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*Research article*

# Local boundedness for $p$-Laplacian with degenerate coefficients[†]

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**Abstract:** We study local boundedness for subsolutions of nonlinear nonuniformly elliptic equations whose prototype is given by $\nabla \cdot (\lambda|\nabla u|^{p-2}\nabla u) = 0$, where the variable coefficient $0 \leq \lambda$ and its inverse $\lambda^{-1}$ are allowed to be unbounded. Assuming certain integrability conditions on $\lambda$ and $\lambda^{-1}$ depending on $p$ and the dimension, we show local boundedness. Moreover, we provide counterexamples to regularity showing that the integrability conditions are optimal for every $p > 1$.

**Keywords:** elliptic regularity; nonuniform ellipticity; local boundedness; unbounded solutions; weak solutions

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## 1. Introduction

In this note, we study local boundedness of weak (sub)solutions of non-uniformly elliptic quasi-linear equations of the form

$$\nabla \cdot a(x, \nabla u) = 0 \qquad \text{in } \Omega, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^d$ with $d \geq 2$ and $a : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ is a Caratheodory function. The main example that we have in mind are $p$-Laplace type operators with variable coefficients, that is, there exist $p > 1$ and $A : \Omega \to \mathbb{R}^{d\times d}$ such that $a(x, \xi) = A(x)|\xi|^{p-2}\xi$ for all $x \in \Omega$ and $\xi \in \mathbb{R}^d$. In order to measure the ellipticity of $a$, we introduce for fixed $p > 1$

$$\lambda(x) := \inf_{\xi\in\mathbb{R}^d\setminus\{0\}} \frac{a(x,\xi)\cdot\xi}{|\xi|^p} \qquad \mu(x) := \sup_{\xi\in\mathbb{R}^d\setminus\{0\}} \frac{|a(x,\xi)|^p}{(a(x,\xi)\cdot\xi)^{p-1}} \tag{1.2}$$

and suppose that $\lambda$ and $\mu$ are nonnegative. In the uniformly elliptic setting, that is that there exists $0 < m \leq M < \infty$ such that $m \leq \lambda \leq \mu \leq M$ in $\Omega$, solution to (1.1) are locally bounded, Hölder continuous and even satisfy Harnack inequality, see e.g., classical results of Ladyzhenskaya & Ural'tseva, Serrin and Trudinger [34, 41, 42].

In this contribution, we are interested in a nonuniformly elliptic setting and assume that $\lambda^{-1} \in L^t(\Omega)$ and $\mu \in L^s(\Omega)$ for some integrability exponents $s$ and $t$. In [7], we studied this in the case of linear nonuniformly elliptic equations, that is $a(x,\xi) = A(x)\xi$ corresponding to the case $p = 2$, and showed local boundedness and Harnack inequality for weak solutions of (1.1) provided it holds $\frac{1}{s} + \frac{1}{t} < \frac{2}{d-1}$. The results of [7] improved classical findings of Trudinger [43, 44] (see also [39]) from the 1970s and are optimal in view of counterexamples constructed by Franchi et al. in [27]. In this manuscript we extend these results to the more general situation of quasilinear elliptic equation with $p$-growth as described above. More precisely, we show

**Theorem 1.** *Let $d \geq 2, p > 1$, and let $\Omega \subset \mathbb{R}^d$. Moreover, let $s \in [1,\infty]$ and $t \in (1/(p-1),\infty]$ satisfy*

$$\frac{1}{s} + \frac{1}{t} < \frac{p}{d-1}. \tag{1.3}$$

*Let $a : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ be a Caratheodory function with $a(\cdot,0) \equiv 0$ such that $\lambda$ and $\mu$ defined in (1.2) satisfy $\mu \in L^s(\Omega)$ and $\frac{1}{\lambda} \in L^t(\Omega)$. Then any weak subsolution of (1.1) is locally bounded from above in $\Omega$.*

**Remark 1.** *Note that Theorem 1, restricted to the case $p = 2$ recovers the local boundedness part of [7, Theorem 1.1].*

**Remark 2.** *In [15], Cupini, Marcellini and Mascolo studied local boundedness of local minimizer of nonuniformly elliptic variational integrals of the form $\int_\Omega f(x,\nabla v)\,dx$ where $f$ satisfies*

$$\lambda(x)|\xi|^p \leq f(x,\xi) \leq \mu(x) + \mu(x)|\xi|^q \qquad \text{with } \lambda^{-1} \in L^t(\Omega) \text{ and } \mu \in L^s(\Omega). \tag{1.4}$$

*They proved local boundedness under the relation $\frac{1}{pt} + \frac{1}{qs} + \frac{1}{p} - \frac{1}{q} < \frac{1}{d}$ (see also [11] for related results). Considering the specific case $f(x,\xi) = \lambda(x)|\xi|^p$, the result of [15] implies local boundedness of solutions to $\nabla\cdot(\lambda(x)|\nabla u|^{p-2}\nabla u) = 0$ provided $\lambda^{-1} \in L^t(\Omega)$ and $\lambda \in L^s(\Omega)$ with $\frac{1}{s} + \frac{1}{t} < \frac{p}{d}$, which is more restrictive compared to assumption (1.3) in Theorem 1. It would be interesting to investigate if the methods of the present paper can be combined with the ones of [15] to obtain local boundedness for minimizer of functionals satisfying (1.4) assuming $\frac{1}{pt} + \frac{1}{qs} + \frac{1}{p} - \frac{1}{q} < \frac{1}{d-1}$. Note that in the specific case $s = t = \infty$, this follows from [32].*

**Remark 3.** *We emphasize that we only impose global integrability conditions on $\lambda^{-1}$ and $\mu$. Assuming additional local conditions on the coefficients in the form $\lambda \sim \mu$ and $\mu$ is in some Muckenhoupt class, local boundedness is proven under weaker integrability conditions in the seminal work [25] in the case $p = 2$ (see also [31] for the case $p > 1$); for further recent results on higher regularity for nonlinear elliptic equations with Muckenhoupt coefficients we refer to [4, 5, 13, 17].*

The proof of Theorem 1 is presented in Section 2 and follows a variation of the well-known Moser-iteration method. The main new ingredient compared to earlier works [15, 43] lies in an optimized choice of certain cut-off functions – an idea that we first used in [7] for linear nonuniformly elliptic equations (see also [1, 10, 45] for recent applications to linear parabolic equations).

As mentioned above, an example constructed in [27] shows that condition (1.3) is optimal for the conclusion of Theorem 1 in the case $p = 2$. In the second main result of this paper, we show – building on the construction of [27] – that condition (1.3) is optimal for the conclusion of Theorem 1 for all $p \in (1, \infty)$. More precisely, we have

**Theorem 2.** *Let $d \geq 3$, $1 + \frac{1}{d-2} < p < \infty$, and let $s \geq 1$ and $t > \frac{1}{p-1}$ be such that $\frac{1}{s} + \frac{1}{t} \geq \frac{p}{d-1}$ and $\frac{p}{1+1/t} < d - 1$. Then there exists $\lambda : B(0, 1) \to (0, \infty)$ satisfying $\lambda \in L^s(B_1)$ and $\lambda^{-1} \in L^t(B_1)$ and an unbounded weak subsolution of*

$$-\nabla \cdot (\lambda |\nabla v|^{p-2} \nabla v) = 0 \tag{1.5}$$

*in $B(0, 1)$. Moreover, the same conclusion is valid for $d \geq 3$, $1 < p \leq 1 + \frac{1}{d-2}$ and $s \geq 1$ and $t > \frac{1}{p-1}$ satisfying the strict inequalities $\frac{1}{s} + \frac{1}{t} > \frac{p}{d-1}$ and $\frac{t}{t+1} p < d - 1$.*

In particular, we see that condition (1.3) is sharp on the scale of Lebesgue-integrability for the conclusion of Theorem 1. We note that in the particularly interesting case $p = 2$ and $d = 3$ the construction in Theorem 2 fails in the critical case $\frac{1}{s} + \frac{1}{t} = \frac{p}{d-1}$, see [1] for counterexamples to local boundedness for related problems in $d = 3$.

Let us now briefly discuss a similar but different instance of non-uniform ellipticity which is one of the many areas within the Calculus of Variations, where G. Mingione made significant contributions. Consider variational integrals

$$\int_\Omega F(x, \nabla u)\, dx, \tag{1.6}$$

where the integrand $F$ satisfies $(p, q)$ growth conditions of the form

$$|\xi|^p \lesssim F(x, \xi) \lesssim 1 + |\xi|^q \qquad 1 < p \leq q < \infty, \tag{1.7}$$

which where first systematically studied by Marcellini in [35, 36]; see also the recent reviews [37, 38]. The focal point in the regularity theory for those functionals is to obtain Lipschitz-bounds on the minimizer. Indeed, once boundedness of $|\nabla u|$ is proven the unbalanced growth in (1.7) becomes irrelevant and there is a huge literature dedicated to Lipschitz estimates under various assumptions on $F$, see e.g., the interior estimates [6, 8, 9, 23] in the autonomous case, [2, 14, 16, 18–20, 22, 30] in the non-autonomous case, [12, 21] for Lipschitz-bounds at the boundary, and also examples where the regularity of minimizer fail [3, 24, 26, 28, 36]. Finally, we explain a link between functionals with $(p, q)$-growth and (linear) equations with unbounded coefficients. Consider the autonomous case that $F(x, \xi) = F(\xi)$ and let $u \in W^{1,p}(\Omega)$ be a local minimizer of (1.6). Linearizing the corresponding Euler-Largrange equation yield (formally)

$$\nabla \cdot D^2 F(\nabla u) \nabla \partial_i u = 0.$$

Assuming $(p, q)$-growth with $p = 2$ of the form $|\zeta|^2 \lesssim D^2 F(\xi)\zeta \cdot \zeta \lesssim (1 + |\xi|)^{q-2} |\zeta|^2$ implies that $|D^2 F(\nabla u)| \in L^{\frac{2}{q-2}}_{\rm loc}(\Omega)$. Hence condition (1.3) with $p = 2$ yield local boundedness of $\partial_i u$ if $\frac{q-2}{2} < \frac{2}{d-1}$, which is the currently best known general bound ensuring Lipschitz-continuity of local minimizer of (1.6) – this reasoning was made rigorous in [8] for $p \geq 2$ (see also [9] for the case $p \in (1, \infty)$).

