into hybrid composite materials, not ordinarily found in nature, whose properties are specifically planned and controlled not only by the constituent phases, but also by their morphology, spatial anisotropy and relative proximity with respect to one other and the host matrix. The worldwide scientific focus on hybrid systems, based on synthetic or natural polymers combined with metal, ceramic or carbon nanosctructures, represents a truly revolutionary change in the society way of thinking and allows us to create high–performance materials, underpinning the development of previously unrealizable applications. Further details can be found in [1] and its references. Indeed, in [1] nonlocal problems have been investigated, with the aim of describing the elastic behavior of complex structures composed by two or more different phases having extremely efficient mechanical features, to employ in a wide range of fields as civil engineering and architecture. The insurgence of nonlocality in composites made of several layer of alternating stiff and soft phases as been shown in [17].

The analysis of a nonlocal elastic medium has been recently tackled with the tools of fractional calculus, which essentially are based on a fractional gradient elasticity model. By means of fractional operators, it is possible to associate a mechanical model to the equation involving fractional terms, consisting of points connected not only to adjacent ones but also to all the others by springs: the springs themselves have stiffness which decrease with the distance according to a power–law.

The mixed problem discussed in this paper involves in its nonlocal part the so called peridynamics, which is a nonlocal continuum model in solid mechanics introduced by Silling [18]; we refer to [4–6] for a detailed discussion. The main difference with the usual Cauchy–Green elasticity relies on the nonlocality, which is reflected in the fact that points separated by a positive distance exert a force upon each other. Mathematically, deformations are not assumed to be weakly differentiable, in contrast with classical continuum mechanics, and in particular hyperelasticity, where they are required to be Sobolev. This makes peridynamics a suitable framework for problems where discontinuities appear naturally, such as fractures, dislocations, or, in general, multiscale materials.

In particular, the balance equation for a one-dimensional solid (a rod) in classical (local) mechanics is given by

$$\sigma'(x) + f(x) = 0$$

where σ is the stress, f are distributed forces possibly depending on x and $'$ denotes first derivative. The constitutive elastic law for the rod affirms that

$$\sigma(x) = E\varepsilon(x)$$

where E is the modulus of elasticity and ε is the deformation, related to the displacements u by the compatibility law

$$\varepsilon(x) = u'(x)$$

Therefore, in classical local mechanics we have

$$E\Delta u(x) + f(x) = 0$$

where Δ is the Laplacian. In [9] Eringen considered the stress depending on the strain not only locally but also on the strain in all the point along the rod length $\mathscr{L}$, introducing the following constitutive law

$$\sigma(x) = \int_{\mathscr{L}} E\alpha(|x-y|)\varepsilon(y)dy$$

where α is a weight (kernel) function that assess the influence of the strain in distance points to the stress. Anyway, this approach still requires the differentiation in the equation of motion, as highlighted by Silling [18], which could be a problem in the case of presence of discontinuities. Therefore, following the approach of Silling with which he introduced the peridynamics, where the equilibrium equation is written including a nonlocal operator L_u as in the following

$$L_u(x,t) + f(x) = 0 \text{ with } L_u(x,t) = \int_{\mathscr{L}} f(u(y,t) - u(x,t), y-x)dy,$$

we assume that the peridynamic equation of motion in statics can be written as

$$-k(-\Delta)^s_{\mathscr{L}} u(x) + f(x) = 0$$

where $(-\Delta)^s_{\mathscr{L}}$ is the regional fractional Laplacian which is a nonlocal operator (see later) and k is a suitable parameter. Anyway, since it has been found (see [20]) that in the case of composites with nonlocal behaviour it is better to consider two phases, a local one and a nonlocal one, the problem can be written as

$$-c\Delta u + k(-\Delta)^s_{\mathscr{L}} u + V(x)u = f$$

where c is a suitable parameter (equivalent to E in the purely local case) and V account for the possible presence of external linear spring forces. It is worth noting that the previous problem is similar to that which appears in preceding papers of some of the authors [2, 3, 10] but with a different operator, the fractional Laplacian defined in an unbounded domain, and with a different approach based on the original work of Eringen using fractional derivatives. The problem of the rod in a peridynamic context has been considered in literature (see for example [13, 19]) but with a different approach. For the regularity properties of the solutions of mixed local and nonlocal problems we refer to the recent paper [8] by De Filippis and Mingione, as well as its wide bibliography.

More specifically, the first problem we consider in details is linear and given by

$$\begin{cases} -c\Delta u + k(-\Delta)^s_{\Omega} u + V(x)u = f(x) & \text{in } \Omega_0, \\ u = 0 & \text{in } \Omega_1, \end{cases} \tag{P_L}$$

where c, $k > 0$ are physical coefficients and usually, but not in this paper, are supposed to satisfy the convex restriction $c + k = 1$, the additional term $V(x)u$ represents external springs whose stiffness is related to the position of the point along Ω_0, and the nonnegative potential V is in $L^\infty(\Omega_0)$, while f is a perturbation of class $L^2(\Omega_0)$. The operator

$$(-\Delta)^s_{\Omega}\varphi(x) = \int_{\Omega} \frac{\varphi(x) - \varphi(y)}{|x-y|^{N+2s}}\, dy \quad \text{for all } \varphi \in C^\infty_c(\Omega),$$

is the so called *regional fractional Laplacian*. Throughout the paper we denote by $B(x_0, r)$ the open ball in $\mathbb{R}^N$ of center x_0 and radius $r > 0$. When $x_0 = 0$ we simply denote $B(0, r)$ by B_r.

Here and in what follows $\Omega \subset \mathbb{R}^N$ is a bounded domain, with smooth boundary $\partial\Omega$. Moreover, Ω is divided into two parts, that is $\Omega = \Omega_0 \cup \Omega_1$, where Ω_0 is an open set, with smooth boundary $\partial\Omega_0$ and $\Omega_0 \cap \Omega_1 = \emptyset$. Furthermore, the open enlargement $\Omega_\delta = \Omega_0 + B_\delta$, for a suitable small radius $\delta > 0$, is assumed to be a subset of Ω. In this way, the remaining set $\Omega_1 \supset \partial\Omega_0$ and Ω_1 can be seen as the nonlocal boundary of Ω, see Figure 1.

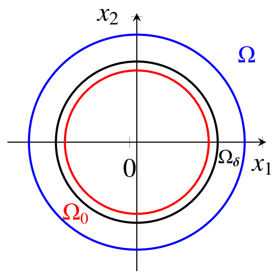

Figure 1. Description of Ω, Ω_0 and Ω_δ.

In the present paper we first interpret the problem (P_L) which models the behavior of an elastic body, with a linear peridynamical approach, and enrich it with additional terms in order to widen its mechanical meaning and give the conditions under which one unique solution exists. Then we prove multiplicity results for the nonlinear version of (P_L), namely for problem

$$\begin{cases} -c\Delta u + k(-\Delta)^s_\Omega u + V(x)u = f(x,u) & \text{in } \Omega_0, \\ u = 0 & \text{in } \Omega_1. \end{cases} \tag{P_N}$$

Finally, we treat (P_L) and (P_N) also numerically, estimating the regional fractional Laplacian by means of its singular integral representation. For both problems, a numerical estimation of the solutions is obtained, using in the nonlinear case an approach involving a random variation of an initial guess of the solution. Moreover, in the linear case a parametric analysis is made in order to study the effects of the parameters involved in the model, with a particular emphasis on the order of the fractional Laplacian and on the mixture law between local and nonlocal behavior.

