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Research article

Fractional KPZ equations with fractional gradient term and Hardy potential†

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Abstract: In this work we address the question of existence and non existence of positive solutions to a class of fractional problems with non local gradient term. More precisely, we consider the problem

$$\begin{cases} (-\Delta)^s u &=& \lambda \dfrac{u}{|x|^{2s}} + (\mathfrak{F}(u)(x))^p + \rho f & \text{in } \Omega, \\ u &>& 0 & \text{in } \Omega, \\ u &=& 0 & \text{in } (\mathbb{R}^N \setminus \Omega), \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a $C^{1,1}$ bounded domain, $N > 2s, \rho > 0$, $0 < s < 1$, $1 < p < \infty$ and $0 < \lambda < \Lambda_{N,s}$, the Hardy constant defined below. We assume that f is a non-negative function with additional hypotheses. Here $\mathfrak{F}(u)$ is a nonlocal "gradient" term. In particular, if $\mathfrak{F}(u)(x) = |(-\Delta)^{\frac{s}{2}} u(x)|$, then we are able to show the existence of a critical exponents $p_+(\lambda, s)$ such that: 1) if $p > p_+(\lambda, s)$, there is no positive solution, 2) if $p < p_+(\lambda, s)$, there exists, at least, a positive supersolution solution for suitable data and ρ small. Moreover, under additional restriction on p, there exists a solution for general datum f.

Keywords: fractional elliptic equations; nonlocal gradient term; Hardy potential; stationary Kardar-Parisi-Zhang equations; existence and nonexistence results

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This work is dedicated to Ireneo in memoriam, for his teachings, friendship and enthusiasm.

1. Introduction

This work deals with the following problem:

$$\begin{cases} (-\Delta)^s u &= \lambda \dfrac{u}{|x|^{2s}} + (\mathfrak{F}(u)(x))^p + \rho f & \text{in } \Omega, \\ u &> 0 & \text{in } \Omega, \\ u &= 0 & \text{in } (\mathbb{R}^N \setminus \Omega), \end{cases} \tag{1}$$

where $\lambda > 0$, $\rho > 0$, $s \in (0,1)$, $2s < N$, $1 < p < \infty$, $\Omega \subset \mathbb{R}^N$ is a bounded regular domain containing the origin and f is a measurable non-negative function satisfying suitable hypotheses.

By $(-\Delta)^s$ we denote the fractional Laplacian of order $2s$ introduced by M. Riesz in [39], that is,

$$(-\Delta)^s u(x) := a_{N,s} \text{ P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{N+2s}}\, dy, \qquad s \in (0,1),$$

where

$$a_{N,s} = 2^{2s-1} \pi^{-\frac{N}{2}} \frac{\Gamma(\frac{N+2s}{2})}{|\Gamma(-s)|},$$

is the normalizing constant that gives the Fourier multiplier identity

$$\mathcal{F}((-\Delta)^s u)(\xi) = |\xi|^{2s} \mathcal{F}(u)(\xi), \text{ for } u \in S(\mathbb{R}^N).$$

See [26] for details. Our goal is to find *natural* conditions on p and f (related to the value of λ), in order to get the existence of positive solutions.

If $\lambda = 0$, the problem (1) can be seen as a Kardar-Parisi-Zhang stationary equation with fractional diffusion and nonlocal gradient term. We refer to [30] for the main model and additional properties of the local case.

The nonlocal case $s \in (0,1)$, but still with the local gradient term, was used recently in order to describe the growing surface in the presence of self-similar hopping surface diffusion. We refer the reader to the papers [29, 32, 33, 35] for a physical rigorous justification.

Existence results for the corresponding problem were obtained in [23] and [11] under suitable hypotheses on f and p. As it was shown in [11], if $p > \frac{1}{1-s}$, then the corresponding problem does not have positive solutions with global regularity of the gradient, even in the case of regular datum f. Existence of a solution, in the viscosity sense, is proved in [9, 16, 17] for some particular cases.

The case $\lambda = 0$, under the presence of a nonlocal gradient term, was analyzed recently in [7]. Without any limitation on the value of p and under suitable hypothesis of f, the author proved the existence of a solution using a priori estimates and fixed point arguments.

The case $\lambda > 0$ with a local gradient term was considered in [10] and [12]. Here the authors showed the existence of a critical exponent related to the existence of solutions. Our work can be seen as the non-local counterpart of [12]. However, the non-local gradient term makes the problem more difficult and fine analysis is needed to determine the existence or non-existence scheme.

Notice that for $\lambda > 0$, problem (1) is related to the Hardy inequality proved in [28], (see also [18] and [38] for equivalent forms.) Namely, for $\phi \in C_0^\infty(\boldsymbol{R}^N)$, we have

$$\int_{\boldsymbol{R}^N} |\xi|^{2s} |\hat{\phi}|^2\, d\xi \geqslant \Lambda_{N,s} \int_{\boldsymbol{R}^N} |x|^{-2s} \phi^2\, dx, \tag{2}$$

where

$$\Lambda_{N,s} := 2^{2s}\frac{\Gamma^2(\frac{N+2s}{4})}{\Gamma^2(\frac{N-2s}{4})} \tag{3}$$

is optimal and not attained.

It is clear that

$$\lim_{s\to 1}\Lambda_{N,s} = \left(\frac{N-2}{2}\right)^2,$$

the Hardy constant in the local case.

Inequality (2) can be also formulated in the following way

$$\frac{a_{N,s}}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\phi(x)-\phi(y)|^2}{|x-y|^{N+2s}}\,dx\,dy \geqslant \Lambda_{N,s}\int_{\mathbb{R}^N}\frac{\phi^2}{|x|^{2s}}\,dx,\ \forall\phi\in C_0^\infty(\mathbb{R}^N).$$

If $\lambda > \Lambda_{N,s}$, then we can prove that problem (1) has no positive supersolution. Hence, we assume throughout this paper that $0 < \lambda < \Lambda_{N,s}$.

