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Research article

Regularizing effect in some Mingione's double phase problems with very singular data[†]

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Abstract: In this paper we study the existence of solutions of the Dirichlet problem associated to the following nonlinear PDE

$$-\operatorname{div}(a(x)\,\nabla u|\nabla u|^{p-2}) - \operatorname{div}(|u|^{(r-1)\lambda+1}\nabla u|\nabla u|^{\lambda-2}) = f$$

where $1 < \lambda \leq p$, $r > 1$ and $f \in L^1(\Omega)$.

Keywords: nonlinear elliptic equations; weak solutions; double phase problems; singular data; regularity

1. Introduction

The topic of this paper is inspired by one of the recent scientific interests of Rosario Mingione, the so–called "double phase" elliptic problem.

The main example of a double phase integral functional is

$$J(v) = \int_\Omega \Big[\frac{1}{p}|\nabla v|^p + \frac{\rho(x)}{q}|\nabla v|^q\Big], \quad \text{with } 1 < p < q,$$

where Ω is an open, bounded subset of $\mathbb{R}^N$ $(N \geq 2)$,

$$1 < p < q, \text{with } \frac{q}{p} \text{ close to 1 in dependence on } N. \tag{1.1}$$

and

$$\rho(x) \geq 0. \tag{1.2}$$

Since it is not assumed that the weight $\rho(x)$ is bounded away from zero (that is, it is not assumed that $\exists\, \rho_0 \in \mathbb{R}^+$ such that $\rho(x) \geq \rho_0 > 0$), it is not possible to say, even under the assumption $p < q$, that the term $\rho(x)|\nabla v|^q$ is dominant, so that the set $\{x : \rho(x) = 0\}$ plays an important role.

Few years ago, R. Mingione found a name for such a problem: double phase problems. Since then, these problems and this terminology have become very popular.

Note that, the functional J exhibits unbalanced growth: the (p, q)-growth in the Marcellini terminology (see [16]).

Nowadays, there is a huge literature concerning double phase elliptic problems. Here we only recall the fundamental papers [2, 3, 12, 13], and recently [14, 17].

The main example of a double phase elliptic nonlinear differential operator is the derivative of J, that is

$$A(v) = -\operatorname{div}(\nabla v|\nabla v|^{p-2}) - \operatorname{div}(\rho(x)\,\nabla v|\nabla v|^{q-2}),$$

In this paper we study the existence of distributional solutions, belonging to some standard Sobolev spaces, of Dirichlet problems with very singular data, and associated to differential operators of double phase type like

$$-\operatorname{div}(a(x)\,\nabla v|\nabla v|^{p-2}) - \operatorname{div}(g(v)\nabla v|\nabla v|^{\lambda-2}),$$

with $g(0) = 0$.

Namely, we deal with the existence of solutions of the following boundary value problem

$$\begin{cases} -\operatorname{div}(a(x)\,\nabla u|\nabla u|^{p-2}) - \operatorname{div}(g(u)\nabla u|\nabla u|^{\lambda-2}) = f, & \text{in } \Omega; \\ u = 0, & \text{on } \partial\Omega; \end{cases} \tag{1.3}$$

where Ω is an open, bounded subset of $\mathbb{R}^N$ $(N \geq 2)$,

$$1 < \lambda \leq p < N, \tag{1.4}$$

$a(x)$ is a measurable function such that

$$\alpha \leq a(x) \leq \beta, \quad \text{with } \alpha, \beta > 0, \tag{1.5}$$

$$g(t) = |t|^{(r-1)\lambda+1}, \quad \text{with } r > 1, \tag{1.6}$$

$$f \in L^1(\Omega). \tag{1.7}$$

We point out that

- in (1.4) the parameters λ, p play the role of p, q in (1.1)
- the operator presented in (1.3) also depends on a power of u ;
- the coefficient $a(x)$ does not need to be smooth.

Our existence results hinge on the presence of the additional term

$$-\operatorname{div}(g(u)\nabla u|\nabla u|^{\lambda-2}),$$

which strongly helps, even if it has a growth (with respect to the gradient) $\lambda \le p$ and despite of the degeneracy due to the factor $|u|^{(r-1)\lambda+1}$.

As a matter of fact, this term provides a strong regularizing property: roughly speaking, we prove that the solution u of (1.3), under a suitable relationship between the parameters p and λ, is more regular (and it even exists) than the solution y of

$$\begin{cases} -\operatorname{div}(a(x)\,\nabla y|\nabla y|^{p-2}) = f, & \text{in } \Omega; \\ y = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.8}$$

studied in [4, 7, 8].

The regularizing effect of some lower orders terms, in the framework of boundary value problems with L^1-data, is already known since the paper [10] by H. Brezis and W. A. Strauss. We also refer to the paper [1, 6, 9, 11], where some Dirichlet problems with lower order terms of order zero or of order one, with natural growth with respect to ∇u are studied.