## 2. Local boundedness, proof of Theorem 1

Before we prove Theorem 1, we introduce the notion of solution that we consider here.

**Definition 1.** *Fix a domain $\Omega \subset \mathbb{R}^d$ and a Caratheodory function $a : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ such that for a fixed $p \in (1,\infty)$ the functions $\lambda, \mu \geq 0$ given in* (1.2) *satisfy $\frac{1}{\lambda} \in L^{\frac{1}{p-1}}(\Omega)$ and $\mu \in L^1(\Omega)$. The spaces $H_0^{1,p}(\Omega, a)$ and $H^{1,p}(\Omega, a)$ are respectively defined as the completion of $C_c^1(\Omega)$ and $C^1(\Omega)$ with respect to the norm $\|\cdot\|_{H^{1,p}(\Omega,a)}$, where*

$$\|u\|_{H^{1,p}(\Omega,a)} := \left(\int_\Omega \lambda|\nabla u|^p + \mu|u|^p\, dx\right)^{\frac{1}{p}}.$$

*We call $u$ a weak solution (subsolution, supersolution) of* (1.1) *in $\Omega$ if and only if $u \in H^{1,p}(\Omega, a)$ and*

$$\forall \phi \in H_0^{1,p}(\Omega, a),\ \phi \geq 0 : \qquad \mathcal{A}(u, \phi) = 0 \quad (\leq 0, \geq 0), \quad \textit{where} \quad \mathcal{A}(u,\phi) := \int_\Omega a(x, \nabla u)\cdot\nabla\phi\, dx. \tag{2.1}$$

*Moreover, we call $u$ a local weak solution of* (1.1) *in $\Omega$ if and only if $u$ is a weak solution of* (1.1) *in $\Omega'$ for every bounded open set $\Omega' \Subset \Omega$. Throughout the paper, we call a solution (subsolution, supersolution) of* (1.1) *in $\Omega$ a-harmonic (a-subharmonic, a-superharmonic) in $\Omega$.*

The above definitions generalize the concepts of weak solutions and the spaces $H^1(\Omega, a)$ and $H_0^1(\Omega, a)$ discussed by Trudinger [43, 44] in the linear case, that is $a(x,\xi) = A(x)\xi$. We stress that the condition $\lambda^{-1} \in L^{\frac{1}{p-1}}(\Omega)$ and Hölder inequality imply

$$\|\nabla u\|_{L^1(\Omega)} \leq \|\lambda^{-1}\|_{L^{\frac{1}{p-1}}(\Omega)} \left(\int_\Omega \lambda|\nabla u|^p\right)^{\frac{1}{p}} \leq \|\lambda^{-1}\|_{L^{\frac{1}{p-1}}(\Omega)} \|u\|_{H^{1,p}(\Omega,a)}$$

and thus, we have that $W^{1,1}(\Omega) \subset H^{1,p}(\Omega, a)$, where we use that by the same computation as above it holds $\|u\|_{L^1(\Omega)} \leq \|\mu^{-1}\|_{L^{\frac{1}{p-1}}(\Omega)} \|u\|_{H^{1,p}(\Omega,a)}$ and that by definition we have $\lambda \leq \mu$. From this, we also deduce that the elements of $H^{1,p}(\Omega, a)$ are strongly differentiable in the sense of [29]. In particular this implies that there holds a chain rule in the following sense

**Remark 4.** *Let $g : \mathbb{R} \to \mathbb{R}$ be uniformly Lipschitz-continuous with $g(0) = 0$ and consider the composition $F := g(u)$. Then, $u \in H_0^{1,p}(\Omega, a)$ (or $\in H^{1,p}(\Omega, a)$) implies $F \in H_0^{1,p}(\Omega, a)$ (or $\in H^{1,p}(\Omega, a)$), and it holds $\nabla F = g'(u)\nabla u$ a.e. (see e.g., [44, Lemma 1.3]). In particular, if $u$ satisfies $u \in H^{1,p}(\Omega, a)$ (or $\in H^{1,p}(\Omega, a)$) then also the truncations*

$$u_+ := \max\{u, 0\}; \quad u_- := -\min\{u, 0\}$$

*satisfy $u_+, u_- \in H^{1,p}(\Omega, a)$ (or $\in H^{1,p}(\Omega, a)$).*

Now we come to the local boundedness from above for weak subsolutions of (1.1). In order to state the estimates in the right dimensionality, we introduce for $v \in W^{1,\gamma}(\Omega)$ with $\gamma \geq 1$ the notation

$$\|v\|_{\underline{W}^{1,\gamma}(\Omega)} := |\Omega|^{-\frac{1}{\gamma}}\|v\|_{L^\gamma(\Omega)} + |\Omega|^{\frac{1}{d}-\frac{1}{\gamma}}\|\nabla v\|_{L^\gamma(\Omega)}, \tag{2.2}$$

where $|\Omega|$ denotes the $d$-dimensional Lebesgue-measure of $\Omega$. Moreover, we denote by $\|v\|_{W^{1,\gamma}(\Omega)}$ the usual Sobolev-norm given by $\|v\|_{W^{1,\gamma}(\Omega)} := \|v\|_{L^\gamma(\Omega)} + \|\nabla v\|_{L^\gamma(\Omega)}$.

**Theorem 3.** *Let $d \geq 3$, $\Omega \subset \mathbb{R}^d$ and $p \in (1,\infty)$. Moreover, let $s \in [1,\infty]$ and $t \in (\frac{1}{p-1},\infty]$ satisfy* (1.3). *Let $a : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ be a Caratheodory function with $a(\cdot,0) \equiv 0$ such that $\lambda$ and $\mu$ defined in* (1.2) *satisfy $\mu \in L^s(\Omega)$ and $\frac{1}{\lambda} \in L^t(\Omega)$ and for every measurable set $S \subset \Omega$, we set*

$$\Lambda(S) := (\fint_S \mu^s)^{1/s}(\fint_S \lambda^{-t})^{1/t}.$$

*Then, there exists $c = c(d,p,s,t) \in [1,\infty)$ such that for any weak subsolution $u$ of* (1.1) *and for any ball $B_R \subset \Omega$ it holds*

$$\sup_{B_{R/2}} u \leq c\Lambda(B_R)^{\frac{1}{p}\frac{1}{\delta}} \|u_+\|_{\underline{W}^{1,\frac{1}{1+1/t}p}(B_R)},$$

*where $\|\cdot\|_{\underline{W}^{1,\gamma}(B_r)}$ is defined in* (2.2)*; and $\delta := \frac{1}{s_*} - (\frac{1}{p} - \frac{1}{pt}) > 0$ (see Lemma 1 for the definition of $s_*$). Moreover, in the case $1 + \frac{1}{t} < \frac{p}{d-1}$, there exists $c = c(d,p,t) \in [1,\infty)$ such that*

$$\sup_{B_{R/2}} u \leq c\|u_+\|_{\underline{W}^{1,\frac{1}{1+1/t}p}(B_R)}.$$

In the two-dimensional case, we have the following

**Proposition 1.** *Let $\Omega \subset \mathbb{R}^2$ and $p \in (1,\infty)$. Let $a : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ be a Caratheodory function with $a(\cdot,0) \equiv 0$ such that $\lambda$ and $\mu$ defined in* (1.2) *satisfy $\mu \in L^1(\Omega)$ and $\frac{1}{\lambda} \in L^{\frac{1}{p-1}}(\Omega)$. Then, there exists $c = c(d,p) \in [1,\infty)$ such that for any weak subsolution $u$ of* (1.1) *and for any ball $B_R \subset \Omega$ it holds*

$$\sup_{B_{R/2}} u \leq c\|u_+\|_{\underline{W}^{1,1}(B_R)}.$$

Before we proof Theorem 3 and Proposition 1, we show that they imply the claim of Theorem 1.

*Proof of Theorem 1.* In view of Theorem 3 and Proposition 1 it remains to show that for any weak subsolution $u$ of (1.1) and for any ball $B_R \subset \Omega$ it holds $\|u_+\|_{\underline{W}^{1,\frac{t}{t+1}p}(B_R)} < \infty$. This is a consequence of Hölder inequality and the concept of weak subsolution, see Definition 1. Indeed, we have

$$\left(\int_{B_R} (|u| + |\nabla u|)^{\frac{tp}{t+1}}\right)^{\frac{t+1}{t}} \leq \left(\int_{B_R} \lambda^{-t}\right)^{\frac{1}{t}} \int_{B_R} \lambda(|u| + |\nabla u|)^p < \infty,$$

where the right-hand side is finite since $u \in H^{1,p}(\Omega, a)$ (note that $\lambda \leq \mu$ by definition). □

For the proof of Theorem 3, we need a final bit of preparation, namely the following optimization lemma

**Lemma 1** (Radial optimization)**.** *Let $d \geq 3$, $p > 1$, $s > 1$, and let $s_* := \max\{1, (\frac{1}{p}(1 - \frac{1}{s}) + \frac{1}{d-1})^{-1}\}$. For $\frac{1}{2} \leq \rho < \sigma \leq 2$, let $v \in W^{1,s_*}(B_\sigma)$ and $\mu \in L^s(B_\sigma)$, $\mu \geq 0$, be such that $\mu|v|^p \in L^1(B_\sigma)$. Then there exists $c = c(d,p,s)$ such that*

$$J(\rho,\sigma,v) := \inf\left\{\int_{B_\sigma} \mu|v|^p|\nabla\eta|^p \,\mathrm{d}x : \eta \in C_0^1(B_\sigma), \eta \geq 0, \eta = 1 \text{ in } B_\rho\right\}$$

*satisfies*

$$J(\rho,\sigma,v) \leq c(\sigma-\rho)^{-\frac{pd}{d-1}} \|\mu\|_{L^s(B_\sigma\setminus B_\rho)} (\|\nabla v\|^p_{L^{s_*}(B_\sigma\setminus B_\rho)} + \rho^{-p}\|v\|^p_{L^{s_*}(B_\sigma\setminus B_\rho)}).$$

Lemma 1 generalizes [7, Lemma 2.1] from $p = 2$ to $p > 1$ and we provide a proof in the appendix.