The presence of the Hardy potential forces the solution to enjoy a singular behavior near the singular point zero and then a loss of regularity is generated.

The paper is organized as follows. In Section 2 we present the functional setting used in order to study our problem. More precisely we describe some related spaces, as the Bessel potential space, and their relationship with the fractional Sobolev space. We introduce also the different forms of the fractional gradient that will be used throughout the paper. In Subsection 2.1 we recall the global regularity results for the Poisson fractional problem proved in [6]. This will be the key in order to show the fractional regularity in our problem.

The analysis of the problem under the presence of the Hardy potential, without the nonlocal gradient term, is considered in Section 3. More precisely, we will consider the semilinear problem

$$\begin{cases} (-\Delta)^s u &= \lambda\dfrac{u}{|x|^{2s}} + f & \text{in } \Omega,\\ u &= 0 & \text{in } \mathbb{R}^N\setminus\Omega,\end{cases}$$

where $f \in L^m(\Omega)$ with $m \geqslant 1$. Some partial regularity results are known in the case where $\lambda < J_{s,m} \equiv \Lambda_{N,s}\frac{4N(m-1)(N-2ms)}{m^2(N-2s)^2}$.

However for $J_{s,m} \leqslant \lambda \leqslant \Lambda_{N,s}$, using a different approach based on weighted spaces, we are able to complete the full picture of regularity. As a consequence, we get a complete classification of the fractional regularity of the solution to the above problem.

The first analysis of the KPZ problem (1) is done in Section 4. We begin by considering the case where $\mathfrak{F}(u)(x) = |(-\Delta)^{\frac{s}{2}}u(x)|$. Using suitable radial computations in the whole space, we derive the existence of a critical exponent $p_+(\lambda, s)$ such that if $p > p_+(\lambda, s)$, then for all $\rho > 0$, the problem (1) has no positive solution in a weak sense. Some other non existence results are proved for ρ large under technical condition on p.

The case $p < p_+(\lambda, s)$ is analyzed in Subsection 4.2. Under the hypothesis that f is bounded, we are able to show the existence of a supersolution for ρ small. Moreover, for $p < \frac{N}{N-s}$, and for all $f \in L^1(\Omega)$ that satisfy a suitable integral condition near the origin, we are able to show the existence of a weak solution for $\rho < \rho^*$.

In Section 5 we treat the KPZ problem, namely equation (1), under the presence of another version of the non local gradient.

More precisely, we consider the case where $\mathfrak{F}(u)(x) = \left(\frac{a_{N,s}}{2}\int_{\boldsymbol{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dy\right)^{\frac{1}{2}}$. Then, also in this case, we are able to show the existence of a critical exponent $p_+(\lambda, s)$ such that non existence holds if $p > p_+(\lambda)$. The proof of the non existence in this case is more technical and need some additional estimates.

Finally, at the end of the section we formulate some interesting open problems that may describe a full picture for the existence in our problem.

2. Regularity results and useful tools

The goal of this section is to establish some useful tools and definitions that will play an important role in what follows.

Definition 2.1. Let $\Omega \subset \boldsymbol{R}^N$ be a bounded domain and $s \in (0,1)$. For $p \in [1,\infty)$, the fractional Sobolev space $W^{s,p}(\Omega)$ is defined by

$$W^{s,p}(\Omega) := \left\{u \in L^p(\Omega) : \iint_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}dxdy < \infty\right\}.$$

$W^{s,p}(\Omega)$ is a Banach space endowed with the norm

$$||u||_{W^{s,p}(\Omega)} := \left(||u||^p_{L^p(\Omega)} + \iint_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}dxdy\right)^{\frac{1}{p}}.$$

The space $W^{s,p}_0(\Omega)$ is defined as follows:

$$W^{s,p}_0(\Omega) := \left\{u \in W^{s,p}(\boldsymbol{R}^N) : u = 0 \text{ in } \boldsymbol{R}^N \setminus \Omega\right\}.$$

This is a Banach space endowed with the norm

$$||u||_{W^{s,p}_0(\Omega)} := \left(\iint_{D_\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}dxdy\right)^{1/p},$$

where

$$D_\Omega := (\boldsymbol{R}^N \times \boldsymbol{R}^N) \setminus (C\Omega \times C\Omega) = (\Omega \times \boldsymbol{R}^N) \cup (C\Omega \times \Omega).$$

Now, for $s \in (0,1)$ and $1 \leqslant p < +\infty$ we define the Bessel potential space by setting

$$L^{s,p}(\mathbb{R}^N) \;:=\; \overline{\{u \in C^\infty_0(\mathbb{R}^N)\}}^{||.||_{L^{s,p}(\mathbb{R}^N)}},$$

where

$$||u||_{L^{s,p}(\mathbb{R}^N)} = ||(1-\Delta)^{\frac{s}{2}}u||_{L^p(\mathbb{R}^N)} \quad \text{and} \quad (1-\Delta)^{\frac{s}{2}}u = \mathcal{F}^{-1}((1+|\cdot|^2)^{\frac{s}{2}}\mathcal{F}u), \quad \forall\, u \in C^\infty_c(\mathbb{R}^N).$$

Let us stress that, in the case where $s \in (0,1)$ and $1 < p < +\infty$,

$$||u||_{L^{s,p}(\mathbb{R}^N)} := ||u||_{L^p(\mathbb{R}^N)} + ||(-\Delta)^{\frac{s}{2}}u||_{L^p(\mathbb{R}^N)}$$

is an equivalent norm for $L^{s,p}(\mathbb{R}^N)$ (see e.g., [1, page 5] for a precise explanation of this fact). Let us as well recall that, for all $0 < \epsilon < s < 1$ and all $1 < p < +\infty$, by [13, Theorem 7.63, (g)], we have

$$L^{s+\epsilon,p}(\mathbb{R}^N) \subset W^{s,p}(\mathbb{R}^N) \subset L^{s-\epsilon,p}(\mathbb{R}^N).$$