2. The case of bounded data

This section deals with the case

$$f \in L^\infty(\Omega).$$

In the sequel, given $k > 0$, we denote by $G_k(s)$ and $T_k(s)$ the classical truncated functions defined by

$$G_k(s) = (|s| - k)^+ \operatorname{sgn} s, \quad T_k(s) = s - G_k(s), \quad s \in \mathbb{R}.$$

Let us introduce the following sequence of boundary value problems

$$\begin{cases} -\operatorname{div}(a(x)\,\nabla u_n|\nabla u_n|^{p-2}) - \operatorname{div}\Big(g(T_n(u_n))\dfrac{\nabla u_n|\nabla u_n|^{\lambda-2}}{1+\frac{1}{n}|\nabla u_n|^{\lambda-1}}\Big) & \\ = f(x), & \text{in } \Omega; \\ u_n = 0, & \text{on } \partial\Omega. \end{cases}$$

As a consequence of the classical result due to J. Leray, J. L. Lions (see [15]) there exists $u_n \in W_0^{1,p}(\Omega)$ which is a weak solution of the above problem in the sense that the following integral identity holds

$$\int_\Omega a(x)\,\nabla u_n|\nabla u_n|^{p-2}\nabla v + \int_\Omega g(T_n(u_n))\frac{\nabla u_n|\nabla u_n|^{\lambda-2}}{1+\frac{1}{n}|\nabla u_n|^{\lambda-1}}\nabla v \tag{2.1}$$

$$= \int_\Omega f v\,, \qquad \text{for any } v \in W_0^{1,p}(\Omega).$$

Moreover, due to the boundedness of f and adapting the well known method used in [18], each u_n is a bounded function and there exists a positive contant C_f, independent on n, such that

$$\|u_n\|_{L^\infty(\Omega)} \le C_f, \quad \forall\, n \in \mathbb{N}.$$

Thus, for any $n > C_f$ it holds $T_n(u_n) = u_n$ and u_n is a weak solution of the following Dirichlet problem

$$\begin{cases} u_n \in W_0^{1,p}(\Omega) : \\ -\operatorname{div}\big(a(x)\,\nabla u_n|\nabla u_n|^{p-2}\big) - \operatorname{div}\Big(g(u_n)\dfrac{\nabla u_n|\nabla u_n|^{\lambda-2}}{1+\frac{1}{n}|\nabla u_n|^{\lambda-1}}\Big) = f(x), \end{cases} \tag{2.2}$$

that is

$$\int_\Omega a(x)\,\nabla u_n|\nabla u_n|^{p-2}\nabla v + \int_\Omega g(u_n)\frac{\nabla u_n|\nabla u_n|^{\lambda-2}}{1+\frac{1}{n}|\nabla u_n|^{\lambda-1}}\nabla v \tag{2.3}$$

$$= \int_\Omega f v\,, \qquad \text{for any } v \in W_0^{1,p}(\Omega).$$

Taking u_n as test function in (2.3) and using the assumption (1.5) we have

$$\alpha\int_\Omega |\nabla u_n|^p + \int_\Omega g(u_n)\frac{|\nabla u_n|^{\lambda}}{1+\frac{1}{n}|\nabla u_n|^{\lambda-1}} \le \int_\Omega f u_n\,.$$

Dropping the second (positive) term in the left–hand side and using the boundedness of f we obtain

$$\|u_n\|_{W_0^{1,p}(\Omega)} \le C_1, \quad \forall\, n \in \mathbb{N}. \tag{2.4}$$

Here, and in the sequel, we denote by C_i positive constants only depending on the data (but not on n).

Thus, there exist a subsequence, not relabelled, and a function $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ such that

$$u_n \rightharpoonup u \quad \text{weakly in } W_0^{1,p}(\Omega), \tag{2.5}$$

$$u_n \to u \quad \text{strongly in } L^p(\Omega), \text{ and a.e. in } \Omega. \tag{2.6}$$

Moreover, using estimate (2.4) we obtain, since $1 < \lambda \le p$,

$$\int_\Omega \frac{1}{n}|\nabla u_n|^{\lambda-1} \le \frac{C_2}{n}, \quad \forall\, n \in \mathbb{N}.$$

Thus,

$$\frac{1}{n}|\nabla u_n|^{\lambda-1} \to 0 \quad \text{strongly in } L^1(\Omega), \text{ and a.e. in } \Omega. \tag{2.7}$$

In order to have

$$u_n \to u \quad \text{strongly in } W_0^{1,p}(\Omega), \tag{2.8}$$

it is enough to prove that

$$\int_\Omega a(x)\,[\nabla u_n|\nabla u_n|^{p-2} - \nabla u|\nabla u|^{p-2}]\nabla(u_n - u) \to 0. \tag{2.9}$$