2. Preliminaries

Let Ω, Ω_0, Ω_1 and Ω_δ be as stated in the Introduction. The natural solution functional space associated to (P_L) is

$$\mathbb{H}^s_{0,\Omega} = \left\{ u \in H^1_0(\Omega_0) \cap H^s_0(\Omega) \,:\, u = 0 \text{ a.e. in } \Omega_1 \right\},$$

where $H^1_0(\Omega_0)$ is the completion of $C^\infty_c(\Omega_0)$ with respect to the norm $\|\nabla\cdot\|_{L^2(\Omega_0)}$ and $H^s_0(\Omega)$ is the completion of $C^\infty_c(\Omega)$, with respect to the Gagliardo seminorm

$$[u]_{2,\Omega} = \left(\iint_{\Omega\times\Omega} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\, dxdy \right)^{1/2}.$$

The canonical Hilbertian norm on $\mathbb{H}^s_{0,\Omega}$ is

$$|||u|||_{\mathbb{H}^s_{0,\Omega}} = \left(\int_{\Omega_0} |\nabla u|^2 dx + \iint_{\Omega\times\Omega} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\, dxdy \right)^{1/2} = \left(\|\nabla u\|^2_{L^2(\Omega_0)} + [u]^2_{2,\Omega} \right)^{1/2},$$

which, since $V \in L^\infty(\Omega_0)$, is equivalent to the Hilbertian norm

$$\|u\|_{\mathbb{H}^s_{0,\Omega}} = \left(\int_{\Omega_0} V(x)|u|^2 dx + \|\nabla u\|^2_{L^2(\Omega_0)} + [u]^2_{2,\Omega}\right)^{1/2} = \left(\|u\|^2_{L^2(\Omega_0,V)} + \|\nabla u\|^2_{L^2(\Omega_0)} + [u]^2_{2,\Omega}\right)^{1/2},$$

being

$$|||u|||^2_{\mathbb{H}^s_{0,\Omega}} \le \|u\|^2_{\mathbb{H}^s_{0,\Omega}} \le \max\{C_P\|V\|_\infty,\ 1\}|||u|||^2_{\mathbb{H}^s_{0,\Omega}},$$

where C_P is the Poincaré constant. It is convenient for later purposes to endow $\mathbb{H}^s_{0,\Omega}$, with the Hilbertian norm

$$\|u\| = \left(\|u\|^2_{L^2(\Omega_0,V)} + c\|\nabla u\|^2_{L^2(\Omega_0)} + k[u]^2_{2,\Omega}\right)^{1/2},$$

which is equivalent to $\|\cdot\|_{\mathbb{H}^s_{0,\Omega}}$, since $c, k > 0$, being $\kappa\|u\|_{\mathbb{H}^s_{0,\Omega}} \le \|u\| \le K\|u\|_{\mathbb{H}^s_{0,\Omega}}$ for all $u \in \mathbb{H}^s_{0,\Omega}$, where $\kappa = \min\{c,\ k,\ 1\}$ and $K = \max\{c,\ k,\ 1\}$.

3. Existence results

3.1. The linear problem

The first problem that we consider is

$$\begin{cases} -c\Delta u + k(-\Delta)^s_\Omega u + V(x)u = f(x) & \text{in } \Omega_0, \\ u = 0 & \text{in } \Omega_1, \end{cases} \tag{P_L}$$

where $c,\ k > 0$ and $f \in L^2(\Omega_0)$.

Definition 3.1. We say that $u \in \mathbb{H}^s_{0,\Omega}$ is a (weak) *solution* of problem (P_L) if

$$c\int_{\Omega_0} \nabla u \cdot \nabla v\, dx + k\iint_{\Omega\times\Omega} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\, dxdy + \int_{\Omega_0} V(x)uv\, dx - \int_{\Omega_0} f(x)v\, dx = 0$$

for every function $v \in \mathbb{H}^s_{0,\Omega}$.

In light of the variational structure of problem (P_L), the critical points of the underlying C^1 functional $J : \mathbb{H}^s_{0,\Omega} \to \mathbb{R}$, defined as

$$J(u) := \frac{1}{2}\|u\|^2 - \int_{\Omega_0} f(x)u\, dx,$$

are exactly the (weak) solutions of (P_L). Thanks to the linearity of (P_L), existence and uniqueness are obtained by standard arguments.

Proposition 3.1. *Let $f \in L^2(\Omega_0)$ and $s \in (0,1)$. Then there exists a unique solution $u \in \mathbb{H}^s_{0,\Omega}$ of problem (P_L). If f is nontrivial, then also the solution is nontrivial.*

Proof. First, it is easy to show that the functional J is coercive, since

$$J(u) \ge \frac{1}{2}\|u\|^2 - \|f\|_{2,\Omega_0}\|u\| \underset{\|u\|\to\infty}{\longrightarrow} \infty.$$

Moreover, J is C^1, strictly convex and coercive in the Hilbert space $\mathbb{H}^s_{0,\Omega}$, so that the Weierstrass Theorem, see Corollary 3.23 of [7], the functional J has a global minimum in $\mathbb{H}^s_{0,\Omega}$, which is also a critical point of J, and hence a solution of (P_L).

Uniqueness of solutions of (P_L), that is uniqueness of critical points of J follows from the strict convexity of J. This completes the proof. □

3.2. The nonlinear model

This subsection deals with the nonlinear problem

$$\begin{cases} -c\Delta u + k(-\Delta)^s_\Omega u + V(x)u = f(x,u) & \text{in } \Omega_0, \\ u = 0 & \text{in } \Omega_1, \end{cases} \tag{P_N}$$

where $f : \Omega_0 \times \mathbb{R} \to \mathbb{R}$ *is a Carathéodory function such that* $f(\cdot, 0) = 0$ *a.e. in* Ω_0. Let us introduce the notation 2^* for the Sobolev exponent, that is

$$2^* = \begin{cases} \dfrac{2N}{N-2} & \text{if } N > 2, \\ \infty & \text{if } N \in \{1,\ 2\}, \end{cases} \quad \text{and its Hölder conjugate is } (2^*)' = \begin{cases} \dfrac{2N}{N+2} & \text{if } N > 2, \\ 1 & \text{if } N \in \{1,\ 2\}. \end{cases}$$

In addition, we assume the following conditions coming from [16], as improvements of those in [15] and [12]:

(f_1) there exist $a \in L^q(\Omega_0)$, $b \in L^\infty(\Omega_0)$, *with* $a \geq 0$, $b > 0$ *and* $q \in ((2^*)', 2)$, *and* $r \in (2, 2^*)$ *such that*

$$|f(x,t)| \leq a(x) + b(x)|t|^{r-1} \quad \text{for a.e. } x \in \Omega_0 \text{ and for all } t \in \mathbb{R};$$

(f_2) $\displaystyle\lim_{t\to\pm\infty} \frac{F(x,t)}{|t|^2} = \infty$ *uniformly for a.e.* $x \in \Omega_0$, *where* $F(x,t) = \int_0^t f(x,\tau)d\tau$*;*

(f_3) there exist $\theta \geq 1$ *and* $\beta \in L^1(\Omega_0)$, $\beta \geq 0$, *such that*

$$\sigma(x,t_1) \leq \theta\sigma(x,t_2) + \beta(x) \quad \text{for a.e. } x \in \Omega_0 \text{ and all } 0 \leq t_1 \leq t_2 \text{ or } t_2 \leq t_1 \leq 0,$$

where $\sigma(x,t) = f(x,t)t - 2F(x,t)$ *in* $\Omega_0 \times \mathbb{R}$*;*

(f_4) $\displaystyle\lim_{t\to 0} \frac{f(x,t)}{|t|} = 0$ *uniformly for a.e.* $x \in \Omega_0$.

Remark 3.1. *An example of a function satisfying conditions* $(f_1) - (f_4)$ *is given by*

$$f(x,t) = b(x)t\log(1+|t|),$$

with $b \in L^\infty(\Omega_0)$ *and* $b(x) > 0$ *a.e. in* Ω_0. *Clearly,* $f(x,0) = 0$ *a.e. in* Ω_0, *and* (f_1) *is satisfied for* $N < 6$ *as*

$$|f(x,t)| \leq b(x)|u|^2$$

for a.e. $x \in \Omega_0$, *recalling that* $\log(1+|t|) \leq |t|$ *for every* $t \in \mathbb{R}$. *To show the validity of* (f_2), *we first compute*

$$F(x,t) = \frac{1}{2}b(x)\left(t^2\log(1+|t|) - \frac{t^2}{2} + |t| - \log(1+|t|)\right),$$

then a simple computation of the limit is enough. Since

$$\sigma(x,t) = b(x)\left(\frac{t^2}{2} - |t| + \log(1+|t|)\right),$$

it is easy to see that (f_3) *is satisfied for* $\theta = 1$ *and* $\beta = 0$. *Finally, the computation of the limit shows that* (f_4) *is satisfied.*

Definition 3.2. With the same assumption on f as above, we say that $u \in \mathbb{H}^s_{0,\Omega}$ is a (weak) *solution* of problem (P_N) if

$$c\int_{\Omega_0} \nabla u\, \nabla v\, dx + k\iint_{\Omega\times\Omega} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\, dxdy + \int_{\Omega_0} V(x)uv\, dx - \int_{\Omega_0} f(x,u)v\, dx = 0$$

for every function $v \in \mathbb{H}^s_{0,\Omega}$.