For $\phi \in C_0^\infty(\boldsymbol{R}^N)$ we define the fractional gradient of order s of ϕ by

$$\nabla^s \phi(x) := \int_{\boldsymbol{R}^N} \frac{\phi(x) - \phi(y)}{|x-y|^s} \frac{x-y}{|x-y|} \frac{dy}{|x-y|^N}, \quad \forall\, x \in \boldsymbol{R}^N. \tag{4}$$

Notice that, as it was proved in [46, Theorem 2] and [42, Theorem 1.7], we have

$$\begin{aligned} L^{s,p}(\boldsymbol{R}^N) &:= \left\{u \in L^p(\boldsymbol{R}^N) \text{ such that } |\nabla^s u| \in L^p(\boldsymbol{R}^N)\right\} \\ &= \left\{u \in L^p(\boldsymbol{R}^N) \text{ such that } |(-\Delta)^{\frac{s}{2}} u| \in L^p(\boldsymbol{R}^N)\right\} \end{aligned}$$

with the equivalent norms

$$|||u|||_{L^{s,p}(\boldsymbol{R}^N)} := ||u||_{L^p(\boldsymbol{R}^N)} + ||\nabla^s u||_{L^p(\boldsymbol{R}^N)} \simeq ||u||_{L^p(\boldsymbol{R}^N)} + ||(-\Delta)^{\frac{s}{2}} u||_{L^p(\boldsymbol{R}^N)}.$$

Another type of "nonlocal gradient" can be defined also by

$$\mathbb{D}_s(u)(x) = \left(\frac{a_{N,s}}{2} \int_{\boldsymbol{R}^N} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} dy\right)^{\frac{1}{2}}. \tag{5}$$

We refer to [20] and [36] for some motivation of this non local version of the gradient.

In this case one has

$$\lim_{s \to 1^-} (1-s)\mathbb{D}_s^2(u(x)) = |\nabla u(x)|^2, \quad \forall\, u \in C_0^\infty(\mathbb{R}^N). \tag{6}$$

If $p > \frac{2N}{N+2s}$, it was proved in [46] that the Bessel potential space $L^{s,p}(\boldsymbol{R}^N)$ can be defined also as the set of functions $u \in L^p(\boldsymbol{R}^N)$ such that $\mathbb{D}_s(u) \in L^p(\boldsymbol{R}^N)$. The space $L^{s,p}(\boldsymbol{R}^N)$ can be equipped with the equivalent norms

$$|||u|||_{L^{s,p}(\boldsymbol{R}^N)} = ||u||_{L^p(\boldsymbol{R}^N)} + ||\mathbb{D}_s(u)||_{L^p(\boldsymbol{R}^N)}.$$

The next Sobolev inequality in $L^{s,p}(\boldsymbol{R}^N)$ is proved in [13], see also [25].

Theorem 2.2. *Let $1 < p < \infty$ and $s \in (0,1)$ be such that $sp < N$. Then there exist two positive constants $S_1 := S_2(N,p,s)$ and $S_2 := S_1(N,p,s)$ such that for all $u \in L^{s,p}(\boldsymbol{R}^N)$, we have*

$$S_1 ||u||_{L^{p_s^*}(\boldsymbol{R}^N)} \leqslant ||\nabla^s u||_{L^p(\boldsymbol{R}^N)},$$

and

$$S_2 ||u||_{L^{p_s^*}(\boldsymbol{R}^N)} \leqslant ||(-\Delta)^{\frac{s}{2}} u||_{L^p(\boldsymbol{R}^N)},$$

with $p_s^* = \frac{pN}{N-ps}$.

If $\Omega \subset \boldsymbol{R}^N$, we define the space $L_0^{s,p}(\Omega)$ as the set of functions $u \in L^{s,p}(\boldsymbol{R}^N)$ with $u = 0$ in $\boldsymbol{R}^N \backslash \Omega$.

From Lemma 1 in [46], if $p > \frac{2N}{N+2s}$ and Ω is a bounded domain, then there exist $C_1 := C_1(\Omega, N, p, s)$ and $C_2 := C_2(\Omega, N, p, s)$, two positive constants, such that for all $u \in L_0^{s,p}(\Omega)$

$$C_1 |||u|||_{L^{s,p}(\boldsymbol{R}^N)} \leqslant ||\mathbb{D}_s(u)||_{L^p(\boldsymbol{R}^N)} \leqslant C_2 |||u|||_{L^{s,p}(\boldsymbol{R}^N)}.$$

Notice that if Ω is a bounded domain, we can endow $L_0^{s,p}(\Omega)$ with the equivalent norms $||\nabla^s u||_{L^p(\boldsymbol{R}^N)}$ or $||(-\Delta)^{\frac{s}{2}} u||_{L^p(\boldsymbol{R}^N)}$. In the same way, by assuming in addition that $p > \frac{2N}{N+2s}$, then we can equip $L_0^{s,p}(\Omega)$ also with the equivalent norms $||\mathbb{D}_s(u)||_{L^p(\boldsymbol{R}^N)}$. We refer to [47] for more details about the properties of the Bessel potential space and its relation with the fractional Sobolev space.

The next Hardy inequality will be useful in order to prove the non existence result above the critical exponent. See [7] for the proof.

Proposition 2.3. *Let $\Omega \subset \mathbb{R}^N$ be a regular domain with $0 \in \Omega$ and $0 < s < 1$. Suppose that $p > \frac{2N}{N+2s}$ with $ps < N$ and define*

$$\mathcal{L}(\Omega) := \inf \left\{ \frac{\displaystyle\int_{\boldsymbol{R}^N} (\mathbb{D}_s(\phi)(x))^p dx}{\displaystyle\int_\Omega \frac{|\phi(x)|^p}{|x|^{ps}} dx} : \phi \in C_0^\infty(\Omega) \setminus \{0\} \right\}. \tag{7}$$

Then $\mathcal{L}(\Omega) > 0$ and $\mathcal{L}(\Omega) = \mathcal{L}$ does not depends on Ω. Moreover, the weight $|x|^{-ps}$ is optimal in the sense that, for all $\varepsilon > 0$ we have

$$\inf \left\{ \frac{\displaystyle\int_{\boldsymbol{R}^N} (\mathbb{D}_s(\phi)(x))^p dx}{\displaystyle\int_\Omega \frac{|\phi(x)|^p}{|x|^{ps+\varepsilon}} dx} : \phi \in C_0^\infty(\Omega) \setminus \{0\} \right\} = 0.$$

Finally, we recall the next standard result from harmonic analysis. See for instance [45, Theorem I, Section 1.2, Chapter V].