Let us take $v = u_n - u$ as test function in (2.3)

$$\int_\Omega a(x)\,[\nabla u_n|\nabla u_n|^{p-2} - \nabla u|\nabla u|^{p-2}]\nabla(u_n - u)$$

$$+\int_\Omega g(u_n)\frac{\nabla u_n|\nabla u_n|^{\lambda-2}-\nabla u|\nabla u|^{\lambda-2}}{1+\frac{1}{n}|\nabla u_n|^{\lambda-1}}\nabla(u_n-u)$$
$$=\int_\Omega f(u_n-u)-\int_\Omega a(x)\,\nabla u|\nabla u|^{p-2}\nabla(u_n-u)$$
$$-\int_\Omega g(u_n)\frac{\nabla u|\nabla u|^{\lambda-2}}{1+\frac{1}{n}|\nabla u_n|^{\lambda-1}}\nabla(u_n-u)$$

Due to the positivity of the second term we get

$$\int_\Omega a(x)\,[\nabla u_n|\nabla u_n|^{p-2}-\nabla u|\nabla u|^{p-2}]\nabla(u_n-u) \tag{2.10}$$
$$\leq \int_\Omega f(u_n-u)-\int_\Omega a(x)\,\nabla u|\nabla u|^{p-2}\nabla(u_n-u)$$
$$-\int_\Omega g(u_n)\frac{\nabla u|\nabla u|^{\lambda-2}}{1+\frac{1}{n}|\nabla u_n|^{\lambda-1}}\nabla(u_n-u)\,.$$

We note that the first and the second integral in the right–hand side converge to 0. Moreover,

$$g(u_n)\frac{\nabla u|\nabla u|^{\lambda-2}}{1+\frac{1}{n}|\nabla u_n|^{\lambda-1}}\to g(u)\nabla u|\nabla u|^{\lambda-2}\quad \text{a.e. in }\ \Omega$$

and (since $|\nabla u|^{\lambda-1}\in L^{p'}$)

$$\left|g(u_n)\frac{\nabla u|\nabla u|^{\lambda-2}}{1+\frac{1}{n}|\nabla u_n|^{\lambda-1}}\right|\leq g(C_f)|\nabla u|^{\lambda-1},\quad \forall\, n\in\mathbb{N}.$$

Thus, by the Lebesgue Theorem we get

$$g(u_n)\frac{\nabla u|\nabla u|^{\lambda-2}}{1+\frac{1}{n}|\nabla u_n|^{\lambda-1}}\to g(u))\nabla u|\nabla u|^{\lambda-2}\quad\text{strongly in }\ L^{p'}(\Omega), \tag{2.11}$$

which in turn implies

$$\int_\Omega g(u_n)\frac{\nabla u|\nabla u|^{\lambda-2}}{1+\frac{1}{n}|\nabla u_n|^{\lambda-1}}\nabla(u_n-u)\ \to 0.$$

Then, (2.9) easily follows taking the limit as $n\to+\infty$ in (2.10) and the strong convergence (2.8) is proved. Finally, we take the limit as $n\to+\infty$ in (2.3) (using (2.9) and (3.6)) and we obtain the following existence theorem.

Theorem 2.1. *Let* $1<\lambda\leq p<N$. *Assume that* (1.5), (1.6) *hold and let*

$$f\in L^\infty(\Omega).$$

Then there exists a weak solution $u\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ *which solves the problem* (1.3) *in the following weak sense*

$$\int_\Omega a(x)\,\nabla u|\nabla u|^{p-2}\nabla v+\int_\Omega g(u)\nabla u|\nabla u|^{\lambda-2}\nabla v\ =\int_\Omega fv \tag{2.12}$$

for any $v\in W_0^{1,p}(\Omega)$.

3. Non regular data

In this section we assume that

$$f \in L^1 \log L^1(\Omega) \tag{3.1}$$

and we will prove the existence of a distributional solution of problem (1.3)

3.1. Approximating problems

Let $\{f_n\}$ be a sequence of bounded functions such that

$$f_n \to f \quad \text{strongly in}\, L^1(\Omega),$$

and

$$\|f_n\|_{L^1(\Omega)} \le \|f\|_{L^1(\Omega)}, \quad \forall\, n \in \mathbb{N}.$$

Classical examples are $f_n = T_n[f]$ and $f_n = \frac{f}{1+\frac{1}{n}|f|}$.

Let us introduce the following approximate boundary value problems

$$\begin{cases} -\operatorname{div}(a(x)\,\nabla u_n|\nabla u_n|^{p-2}) - \operatorname{div}(g(u_n)\nabla u_n|\nabla u_n|^{\lambda-2}) = f_n(x), & \text{in } \Omega; \\ u_n = 0, & \text{on } \partial\Omega. \end{cases} \tag{3.2}$$

By Theorem 2.1, there exists $u_n \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ such that, for any $v \in W_0^{1,p}(\Omega)$,

$$\int_\Omega a(x)\,\nabla u_n|\nabla u_n|^{p-2}\nabla v + \int_\Omega g(u_n)\nabla u_n|\nabla u_n|^{\lambda-2}\nabla v = \int_\Omega f_n v. \tag{3.3}$$

Let $k > 0$; by taking $T_k(u_n)$ as test function in the weak formulation (3.3) of problem (3.2) and dropping the positive second term, we can proceed as in [4] and the following lemma holds.