From this definition and from the variational nature of (P_N), the critical points of the corresponding functional $I : \mathbb{H}^s_{0,\Omega} \to \mathbb{R}$, defined as

$$I(u) = \frac{1}{2}\|u\|^2 - \int_{\Omega_0} F(x,u)\, dx,$$

are exactly the (weak) solutions of (P_N).

Before proving the main existence theorem for (P_N), let us give some preliminaries. It is useful to introduce the functionals

$$I_\pm(u) = \frac{1}{2}\|u\|^2 - \int_{\Omega} F(x,u^\pm)\, dx,$$

where u^+ and u^- are the classical positive part and negative part of u.

Our aim is to prove that both $I_\pm$ satisfy the Cerami condition, (C) for short, which states that any sequence $(u_n)_n$ in $\mathbb{H}^s_{0,\Omega}$ such that $(I_\pm(u_n))_n$ is bounded and $(1+\|u_n\|)I'_\pm(u_n) \to 0$ as $n \to \infty$ admits a convergent subsequence.

Proposition 3.2. *Under assumptions* (f_1)*–*(f_3)*, the functionals* $I_\pm$ *satisfy the (C) condition.*

Proof. We give the proof for I_+, the proof for I_- being analogous.

Let $(u_n)_n$ in $\mathbb{H}^s_{0,\Omega}$ be such that

$$|I_+(u_n)| \le M_1 \tag{3.1}$$

for some $M_1 > 0$ and all n, and

$$(1+\|u_n\|)I'_+(u_n) \to 0 \text{ in } (\mathbb{H}^s_{0,\Omega})' \text{ as } n \to \infty. \tag{3.2}$$

From (3.2) we have

$$|I'_+(u_n)h| \le \frac{\varepsilon_n h}{1+\|u_n\|}$$

for every $h \in \mathbb{H}^s_{0,\Omega}$, where $\varepsilon_n \to 0$ as $n \to \infty$. That is,

$$\begin{aligned}\Bigg| c\int_{\Omega_0} \nabla u_n \nabla h\, dx + k\iint_{\Omega\times\Omega} &\frac{(u_n(x)-u_n(y))(h(x)-h(y))}{|x-y|^{N+2s}}\, dxdy \\ &+ \int_{\Omega_0} V(x)u_n h\, dx - \int_{\Omega_0} f(x,u_n^+)h\, dx \Bigg| \le \frac{\varepsilon_n h}{1+\|u_n\|}.\end{aligned} \tag{3.3}$$

Taking $h = u_n^-$ in (3.3), we get for all n

$$\left| c\int_{\Omega_0} |\nabla u_n^-|^2 dx + k\iint_{\Omega\times\Omega} \frac{(u_n(x)-u_n(y))(u_n^-(x)-u_n^-(y))}{|x-y|^{N+2s}}\, dxdy + \int_{\Omega_0} V(x)|u_n^-|^2 dx \right| \le \varepsilon_n \tag{3.4}$$

Since

$$|u_n^-(x) - u_n^-(y)|^2 \le (u_n(x) - u_n(y))(u_n^-(x) - u_n^-(y)),$$

we get

$$\|u_n^-\| \le \varepsilon_n,$$

and so

$$u_n^- \to 0 \text{ in } (\mathbb{H}^s_{0,\Omega})' \text{ as } n \to \infty. \tag{3.5}$$

Now, taking $h = u_n^+$ in (3.3), we obtain

$$\begin{aligned} -c\int_{\Omega_0} |\nabla u_n^+|^2 dx - k\iint_{\Omega\times\Omega} \frac{(u_n(x) - u_n(y))(u_n^+(x) - u_n^+(y))}{|x-y|^{N+2s}}\,dxdy \\ - \int_{\Omega_0} V(x)|u_n^+|^2 dx + \int_{\Omega_0} f(x, u_n^+)u_n^+ dx \le \varepsilon_n. \end{aligned} \tag{3.6}$$

From (3.1) we have

$$c\int_{\Omega_0} |\nabla u_n|^2 dx + k\iint_{\Omega\times\Omega} \frac{|u_n(x) - u_n(y)|^2}{|x-y|^{N+2s}}\,dxdy + \int_{\Omega_0} V(x)|u_n(x)|^2 dx - 2\int_{\Omega_0} F(x, u_n^+)\,dx \le 2M_1$$

for $M_1 > 0$ and all n. Hence, together with (3.4), this leads to

$$\begin{aligned} c\int_{\Omega_0} |\nabla u_n^+|^2 dx + k\iint_{\Omega\times\Omega} \frac{(u_n(x) - u_n(y))(u_n^+(x) - u_n^+(y))}{|x-y|^{N+2s}}\,dxdy \\ + \int_{\Omega_0} V(x)|u_n^+|^2 dx - 2\int_{\Omega_0} F(x, u_n^+)\,dx \le M_2 \end{aligned} \tag{3.7}$$

for some $M_2 > 0$ and all n. Adding (3.6) to (3.7), we obtain

$$\int_{\Omega_0} f(x, u_n^+)u_n^+ dx - 2\int_{\Omega_0} F(x, u_n^+)\,dx \le M_3$$

for some $M_3 > 0$ and all n, that is

$$\int_{\Omega_0} \sigma(x, u_n^+)\,dx \le M_3. \tag{3.8}$$

To prove that $(u_n^+)_n$ is bounded in $\mathbb{H}^s_{0,\Omega}$, we argue by contradiction. Passing to a subsequence if necessary, we assume that $\|u_n^+\| \to \infty$ as $n \to \infty$ and that for some $v \ge 0$

$$v_n \rightharpoonup v \text{ in } \mathbb{H}^s_{0,\Omega} \text{ and } v_n \to v \text{ in } L^q(\Omega_0), \quad v_n = u_n^+/\|u_n^+\|, \tag{3.9}$$

for every $q \in (2, 2_s^*)$.

First, we deal with the case $v \ne 0$. We define the set

$$Z(v) = \{x \in \Omega_0 : v(x) = 0\},$$

so that $|\Omega_0 \setminus Z(v)| > 0$ and $u_n^+ \to \infty$ for a.e. $x \in \Omega_0 \setminus Z(v)$ as $n \to \infty$. By hypothesis (f_2) we have

$$\lim_{n\to\infty} \frac{F(x, u_n^+(x))}{\|u_n^+\|^2} = \lim_{n\to\infty} \frac{F(x, u_n^+(x))}{u_n^+(x)^2} v_n(x)^2 = \infty$$

for almost every $x \in \Omega_0 \setminus Z(v)$. On the other hand, by Fatou's Lemma

$$\int_{\Omega_0} \liminf_{n\to\infty} \frac{F(x, u_n^+(x))}{\|u_n^+\|^2}\, dx \le \liminf_{n\to\infty} \int_{\Omega_0} \frac{F(x, u_n^+(x))}{\|u_n^+\|^2}\, dx,$$

which leads to

$$\lim_{n\to\infty} \int_{\Omega_0} \frac{F(x, u_n^+(x))}{\|u_n^+\|^2}\, dx = \infty. \tag{3.10}$$

From (3.1) we have

$$-\frac{1}{2}\|u_n\|^2 + \int_{\Omega_0} F(x, u_n^+(x))\, dx \le M_4$$

for some $M_4 > 0$ and all n. Recalling that $\|u_n\|^2 \le 2(\|u_n^+\|^2 + \|u_n^-\|^2)$, from (3.5) we obtain

$$-\|u_n^+\|^2 + \int_{\Omega_0} F(x, u_n^+(x))\, dx \le M_5$$

for some $M_5 > 0$. Dividing by $\|u_n^+\|^2$,

$$-1 + \int_{\Omega_0} \frac{F(x, u_n^+(x))}{\|u_n^+\|^2}\, dx \le \frac{M_5}{\|u_n^+\|^2}.$$

Passing to the limit, we get

$$\limsup_{n\to\infty} \int_{\Omega_0} \frac{F(x, u_n^+(x))}{\|u_n^+\|^2}\, dx \le M_6$$

for some $M_6 > 0$. This contradicts (3.10), and concludes the case $v \neq 0$.