*Proof of Theorem 3.* By standard scaling and translation arguments it suffices to suppose that $B_1 \Subset \Omega$ and $u$ is locally bounded in $B_{\frac{1}{2}}$. Hence, we suppose from now on that $B_1 \Subset \Omega$. In Steps 1–4 below, we consider the case $s > 1$. We first derive a suitable Caccioppoli-type inequality for powers of $u_+$ (Step 1) and perform a Moser-type iteration (Steps 2–4). In Step 5, we consider the case $1 + \frac{1}{t} < \frac{p}{d-1}$ which includes the case $s = 1$.

STEP 1. Caccioppoli inequality.
Assuming $B \subset \Omega$, for any cut-off function $\eta \in C_0^1(B)$, $\eta \geq 0$ and any $\beta \geq 1$, there holds

$$\int \eta^p \lambda(x) u_+^{\beta-1} |\nabla u_+|^p \leq \left(\frac{p}{\beta}\right)^p \int u_+^{p+\beta-1} \mu(x) |\nabla \eta|^p. \tag{2.3}$$

For $\beta \geq 1$, we use the weak formulation (2.1) with $\phi := \eta^p u_+^\beta$: *

$$\int a(x, \nabla u) \cdot \nabla(\eta^p u_+^\beta) \leq 0.$$

We have $\int (a(x, \nabla u) - a(x, \nabla u_+)) \cdot \nabla(\eta^p u_+) = 0$, so that we were able to replace $u$ with $u_+$ inside $a(x, \cdot)$. Applying Leibniz rule we get from the previous display

$$\beta \int \eta^p u^{\beta-1} a(x, \nabla u) \cdot \nabla u \leq - \int p \eta^{p-1} u^\beta a(x, \nabla u) \cdot \nabla \eta, \tag{2.4}$$

where to simplify the notation for the rest of this proof we write $u$ instead of $u_+$. Using definition of $\mu$ in (1.2) in form of $|a(x, \xi)| \leq \mu(x)^{\frac{1}{p}} (a(x, \xi) \cdot \xi)^{\frac{p-1}{p}}$ for any $\xi \in \mathbb{R}^d$ (in fact we use (1.2) for $\xi \neq 0$ and for $\xi = 0$ the inequality follow from the assumption $a(x, 0) = 0$), we can bound the r.h.s. in the last math display from above by

$$\begin{aligned} p \int \eta^{p-1} u^\beta \mu(x)^{\frac{1}{p}} (a(x, \nabla u) \cdot \nabla u)^{\frac{p-1}{p}} |\nabla \eta| &= p \int u^{\beta - (\beta-1)\frac{p-1}{p}} \mu(x)^{\frac{1}{p}} |\nabla \eta| (\eta^p u^{\beta-1} a(x, \nabla u) \cdot \nabla u)^{\frac{p-1}{p}} \\ &\leq p \Big( \int u^{p+\beta-1} \mu(x) |\nabla \eta|^p \Big)^{\frac{1}{p}} \Big( \int \eta^p u^{\beta-1} a(x, \nabla u) \cdot \nabla u \Big)^{\frac{p-1}{p}}, \end{aligned}$$

where in the second step we applied Hölder inequality with exponents $p$ and $\frac{p}{p-1}$, respectively. Observe that the last term on the r.h.s. appears on the l.h.s. in (2.4), so that after absorbing it we get from (2.4)

$$\beta \Big( \int \eta^p u^{\beta-1} a(x, \nabla u) \cdot \nabla u \Big)^{\frac{1}{p}} \leq p \Big( \int u^{p+\beta-1} \mu(x) |\nabla \eta|^p \Big)^{\frac{1}{p}},$$

which after taking the $p$-th power turns into

$$\int \eta^p u^{\beta-1} a(x, \nabla u) \cdot \nabla u \leq \left(\frac{p}{\beta}\right)^p \int u^{p+\beta-1} \mu(x) |\nabla \eta|^p.$$

*Rigorously, we are a priori not allowed to test with $u^\beta$. Instead, for $N \geq 1$ one should modify $u^\beta$ by replacing $u^\beta$ with affine $\alpha N^{\alpha-1} u - (\alpha - 1) N^\beta$ in the set $u \geq N$, obtain the conclusion by testing the weak formulation with this modified function, and subsequently sends $N \to \infty$ – for details, see [7, Page 460].

By definition of $\lambda$ in (1.2) in form of $\lambda(x)|\xi|^p \leq a(x,\xi)\cdot\xi$ for any $\xi \in \mathbb{R}^d$, one has $\lambda(x)|\nabla u|^p \leq a(x,\nabla u)\cdot\nabla u$, thus implying the claimed Caccioppoli inequality (2.3).

Step 2. Improvement of integrability.
We claim that there exists $c = c(d,p,s) \in [1,\infty)$ such that for $\frac{1}{2} \leq \rho < \sigma \leq 1$ and $\alpha \geq 1$ it holds

$$\|\nabla(u^\alpha)\|_{L^{\frac{pt}{t+1}}(B_\rho)} \leq c(\sigma-\rho)^{-\frac{d}{d-1}}\Lambda(B_\sigma)^{\frac{1}{p}}\|u^\alpha\|_{W^{1,s_*}(B_\sigma\setminus B_\rho)}. \tag{2.5}$$

Let $\eta \in C_0^1(B_\sigma)$, $\eta \geq 0$, with $\eta = 1$ in $B_\rho$. First, we rewrite the Caccioppoli inequality (2.3) from Step 1 as inequality for $u^{1+\frac{\beta-1}{p}}$:

$$\left(\frac{p}{p+\beta-1}\right)^p \int \eta^p \lambda(x)|\nabla(u^{1+\frac{\beta-1}{p}})|^p \leq \left(\frac{p}{\beta}\right)^p \int \mu(x)(u^{1+\frac{\beta-1}{p}})^p|\nabla\eta|^p. \tag{2.6}$$

Calling $v := u^{1+\frac{\beta-1}{p}}$, we can estimate the r.h.s. with the help of Lemma 1, yielding

$$\int \eta^p\lambda(x)|\nabla v|^p \leq c\left(\frac{p+\beta-1}{\beta}\right)^p (\sigma-\rho)^{-\frac{pd}{d-1}}\|\mu\|_{L^s(B_\sigma\setminus B_\rho)}\big(\|\nabla v\|^p_{L^{s_*}(B_\sigma\setminus B_\rho)} + \rho^{-p}\|v\|^p_{L^{s_*}(B_\sigma\setminus B_\rho)}\big).$$

Using Hölder inequality with exponents $(\frac{t+1}{t}, t+1)$ and the fact that $\eta = 1$ in $B_\rho$, we see that

$$\|\nabla v\|^p_{L^{\frac{pt}{t+1}}(B_\rho)} \leq \|\lambda^{-1}\|_{L^t(B_\rho)}\|\lambda|\nabla v|^p\|_{L^1(B_\rho)} \leq \|\lambda^{-1}\|_{L^t(B_\rho)}\int \eta^p\lambda(x)|\nabla v|^p.$$

Using that $\frac{1}{2} \leq \rho \leq \sigma \leq 1$, combination of two previous relations yields

$$\|\nabla v\|^p_{L^{\frac{pt}{t+1}}(B_\rho)} \leq c\left(\frac{p+\beta-1}{\beta}\right)^p (\sigma-\rho)^{-\frac{pd}{d-1}}\Lambda(B_\sigma)\|v\|^p_{W^{1,s_*}(B_\sigma\setminus B_\rho)},$$

which after taking $p$-root turns into

$$\|\nabla(u^\alpha)\|_{L^{\frac{pt}{t+1}}(B_\rho)} \leq c(\sigma-\rho)^{-\frac{d}{d-1}}\Lambda(B_\sigma)^{\frac{1}{p}}\|u^\alpha\|_{W^{1,s_*}(B_\sigma\setminus B_\rho)},$$

with $\alpha := 1 + \frac{\beta-1}{p}$.

Step 3. One-step improvement.
First, we note that (1.3) and $t > \frac{1}{p-1}$ imply $\delta := \frac{1}{s_*} - \frac{1}{p}(1+\frac{1}{t}) > 0$. In particular it holds $s_* < \frac{tp}{t+1}$. We claim that there exists $c = c(d,s,t,p)$ such that for $\frac{1}{2} \leq \rho < \sigma \leq 1$ there holds

$$\|u^{\chi\alpha}\|^{\frac{1}{\chi\alpha}}_{W^{1,s_*}(B_\rho)} \leq \left(\frac{c\Lambda(B_\sigma)^{\frac{1}{p}}}{(\sigma-\rho)^{\frac{d}{d-1}}}\right)^{\frac{1}{\chi\alpha}}\|u^\alpha\|^{\frac{1}{\alpha}}_{W^{1,s_*}(B_\sigma)}, \tag{2.7}$$

where $\chi := 1+\delta > 1$. Using Hölder inequality with exponent $\frac{pt}{(t+1)s_*} > 1$ and its dual exponent $\frac{pt}{pt-(t+1)s_*} = \frac{1}{\delta s_*}$ we get

$$\left(\int_{B_\rho}|\nabla(u^{(1+\delta)\alpha})|^{s_*}\right)^{\frac{1}{s_*}} = (1+\delta)\alpha\left(\int_{B_\rho}|\nabla u|^{s_*}u^{(\alpha-1)s_*}u^{\alpha\delta s_*}\right)^{\frac{1}{s_*}} = (1+\delta)\left(\int_{B_\rho}|\nabla(u^\alpha)|^{s_*}u^{\alpha\delta s_*}\right)^{\frac{1}{s_*}}$$

$$\leq (1+\delta)\Big(\int_{B_\rho} |\nabla(u^\alpha)|^{\frac{pt}{t+1}}\Big)^{\frac{t+1}{pt}}\Big(\int_{B_\rho} u^\alpha\Big)^\delta.$$

Combining the above estimate with (2.5) from Step 2, we get (recall $\chi = 1+\delta$)

$$\|\nabla(u^{\chi\alpha})\|_{L^{s_*}(B_\rho)} \leq c(\sigma-\rho)^{-\frac{d}{d-1}}\Lambda(B_\sigma)^{\frac{1}{p}}\|u^\alpha\|^{\chi}_{W^{1,s_*}(B_\sigma)},$$

where we hided $\chi = 1+\delta < \frac{d}{d-1}$ into $c$. In order to have full $W^{1,s_*}(B_\rho)$-norm also on the l.h.s., using $s_* \geq 1$ as well as $\chi < \frac{d}{d-1}$ we can use Sobolev inequality to the effect

$$\|u^{\chi\alpha}\|_{L^{s_*}(B_\rho)} \leq c\|u^\alpha\|_{W^{1,s_*}(B_\rho)},$$

thus obtaining the claim.