Lemma 3.1. *Let $1 < \lambda \le p < N$. Assume that the hypotheses* (1.5)*,* (1.6)*,* (3.1) *are satisfied. Then, for any $k > 0$ it holds*

$$\int_\Omega |\nabla T_k(u_n)|^p \le k \int_\Omega |f|, \quad \forall\, n \in \mathbb{N}. \tag{3.4}$$

Moreover, there exists $C_0 > 0$ such that

$$\int_\Omega |\nabla u_n|^s \le C_0, \quad s < \frac{(p-1)N}{N-1}, \tag{3.5}$$

and

$$\{a(x)\,\nabla u_n|\nabla u_n|^{p-2}\} \textit{ is bounded in } L^t(\Omega),\ 1 < t < \frac{N}{N-1}. \tag{3.6}$$

Next, we will prove the following lemma.

Lemma 3.2. *Let $1 < \lambda \le p < N$. Assume that the hypotheses* (1.5)*,* (1.6)*,* (3.1) *are satisfied. Then there exists a positive constant R, independent on n, such that*

$$\int_\Omega |\nabla u_n|^\lambda \le R, \quad \forall\, n \in \mathbb{N}. \tag{3.7}$$

Proof. We set $\eta = (r-1)\lambda + 1$ (note that $\eta > 1$ since $r > 1$) and we take

$$v = \left[1 - \frac{1}{(1+|u_n|)^{\eta-1}}\right]\frac{u_n}{|u_n|}$$

as test function in (3.3). Dropping the positive term resulting by the principal part, we obtain

$$(\eta - 1)\int_\Omega \frac{|u_n|^\eta}{(1+|u_n|)^\eta}|\nabla u_n|^\lambda \le \int_\Omega |f_n(x)|[1-(1+|u_n|)^{1-\eta}] \le \int_\Omega |f(x)|.$$

We fix $k > 0$. By the above estimate we have

$$(\eta - 1)\frac{k^\eta}{(1+k)^\eta}\int_{\{|u_n|>k\}} |\nabla u_n|^\lambda \le \|f\|_{L^1(\Omega)}. \tag{3.8}$$

Thus, putting together estimates (3.4) and (3.8), it follows (3.7).

Further improvements on the boundedness of u_n and ∇u_n, depending on the relationship between the parameters p, λ and r, can be derived from the following lemma

Lemma 3.3. *Let* $1 < \lambda \le p < N$. *Assume that the hypotheses* (1.5)*,* (1.6)*,* (3.1) *are satisfied. Then there exist two positive constants* R_1, R_2 *independent of* n *such that*

$$\int_\Omega |u_n|^{r\lambda^*} \le R_1, \quad \forall\, n \in \mathbb{N} \tag{3.9}$$

and

$$\int_\Omega |\nabla u_n|^\sigma \le R_2, \quad \forall\, n \in \mathbb{N} \tag{3.10}$$

with

$$\sigma = \frac{r\,p\,\lambda^*}{1 + r\,\lambda^*}.$$

Proof. By taking $v = \log(1+|u_n|)\frac{u_n}{|u_n|}$ as test function in the weak formulation (3.3) of problem (3.2) (see [8]), it is easy to see that

$$\alpha\int_\Omega \frac{|\nabla u_n|^p}{1+|u_n|} + \int_\Omega \frac{|u_n|^{(r-1)\lambda+1}}{1+|u_n|}|\nabla u_n|^\lambda \le \int_\Omega |f|\log(1+|u_n|).$$

Using in the right–hand side the inequality

$$st \le s\log(1+s) + e^t, \quad \forall\, s, t > 0$$

and the boundedness of $\{u_n\}$ in $L^1(\Omega)$ and, taking into account the positivity of each of the two integrals in the left-hand side and (3.7), the following two estimates hold

$$\int_\Omega \frac{|u_n|^{(r-1)\lambda+1}}{1+|u_n|}|\nabla u_n|^\lambda \le C_3, \quad \forall\, n \in \mathbb{N} \tag{3.11}$$

and

$$\int_\Omega \frac{|\nabla u_n|^p}{1+|u_n|} \le C_4, \quad \forall\, n \in \mathbb{N} \tag{3.12}$$

From estimate (3.11) we also deduce

$$\frac{1}{2}\int_{|u_n|>1} |u_n|^{(r-1)\lambda}|\nabla u_n|^{\lambda} \leq C_5, \quad \forall\, n \in \mathbb{N}$$

which, together with inequality (3.4), implies

$$\int_{\Omega} |u_n|^{(r-1)\lambda}|\nabla u_n|^{\lambda} \leq C_6, \quad \forall\, n \in \mathbb{N}. \tag{3.13}$$

Now, we can use Sobolev inequality

$$\frac{1}{(rS)^{\lambda}}\Big(\int_{\Omega} |u_n|^{r\lambda^*}\Big)^{\frac{\lambda}{\lambda^*}} \leq \frac{1}{r^{\lambda}}\int_{\Omega} |\nabla |u_n|^r|^{\lambda} =$$

$$\int_{\Omega} |u_n|^{(r-1)\lambda}|\nabla u_n|^{\lambda} \leq C_7, \quad \forall\, n \in \mathbb{N}$$

and the estimate (3.9) follows.