Now we deal with the case $v \equiv 0$. We consider the continuous functions $\gamma_n : [0, 1] \to \mathbb{R}$, defined as $\gamma_n(t) = I_+(tu_n^+)$, with all $t \in [0, 1]$ and all n. Thus, we can define t_n such that

$$\gamma_n(t_n) = \max_{t\in[0,1]} \gamma_n(t). \tag{3.11}$$

Now we define $w_n = (2\lambda)^{\frac{1}{2}} v_n \in \mathbb{H}^s_{0,\Omega}$ for $\lambda > 0$. From (3.9), $w_n \to 0$ in $L^q(\Omega_0)$ for all $q \in (2, 2_s^*)$. Performing some integration from (f_1) we have

$$\int_{\Omega_0} F(x, w_n(x))\, dx \le \int_{\Omega_0} a(x)|w_n(x)|\, dx + C \int_{\Omega_0} |w_n(x)|^r dx,$$

which implies

$$\lim_{n\to\infty} \int_{\Omega_0} F(x, w_n(x))\, dx = 0. \tag{3.12}$$

Since $\|u_n^+\| \to \infty$, there exists n_0 such that $(2\lambda)^{\frac{1}{2}}/\|u_n^+\| \in (0, 1)$ for all $n \ge n_0$. Then, from (3.11),

$$\gamma_n(t_n) \ge \gamma_n \left(\frac{(2\lambda)^{\frac{1}{2}}}{\|u_n^+\|} \right)$$

for all $n \ge n_0$. Thus,

$$I_+(t_n u_n^+) \ge I_+((2\lambda)^{\frac{1}{2}} v_n) = \lambda \|v_n\|^2 - \int_{\Omega_0} F(x, w_n(x))\, dx.$$

Then (3.12) implies that

$$I_+(t_n u_n^+) \geq \lambda \|v_n\|^2 + o(1),$$

and since λ is arbitrary we have

$$\lim_{n\to\infty} I_+(t_n u_n^+) = \infty. \tag{3.13}$$

Clearly, $0 \leq t_n u_n^+ \leq u_n^+$ for all n, so that from (f_3) we know that

$$\int_{\Omega_0} \sigma(x, t_n u_n^+)\, dx \leq \theta \int_{\Omega_0} \sigma(x, u_n^+)\, dx + \|\beta\|_1 \tag{3.14}$$

for all n. Clearly, $I_+(0) = 0$. In addition, (3.1) and (3.4) imply that $I_+(u_n^+) \leq M_7$ for some $M_7 > 0$. Together with (3.13) this implies that $t_n \in (0, 1)$ for all $n \geq n_1 \geq n_0$. Since t_n is a maximum point, we get

$$\begin{aligned} 0 = t_n \gamma_n'(t_n) = k \iint_{\Omega\times\Omega} &\frac{(t_n u_n(x) - t_n u_n(y))(t_n u_n^+(x) - t_n u_n^+(y))}{|x-y|^{N+2s}}\, dxdy \\ &+ c \int_{\Omega_0} |\nabla t_n u_n^+|^2 dx + \int_{\Omega_0} V(x)|t_n u_n^+|^2 dx - \int_{\Omega_0} f(x, t_n u_n^+) t_n u_n^+\, dx, \end{aligned}$$

and recalling that

$$|t_n u_n^+(x) - t_n u_n^+(y))|^2 \leq (t_n u_n(x) - t_n u_n(y))(t_n u_n^+(x) - t_n u_n^+(y)),$$

we have

$$\|t_n u_n^+\|^2 - \int_{\Omega_0} f(x, t_n u_n^+) t_n u_n^+ dx \leq 0. \tag{3.15}$$

Adding (3.15) to (3.14), we obtain

$$\|t_n u_n^+\|^2 - 2\int_{\Omega_0} F(x, t_n u_n^+)\, dx \leq \theta \int_{\Omega_0} \sigma(x, u_n^+)\, dx + \|\beta\|_1,$$

that is

$$2 I_+(t_n u_n^+) \leq \theta \int_{\Omega_0} \sigma(x, u_n^+)\, dx + \|\beta\|_1.$$

Hence, (3.13) implies that

$$\lim_{n\to\infty} \int_{\Omega_0} \sigma(x, u_n^+)\, dx = \infty. \tag{3.16}$$

Combining (3.8) and (3.16) we obtain the contradiction, which concludes the case $v \equiv 0$.

In conclusion, $(u_n^+)_n$ is bounded in $\mathbb{H}^s_{0,\Omega}$, and (3.5) gives that $(u_n)_n$ is bounded in $\mathbb{H}^s_{0,\Omega}$. Hence, upto a subsequence, if necessary, there exists $u \in \mathbb{H}^s_{0,\Omega}$ such that

$$u_n \rightharpoonup u \text{ in } \mathbb{H}^s_{0,\Omega} \text{ and } u_n \to u \text{ in } L^q(\Omega_0), \quad q \in (2, 2^s). \tag{3.17}$$

Taking $h = u_n - u$ in (3.3), we have

$$\begin{aligned} \|u_n\|^2 - c\int_{\Omega_0} \nabla u_n \nabla u\, dx - k \iint_{\Omega\times\Omega} &\frac{(u_n(x) - u_n(y))(u(x) - u(y))}{|x-y|^{N+2s}}\, dxdy \\ &- \int_{\Omega_0} V(x) u_n u\, dx - \int_{\Omega_0} f(x, u_n^+)(u_n - u)\, dx \leq \varepsilon_n \end{aligned} \tag{3.18}$$

From (f_1) and (3.17) we know that

$$\lim_{n\to\infty}\int_{\Omega_0}|f(x,u_n^+)(u_n-u)|\,dx=0.$$

Passing to the limit in (3.18) we obtain

$$\lim_{n\to\infty}\left(\|u_n\|^2-k\iint_{\Omega\times\Omega}\frac{(u_n(x)-u_n(y))(u(x)-u(y))}{|x-y|^{N+2s}}\,dxdy-c\int_{\Omega_0}\nabla u_n\nabla u\,dx-\int_{\Omega_0}V(x)u_nu\,dx\right)=0.$$

This implies that $\|u_n\|\to\|u\|$, so that $u_n\to u$ in $\mathbb{H}^s_{0,\Omega}$. Then I_+ satisfies the (C) condition, which concludes the proof. □

We are now able to give the proof of the main existence theorem for (P_N).

Theorem 3.3. *If* (f_1)–(f_4) *hold, then problem* (P_N) *admits at least two nontrivial constant sign solutions.*

Proof. Let us apply the Mountain Pass Theorem to I_+. From Proposition 3.2 we know that I_+ satisfies the (C) condition, so that we only have to verify the geometric conditions.

From (f_1) and (f_4), for every $\varepsilon>0$ there exists C_ε such that

$$F(x,t)\le\frac{\varepsilon}{2}|t|^2+C_\varepsilon|t|^r$$

for a.e. $x\in\Omega_0$ and all $t\in\mathbb{R}$. Then

$$I_+(u)=\frac{1}{2}\|u\|^2-\int_{\Omega_0}F(x,u^+)\,dx\ge\frac{1}{2}\|u\|^2-\frac{\varepsilon}{2}\|u\|_2^2-C_\varepsilon\|u\|_r^r\ge\frac{1-\varepsilon C_1}{2}\|u\|^2-C_2\|u\|_r^r.$$

From this we kwon that, if $\|u\|=\rho$ is small enough,

$$\inf_{\|u\|=\rho}I_+(u)>0.$$

Now, let $u\in\mathbb{H}^s_{0,\Omega}$ be positive in Ω_0 and let $t>0$. Then

$$I_+(u)=\frac{t^2}{2}\|u\|^2-\int_{\Omega_0}F(x,tu)\,dx=\frac{t^2}{2}\|u\|^2-t^2\int_{\Omega_0}\frac{F(x,tu)}{|tu|^2}u^2\,dx.$$

By Fatou's Lemma

$$\int_{\Omega_0}\liminf_{t\to\infty}\frac{F(x,tu)}{|tu|^2}u^2dx\le\liminf_{t\to\infty}\int_{\Omega_0}\frac{F(x,tu)}{|tu|^2}u^2dx,$$

so that (f_2) implies that

$$\lim_{t\to\infty}\int_{\Omega_0}\frac{F(x,tu)}{|tu|^2}u^2dx=\infty.$$

Consequently,

$$\lim_{t\to\infty}I_+(tu)=-\infty.$$

Therefore, there exists $e\in\mathbb{H}^s_{0,\Omega}$ such that $\|e\|>\rho$ and $I_+(e)<0$.