Step 4. Iteration.
We iterate the outcome of Step 3. For $\bar\alpha \geq 1$ and $n \in \mathbb{N}$ let $\alpha_n := \bar\alpha\chi^{n-1}$, $\rho_n := \frac{1}{2} + \frac{1}{2^n+1}$, $\sigma_n := \rho_n + \frac{1}{2^{n+1}} = \rho_{n-1}$. Then (2.8) from Step 4 with $\alpha := \alpha_n$ has the form

$$\|u^{\alpha_{n+1}}\|^{\frac{1}{\alpha_{n+1}}}_{W^{1,s_*}(B_{\rho_n})} \leq (c\Lambda(B_1)^{\frac{1}{p}}4^n)^{\frac{1}{\bar\alpha\chi^n}}\|u^{\alpha_n}\|^{\frac{1}{\alpha_n}}_{W^{1,s_*}(B_{\rho_{n-1}})}. \tag{2.8}$$

Using that $L^p$ approximates $L^\infty$ as $p \to \infty$, we see that

$$\begin{aligned}\|u\|_{L^\infty(B_{1/2})} &\leq \Big(\prod_{n=1}^{\infty}(c\Lambda(B_\sigma)^{\frac{1}{p}}4^n)^{\frac{1}{\bar\alpha\chi^n}}\Big)\|u^{\bar\alpha}\|^{\frac{1}{\bar\alpha}}_{W^{1,s_*}(B_1)}\\ &\leq c\Lambda(B_\sigma)^{\frac{1}{p\bar\alpha}\frac{1}{\chi-1}}\|u^{\bar\alpha}\|^{\frac{1}{\bar\alpha}}_{W^{1,s_*}(B_1)},\end{aligned} \tag{2.9}$$

which for $\bar\alpha = 1$ yields the desired claim where we use $\chi = 1+\delta$ and $s_* \leq \frac{tp}{t+1}$.

Step 5. The remaining case $1+\frac{1}{t} < \frac{p}{d-1}$.
Using Fubini theorem, we can choose a generic radius $r_0 \in (\frac{1}{2}, 1)$ such that

$$\|u_+\|^{\frac{pt}{t+1}}_{W^{1,\frac{pt}{t+1}}(S_{r_0})} \leq 2\|u_+\|^{\frac{pt}{t+1}}_{W^{1,\frac{pt}{t+1}}(B_1)}.$$

We test the weak formulation of $-\nabla\cdot a(x,\nabla u) \leq 0$ see (2.1) with the non-negative test function $\phi := (u_+ - \sup_{S_{r_0}} u_+)_+$, which obviously vanishes on $S_{r_0}$ and can be therefore trivially extended by zero to the whole domain $\Omega$. This yields

$$0 \overset{(2.1)}{\geq} \int_{B_{r_0}} a(x,\nabla u)\cdot\nabla\phi = \int_{B_{r_0}} a(x,\nabla\phi)\cdot\nabla\phi \overset{(1.2)}{\geq} \int_{B_{r_0}} \lambda(x)|\nabla\phi|^p.$$

In particular, we see that $\nabla\phi = 0$ a.e. in $B_{r_0}$, hence $\phi \equiv 0$ and thus

$$\|u_+\|_{L^\infty(B_{\frac{1}{2}})} \leq \|u_+\|_{L^\infty(B_{r_0})} \leq \sup_{S_{r_0}} u_+.$$

Using that $\frac{pt}{t+1} > d-1$, which follows from $1+\frac{1}{t} < \frac{p}{d-1}$, we have by Sobolev embedding that $\sup_{S_{r_0}} u_+ \leq c\|u_+\|_{W^{1,\frac{pt}{t+1}}(S_{r_0})}$ for some $c = c(d,p,t) > 0$ which by the above choice of $r_0$ completes the claim.

□

*Proof of Proposition 1.* This follows exactly as in Step 5 of the proof of Theorem 3 using that for $d = 2$ it holds $\sup_{S_{r_0}} u_+ \leq c\|u_+\|_{W^{1,1}(S_{r_0})}$. □

We close this section by deriving from Theorem 3 in the case $s > 1$ an $L^\infty - L^\gamma$ estimate.

**Corollary 1.** *Let $d \geq 2$, $\Omega \subset \mathbb{R}^d$ and $p \in (1, \infty)$. Moreover, let $s \in (1, \infty]$ and $t \in (\frac{1}{p-1}, \infty]$ satisfy* (1.3). *Let $a : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ be a Caratheodory function with $a(\cdot, 0) \equiv 0$ such that $\lambda$ and $\mu$ defined in* (1.2) *satisfy $\mu \in L^s(\Omega)$ and $\frac{1}{\lambda} \in L^t(\Omega)$. Then, any weak subsolution $u$ of* (1.1) *and any $\gamma > 0$ there exists $c = c(\gamma, d, p, s, t) \in [1, \infty)$ such that for any ball $B_R \subset \Omega$*

$$\sup_{B_{R/2}} u \leq c\Lambda(B_R)^{\frac{1}{\gamma}\frac{s}{s-1}(1+\frac{1}{\delta})}\Big(\fint_{B_R} u_+^\gamma\Big)^{\frac{1}{\gamma}}.$$

*Proof.* Without loss of generality we consider $R = 1$ and suppose that $B_1 \Subset \Omega$. Caccioppoli inequality (2.6) with $\beta = 1 + p(\alpha - 1)$ for $\alpha \geq 1$ and $\eta \in C_c^1(B_1)$ with $\eta = 1$ on $B_{\frac{1}{2}}$ and $|\nabla\eta| \leq 2$ and Hölder inequality yield

$$\begin{aligned}\|\nabla(u_+^\alpha)\|^p_{L^{\frac{pt}{t+1}}(B_{1/2})} &\leq \|\lambda^{-1}\|_{L^t(B_1)}\int_{B_1}\eta^p\lambda|\nabla(u_+^\alpha)|^p \leq (2p)^p\|\lambda^{-1}\|_{L^t(B_1)}\int_{B_1}\mu u_+^{\alpha p}\\ &\leq (2p)^p\|\lambda^{-1}\|_{L^t(B_1)}\|\mu\|_{L^s(B_1)}\|u_+^\alpha\|_{L^{\frac{s}{s-1}p}(B_1)}.\end{aligned}$$

The above inequality combined with $\frac{tp}{t+1} \leq p \leq \frac{sp}{s-1}$ implies $\|u_+^\alpha\|^{\frac{1}{\alpha}}_{W^{1,\frac{tp}{t+1}}(B_{1/2})} \leq c\Lambda(B_1)^{\frac{1}{\alpha p}}\|u_+\|_{L^{\frac{\alpha ps}{s-1}}(B_1)}$ (note that $1 \leq \Lambda(B_r)$) for some $c = c(d, p) \in [1, \infty)$. Hence, we have in combination with (2.9) that

$$\|u_+\|_{L^\infty(B_{1/4})} \leq c\Lambda(B_1)^{\frac{1}{\alpha p}(1+\frac{1}{\delta})}\|u_+\|_{L^{\frac{\alpha ps}{s-1}}(B_1)}, \tag{2.10}$$

where $c = c(\alpha, d, p, t, s) \in [1, \infty)$.

From estimate (2.10) the claim follows by routine arguments and we only sketch the idea (see [7, Proof of Theorem 3.3, Step 2] for precise arguments in the case $p = 2$). By scaling and translation, we deduce from (2.10) that for all $\rho > 0$ and $x \in B_1$ such that $B_\rho(x) \subset B_1$ it holds for $\alpha \geq 1$

$$\|u_+\|_{L^\infty(B_{\rho/4}(x))} \leq c\Lambda(B_\rho(x))^{\frac{1}{\alpha p}(1+\frac{1}{\delta})}\rho^{-\frac{d}{p}(1-\frac{1}{s})}\|u_+\|_{L^{\frac{\alpha ps}{s-1}}(B_\rho(x))},$$

where $c$ is as in (2.10). Combining the above estimate with a simple covering argument, we obtain that there exists $c = c(\alpha, d, p, s, t) \in [1, \infty)$ such that for all $\theta \in (0, 1)$ and $r \in (0, 1]$ it holds

$$\|u_+\|_{L^\infty(B_{\theta r})} \leq c\Lambda(B_r)^{\frac{1}{\alpha p}(1+\frac{1}{\delta})}(1-\theta)^{-\kappa}r^{-d\frac{s-1}{\alpha ps}}\|u_+\|_{L^{\frac{\alpha ps}{s-1}}(B_r)},$$

where $\kappa := \frac{d}{\alpha p}((\frac{1}{t} + \frac{1}{s})(1 + \frac{1}{\delta}) + 1 - \frac{1}{s})$ which is the claim for all $\gamma \geq \frac{ps}{s-1}$ (by choosing $\alpha = \frac{s-1}{ps}\gamma$). The claim for $\gamma \in (0, \frac{ps}{s-1})$ follows by a standard interpolation and iteration argument see e.g., the textbook reference [33, p. 75] in the uniformly elliptic case or as mentioned above [7, Proof of Theorem 3.3, Step 2] for a closely related setting. □

## 3. Counterexample, proof of Theorem 2

*Proof of Theorem 2.* The following construction is very much inspired by a construction in [27] in the linear case, that is $p = 2$, and $d = 4$ (which was already extended to $d \geq 3$ in [40]).