Next, let us prove (3.10). We follow the outline of [8]. Note that since $\sigma < p$, by Hölder inequality with exponents $\frac{p}{\sigma}$, $\frac{p}{p-\sigma}$ and inequality (3.12), we have

$$\int_{\Omega} |\nabla u_n|^{\sigma} = \int_{\Omega} \frac{|\nabla u_n|^{\sigma}}{(1+|u_n|)^{\frac{\sigma}{p}}}\,(1+|u_n|)^{\frac{\sigma}{p}} \leq C_8\Big[\int_{\Omega} (1+|u_n|)^{\frac{\sigma}{p-\sigma}}\Big]^{\frac{p-\sigma}{p}}$$

and the proof is concluded, since, by the choice of σ, it follows $\frac{\sigma}{p-\sigma} = r\,\lambda^*$. □

Remark 3.4. Note that in Lemmas 3.1 and 3.7 we only use the assumption $f \in L^1(\Omega)$, while Lemma 3.3 requires the additional hypothesis $f \in L^1 \log L^1(\Omega)$. However, if f is merely summable, the proof of Lemma 3.3 can be repeated in order to obtain the boundedness of $\{\nabla u_n\}$ in $W_0^{1,\sigma}(\Omega)$, for any $1 \leq \sigma < \frac{r\,p\,\lambda^*}{1+r\,\lambda^*}$.

Remark 3.5. We point out that

$$\max\{\lambda, \sigma\} = \begin{cases} \lambda & \text{if} \quad \lambda \geq N\frac{pr-1}{Nr-1} \\ \sigma & \text{if} \quad \lambda < N\frac{pr-1}{Nr-1}. \end{cases}$$

Moreover

$$N\frac{pr-1}{Nr-1} > 1 \quad \iff \quad p > 1 + \frac{N-1}{r}.$$

Thus, Lemma 3.3 improves Lemma 3.7 if $1 \leq \lambda < N\frac{pr-1}{Nr-1}$ and $p > 1+\frac{N-1}{r}$. (Note that $1+\frac{N-1}{r} \in]1, 2-\frac{1}{N}[$ since $r > 1$).

Remark 3.6. Let $2 - \frac{1}{N} < p < N$. Taking into account only the contribution of the principal part and applying the results of [7] we deduce that the sequence $\{u_n\}$ is bounded in $W_0^{1,\frac{N(p-1)}{N-1}}(\Omega)$.

Thus the term

$$-\operatorname{div}(g(u_n)\nabla u_n|\nabla u_n|^{\lambda-2})$$

has a regularizing effect in the following two cases

i) $2 - \frac{1}{N} < p < N$ and $\frac{(p-1)N}{N-1} < \lambda \leq p$,

ii) $1 < p \leq 2 - \frac{1}{N}$ and $1 < \lambda \leq p$.

As a consequence of previous lemmas we prove the following two existence results.

Theorem 3.7. *Let* $1 < \lambda \leq p < N$. *Assume that hypotheses* (1.5), (1.6) *and* (3.1) *hold.*

Then there exists $u \in W_0^{1,\lambda}(\Omega)$*, such that* $g(u)|\nabla u|^{\lambda-1} \in L^1(\Omega)$*, which solves the problem* (1.3) *in the following distributional sense*

$$\int_\Omega a(x)\,\nabla u|\nabla u|^{p-2}\nabla v + \int_\Omega g(u)\nabla u|\nabla u|^{\lambda-2}\nabla v = \int_\Omega fv, \tag{3.14}$$

for any $v \in C_0^\infty(\Omega)$.

Theorem 3.8. *Let* $1 < \lambda \leq p < N$. *Assume that hypotheses* (1.5), (1.6) *and* (3.1) *hold.*

Then there exists $u \in W_0^{1,\sigma}(\Omega)$*, such that* $g(u)|\nabla u|^{\lambda-1} \in L^1(\Omega)$*, which solves the problem* (1.3) *in the distributional sense* (3.14).

Remark 3.9. We explicitly remark that, in the case $\lambda = p$, Theorem 3.7 gives the existence of at least one solution with finite energy without any additional assumption on the summability of f. A similar regularizing effect occurs for the solution of the Dirichlet problem associated to the equation

$$-\operatorname{div}(a(x)\,\nabla u|\nabla u|^{p-2}) + u|u|^{s-1} = f,$$

where $f \in L^m(\Omega)$ with $1 < m < (p^*)'$, when a suitable balance between m and s holds, (see [11]) or to the equation

$$-\operatorname{div}(a(x)\,\nabla u|\nabla u|^{p-2}) + u|u|^s|\nabla u|^p = f$$

with $f \in L^1(\Omega)$ (see [9]).