Now, thanks to Proposition 3.2 and the Mountain Pass Theorem, the functional I_+ possesses a nontrivial critical point u at the minimax level

$$c = \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} I_+(\gamma(t)) \geq \rho > 0, \quad \Gamma = \{\gamma \in C([0,1], \mathbb{H}^s_{0,\Omega}) \,:\, \gamma(0) = 0,\ \gamma(1) = e\}.$$

In particular, u is a nontrivial solution of (P_N) and, taking $v = u^- \in \mathbb{H}^s_{0,\Omega}$ as test function

$$\begin{aligned} 0 &= c\int_{\Omega_0} |\nabla u^-|^2 dx + k\iint_{\Omega\times\Omega} \frac{(u(x)-u(y))(u^-(x)-u^-(y))}{|x-y|^{N+2s}}\,dxdy + \int_{\Omega_0} V(x)|u^-|^2 dx - \int_{\Omega_0} f(x,u^+)u^- dx \\ &= c\int_{\Omega_0} |\nabla u^-|^2 dx + k\iint_{\Omega\times\Omega} \frac{(u(x)-u(y))(u^-(x)-u^-(y))}{|x-y|^{N+2s}}\,dxdy + \int_{\Omega_0} V(x)|u^-|^2 dx. \end{aligned}$$

Recalling that

$$|u^-(x) - u^-(y)|^2 \leq (u(x)-u(y))(u^-(x)-u^-(y))$$

we obtain

$$0 \geq \|u^-\|^2,$$

and so $u^- \equiv 0$. Hence, $I_+(u) = I(u)$. This gives at once that $u \geq 0$ is a nontrivial solution of (P_N).

Arguing in the same way for I_-, we find a second nontrivial nonpositive solution of (P_N). □

4. Numerical approximation of the problems

For an arbitrary function f it could be not simple to find a closed-form solution for the problems (P_L) and (P_N), therefore we resort to find a numerical approximation of the solution itself.

In particular, if the domain $\Omega_0 \cup \Omega_1$ is the interval $(-L, L)$ and Ω_0 is $(-L_0, L_0)$, we discretize it in a finite number, n, of points denoted as x_i with $i = 0, 2, \ldots, n-1$, as shown in Figure 2. we have that

$$x_i = -L + ih, \quad i = 0, 1, \ldots, n-1,$$

with

$$h = \frac{2L}{n-1}$$

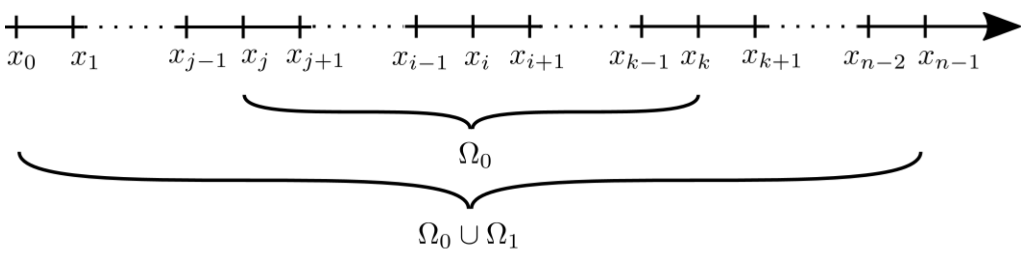

Figure 2. Discretized interval.

The corresponding value of u in the points x_i will be denoted $u_i = u(x_i)$.

The Laplacian Δu is approximated by means of the central difference formula

$$(\Delta u)_i = \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}.$$

in all the points except when $i = 0$ and $i = n - 1$ forward difference and backward difference formulas, respectively, are used.

In order to approximate $(-\Delta)^s_\Omega u$, the approach proposed in [11] is used. We recall that, in the present case

$$(-\Delta)^s_\Omega u(x) = C_{1,2s} \int_{-L}^{L} \frac{u(x) - u(y)}{|x - y|^{1+2s}}\, dy,$$

with

$$C_{1,2s} = \frac{2s 2^{2s-1} \Gamma(\frac{2s+1}{2})}{\pi^{1/2} \Gamma(\frac{2-2s}{2})}.$$

The idea of the approach is to split the integral over Ω in two contributions, the first is the sum of the integrals in the intervals $(-L, x_i - h)$ and $(x_i + h, L)$ and the second is the (improper) integral in the interval $(x_i - h, x_i + h)$ which contains the singularity.

$$\begin{aligned}
(-\Delta)^s_\Omega u(x_i) &= C_{1,2s} \left[\int_{-L}^{x_i - h} \frac{u(x_i) - u(y)}{(x_i - y)^{1+2s}}\, dy + \int_{x_i + h}^{L} \frac{u(x_i) - u(y)}{(y - x_i)^{1+2s}}\, dy \right] + C_{1,2s} \int_{x_i - h}^{x_i + h} \frac{u(x_i) - u(y)}{|x_i - y|^{1+2s}}\, dy \\
&= C_{1,2s} \left[\int_{h}^{x_i - (-L)} \frac{u(x_i) - u(x_i - t)}{t^{1+2s}}\, dt + \int_{h}^{L - x_i} \frac{u(x_i) - u(x_i + t)}{t^{1+2s}}\, dt \right] \\
&\quad + C_{1,2s} \int_{-h}^{h} \frac{u(x_i) - u(x_i - t)}{|t|^{1+2s}}\, dt
\end{aligned}$$

For the second contribution, we use the result in Section 2.2 of [11] and therefore

$$C_{1,2s} \int_{-h}^{h} \frac{u(x_i) - u(x_i - t)}{|t|^{1+2s}}\, dt = -C_{1,2s} \frac{u(x_i + h) - 2u(x_i) + u(x_i + h)}{(2 - 2s) h^{2s}}$$

For the first contribution, as suggested in [11] an exact integration is made using the following interpolant of the terms $u(x_i) - u(x_i - y)$:

$$u(x_i) - u(x_i \pm t) = \sum_{j \in \mathbb{N}} \left[u(x_i) - u(x_{i \pm j}) \right] T_h(t - x_j)$$

with

$$T_h(t) = \begin{cases} 1 - \dfrac{|t|}{h} & |t| \leq h, \\ 0 & \text{otherwise} \end{cases}$$

Combining the two contributions, the regional Fractional Laplacian can be evaluated numerically as

$$(-\Delta)^s_\Omega u_i = \sum_{j=1}^{i} \left(u_i - u_{i-j} \right) w_j + \sum_{j=1}^{n-1-i} \left(u_i - u_{i+j} \right) w_j,$$

with

$$w_j = h^{-2s} \begin{cases} \dfrac{C_{1,2s}}{2 - 2s} - F'(1) + F(2) - F(1,) & j = 1, \\ F(j+1) - 2F(j) + F(j-1), & j = 2, 3, \ldots \end{cases}$$

where

$$F(t) = \begin{cases} \dfrac{C_{1,2s}}{(2s-1)2s}|t|^{1-2s}, & 2s \neq 1, \\ -C_{1,2s}\log|t|, & 2s = 1. \end{cases}$$

The function F is such that

$$F''(t) = C_{1,2s}\frac{1}{|t|^{1+2s}}$$

The discretized form of (P_L) is therefore

$$\begin{cases} -c(\Delta u)_i + k((-\Delta)^s_\Omega u)_i + u_i = f_i & \text{for } i = j, j+1, \dots, k \\ u_i = 0 & \text{for } i = 0, 1, \dots, j, k, k+1, \dots, n-1, \end{cases}$$

where the points in Ω_0 are x_i, with $i = j, j+1, \dots, k$.

Now the problem is to find the values of u_i for each i in order to find the zeros of the following function g expressed in discretized form

$$g(x_i) = \begin{cases} -c(\Delta u)_i + k((-\Delta)^s_\Omega u)_i + u_i - f_i & \text{for } i = j, j+1, \dots, k \\ u_i & \text{for } i = 0, 1, \dots, j, k, k+1, \dots, n-1 \end{cases}$$

This problem can be solved numerically. In particular, we use the Python programming language and its procedure "fsolve", based on the Powell hybrid method, as implemented in MINPACK, see [14].