Theorem 2.4. *Let $0 < \nu < N$ and $1 \leqslant p < \ell < \infty$ be such that $\dfrac{1}{\ell} + 1 = \dfrac{1}{p} + \dfrac{\nu}{N}$. For $g \in L^p(\boldsymbol{R}^N)$, we define*

$$J_\nu(g)(x) = \int_{\boldsymbol{R}^N} \frac{g(y)}{|x-y|^\nu} dy.$$

Then, it follows that:

a) J_ν is well defined in the sense that the integral converges absolutely for almost all $x \in \mathbb{R}^N$.

b) If $p > 1$, then $||J_\nu(g)||_{L^\ell(\boldsymbol{R}^N)} \leqslant c_{p,l} ||g||_{L^p(\boldsymbol{R}^N)}$.

c) If $p = 1$, then $\Big|\{x \in \mathbb{R}^N \,|J_\nu(g)(x) > \sigma\}\Big| \leqslant \left(\dfrac{A||g||_{L^1(\boldsymbol{R}^N)}}{\sigma}\right)^\ell$.

2.1. Regularity and useful estimates

The goal of this section is to state some well known results about the regularity of the Poisson equation

$$\begin{cases} (-\Delta)^s u &= g \quad \text{in } \Omega, \\ \quad\; u &= 0 \quad \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{8}$$

where Ω is a bounded regular domain of $\boldsymbol{R}^N$ and $g \in L^m(\Omega)$ with $m \geqslant 1$. We begin by the sense for which solutions are defined.

Definition 2.5. We define the class of test functions

$$\mathcal{T}(\Omega) = \{\phi \mid (-\Delta)^s(\phi) = \psi \text{ in } \Omega, \quad \phi = 0 \text{ in } \mathbb{R}^N \setminus \Omega, \quad \psi \in C_0^\infty(\Omega)\}. \tag{9}$$

Notice that if $v \in \mathcal{T}(\Omega)$ then, using the results in [34], $v \in H_0^s(\Omega) \cap L^\infty(\Omega)$. Moreover, according to the regularity theory developed in [43], if Ω is smooth enough, there exists a constant $\beta > 0$ (that depends only on the structural constants) such that $v \in C^\beta(\Omega)$ (see also [31]).

Definition 2.6. We say that $u \in L^1(\Omega)$ is a *weak solution* to (8) if for $g \in L^1(\Omega)$ we have that

$$\int_\Omega u\psi dx = \int_\Omega g\phi dx,$$

for any $\phi \in \mathcal{T}(\Omega)$ with $\psi \in C_0^\infty(\Omega)$.

Recall also the definition of the truncation operator T_k,

$$T_k(\sigma) = \max\{-k; \min\{k, \sigma\}\} \text{ and } G_k(\sigma) = \sigma - T_k(\sigma). \tag{10}$$

From [2, 22, 34] we have the next existence result.

Theorem 2.7. *Suppose that $g \in L^1(\Omega)$, then problem* (8) *has a unique weak solution u obtained as the limit of $\{u_n\}_{n\in\mathbb{N}}$, the sequence of unique solutions to the approximating problems*

$$\begin{cases} (-\Delta)^s u_n &= g_n(x) \quad \text{in } \Omega, \\ \quad\; u_n &= 0 \qquad\;\; \text{in } \boldsymbol{R}^N \backslash \Omega, \end{cases} \tag{11}$$

with $g_n = T_n(g)$. Moreover,

$$T_k(u_n) \to T_k(u) \text{ strongly in } H_0^s(\Omega), \quad \forall k > 0, \tag{12}$$

$$u \in L^q(\Omega), \qquad \forall\, q \in [1, \frac{N}{N-2s}) \tag{13}$$

and

$$\left|(-\Delta)^{\frac{s}{2}} u\right| \in L^r(\Omega), \qquad \forall\, r \in [1, \frac{N}{N-s}). \tag{14}$$

In addition, if $s > \frac{1}{2}$, then $u \in W_0^{1,q}(\Omega)$ for all $1 \leqslant q < \frac{N}{N-(2s-1)}$ and $u_n \to u$ strongly in $W_0^{1,q}(\Omega)$.

In what follows we denote $\mathcal{G}_s$ the Green function associated to the fractional laplacian $(-\Delta)^s$.
Notice that $\mathcal{G}_s(x,y)$ solves the problem

$$\begin{cases} (-\Delta)^s_y \mathcal{G}_s(x,y) = \delta_x(y) & \text{if } y \in \Omega, \\ \mathcal{G}_s(x,y) = 0 & \text{if } y \in \mathbb{R}^N \backslash \Omega, \end{cases} \tag{15}$$

where $x \in \Omega$ is fixed and δ_x is Dirac's delta function.
It is clear that if u is the unique weak solution to problem (8), then

$$u(x) = \int_\Omega \mathcal{G}_s(x,y) g(y) dy.$$

We collect in the next Proposition some useful properties of the Green function $\mathcal{G}_s$ (See [21] and [19] for the proof).