Let $d \geq 3$. Throughout the proof, we set

$$x = (x_1, \dots, x_d) = (x_1, x') \quad \text{and} \quad |x'| = \sqrt{\sum_{j=2}^{d} x_j^2}.$$

For any $p \in (1, \infty)$ and $\theta \in [0, 1]$, we define $\lambda_\theta(x) := \omega_\theta(|x'|)$ where $\omega_\theta : (0, 1) \to \mathbb{R}_+$ is defined as

$$\omega_\theta(r) = \begin{cases} (i+1)^{(p-1)\theta} 4^{-pi\theta} & \text{when } r \in [\frac{1}{2}4^{-i}, 4^{-i}), \\ ((i+1)^{-(p-1)} 4^{pi})^{1-\theta} & \text{when } r \in [\frac{1}{4}4^{-i}, \frac{1}{2}4^{-i}) \end{cases} \tag{3.1}$$

for $i \in \mathbb{N}$. We will construct an explicit subsolution to $-\nabla \cdot (\lambda_\theta |\nabla v|^{p-2} \nabla v) = 0$, which is of the form

$$v(x) = e^{\alpha x_1} \phi(|x'|) \tag{3.2}$$

for some parameter $\alpha = \alpha(d, p) > 0$ and $\phi : (0, 1) \to \mathbb{R}$ is defined by

$$\phi(r) = \begin{cases} i + \frac{\eta_i}{2^Q - 1}((4^i r)^{-Q} - 1) & \text{when } r \in [\frac{1}{2}4^{-i}, 4^{-i}), \\ (i+1) - (1-\eta_i)(4^{i+1} r - 1)^2 & \text{when } r \in [\frac{1}{4}4^{-i}, \frac{1}{2}4^{-i}) \end{cases}, \quad \text{with} \quad Q = \begin{cases} \max\{d-3, 1\} & \text{if } p \geq 2 \\ \frac{d-2}{p-1} - 1 & \text{if } 1 < p < 2 \end{cases} \tag{3.3}$$

where $\eta_i \in [0, 1]$ will be specified below. Note that $Q > 0$ and $\phi$ is continuous by definition. We choose $\eta_i \in (0, 1)$ such that the flux $\lambda_\theta |\nabla v|^{p-2} \nabla v$ is continuous at $|x'| = \frac{1}{2}4^{-i}$ for every $i \in \mathbb{N}$. More precisely, we set $\eta_i$ to be the largest constant (in $[0, 1]$) satisfying

$$F_i(\eta_i) = 0, \tag{3.4}$$

where $F_i : (0, 1] \to \mathbb{R}$ is given by

$$\begin{aligned} F_i(\eta) := & \sqrt{(\alpha(i+\eta)4^{-i})^2 + (C_Q \eta)^2}^{\,p-2} C_Q \eta \\ & - \sqrt{(\alpha(i+\eta)(i+1)^{-1})^2 + (8(1-\eta)4^i (i+1)^{-1})^2}^{\,p-2} 8(1-\eta) 4^{2i} (i+1)^{-1} \end{aligned}$$

with

$$C_Q = Q \frac{2^{Q+1}}{2^Q - 1}.$$

Note that $\eta_i$ is well-defined since $F_i : (0, 1) \to \mathbb{R}$ is continuous with

$$\lim_{\eta \to 0} F_i(\eta) = -\sqrt{(\alpha i)^2 + (2 \cdot 4^{i+1})^2}^{\,p-2} 8 \cdot 4^{2i} (i+1)^{-(p-1)} < 0$$

and

$$\lim_{\eta \to 1} F_i(\eta) = \sqrt{(\alpha(i+1)4^{-i})^2 + C_Q^2}^{\,p-2} C_Q > 0.$$

The definition of $\eta_i$ is rather implicit and we provide now some explicit bounds on $\eta_i$ which will be useful for later computations. We distinguish two cases. For $p \geq 2$ and $\alpha \geq C_Q$, we have that

$$\exists j = j(d,p) \geq 2 \text{ such that } \forall i \geq j\colon \quad \eta_i \geq 1 - 8^{-1}(4^{p-2}C_Q)4^{-2i}(i+1) =: \underline{\eta}_i. \tag{3.5}$$

Indeed, let $j = j(d,p) \geq 2$ be such that $\underline{\eta}_i \in (0,1)$ for all $i \geq j$. By definition of $\eta_i$, it suffices to show that $F_i(\underline{\eta}_i) \leq 0$ for $i \geq j$. We have

$$\begin{aligned} F_i(\underline{\eta}_i) \leq &\ \sqrt{(\alpha(i+1)4^{-i})^2 + C_Q^2}^{\,p-2} C_Q - \sqrt{(\alpha i/(i+1))^2}^{\,p-2}(4^{p-2}C_Q) \\ = &\ \sqrt{((i+1)4^{-i})^2 + (C_Q/\alpha)^2}^{\,p-2}\alpha^{p-2}C_Q - \alpha^{p-2}\sqrt{(i/(i+1))^2}^{\,p-2}(4^{p-2}C_Q) \\ \leq &\ \alpha^{p-2}(2^{p-2}C_Q - 2^{-(p-2)}(4^{p-2}C_Q)) = 0, \end{aligned}$$

where we used for the last inequality $(i+1)4^{-i} \leq 1$ and $i/(i+1) \geq \frac{1}{2}$ for $i \geq 1$ and $\alpha \geq C_Q$.

In the case $p \in (1,2)$, we have for $\alpha \geq 2^{\frac{2-p}{p-1}}C_Q$ that

$$\exists j = j(\alpha,d,p) \geq 2 \text{ such that } \forall i \geq j\colon \quad \eta_i \geq 1 - 8^{-1}\alpha 4^{-2i}(i+1) =: \overline{\eta}_i. \tag{3.6}$$

Indeed, this follows as above from

$$F_i(\overline{\eta}_i) \leq C_Q^{p-1} - \sqrt{\alpha^2 + (\alpha 4^{-i})^2}^{\,p-2}\alpha \leq C_Q^{p-1} - \alpha^{p-1}2^{p-2} \leq 0.$$

Step 1. We show that for every $\alpha \geq \max\{1, 2^{\frac{2-p}{p-1}}\}C_Q$, the function $v$ defined in (3.2) has finite energy, that is $\int_{B_1} \lambda_\theta(|v|^p + |\nabla v|^p) < \infty$ provided $(1-\theta)p < d-1$.

We show first $\int_{B_1} \lambda_\theta |v|^p < \infty$. For this, we observe that $0 \leq \phi(r) \leq \log(4/r)$ for all $r \in (0,1)$. Indeed, $\phi \geq 0$ is clear from the definition (3.3) and for $r \in [\frac{1}{4}4^{-i}, 4^{-i})$, we have

$$\phi(r) \leq i+1 = \log_4(4^{i+1}) \leq \log_4(\tfrac{4}{r}) \leq \log(\tfrac{4}{r}).$$

Similarly, we get

$$\omega_\theta(r) \leq \begin{cases} ((2r)^p \log(4/r)^{p-1})^\theta & \text{when } r \in [\frac{1}{2}4^{-i}, 4^{-i}), \\ (r^p \log(2/r)^{p-1})^{-(1-\theta)} & \text{when } r \in [\frac{1}{4}4^{-i}, \frac{1}{2}4^{-i}) \end{cases}. \tag{3.7}$$

Hence, there exists $C = C(\alpha,d,p) > 0$ such that

$$\int_{B_1} \lambda_\theta v^p \, dx \leq C \int_0^1 r^{-(1-\theta)p} \log(2/r)^{p-(1-\theta)(p-1)} r^{d-2} \, dr < \infty,$$

where the last integral is finite since $(1-\theta)p < d-1$.

Next, we show $\int_{B_1} \lambda_\theta |\nabla v|^p < \infty$. For this we compute the gradient of $v$:

$$\nabla v = \begin{pmatrix} \alpha\phi \\ \phi' \frac{x'}{|x'|} \end{pmatrix} e^{\alpha x_1} \quad \text{and} \quad |\nabla v| = \sqrt{\alpha^2\phi^2 + \phi'^2}\, e^{\alpha x_1}. \tag{3.8}$$

Moreover, we compute

$$\phi'(r) = \begin{cases} -Q\frac{\eta_i}{2^Q - 1}(4^i r)^{-Q} r^{-1} & \text{when } r \in (\frac{1}{2}4^{-i}, 4^{-i}), \\ -2(1-\eta_i)4^{i+1}(4^{i+1}r - 1) & \text{when } r \in (\frac{1}{4}4^{-i}, \frac{1}{2}4^{-i}) \end{cases} \tag{3.9}$$

and for later usage

$$\phi''(r) = \begin{cases} Q(Q+1)\frac{\eta_i}{2^Q-1}(4^i r)^{-Q} r^{-2} & \text{when } r \in (\frac{1}{2}4^{-i}, 4^{-i}), \\ -2(1-\eta_i)4^{2(i+1)} & \text{when } r \in (\frac{1}{4}4^{-i}, \frac{1}{2}4^{-i}) \end{cases}. \tag{3.10}$$

From (3.5) and (3.6), we obtain that there exists $C = C(\alpha, d, p) > 0$ such that $0 \leq 1 - \eta_i \leq C4^{-2i}(i+1)$ for $i \geq j(\alpha, d, p)$ and thus in combination with (3.9) there exists $C = C(\alpha, d, p) > 0$ such that

$$|\phi'(r)| \leq C \begin{cases} r^{-1} & \text{when } r \in (\frac{1}{2}4^{-i}, 4^{-i}), \\ \log(2/r)r & \text{when } r \in (\frac{1}{4}4^{-i}, \frac{1}{2}4^{-i}) \end{cases}$$

for all $i \geq j$. Hence, we find $C = C(\alpha, d, p) > 0$ such that

$$\int_{B_1} \lambda_\theta |\nabla v|^p \leq C + C\int_0^1 \Big((r^p \log(2/r)^{p-1})^\theta r^{-p} + (r^p \log(2/r)^{p-1})^{-(1-\theta)}(\log(2/r)r)^p\Big) r^{d-2}\, dr < \infty,$$

where we use again $(1-\theta)p < d-1$. Finally, it is easy to check that the sequence $(v_k)_k$ defined by $v_k(x) = e^{\alpha x_1}\phi_k(|x'|)$ with $\phi_k(x) = \phi(x)$ if $|x| > 4^{-k}$ and $\phi_k(x) = k$ if $|x'| \leq 4^{-k}$ is a sequence of Lipschitz functions satisfying $\lim_{k\to\infty} \int_{B_1} \lambda_\theta(|v - v_k|^p + |\nabla v - \nabla v|^p) \to 0$ as $k \to \infty$ and a straightforward regularization shows that $v$ in $H^{1,p}(B_1, a)$ with $a(x, \xi) := \lambda_\theta(x)|\xi|^{p-2}\xi$.