4.1. Results for linear case

The following values have been taken:

$$c = 15000\,\mathrm{N/mm^2}, \quad \kappa = 15000\,\mathrm{N/mm^{4-2s}}, \quad s = 0.75,$$
$$L = 100\,\mathrm{mm}, \quad L_0 = 80\,\mathrm{mm}.$$

The value of κ is chosen so that for $s = 1$ the contribution due to the ordinary Laplacian and the regional Fractional Lapalacian are equal, with c equal to the half of modulus of elasticity of the material.

The first case studied consist in a rod loaded by two forces in opposite direction symmetrically with respect to the midspan, as shown in Figure 3.

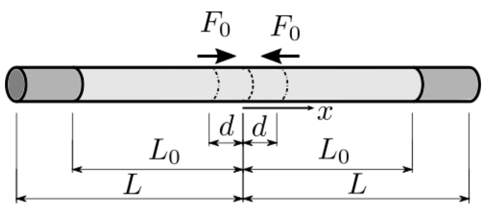

Figure 3. Rod loaded by two forces in opposite direction symmetrically with respect to the midspan.

The response of the rod when $d = \pm 2$ mm and magnitude $F_0 = 1000$ N is shown in Figure 4.

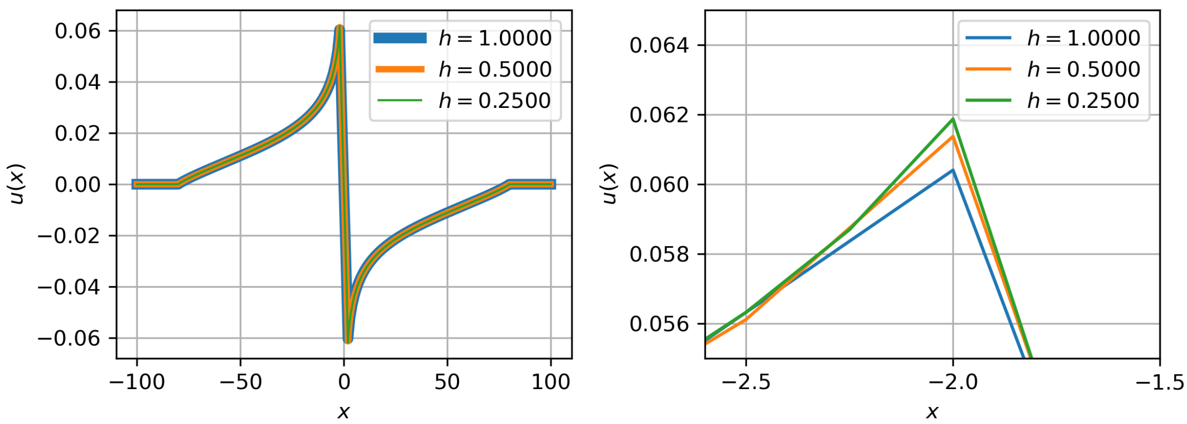

Figure 4. Response of the rod under two opposite forces: left, displacements in all the rod; right, displacement near $x = -d$.

The effect of the variation of the step h is shown, and it can be appreciated that even for quite large values of h the solution is sufficiently accurate.

It is interesting to note that for $s \to 1$ the purely local response is obtained, as shown in Figure 5.

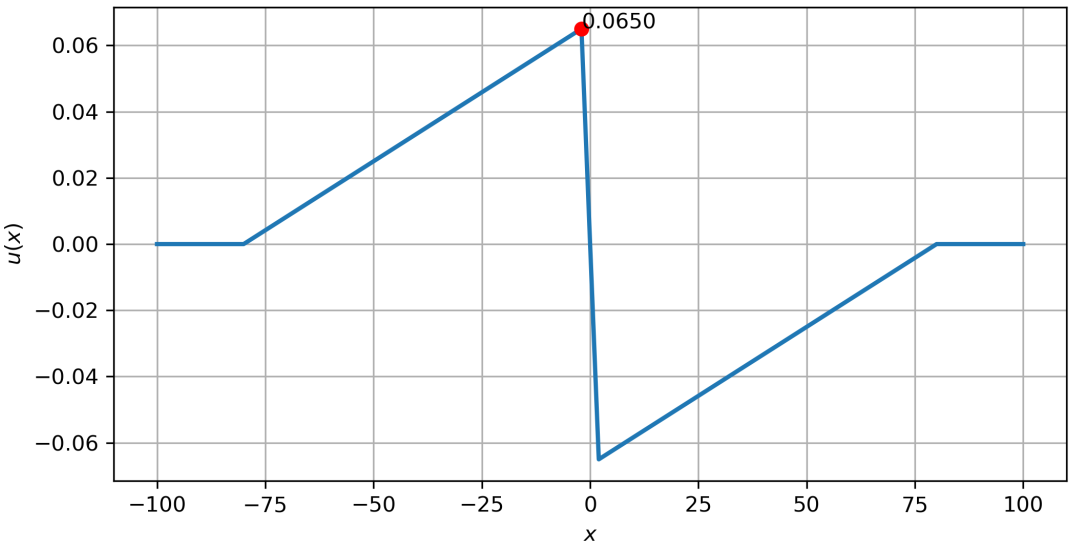

Figure 5. Response of the rod under two opposite forces in the purely local case.

In this case, the displacements are linear in each interval $(-L_0, -d)$, $(-d, d)$ and (d, L_0) and the maximum displacement for $x = -d$ is given by

$$u(-d) = \frac{F_0 d}{c + \kappa} - \frac{F_0 d^2}{(c + \kappa)\, L_0}.$$

With the value recalled before, this gives $u(-d) = 0.065$ mm, as obtained numerically.

Subsequently, we study the response of the rod under the effect of a distributed load f given by the following expression

$$f(x) = f_0 e^{\left(-\frac{x^2}{2\left(\frac{L_0}{16}\right)^2}\right)}$$

as shown in Figure 6.

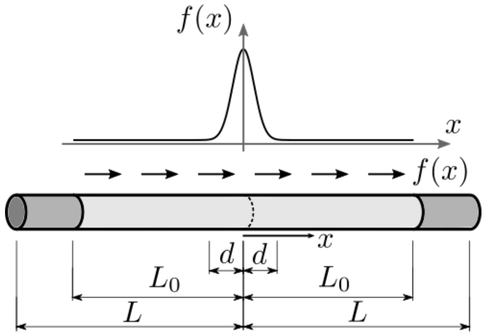

Figure 6. Rod under the effect of a distributed load $f(x)$.

The results are shown in Figure 7.

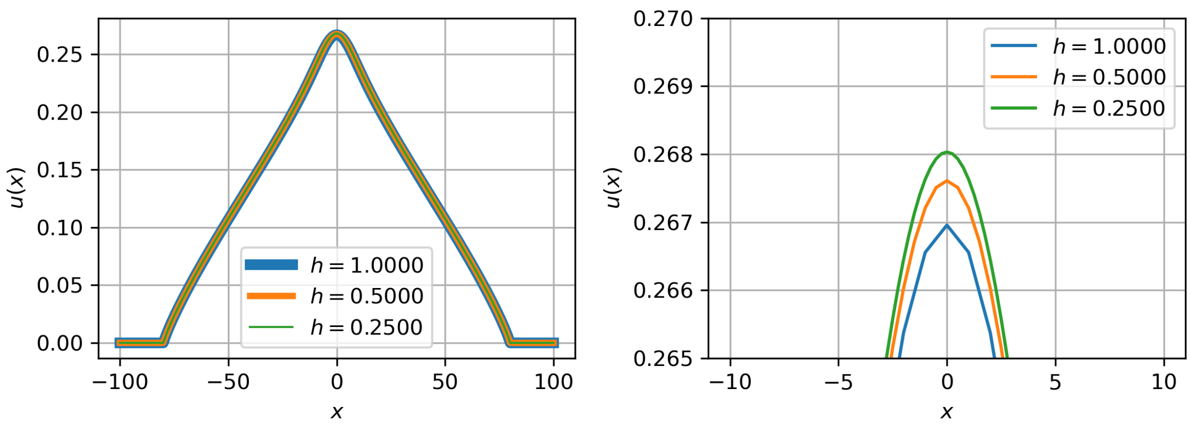

Figure 7. Response of the rod under a distributed load: left, displacements in all the rod; right, displacement near the midspan.