Proposition 2.8. *Assume that $s \in (0,1)$. Then, for almost every x, $y \in \Omega$, we have*

$$\begin{aligned} \mathcal{G}_s(x,y) &\simeq \frac{1}{|x-y|^{N-2s}} \Big(\frac{\delta^s(x)}{|x-y|^s} \wedge 1 \Big) \Big(\frac{\delta^s(y)}{|x-y|^s} \wedge 1 \Big) \\ &\simeq \frac{1}{|x-y|^{N-2s}} \Big(\frac{\delta^s(x)\delta^s(y)}{|x-y|^{2s}} \wedge 1 \Big). \end{aligned} \tag{16}$$

In particular, we have

$$\mathcal{G}_s(x,y) \leqslant C_1 \min\{ \frac{1}{|x-y|^{N-2s}}, \frac{\delta^s(x)}{|x-y|^{N-s}}, \frac{\delta^s(y)}{|x-y|^{N-s}} \} \quad \text{for a.e. } x, y \in \Omega. \tag{17}$$

In the case where $g \in L^m(\Omega)$, we can improve the regularity results of Theorem 2.7. More precisely from [11], we have the next theorem.

Theorem 2.9. *Assume that $g \in L^m(\Omega)$ with $m > 1$ and let u be the unique solution to problem* (8)*, then there exists a positive constant $C := C(N, s, m, \Omega)$ (that can change from a line to another one), such that*

1) If $1 < m < \frac{N}{2s}$, then $u \in L^{\frac{mN}{N-2ms}}(\Omega)$, $\dfrac{u}{\delta^s} \in L^{\frac{mN}{N-ms}}(\Omega)$ and

$$||u||_{L^{\frac{mN}{N-2ms}}(\Omega)} + ||\frac{u}{\delta^s}||_{L^{\frac{mN}{N-ms}}(\Omega)} \leqslant C||g||_{L^m(\Omega)}.$$

2) If $m = \frac{N}{2s}$, then $u \in L^r(\Omega)$ for all $r < \infty$, $\dfrac{u}{\delta^s} \in L^{\frac{mN}{N-ms}}(\Omega)$ and

$$||u||_{L^r(\Omega)} + ||\frac{u}{\delta^s}||_{L^{\frac{mN}{N-ms}}(\Omega)} \leqslant C||g||_{L^m(\Omega)}.$$

3) If $\frac{N}{2s} < m < \frac{N}{s}$, then $u \in L^\infty(\Omega)$, $\dfrac{u}{\delta^s} \in L^{\frac{mN}{N-ms}}(\Omega)$ and

$$||u||_{L^\infty(\Omega)} + ||\frac{u}{\delta^s}||_{L^{\frac{mN}{N-ms}}(\Omega)} \leqslant C||g||_{L^m(\Omega)}.$$

4) If $m = \frac{N}{s}$, *then* $u \in L^\infty(\Omega)$, $\dfrac{u}{\delta^s} \in L^p(\Omega)$ *for all* $p < \infty$ *and*

$$||u||_{L^\infty(\Omega)} + ||\frac{u}{\delta^s}||_{L^p(\Omega)} \leqslant C||g||_{L^m(\Omega)}.$$

5) If $m > \frac{N}{s}$, *then* $u \in L^\infty(\Omega)$, $\dfrac{u}{\delta^s} \in L^\infty(\Omega)$ *and*

$$||u||_{L^\infty(\Omega)} + ||\frac{u}{\delta^s}||_{L^\infty(\Omega)} \leqslant C||g||_{L^m(\Omega)}.$$

Related to the fractional regularity of the solution to problem (8), a global fractional Calderon-Zygmund regularity result was obtained recently in [6].

Theorem 2.10. *Let* $s \in (0, 1)$ *and consider* u *to be the (unique) weak solution to problem* (8) *with* $f \in L^m(\Omega)$. *Then we have*

1) If $m \geqslant \frac{N}{s}$, *then for all* $1 \leqslant p < \infty$, *there exists a positive constant* $C = C(N, s, p, m, \Omega)$ *such that*

$$||(-\Delta)^{\frac{s}{2}} u||_{L^p(\mathbb{R}^N)} \leqslant C||g||_{L^m(\Omega)}.$$

Moreover $u \in L^{s,p}(\mathbb{R}^N)$ *for all* $1 \leqslant p < \infty$ *and*

$$||u||_{L^{s,p}(\mathbb{R}^N)} \leqslant C||g||_{L^m(\Omega)}.$$

2) $1 \leqslant m < \frac{N}{s}$, *then, for all* $1 \leqslant p < \frac{mN}{N-ms}$, *there exists a positive constant* $C = C(N, s, p, m, \Omega)$ *such that*

$$||(-\Delta)^{\frac{s}{2}} u||_{L^p(\mathbb{R}^N)} \leqslant C\, ||g||_{L^m(\Omega)}\, .$$

Hence $u \in L^{s,p}(\mathbb{R}^N)$ *for all* $1 \leqslant p < \frac{mN}{N-ms}$ *and*

$$||u||_{L^{s,p}(\mathbb{R}^N)} \leqslant C||g||_{L^m(\Omega)}.$$

As a direct consequence of the relation between the fractional Sobolev space $W^{s,p}(\mathbb{R}^N)$ and the Bessel potential space $L^{s,p}(\mathbb{R}^N)$, we get the next result.

Corollary 2.11. *Let* $s \in (0, 1)$. *Consider* u *to be the unique solution of problem* (8) *with* $g \in L^m(\Omega)$. *Then*

1) If $1 \leqslant m < \frac{N}{s}$,
we have, for all $1 < p < \frac{mN}{N-ms}$, *that there exists* $C = C(N, s, m, p, \Omega)$ *such that*

$$||u||_{W^{s,p}(\mathbb{R}^N)} \leqslant C||g||_{L^m(\Omega)}.$$

2) If $m \geqslant \frac{N}{s}$ *then, for all* $1 < p < \infty$, *there exists* $C = C(N, s, m, p, \Omega)$ *such that*

$$||u||_{W^{s,p}(\mathbb{R}^N)} \leqslant C||g||_{L^m(\Omega)}.$$

Let us recall that another version of the nonlocal gradient is given by

$$\mathbb{D}_s(u)(x) = \left(\frac{a_{N,s}}{2} \int_{R^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dy\right)^{\frac{1}{2}}.$$

Taking into consideration the result of [46], we get the following corollary.