The following Figure 1 summarizes the different regularity results in dependence of p and λ.

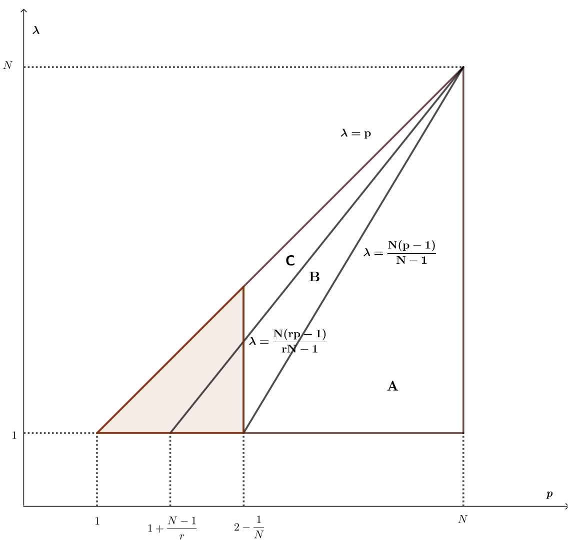

Figure 1. Regularity results in dependence of p and λ.

If (p, λ) belongs to the region A, the better regularity is the one obtained in [7], i.e., $u \in W_0^{1,\frac{N(p-1)}{N-1}}(\Omega)$; otherwise the better regularity is the one proved here.

If (p, λ) belongs to the region B, Theorem 3.8 gives the existence of a distributional solution $u \in W_0^{1,\sigma}(\Omega)$; while in the region C the better regularity is the one stated in Theorem 3.7, i.e., $u \in W_0^{1,\lambda}(\Omega)$.

At last, if (p, λ) belongs to the colored region the result stated in Theorem 3.7 is new.

3.2. Proof of Theorems 3.7 and 3.8

We begin with the proof of Theorem 3.7.

As a consequence of Lemma 3.1 and Lemma 3.7 there exist a subsequence, not relabelled, and a function $u \in W_0^{1,\lambda}(\Omega)$ such that

$$\begin{cases} u_n \rightharpoonup u \quad \text{weakly in } W_0^{1,\lambda}(\Omega), \\ u_n \to u \quad \text{strongly in } L^\lambda(\Omega) \text{ and a.e. in } \Omega, \\ T_k(u_n) \rightharpoonup T_k(u) \quad \text{weakly in } W_0^{1,p}(\Omega). \end{cases} \tag{3.15}$$

In order to take the limit as $n \to +\infty$ in (3.2) we have to prove that

$$\nabla u_n \to \nabla u \quad \text{a.e. in } \Omega.$$

We follow some techniques of [5]. For any $\xi \in \mathbb{R}^N$, we set

$$A(x, \xi) = a(x)\xi|\xi|^{p-2}, \quad B_n(\xi) = \frac{\xi|\xi|^{\lambda-2}}{1 + \frac{1}{n}|\xi|^{\lambda-1}}.$$

Let $j, k > 0$; using $v = T_j[u_n - T_k(u)]$ as test function in (3.3) we have

$$\begin{aligned} &\int_\Omega [A(x, \nabla u_n) - A(x, \nabla T_k(u))]\nabla T_j[u_n - T_k(u)] \\ &\qquad + \int_\Omega A(x, \nabla T_k(u))\nabla T_j[u_n - T_k(u)] \\ &\qquad + \int_\Omega g(u_n)[B_n(\nabla u_n) - B_n(\nabla T_k(u))]\nabla T_j[u_n - T_k(u)] \\ &\qquad + \int_\Omega g(u_n)B_n(\nabla T_k(u))\nabla T_j[u_n - T_k(u)] = \int_\Omega f T_j[u_n - T_k(u)]. \end{aligned} \tag{3.16}$$

We note that

$$\int_\Omega g(u_n)[B_n(\nabla u_n) - B_n(\nabla T_k(u))]\nabla T_j[u_n - T_k(u)] \geq 0.$$