Again, the effect of various values of h is shown and the same considerations as before are valid.

4.1.1. Parametric analysis

A parametric analysis is performed to show the effect of the variation of mechanical parameters on the response.

At first the values of c and κ are varied according to the following rules:

$$c = \beta_1 \cdot 30000$$
$$\mathrm{N/mm^2}\kappa = -2 \cdot \beta_2 \cdot k \cdot \cos(s \cdot \pi) \text{ with } k = 0.5 \cdot 30000\,\mathrm{N/mm}^{4-2s},$$

with $\beta_1 + \beta_2 = 1$. The values of β_1 and β_2 can be though of as the weight of the contribution of the local and nonlocal behavior respectively. For $\beta_1 = \beta_2 = 0.5$ the preceding case is recovered.

The results are shown in Figure 8.

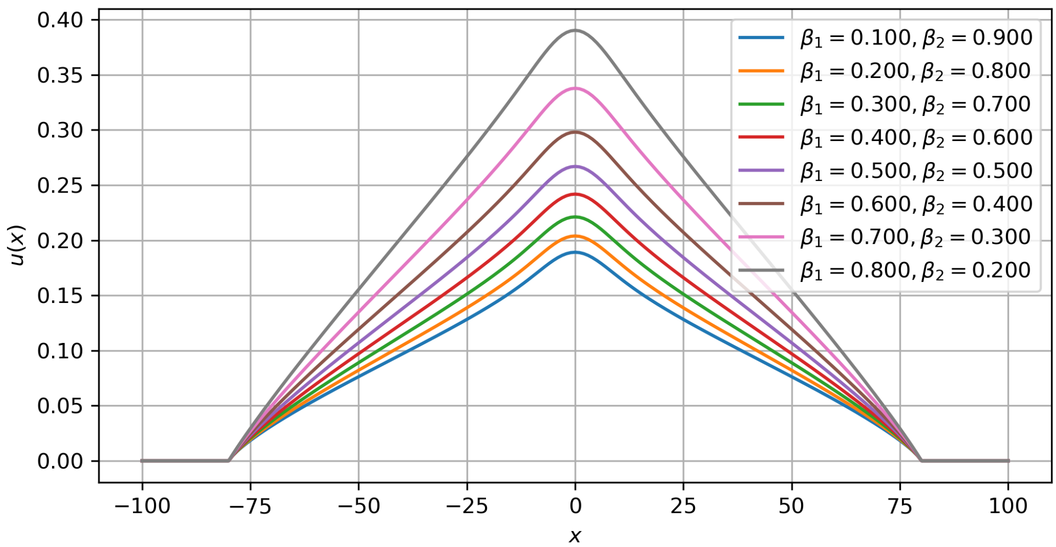

Figure 8. Effect of variation of the relative local and nonlocal contribution to the response.

As can be appreciated, as the value of β_1 increases, the behavior approximates that of a purely local material, as expected. Moreover, the variation of the maximum displacement with β_1 is no monotonic, having and initial decrement followed by a successive pronounced increment.

The values of s vary in the interval $(0.5, 1)$ and produce the results shown in Figure 9.

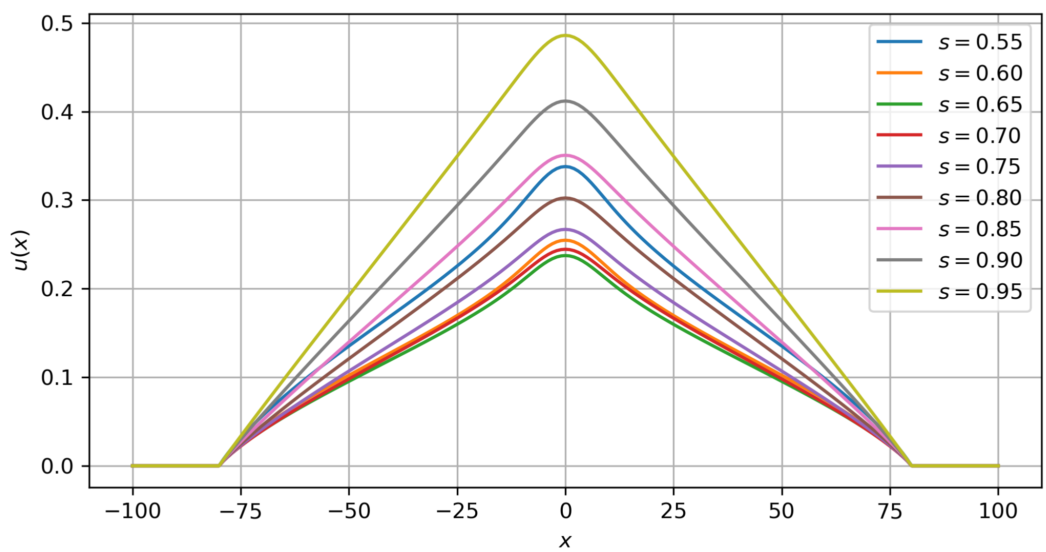

Figure 9. Effect of the order s of the regional fractional Laplacian on the response.

Also in this case a non-monotonic variation of the displacement with respect to s can be appreciated. Subsequently, the effect of the contribution of $V(x)u$ is analyzed. The results are shown in Figure 10.

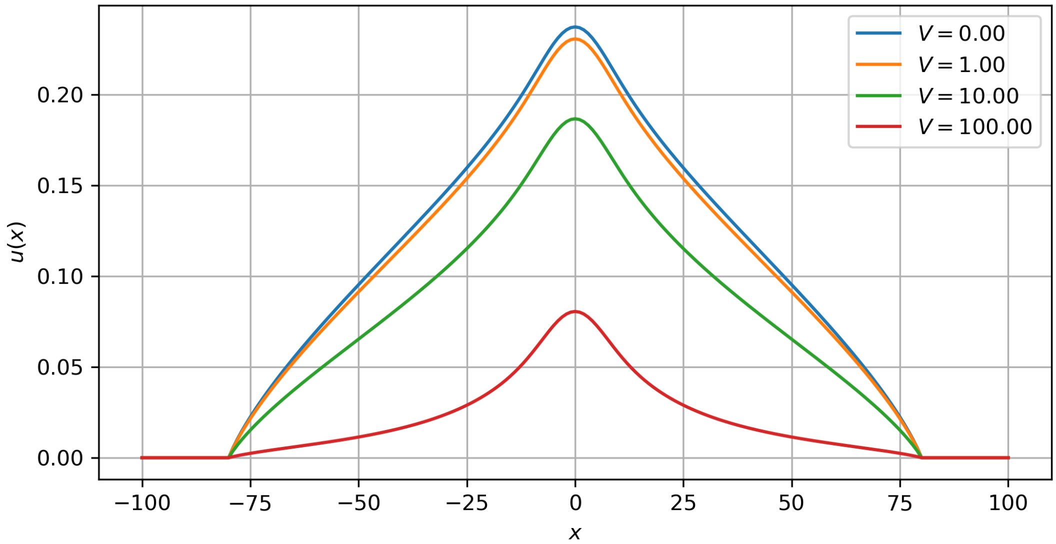

Figure 10. Effect of the presence of external spring forces (V is expressed in $\mathrm{N/mm^4}$).

4.2. Results for nonlinear case

The same mechanical parameters of the linear case have been assumed. In this case, we choose a non symmetric function

$$f(x,u) = (|x+1|+1)u \cdot \log(1+|u|).$$

Anyway, in the nonlinear case, we note that the trivial function $u \equiv 0$ is always a solution of the problem.

In order to find a different non-trivial solution, the numerical procedure starts with an initial guess u_0 which, in the applications, we assume as

$$u_0(x) = \delta_1 \sin\left(\frac{2\pi}{L_0}x\right) + \delta_2 W(x),$$

where W is a function which gives Gaussian white noise.

The effect of the initial guess in retrieving the non-trivial solution is shown in Figure 11, where the blue curve (the trivial solution) was obtained with $\delta_1 = 1$ mm and $\delta_2 = 0.1$ mm, while the orange curve (the nontrivial solution) was obtained with $\delta_1 = 0$ mm and $\delta_2 = 0.1$ mm.