Corollary 2.12. *Assume that the conditions of Theorem 2.10 hold. Then we have*

1) If $m > \frac{N}{s}$, then for all $\frac{2N}{N+2s} < p < \infty$, there exists $C = C(N, s, m, p, \Omega)$ such that

$$\|\mathbb{D}_s(u)\|_{L^p(\mathbb{R}^N)} \leqslant C\|g\|_{L^m(\Omega)}.$$

2) If $\frac{2N}{N+4s} < m \leqslant \frac{N}{s}$, then for all $\frac{2N}{N+2s} < p < \frac{mN}{N-ms}$, there exists $C = C(N, s, m, p, \Omega)$ such that

$$\|\mathbb{D}_s(u)\|_{L^p(\mathbb{R}^N)} \leqslant C\|g\|_{L^m(\Omega)}.$$

3. Regularity results under the presence of the Hardy potential

In this subsection we analyze the question of regularity of the solution to the problem

$$\begin{cases} (-\Delta)^s u &= \lambda \dfrac{u}{|x|^{2s}} + f \quad \text{in } \Omega, \\ u &= 0 \qquad\qquad\quad \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{18}$$

in Lebesgue spaces and fractional Sobolev spaces according to the regularity of the datum f. Here $\Omega \subset \boldsymbol{R}^N$ is a bounded regular domain containing the origin and $s \in (0, 1)$. We will suppose that $f \in L^m(\Omega)$ with $m \geqslant 1$ and $0 < \lambda < \Lambda_{N,s}$.

If $f = 0$, we define the radial *potential* $v_{\pm\alpha_\lambda}(x) = |x|^{-\frac{N-2s}{2} \pm \alpha_\lambda}$ with α_λ given by

$$\lambda = \lambda(\alpha_\lambda) = \lambda(-\alpha_\lambda) = \frac{2^{2s}\,\Gamma(\frac{N+2s+2\alpha_\lambda}{4})\Gamma(\frac{N+2s-2\alpha_\lambda}{4})}{\Gamma(\frac{N-2s+2\alpha_\lambda}{4})\Gamma(\frac{N-2s-2\alpha_\lambda}{4})}. \tag{19}$$

From [8], we obtain that $v_{\pm\alpha_\lambda}$ solves the homogeneous equation

$$(-\Delta)^s u = \lambda \frac{u}{|x|^{2s}} \text{ in } \mathbb{R}^N \setminus \{0\}. \tag{20}$$

It is clear that $\lambda(\alpha) = \lambda(-\alpha) = m_{\alpha_\lambda} m_{-\alpha_\lambda}$, with $m_{\alpha_\lambda} = 2^{\alpha_\lambda + s} \dfrac{\Gamma(\frac{N+2s+2\alpha_\lambda}{4})}{\Gamma(\frac{N-2s-2\alpha_\lambda}{4})}$.

Notice that

$$0 < \lambda(\alpha_\lambda) = \lambda(-\alpha_\lambda) \leqslant \Lambda_{N,s} \text{ if and only if } 0 \leqslant \alpha_\lambda < \frac{N-2s}{2}.$$

Define

$$\mu(\lambda) = \frac{N-2s}{2} - \alpha_\lambda \text{ and } \bar{\mu}(\lambda) = \frac{N-2s}{2} + \alpha_\lambda. \tag{21}$$

For $0 < \lambda < \Lambda_{N,s}$, then $0 < \mu(\lambda) < \dfrac{N-2s}{2} < \bar{\mu} < (N-2s)$. Since $N - 2\mu(\lambda) - 2s = 2\alpha_\lambda > 0$ and $N - 2\bar{\mu}(\lambda) - 2s = -2\alpha_\lambda < 0$, then $(-\Delta)^{s/2}(|x|^{-\mu(\lambda)}) \in L^2(\Omega)$, but $(-\Delta)^{s/2}(|x|^{-\bar{\mu}(\lambda)})$ does not.

As it was proved in [8], if $f \in L^1(\Omega)$, then the existence of a solution to problem (18) is guaranteed under the necessary and sufficient condition $\displaystyle\int_{B_r(0)} f|x|^{-\mu(\lambda)}dx < \infty$. Hence, throughout this section this condition will be assumed.

The first result concerning the behavior in the neighborhood of zero is given by the next Proposition proved in [8].

Proposition 3.1. *Let $u \in L^1_{loc}(\boldsymbol{R}^N)$ be such that $u \geqslant 0$ in $\boldsymbol{R}^N$ and $(-\Delta)^s u \in L^1_{loc}(\Omega)$. Assume that*

$$(-\Delta)^s u \geqslant \lambda \frac{u}{|x|^{2s}} \text{ in } \Omega, \quad 0 < \lambda < \Lambda_{N,s}.$$

Then, there exists $r > 0$ and a positive constant $C \equiv C(r, N, \lambda)$ such that

$$u(x) \geqslant C|x|^{-\mu(\lambda)} = C|x|^{-\frac{N-2s}{2}+\alpha_\lambda} \text{ in } B_r(0) \subset\subset \Omega.$$

We are now in position to prove the first regularity results, in fractional Sobolev space, to the solution of problem (18).