Moreover (since $A(x, \nabla T_k(u))\nabla T_j[u - T_k(u)] = 0$)

$$\lim_{n\to\infty} \int_\Omega A(x, \nabla T_k(u))\nabla T_j[u_n - T_k(u)] = 0$$

and

$$\lim_{n\to\infty}\int_\Omega g(u_n)B_n(\nabla T_k(u))\nabla T_j[u_n - T_k(u)]$$
$$= \lim_{n\to\infty}\int_{\{|u_n-T_k(u)|<j\}} g(u_n)\, B_n(\nabla T_k(u))\nabla[u_n - T_k(u)] = 0$$

since $B_n(0) = 0$. Thus, from (3.16) we deduce

$$0 \le \int_\Omega [A(x,\nabla u_n) - A(x,\nabla T_k(u))]\nabla T_j[u_n - T_k(u)]$$
$$\le \epsilon_n^1(k) + \epsilon_n^2(k) + \omega_n(k), \quad (3.17)$$

where we have denoted by $\epsilon_n^1(k)$ and $\epsilon_n^2(k)$ two functions which go to 0 as $n \to +\infty$, for any $k > 0$ and

$$\omega_n(k) = \int_\Omega f T_j[u_n - T_k(u)].$$

Now, we use the above inequality in order to prove the L^1 compactness of the sequence $\{\nabla u_n\}$. Let $0 < \theta < \dfrac{\lambda}{p}$ $(0 < \theta < 1)$ and $k > 0$. Let us define

$$I_{n,\Omega} = \int_\Omega \{[A(x,\nabla u_n) - A(x,\nabla u)]\nabla(u_n - u)\}^\theta.$$

and let us prove that the previous integral converges to zero.

Indeed, it holds

$$I_{n,\Omega} = I_{n,C_k} + I_{n,A_k}$$

where

$$I_{n,C_k} = \int_{C_k} \{[A(x,\nabla u_n) - A(x,\nabla u)]\nabla(u_n - u)\}^\theta$$

and

$$I_{n,A_k} = \int_{A_k} \{[A(x,\nabla u_n) - A(x,\nabla u)]\nabla(u_n - u)\}^\theta$$

with

$$C_k = \{x : |u(x)| \le k\}, \qquad A_k = \{x : |u(x)| > k\}.$$

We observe that

$$I_{n,C_k} \le \int_\Omega \{[A(x,\nabla u_n) - A(x,\nabla T_k(u))]\nabla(u_n - T_k(u))\}^\theta = J_{n,\Omega}.$$

Using the Hölder inequality, with exponents $\frac{\lambda}{p\theta}$ and $\frac{\lambda}{\lambda-p\theta}$, and the a priori estimate (3.10), we have

$$I_{n,\Omega} \le J_{n,\Omega} + I_{n,A_k}$$
$$\le J_{n,\Omega} + C_{11}\left[\int_{A_k} (|\nabla u_n| + |\nabla u|)^\lambda\right]^{\frac{p\theta}{\lambda}} |A_k|^{1-\frac{p\theta}{\lambda}}$$
$$\le J_{n,\Omega} + C_{12}\,|A_k|^{1-\frac{p\theta}{\lambda}} = J_{n,\Omega} + \omega^1(k). \quad (3.18)$$

Here and in the sequel, for any measurable set $E \subset \mathbb{R}^N$, $|E|$ denotes its $N-$ dimensional measure. Moreover, by $\omega^i(k)$ we denote some quantities such that $\lim_{k\to\infty} \omega^i(k) = 0$. Now, we have to study the behavior of $J_{n,\Omega}$; it can be splitted as ($j \in \mathbb{N}$)

$$J_{n,\Omega} = \int_\Omega \{[A(x, \nabla u_n) - A(x, \nabla T_k(u))]\nabla T_j[u_n - T_k(u)]\}^\theta$$

$$+ \int_{\{|u_n - T_k(u)|>j\}} \{[A(x, \nabla u_n) - A(x, \nabla T_k(u))]\nabla[u_n - T_k(u)]\}^\theta = J^1_{n,\Omega} + J^2_{n,\Omega}.$$

We estimate $J^1_{n,\Omega}$ and $J^2_{n,\Omega}$ by means of Hölder inequality with exponents $\frac{1}{\theta}$, $\frac{1}{1-\theta}$ and $\frac{\lambda}{p\theta}$, $\frac{\lambda}{\lambda-p\theta}$, respectively and we use inequalities (3.17) and (3.7), getting

$$J_{n,\Omega} = \left[\int_\Omega [A(x, \nabla u_n) - A(x, \nabla T_k(u))]\nabla T_j[u_n - T_k(u)] \right]^\theta |\Omega|^{1-\theta}$$

$$+C_R \,|\{x : |u_n - T_k(u)| > j\}|^{1-\frac{p\theta}{\lambda}}.$$

$$\leq C_{13} \,[\epsilon^1_n(k) + \epsilon^2_n(k) + \omega_n(k)]^\theta + C_R \,|\{x : |u_n - T_k(u)| > j\}|^{1-\frac{p\theta}{\lambda}}.$$

Since

$$\omega_n(k) \to \int_\Omega f T_j[u - T_k(u)] = \omega^2(k)$$

and

$$\limsup_{n\to\infty} |\{|u_n - T_k(u)| > j\}|^{1-\frac{p\theta}{\lambda}} \leq |\{|u - T_k(u)| \geq j\}|^{1-\frac{p\theta}{\lambda}} = \omega^3(k),$$

we obtain

$$\limsup_{n\to\infty} J_{n,\Omega} \leq C_{14} \,[\omega^2(k)]^\theta + C_{15}\, \omega^3(k).$$