Step 2. We claim that there exist $\alpha_0 = \alpha_0(d, p) \geq 1$ such that for every $\alpha \geq \alpha_0$ there exists $\rho = \rho(\alpha, d, p) \in (0, 1]$ such that $v$ defined in (3.2) is a weak subsolution in $\{x \in B_1 : \delta < |x'| < \rho\}$ for all $\delta > 0$.

For this, we observe first that by (3.8) the nonlinear strain $|\nabla v|^{p-2}\nabla v$ of $v$ is given by

$$|\nabla v|^{p-2}\nabla v = \sqrt{\alpha^2\phi^2 + \phi'^2}^{p-2} \begin{pmatrix} \alpha\phi \\ \phi' \frac{x'}{|x'|} \end{pmatrix} e^{\alpha(p-1)x_1}. \tag{3.11}$$

Introducing the notation $M_{2i} = B_1 \cap \{\frac{1}{2}4^{-i} < |x'| < 4^{-i}\}$ and $M_{2i+1} = B_1 \cap \{\frac{1}{4}4^{-i} < |x'| < \frac{1}{2}4^{-i}\}$, we obtain with help of integrating by parts

$$\begin{aligned} \int_{B_1} \lambda_\theta |\nabla v|^{p-2}\nabla v \cdot \nabla\varphi &= \sum_{i\in\mathbb{N}} \int_{M_i} \omega_\theta |\nabla v|^{p-2}\nabla v \cdot \nabla\varphi \\ &= \sum_{i\in\mathbb{N}} -\int_{M_i} \omega_\theta \nabla \cdot (|\nabla v|^{p-2}\nabla v)\varphi + \int_{\partial M_i} \omega_\theta |\nabla v|^{p-2}\nabla v \cdot \nu\varphi \\ &= \sum_{i\in\mathbb{N}} -\int_{M_i} \omega_\theta \nabla \cdot (|\nabla v|^{p-2}\nabla v)\varphi + \int_{\partial M_i} \omega_\theta \sqrt{\alpha^2\phi^2 + \phi'^2}^{p-2}\phi' e^{(p-1)\alpha x_1}\varphi, \end{aligned}$$

where $\nu$ denotes the outer unit normal to $M_i$ that is $\nu = (0, x'/|x'|)$. Hence, it suffices to show that there exists $\alpha_0 > 0$ such that for all $\alpha \geq \alpha_0$ there exists $j = j(\alpha, d, p) \geq 2$ such that

(i) $v$ satisfies $\nabla \cdot (|\nabla v|^{p-2}\nabla v) \geq 0$ in the classical sense in each shell $M_i$ for all $i \geq j$;

(ii) the flux has only nonnegative jumps at the interfaces, that is

$$(\omega_\theta \sqrt{\alpha^2\phi^2 + \phi'^2}^{p-2}\phi')(\gamma_-) := \lim_{\substack{r\to\gamma \\ r<\gamma}} (\omega_\theta \sqrt{\alpha^2\phi^2 + \phi'^2}^{p-2}\phi')(r)$$

$$\leq \lim_{\substack{r\to\gamma\\ r>\gamma}} (\omega_\theta \sqrt{\alpha^2\phi^2 + \phi'^2}^{p-2}\phi')(r) =: (\omega_\theta|\nabla v|^{p-2}\phi')(\gamma_+)$$

for all $\gamma \in \bigcup_{i\in\mathbb{N}, i\geq j}\{4^{-i}\} \cup \{\frac{1}{2}4^{-i}\}$.

*Substep 2.1.* Argument for (i). Let $\alpha \geq 1$ be such that

$$\alpha \geq \alpha_0(p,d) := \max\Big\{1, C_Q, 2^{\frac{2-p}{p-1}}C_Q, 2^p\sqrt{C_Q(1+\frac{d-2}{p-1})}, 8\frac{d-1}{p-1}\Big\} \tag{3.12}$$

and let $j = j(\alpha,d,p) \geq 2$ be such that the estimates (3.5) and (3.6) are valid.

We show that $v$ with $\alpha$ as above, satisfies $\nabla\cdot(|\nabla v|^{p-2}\nabla v) \geq 0$ in the classical sense in each shell $M_i$ for all $i \geq j$. We compute with help of (3.11) on $M_i$

$$\begin{aligned}
&\nabla\cdot(|\nabla v|^{p-2}\nabla v)\\
&=\Big(\alpha^2(p-1)\sqrt{\alpha^2\phi^2+\phi'^2}^{p-2}\phi + (p-2)\sqrt{\alpha^2\phi^2+\phi'^2}^{p-4}|\phi'|^2(\alpha^2\phi+\phi'')\\
&\quad + \sqrt{\alpha^2\phi^2+\phi'^2}^{p-2}(\phi''+(d-2)\frac{\phi'}{|x'|})\Big)e^{\alpha(p-1)x_1}\\
&=\sqrt{\alpha^2\phi^2+\phi'^2}^{p-4}\Big(\alpha^2(p-1)(\alpha^2\phi^2+\phi'^2)\phi + (p-2)\phi'^2(\alpha^2\phi+\phi'')\\
&\qquad + (\alpha^2\phi^2+\phi'^2)(\phi''+(d-2)\frac{\phi'}{|x'|})\Big)e^{\alpha(p-1)x_1}.
\end{aligned} \tag{3.13}$$

We show that $v$ is a classical subsolution in $M_{2i+1}$. Note that $\phi > 0$ and $\phi', \phi'' < 0$ on $(\frac{1}{4}4^{-i}, \frac{1}{2}4^{-i})$.

We consider first the case $p \geq 2$. From $\phi > 0$, $\phi'' < 0$ and $\phi'^2 \leq \alpha^2\phi^2 + \phi'^2$, we deduce

$$(p-2)\phi'^2(\alpha^2\phi+\phi'') \geq (p-2)(\alpha^2\phi^2+\phi'^2)\phi''$$

and in combination with (3.13) that

$$\nabla\cdot(|\nabla v|^{p-2}\nabla v) \geq \sqrt{\alpha^2\phi^2+\phi'^2}^{p-2}(\alpha^2(p-1)\phi + (p-1)\phi'' + (d-2)\frac{\phi'}{|x'|})e^{\alpha(p-1)x_1}.$$

Hence, $\nabla\cdot(|\nabla v|^{p-2}\nabla v) \geq 0$ on $M_{2i+1}$ is equivalent to

$$\alpha^2(p-1)\phi(r) + (p-1)\phi''(r) + (d-2)\frac{\phi'(r)}{r} \geq 0 \quad \text{for all } r \in (\tfrac{1}{4}4^{-i}, \tfrac{1}{2}4^{-i}),$$

which is by (3.3), (3.9) and (3.10) valid if and only if

$$\alpha^2(p-1)\Big(i+1-(1-\eta_i)(4^{i+1}r-1)^2\Big) - 2(1-\eta_i)4^{i+1}((p-1)4^{i+1} + r^{-1}(d-2)(4^{i+1}r-1)) \geq 0$$

for all $r \in (\frac{1}{4}4^{-i}, \frac{1}{2}4^{-i})$. We estimate with help of $\eta_i \in [0,1]$,

$$\begin{aligned}
&\alpha^2(p-1)\Big((i+1)-(1-\eta_i)(4^{i+1}r-1)^2\Big) - 2(1-\eta_i)4^{i+1}((p-1)4^{i+1} + r^{-1}(d-2)(4^{i+1}r-1))\\
&\geq \alpha^2(p-1)i - 2(1-\eta_i)4^{2(i+1)}(p-1+d-2).
\end{aligned}$$

The lower bound on $\eta_i \geq \underline{\eta}_i$, see (3.5), implies $1-\eta_i \leq 1-\underline{\eta}_i \leq 8^{-1}(4^{p-2}C_Q)4^{-2i}(i+1)$ and thus

$$\alpha^2(p-1)i - 2(1-\eta_i)4^{2(i+1)}(p-1+d-2) \geq \alpha^2(p-1)i - 4^{p-1}C_Q(i+1)(p+d-3) \geq 0, \qquad (3.14)$$

where the last inequality is valid since $(i+1)/i \leq 2$ for $i \geq 1$ and $\alpha^2 \geq 4^{p-1}C_Q 2(1+\frac{d-2}{p-1})$ (which is ensured by $\alpha \geq \alpha_0$, see (3.12)).