Theorem 3.2. *Assume that $f \in L^m(\Omega)$ with $m > 1$ satisfying the condition $\int_{B_r(0)} f|x|^{-\mu(\lambda)}dx < \infty$. Let $u \in L^1(\Omega)$ to be the unique weak solution to (18) with $\lambda < \Lambda_{N,s}$. Then there exists a positive constant $C = C(N, m, p, s, \Omega)$ such that*

1) *If $m \geqslant \frac{N}{2s}$, then $\frac{u}{|x|^{2s}} \in L^\sigma(\Omega)$ for all $1 \leqslant \sigma < \frac{N}{\mu(\lambda)+2s}$ and $\left|(-\Delta)^{\frac{s}{2}} u\right| \in L^p(\boldsymbol{R}^N)$ for all $1 \leqslant p < \frac{N}{\mu(\lambda)+s}$. Moreover we have*
$$||(-\Delta)^{\frac{s}{2}} u||_{L^p(\boldsymbol{R}^N)} \leqslant C||f||_{L^m(\Omega)}.$$
2) *If $1 < m < \frac{N}{2s}$ and $\lambda < J_{s,m} \equiv \Lambda_{N,s} \dfrac{4N(m-1)(N-2ms)}{m^2(N-2s)^2}$, then $\left|(-\Delta)^{\frac{s}{2}} u\right| \in L^p(\boldsymbol{R}^N)$ for all $1 \leqslant p < \frac{Nm}{N-ms}$. Moreover we have*
$$||(-\Delta)^{\frac{s}{2}} u||_{L^p(\mathbb{R}^N)} \leqslant C||f||_{L^m(\Omega)}.$$

Proof. We begin by analyzing the first case. Assume that $f \in L^m(\Omega)$ with $m > \frac{N}{2s}$. From Theorem 4.1 in [8], we obtain that $u(x) \leqslant C|x|^{-\mu(\lambda)}\chi_\Omega$. Hence $\frac{u}{|x|^{2s}} \leqslant C|x|^{-\mu(\lambda)-2s}\chi_\Omega$. As a consequence, we deduce that $\frac{u}{|x|^{2s}} \in L^\sigma(\Omega)$ for all $1 \leqslant \sigma < \frac{N}{\mu(\lambda)+2s}$.

Setting $g \equiv \frac{u}{|x|^{2s}} + f$, it follows that $g \in L^\sigma(\Omega)$ for all $\sigma < \frac{N}{\mu(\lambda)+2s}$. Using the regularity result in Theorem 2.9, we conclude that $u \in L^t(\Omega)$ for all $t < \frac{N}{\mu(\lambda)}$. Now by Theorem 2.10, it holds that $\left|(-\Delta)^{\frac{s}{2}} u\right| \in L^p(\boldsymbol{R}^N)$ for all $1 \leqslant p < \frac{N}{\mu(\lambda)+s}$ and

$$||(-\Delta)^{\frac{s}{2}} u||_{L^p(\boldsymbol{R}^N)} \leqslant C||f||_{L^m(\Omega)}.$$

Hence we conclude.

We treat now the case $1 < m < \frac{N}{2s}$ and $0 < \lambda < J_{s,m} \equiv \Lambda_{N,s} \dfrac{4N(m-1)(N-2ms)}{m^2(N-2s)^2}$.

Recall that u solves problem (18). Then by Theorem 4.2 of [8], we get the existence of positive constant $C(N, s, m)$ such that

$$||u||_{L^{m_s^{**}}(\Omega)} \leqslant C||f||_{L^m(\Omega)} \text{ where } m_s^{**} = \frac{mN}{N-2sm}. \tag{22}$$

Since $p < \frac{Nm}{N-ms}$, then we get the existence of $m_1 < m$ such that $p < \frac{Nm_1}{N-m_1 s}$. Fixed $m_1 < m$, using Hölder inequality we deduce that

$$\int\limits_\Omega \frac{u^{m_1}}{|x|^{2sm_1}} dx \leqslant C.$$

Since $m_1 < m$, it follows that $g \equiv \lambda \frac{u}{|x|^{2s}} + f \in L^{m_1}(\Omega)$.
On the other hand $m_1 < m < \frac{N}{2s} < \frac{N}{s}$, therefore using the regularity result in Theorem 2.10, we deduce that

$$||(-\Delta)^{\frac{s}{2}} u||_{L^p(\mathbb{R}^N)} \leqslant C||g||_{L^{m_1}(\Omega)} \text{ for all } p < \frac{Nm_1}{N - m_1 s}.$$

Thus

$$||(-\Delta)^{\frac{s}{2}} u||_{L^p(\mathbb{R}^N)} \leqslant C||f||_{L^{m}(\Omega)} \text{ for all } p < \frac{Nm}{N - ms},$$

and the result follows in this case. □

In order to treat the general case $J_{s,m} \equiv \Lambda_{N,s} \dfrac{4N(m-1)(N-2ms)}{m^2(N-2s)^2} \leqslant \lambda < \Lambda_{N,s}$, we need to develop a new approach.

Let u be the unique weak solution to problem (18). Setting $v(x) := |x|^{\mu(\lambda)} u(x)$, it follows that v solves the problem

$$\begin{cases} L_{\mu(\lambda)} v &= |x|^{-\mu(\lambda)} f(x) =: \widetilde{f}(x) \quad \text{in } \Omega, \\ v &= 0 \quad \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{23}$$

with

$$L_\gamma v := a_{N,s} \text{ P.V.} \int_{\mathbb{R}^N} \frac{v(x) - v(y)}{|x - y|^{N+2s}} \frac{dy}{|x|^\gamma |y|^\gamma}. \tag{24}$$

Since $\displaystyle\int_{B_r(0)} f|x|^{-\mu(\lambda)} dx < \infty$, then $\widetilde{f} \in L^1(\Omega)$. Thus v can be seen as the unique entropy solution to problem (23) as defined in [2]. Following closely the argument used in [4], we get the next general regularity result.

Theorem 3.3. *Let $s \in (0,1)$ and $0 < \lambda < \Lambda_{N,s}$. Assume that $\widetilde{f} \in L^q(\Omega, |x|^{\beta(q-1)} dx)$ with $q > 1$ and $\frac{2N\mu(\lambda)}{N-2s} \leqslant \beta \leqslant 2(\mu(\lambda) + s)$. Let us denote by $C := C(N, \beta, \lambda, s, q, \Omega)$ a positive constant that may change from line to other.*

Then if v solves problem (23), we have

1) If $\beta < 2(\mu(\lambda) + s)$ and $q > \frac{(N-\beta)}{2(\mu(\lambda)+s)-\beta}$, then $v \in L^\infty(\Omega)$. Moreover,

$$||v||_{L^\infty(\Omega)} \leqslant C||\widetilde{f}||_{L^q(\Omega, |x|^{\beta(q-1)} dx)}.$$