Summing up the above inequality and (3.18) we have

$$\limsup_{n\to\infty}[I_{n,C_k} + I_{n,A_k}] \leq \omega^1(k) + C_{16} \,[\omega^2(k)]^\theta + C_{17}\, \omega^3(k).$$

Therefore,

$$\int_\Omega \{[A(x, \nabla u_n) - A(x, \nabla u)]\nabla(u_n - u)\}^\theta \to 0,$$

which gives (for a suitable subsequence, still denoted by u_n)

$$\{[A(x, \nabla u_n) - A(x, \nabla u)]\nabla(u_n - u)\}^\theta \to 0 \quad \text{a.e.},$$

and also (since θ is positive)

$$\{[A(x, \nabla u_n) - A(x, \nabla u)]\nabla(u_n - u)\} \to 0 \quad \text{a.e.}.$$

In [15], it is proved that, under our assumptions on the function $A(x, \xi)$, the previous limit implies that

$$\nabla u_n(x) \to \nabla u(x) \qquad \text{a.e.}. \tag{3.19}$$

Thus

$$a(x)\nabla u_n(x)|\nabla u_n(x)|^{p-2} \to a(x)\nabla u(x)|\nabla u(x)|^{p-2}, \text{ a.e.} \tag{3.20}$$

and, thanks to (3.6) we have

$$a(x)\nabla u_n(x)|\nabla u_n(x)|^{p-2} \to a(x)\nabla u(x)|\nabla u(x)|^{p-2}, \text{ in } L^\tau(\Omega)\,,\ 1 < \tau < t < \frac{N}{N-1}. \tag{3.21}$$

Next, we will prove that

$$g(u_n)\nabla u_n|\nabla u_n|^{\lambda-2} \to g(u)\nabla u|\nabla u|^{\lambda-2} \quad \text{in } L^1(\Omega). \tag{3.22}$$

Thanks to (3.19) we also deduce

$$g(u_n)\nabla u_n|\nabla u_n|^{\lambda-2} \to g(u)\nabla u|\nabla u|^{\lambda-2} \quad \text{a.e. in } \Omega.$$

Moreover, for any measurable set $E \subset \Omega$ we have

$$\begin{aligned} \int_E g(u_n)|\nabla u_n|^{\lambda-1} &= \int_E |u_n|^r \left(|u_n|^{r-1}|\nabla u_n|\right)^{\lambda-1} \\ &\le C_{20}\Big[\int_E |u_n|^{\lambda r}\Big]^{\frac{1}{\lambda}}\Big[\int_E |u_n|^{(r-1)\lambda}|\nabla u_n|^\lambda\Big]^{1-\frac{1}{\lambda}} \\ &\le C_{21}\Big[\int_\Omega |u_n|^{\lambda^* r}\Big]^{\frac{1}{\lambda^*}}|E|^{\frac{1}{\lambda}-\frac{1}{\lambda^*}}\Big[\int_\Omega |u_n|^{(r-1)\lambda}|\nabla u_n|^\lambda\Big]^{1-\frac{1}{\lambda}} \end{aligned} \tag{3.23}$$

and the right–hand side goes to 0 as $|E| \to 0$ uniformly w.r.t. n, since estimates (3.9) and (3.13) hold. Thus

$$\lim_{|E|\to 0}\int_E g(u_n)|\nabla u_n|^{\lambda-1} = 0, \text{ uniformly w.r.t. } n.$$

Thanks to Vitali Theorem the convergence (3.22) is proved and

$$\int_\Omega g(u_n)\nabla u_n|\nabla u_n|^{\lambda-2}\nabla v \to \int_\Omega g(u)\nabla u|\nabla u|^{\lambda-2}\nabla v, \quad \forall\ v \in C_0^\infty(\Omega).$$

The above limit and the limit (3.21), allow us to take the limit as $n \to +\infty$ in (3.3) and the proof of Theorem 3.7 follows. □

In order to prove Theorem 3.8 we note that as a consequence of Lemma 3.1 and Lemma 3.3. there exist a subsequence, not relabelled, and a function $u \in W_0^{1,\sigma}(\Omega)$ such that

$$\begin{cases} u_n \rightharpoonup u & \text{weakly in } W_0^{1,\sigma}(\Omega) \\ u_n \to u & \text{strongly in } L^\sigma(\Omega) \text{ and a.e. in } \Omega, \\ T_k(u_n) \rightharpoonup T_k(u) & \text{weakly in } W_0^{1,p}(\Omega). \end{cases} \tag{3.24}$$

and the proof can be performed as above. Precisely, in order to obtain the a.e. convergence of $\{\nabla u_n\}$ we just have to replace λ with σ.

□

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Conflict of interest

The authors declare no conflict of interest.

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