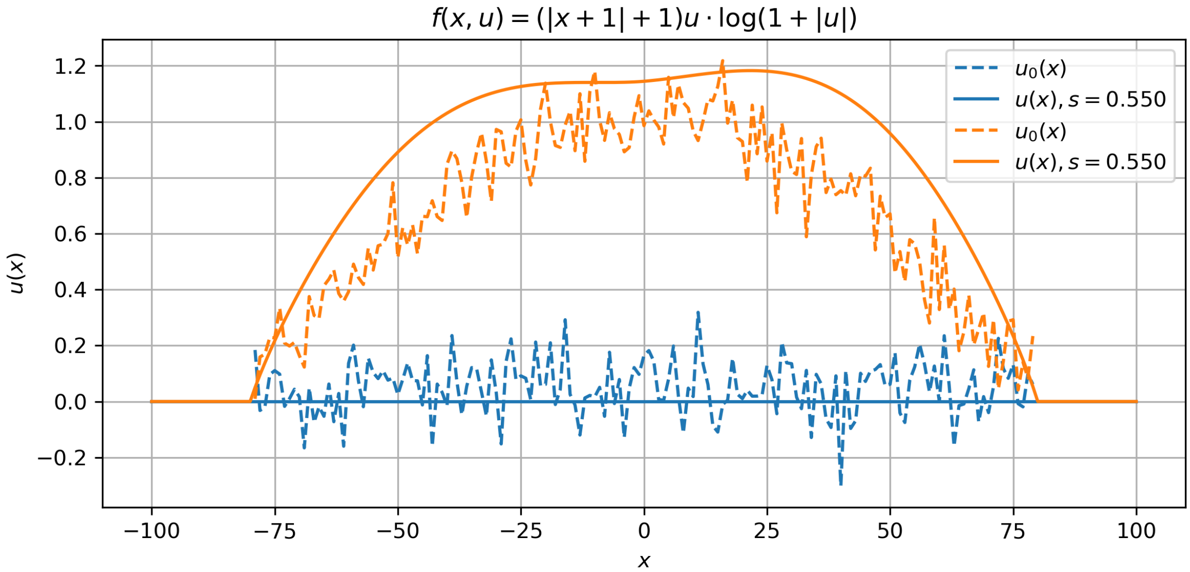

Figure 11. Trivial (blue solid line) and nontrivial (orange solid line) solutions for the nonlinear case; the initial guesses are shown in dashed lines.

It is worth noting that W is different in the two curves since they are from two different random generation.

The effect of different values of s is highlighted in Figure 12.

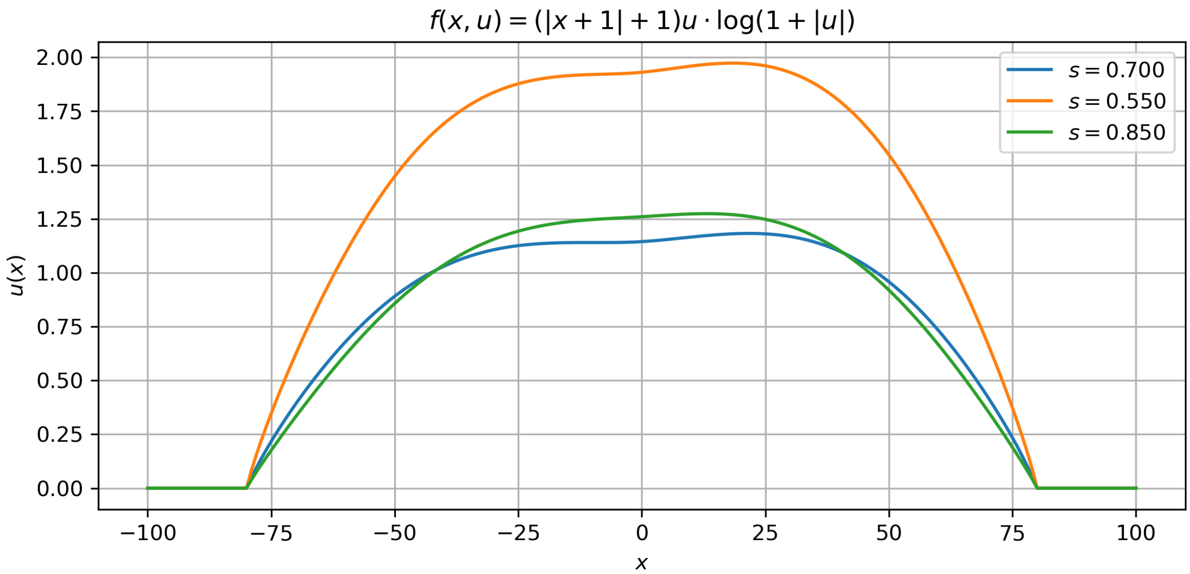

Figure 12. Effect of s on the solution of the nonlinear problem.

Moreover, since we know that at least two different solutions exist, we give an estimation of these solutions in Figure 13. The solutions were obtained using a suitable choice of u_0, in particular the blue curve was obtained with $\delta_1 = 1$ mm and $\delta_2 = 0.1$ mm, while the orange curve was obtained with

$\delta_1 = -1$ mm and $\delta_2 = 0.1$ mm.

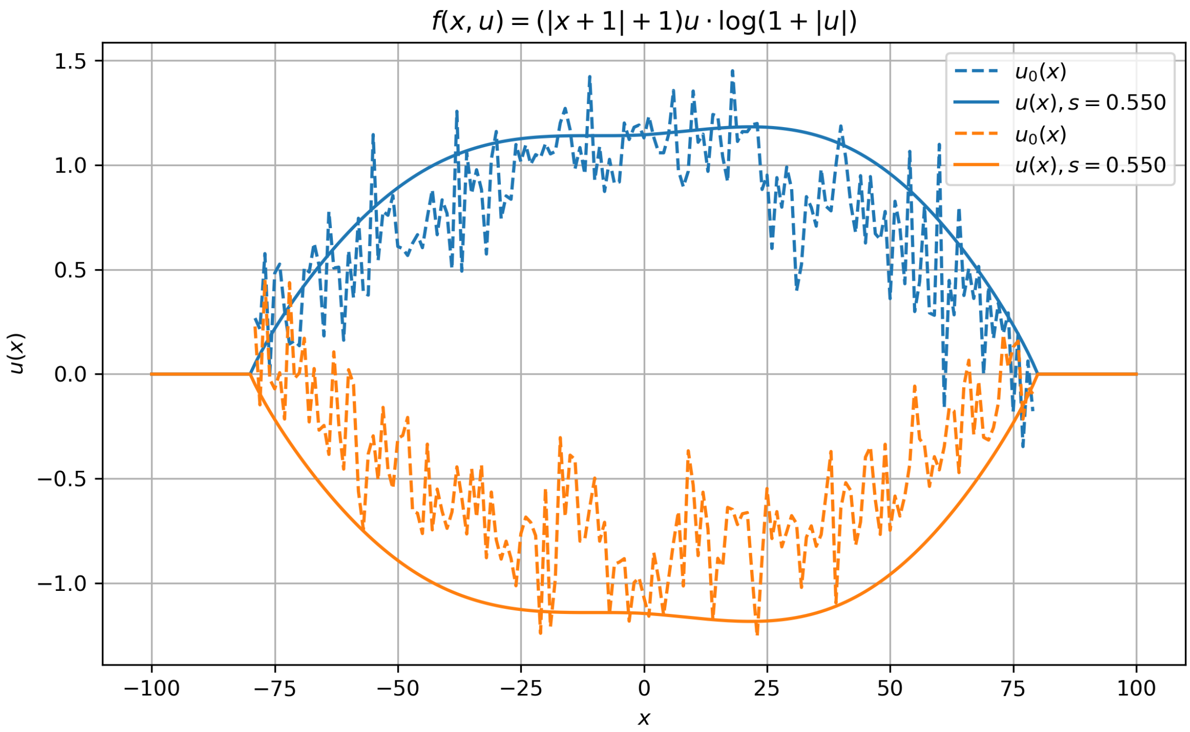

Figure 13. Two different (nontrivial) solutions for the nonlinear problem; the initial guesses are shown in dashed lines.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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