Next, we consider the case $p \in (1,2)$. We deduce from (3.13) with $p-2<0$ and $\phi>0$, $\phi', \phi''<0$ that

$$\begin{aligned}&\nabla\cdot(|\nabla v|^{p-2}\nabla v)\\ &\geq \sqrt{\alpha^2\phi^2+\phi'^2}^{p-4}\left((\alpha^2\phi^2+\phi'^2)\left(\alpha^2(p-1)\phi+\phi''+(d-2)\frac{\phi'}{|x'|}\right)-(2-p)\phi'^2\phi\right)e^{\alpha(p-1)x_1}. \qquad (3.15)\end{aligned}$$

Similar computations as above yield for all $r \in (\frac14 4^{-i}, \frac12 4^{-i})$ and $p \in (1,2)$

$$\begin{aligned}\alpha^2(p-1)\phi(r)+\phi''(r)+(d-2)\frac{\phi'(r)}{r} &\geq \alpha^2(p-1)(i+\eta_i)-2(1-\eta_i)4^{2(i+1)}(d-1)\\ &\overset{(3.6)}{\geq} \alpha^2(p-1)i-4\alpha(i+1)(d-1)\geq 1\end{aligned}$$

where the last inequality is valid for all $i \geq 1$ and $\alpha \geq 8\frac{d-1}{p-1}$ (see (3.12)). Inserting this into (3.15), we obtain (using $2-p \leq 1$)

$$\begin{aligned}\nabla\cdot(|\nabla v|^{p-2}\nabla v) &\geq \sqrt{\alpha^2\phi^2+\phi'^2}^{p-4}\left(\alpha^2\phi^2-\phi'^2\phi\right)e^{\alpha(p-1)x_1}\\ &\overset{(3.9),(3.6)}{\geq} \sqrt{\alpha^2\phi^2+\phi'^2}^{p-4}\phi\left(\alpha^2\phi-(\alpha(i+1)4^{-i})^2\right)e^{\alpha(p-1)x_1}\geq 0,\end{aligned}$$

where we use in the last inequality that $4^{-2i}(i+1)^2 \leq 1$ and $\phi \geq 1$ on $(\frac14 4^{-i}, \frac12 4^{-i})$ with $i \geq 1$.

Now, we show that $v$ is a classical subsolution in $M_{2i}$. In view of (3.13) it suffices to show that for all $r \in (\frac12 4^{-i}, 4^{-i})$ it holds

$$\alpha^4(p-1)\phi^3(r)+\alpha^2(2p-3)\phi(r)\phi'^2(r)+\phi'^2((p-1)\phi''(r)+\frac{d-2}{r}\phi'(r))+\alpha^2\phi^2(r)(\phi''(r)+\frac{d-2}{r}\phi'(r))\geq 0 \qquad (3.16)$$

For $p \geq \frac32$, we obviously have

$$\alpha^4(p-1)\phi^3(r)+\alpha^2(2p-3)\phi(r)\phi'^2(r)\geq 0 \qquad \text{for all } r\in(\tfrac12 4^{-i}, 4^{-i}).$$

Let us first consider $p \geq 2$. In the case $d \geq 4$, the choice of $\phi$ ensures

$$\forall r\in(\frac12 4^{-i},4^{-i}):\quad \phi''(r)+\frac{d-2}{r}\phi'(r)=0 \quad\text{and}\quad (p-1)\phi''(r)+\frac{d-2}{r}\phi'(r)=(p-2)\phi''(r)\geq 0$$

and similarly for $d=3$ that $\phi''(r)+\frac{d-2}{r}\phi'(r)=\frac12\phi''(r)\geq 0$ and $(p-1)\phi''(r)+\frac{d-2}{r}\phi'(r)\geq 0$. Altogether, we have that (3.16) is valid for all $r \in (\frac12 4^{-i}, 4^{-i})$ provided $p \geq 2$.

Next, we consider the case $p \in (1,2)$. The choice of $\phi$ ensures

$$\forall r\in(\frac12 4^{-i},4^{-i}):\quad (p-1)\phi''(r)+\frac{d-2}{r}\phi'(r)=0 \quad\text{and}\quad \phi''(r)+\frac{d-2}{r}\phi'(r)=(2-p)\phi''(r)\geq 0.$$

Using the above two identities, we see that (3.16) is equivalent to

$$\alpha^4(p-1)\phi^3(r) + \alpha^2(2p-3)\phi(r)\phi'^2(r) + \alpha^2\phi^2(r)(2-p)\phi''(r) \geq 0$$

and thus it suffices to show

$$\alpha^2(2p-3)\phi\phi'^2 + \alpha^2\phi^2(2-p)\phi'' \geq 0.$$

For $p \in [\frac{3}{2}, 2]$ the above inequality directly follows from $\phi, \phi'' \geq 0$ and it is left to consider $p \in (1, \frac{3}{2})$ in which case the above inequality is equivalent to

$$\frac{3-2p}{2-p}\frac{\phi'^2}{\phi''} \leq \phi.$$

The above inequality is valid on $(\frac{1}{2}4^{-i}, 4^{-i})$ provided $i \geq 2$. Indeed, this follows from $\phi \geq i$ on $(\frac{1}{2}4^{-i}, 4^{-i})$ and

$$\frac{3-2p}{2-p}\frac{\phi'^2}{\phi''} \leq \frac{3-2p}{2-p}\frac{Q}{Q+1}\frac{\eta_i}{2^Q-1}2^Q \leq 2\frac{Q}{Q+1} \leq 2.$$

*Substep 2.2.* Argument for (ii). Let $\alpha \geq 1$ and $j = j(\alpha, d, p) \geq 2$ be as in Substep 2.1.

In view of (3.8), we need to show that for all $\gamma \in \bigcup_{i\in\mathbb{N}, i\geq j}\{4^{-i}\} \cup \{\frac{1}{2}4^{-i}\}$ it holds

$$(\omega_\theta \sqrt{\alpha^2\phi^2 + \phi'^2}^{p-2}\phi')(\gamma_+) \geq (\omega_\theta \sqrt{\alpha^2\phi^2 + \phi'^2}^{p-2}\phi')(\gamma_-). \tag{3.17}$$

For $\gamma \in \bigcup_{i\in\mathbb{N}}\{4^{-i}\}$, we directly observe that

$$(\omega_\theta \sqrt{\alpha^2\phi^2 + \phi'^2}^{p-2}\phi')(\gamma_+) = 0 > (\omega_\theta \sqrt{\alpha^2\phi^2 + \phi'^2}^{p-2}\phi')(\gamma_-).$$

Moreover, the definition of $\eta_i$ via (3.4) ensures that (3.17) holds as an equality for all $\gamma \in \bigcup_{i\in\mathbb{N}, i\geq j}\{\frac{1}{2}4^{-i}\}$ which finishes the argument.

Step 3. Let $1 < p < \infty$ and $\theta \in [0, 1]$ be such that $(1-\theta)p < d-1$. Let $\alpha \geq \alpha_0$ and $\rho = \rho(\alpha, d, p) \in (0, 1)$ be as in Step 2. We show that $v$ is a weak subsolution on $\Omega_\rho := B_1 \cap \{|x'| < \rho\}$.

We follow a similar reasoning as in [27]. For $k \in \mathbb{N}$, let $\psi_k \in C^1(\mathbb{R}; [0, 1])$ be a cut-off function satisfying

$$\psi_k = 0 \quad \text{on } [0, \tfrac{1}{2}4^{-k}], \quad \psi_k \equiv 1 \quad \text{on } [4^{-k}, 1], \quad \|\psi_k'\|_{L^\infty(0,1)} \leq 4^{k+1}$$

and we define $\varphi_k \in C^1(B_1)$ by $\varphi_k(x) = \psi_k(|x'|)$. For every $\eta \in C_c^1(\Omega_\rho)$ with $\eta \geq 0$, we have

$$\begin{aligned}\int_{\Omega_\rho} \lambda_\theta |\nabla v|^{p-2}\nabla v \cdot \nabla\phi \, dx &= \int_{\Omega_\rho} \lambda_\theta |\nabla v|^{p-2}\nabla v \cdot (\nabla((1-\varphi_k)\eta) + \nabla(\varphi_k\eta)) \, dx \\ &\leq \int_{\Omega_\rho} \lambda_\theta |\nabla v|^{p-2}\nabla v \cdot \nabla((1-\varphi_k)\eta) \, dx,\end{aligned} \tag{3.18}$$

where we use that $0 \leq \varphi_k\eta \in C_c^1(\Omega_\rho \setminus \Omega_{4^{-k-1}})$ and that by Step 2 $v$ is a subsolution on $\Omega_\rho \setminus \Omega_\delta$ for every $\delta \in (0, \rho)$. It remains to show that the integral on the right-hand side in (3.18) vanishes as $k \to \infty$. Note that $0 \leq 1 - \varphi_k \leq 1$ and $1 - \varphi_k \equiv 0$ on $\Omega_\rho \setminus \Omega_{4^{-k}}$. Hence, with help of the product rule, we obtain

$$\left|\int_{\Omega_\rho} \lambda_\theta |\nabla v|^{p-2}\nabla v \cdot \nabla((1-\varphi_k)\eta) \, dx\right| \leq \int_{\Omega_{4^{-k}}} \lambda_\theta |\nabla v|^{p-1}|\nabla\eta| \, dx + \int_{\Omega_\rho} \eta\lambda_\theta |\nabla v|^{p-2}|\nabla v \cdot \nabla\varphi_k| \, dx.$$

By dominated convergence, the first term on the right-hand side converges to zero as $k$ tends to $\infty$ (recall that we showed in Step 1 that $\lambda_\theta|\nabla v|^p \in L^1(B_1)$). To estimate the remaining integral we use $|\nabla v\cdot\nabla\varphi_k| = |\phi'||\nabla\varphi_k|e^{\alpha x_1} \le C4^{k+1}|\phi'|$ for some $C = C(\alpha) > 0$ on the set $\{|x'| \in (\frac12 4^{-k}, 4^{-k})\}$ and $\nabla v\cdot\nabla\varphi_k = 0$ otherwise. Hence, we have that $|\nabla v|^{p-2}|\nabla v\cdot\nabla\varphi_k| \le C4^{k+1}|x'|^{-(p-1)}$ on $\{|x'| \in (\frac12 4^{-k}, 4^{-k})\}$ and thus we obtain (using $\lambda_\theta = (k+1)^{\theta(p-1)}(2|x'|)^{p\theta}$ on $\{|x'| \in (\frac12 4^{-k}, 4^{-k})\}$, see (3.1))

$$
\begin{aligned}
&\int_{\Omega_\rho} \eta\lambda_\theta|\nabla v|^{p-2}|\nabla v\cdot\nabla\varphi_k|\,dx\\
\le& C\|\eta\|_{L^\infty(B_1)}4^{k+1}(k+1)^{\theta(p-1)}\int_{\frac12 4^{-k}}^{4^{-k}} r^{-(p-1)}r^{p\theta}r^{d-2}\,dr\\
=& C\|\eta\|_{L^\infty(B_1)}4^{k+1}(k+1)^{\theta(p-1)}\frac{1}{d-p(1-\theta)}4^{-k(d-p(1-\theta))}\Big(1-2^{-(d-p(1-\theta)}\Big)\overset{k\to\infty}{\to} 0,
\end{aligned}
$$

where we use $p(1-\theta) < d-1$ the assumption and thus $d - p(1-\theta) > 1$.

Step 4. Conclusion.

*Substep 4.1.* We consider the case $1 + \frac{1}{d-2} < p < \infty$. Let $s > 1$ and $t > \frac{1}{p-1}$ be such that $\frac1s + \frac1t = \frac{p}{d-1}$ and $\frac{t}{t+1}p < d-1$. We claim that there exist $0 \le \lambda \in L^s(B_1)$ with $\lambda^{-1} \in L^t(B_1)$ and an unbounded weak subsolution to (1.5). We set $\theta = \frac1t\frac{d-1}{p}$ and observe that $\frac1s + \frac1t = \frac{p}{d-1}$ implies $\theta \in [0,1]$ and $1-\theta = \frac1s\frac{d-1}{p}$. Moreover, the restriction $\frac{t}{t+1}p < d-1$ in the form $p < (1+\frac1t)(d-1)$ ensures

$$
(1-\theta)p = (1-\frac1t\frac{d-1}{p})p = (p - \frac1t(d-1)) < d-1.
$$

Hence, in view of Steps 1–3, there exist the function $v$ defined in (3.2) with $\alpha = \alpha_0 = \alpha_0(p,d) \ge 1$ such that $v$ is an unbounded weak subsolution to

$$
-\nabla\cdot(\lambda_\theta|\nabla v|^{p-2}\nabla v) = 0 \qquad \text{in } B(0,\rho) \text{ with } \rho = \rho(d,p) \in (0,1],
$$

where $\lambda_\theta(x) = \omega_\theta(|x'|)$, *cf.* (3.1). Appealing to (3.7), we have that there exists $C = C(d,p) > 0$ such that

$$
\begin{aligned}
\|\lambda_\theta\|_{L^s(B_1)} &\le C\Big(\int_0^1 (r^{-p}\log(2/r)^{-(p-1)})^{\frac{d-1}{p}}r^{d-2}\,dr\Big)^{\frac1s}\\
&= C\Big(\int_0^1 r^{-1}\log(2/r)^{-(1-\frac1p)(d-1)}\,dr\Big)^{\frac1s} < \infty
\end{aligned}
$$

where we use that $p > 1 + \frac{1}{d-2}$ implies $(1-\frac1p)(d-1) > 1$. Similarly, we have

$$
\|\lambda_\theta^{-1}\|_{L^t(B_1)} \le C\Big(\int_0^1 r^{-1}\log(2/r)^{-(1-\frac1p)(d-1)}\,dr\Big)^{\frac1t} < \infty.
$$

Finally, we observe that by a simple scaling argument namely considering $\tilde v(x) = v(x/\rho)$ and $\lambda(x) := \lambda_\theta(x/\rho)$ we find that $\tilde v$ is a weak subsolution to (1.5) in $B_1$ and $\lambda$ satisfies $\lambda \in L^s(B_1)$ and $\lambda^{-1} \in L^t(B_1)$.

*Substep 4.2.* We consider $1 < p \le 1 + \frac{1}{d-2}$. Let $s$ and $t$ be as in the statement of the theorem. Clearly, we find $\bar s > s$ and $\bar t > t$ such that $\frac1{\bar s} + \frac1{\bar t} = \frac{p}{d-1}$. Hence, for $\lambda_\theta$ with $\theta = \frac1t\frac{d-1}{p}$, we obtain as in Substep 4.1, an unbounded subsolution. It remains to check if $\lambda_\theta \in L^s(B_1)$ and $\lambda^{-1} \in L^t(B_1)$. By construction, we have $1 - \theta = \frac{1}{\overline{s}}\frac{d-1}{p}$ and thus

$$\begin{aligned}\|\lambda_\theta\|_{L^s(B_1)} &\leq C\Big(\int_0^1 (r^{-p}\log(2/r)^{-(p-1)})^{\frac{d-1}{p}\frac{s}{\overline{s}}} r^{d-2}\,dr\Big)^{\frac{1}{s}} \\ &= C\Big(\int_0^1 r^{-(d-1)\frac{s}{\overline{s}}+d-2}\log(2/r)^{-(1-\frac{1}{p})(d-1)\frac{s}{\overline{s}}}\,dr\Big)^{\frac{1}{s}} < \infty,\end{aligned}$$

where we use $s/\overline{s} < 1$ and thus $-(d-1)\frac{s}{\overline{s}} + d - 2 > -1$. A similar argument shows $\lambda_\theta^{-1} \in L^t(B_1)$ which finishes the argument. □

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## Conflict of interest

The authors declare no conflict of interest.

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## A. Proof of Lemma 1

*Proof of Lemma 1.* As a starting point we use [32, Lemma 2.1], which states for any $\delta \in (0,1]$

$$J(\rho,\sigma,v) \le (\sigma-\rho)^{-(p-1+\frac{1}{\delta})}\Big(\int_\rho^\sigma \Big(\int_{S_r} \mu|v|^p \,\mathrm{d}\mathcal{H}^{d-1}\Big)^\delta \mathrm{d}r\Big)^{\frac{1}{\delta}}.$$

With this at hand, we proceed in analogy to the Step 2 of Proof of [7, Lemma 2.1]:

Observe that the assumption $s > 1$ implies $s_* \in [1, d-1)$. To estimate the right-hand side, on each sphere we will use "scale-invariant" Sobolev inequality with $\alpha := s_*$ in the form

$$\Big(\int_{S_r} |\phi|^{\alpha^*}\Big)^{\frac{1}{\alpha^*}} \le c\Big(\Big(\int_{S_r} |\nabla\phi|^\alpha\Big)^{\frac{1}{\alpha}} + \frac{1}{r}\Big(\int_{S_r} |\phi|^\alpha\Big)^{\frac{1}{\alpha}}\Big),$$

which holds with $c = c(d,\alpha)$ with $1 \le \alpha < d-1$, $\frac{1}{\alpha^*} = \frac{1}{\alpha} - \frac{1}{d-1}$ and any $r > 0$. Moreover, observe that by Jensen inequality the previous estimate holds also if we change the exponent $\alpha^*$ on the l.h.s. to a smaller exponent $\alpha' \in [1, \alpha^*)$, while picking up a dimensional factor of $|S_r|^{\frac{1}{\alpha'}-\frac{1}{\alpha^*}}$. Since by assumption $r \in (\rho,\sigma) \subset [\frac{1}{2}, 2]$, we can hide this factor into the constant $c$ on the r.h.s.

The definition of $s_*$ implies that for $\alpha = s_*$ holds $\frac{ps}{s-1} \le \alpha^*$. Hence, for any $\delta \in (0,1]$ we estimate

$$\begin{aligned}\Big(\int_\rho^\sigma \Big(\int_{S_r} \mu|v|^p\Big)^\delta \mathrm{d}r\Big)^{\frac{1}{\delta}} &\le \Big(\int_\rho^\sigma \Big(\int_{S_r} \mu^s\Big)^{\frac{\delta}{s}} \Big(\int_{S_r} |v|^{p\frac{s}{s-1}}\Big)^{\delta\frac{s-1}{s}} \mathrm{d}r\Big)^{\frac{1}{\delta}} \\ &\le c\Big(\int_\rho^\sigma \Big(\int_{S_r} \mu^s\Big)^{\frac{\delta}{s}} \Big[\Big(\int_{S_r} |\nabla v|^{s_*}\Big)^{\frac{p\delta}{s_*}} + \frac{1}{r^{p\delta}}\Big(\int_{S_r} |v|^{s_*}\Big)^{\frac{p\delta}{s_*}}\Big] \mathrm{d}r\Big)^{\frac{1}{\delta}},\end{aligned}$$

with $s_*$ defined above. To be able to apply Hölder inequality in $r$ to get two bulk integrals, we require $\frac{\delta}{s} + \frac{p\delta}{s_*} = 1$. By choosing $\delta = (1 + \frac{p}{d-1})^{-1} \in (0,1)$ in the case $s_* > 1$ and $\delta := (\frac{1}{s} + p)^{-1}$ if $s_* = 1$, we obtain

$$J((\rho,\sigma,v) \le \frac{c}{(\sigma-\rho)^{\frac{pd}{d-1}}}\Big(\int_{B_\sigma\setminus B_\rho} \mu^s\Big)^{\frac{1}{s}} \Big[\Big(\int_{B_\sigma\setminus B_\rho} |\nabla v|^{s_*}\Big)^{\frac{p}{s_*}} + \frac{1}{\rho^p}\Big(\int_{B_\sigma\setminus B_\rho} |v|^{s_*}\Big)^{\frac{p}{s_*}}\Big]$$

Observe that in the latter case of $s_* = 1$ and $\delta = (\frac{1}{s} + p)$ the correct prefactor is actually $c(\sigma-\rho)^{-(2p-1+\frac{1}{s})}$. Nevertheless, the estimate farther holds thanks to $2p - 1 + \frac{1}{s} \ge \frac{pd}{d-1}$, which in turn is equivalent to $1 \le \frac{1}{p}(1 - \frac{1}{s}) + \frac{1}{d-1}$ – the condition which is exactly fulfilled in this case. □