2) If $\beta < 2(\mu(\lambda) + s)$ and $q = \frac{(N-\beta)}{2(\mu(\lambda)+s)-\beta}$, then $v \in L^r(\Omega, |x|^{-\beta} dx)$, for all $1 \leqslant r < +\infty$. Moreover

$$\left(\int_\Omega |v|^r \, |x|^{-\beta} dx \right)^{\frac{1}{r}} \leqslant C||\widetilde{f}||_{L^q(\Omega, |x|^{\beta(q-1)} dx)}.$$

3) If either $\beta = 2(\mu(\lambda) + s)$ or $\beta < 2(\mu(\lambda) + s)$ and $1 < q < \frac{N-\beta}{2(\mu(\lambda)+s)-\beta}$, then $|v|^r \in L^1(\Omega, |x|^{-\beta} dx)$, for all $1 \leqslant r \leqslant r^* = \frac{(N-\beta)q}{N-\beta-q(2(\mu(\lambda)+s)-\beta)}$. Moreover

$$\left(\int_\Omega |v|^r \, |x|^{-\beta} dx \right)^{\frac{1}{r}} \leqslant C||\widetilde{f}||_{L^q(\Omega, |x|^{\beta(q-1)} dx)}.$$

Before proving the previous Theorem, we recall the following weighted fractional Caffarelli-Khon-Nirenberg inequality, whose proof can be found in [3, 15, 37].

Theorem 3.4. *Assume that* $s \in (0,1)$ *and* $-2s < \gamma < \frac{N-2s}{2}$. *Let* $\theta \in [\gamma, \gamma + s]$, *then there exists a positive constant C:=C(N, s ,γ, θ), such that for all* $\phi \in C_0^\infty(\boldsymbol{R}^N)$, *we have*

$$C\left(\int_{\boldsymbol{R}^N} \frac{|\phi|^{\widehat{\sigma}}}{|x|^{\widehat{\sigma}\theta}}dx\right)^{\frac{2}{\widehat{\sigma}}} \leqslant \int_{\boldsymbol{R}^N}\int_{\boldsymbol{R}^N} \frac{|\phi(x)-\phi(y)|^2}{|x-y|^{N+2s}|x|^{\gamma}|y|^{\gamma}}dxdy,$$

with $\widehat{\sigma} = \frac{2N}{N-2s+2(\theta-\gamma)}$.

Setting $\beta = \widehat{\sigma}\theta$, we obtain that $\frac{2N\gamma}{N-2s} \leqslant \beta \leqslant 2(\gamma + s)$ and

$$C\left(\int_{\boldsymbol{R}^N} \frac{|\phi|^{\widehat{\sigma}}}{|x|^{\beta}}dx\right)^{\frac{2}{\widehat{\sigma}}} \leqslant \int_{\boldsymbol{R}^N}\int_{\boldsymbol{R}^N} \frac{|\phi(x)-\phi(y)|^2}{|x-y|^{N+2s}|x|^{\gamma}|y|^{\gamma}}dxdy. \tag{25}$$

Notice that by substituting the value of θ in the formula of $\widehat{\sigma}$, we reach that $\widehat{\sigma} = \frac{2(N-\beta)}{N-2(\gamma+s)}$.

Proof of Theorem 3.3. Notice that, using the notation of Theorem 3.4, then, in our case, we have $\gamma = \mu(\lambda) \in (0, \frac{N-2s}{2})$.

The main idea of the proof is to use a suitable test function and an approximation argument. To make the paper self contained as possible, we include here all the details.

Without loss of generality we can assume that $q > 1$ and $\widetilde{f} \gneqq 0$. Thus $v \gneqq 0$ in $\boldsymbol{R}^N$.

Consider the following approximating problem

$$\begin{cases} L_{\mu(\lambda)} v_n &= \widetilde{f}_n(x) \quad \text{in } \Omega, \\ v_n &= 0 \quad \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{26}$$

where $\widetilde{f}_n(x) = T_n(\widetilde{f}(x))$ is the truncation of $\widetilde{f}_n$ as defined in (10).

Since v is the unique solution to problem (23), at least in the entropy sense, then

$$v_n \uparrow v \text{ a.e. in } \boldsymbol{R}^N \text{ and } v_n \uparrow v \text{ strongly in } L^1(\boldsymbol{R}^N).$$

In the rest of the proof, we denote by C any positive constant that depends only on N, s, q, r, Ω, and is independent of $n, \widetilde{f}, v$, that may change from line to other.

It is not difficult to show that v_n is bounded. Thus, for $\alpha > 0$ fixed, to be chosen later, using v_n^α as a test function in (26), it holds that

$$\frac{1}{2}\iint_{D_\Omega} \frac{(v_n(x)-v_n(y))(v_n^\alpha(x)-v_n^\alpha(y))}{|x-y|^{N+2s}|x|^{\mu(\lambda)}|y|^{\mu(\lambda)}}dxdy = \int\limits_\Omega \widetilde{f}_n v_n^\alpha(x)\,dx.$$

By the algebraic inequality

$$(a-b)(a^\alpha - b^\alpha) \geqslant C(a^{\frac{\alpha+1}{2}} - b^{\frac{\alpha+1}{2}})^2,$$

we reach that

$$C\iint_{D_\Omega} \frac{(v_n^{\frac{\alpha+1}{2}}(x) - v_n^{\frac{\alpha+1}{2}}(y))^2}{|x-y|^{N+2s}|x|^{\mu(\lambda)}|y|^{\mu(\lambda)}}dxdy \leqslant \int\limits_\Omega \widetilde{f}_n(x)v_n^\alpha(x)dx.$$

Using the weighted fractional Caffarelli-Khon-Nirenberg inequality in Theorem 3.4, we get

$$\iint_{D_\Omega} \frac{(v_n^{\frac{\alpha+1}{2}}(x) - v_n^{\frac{\alpha+1}{2}}(y))^2}{|x-y|^{N+2s}|x|^{\mu(\lambda)}|y|^{\mu(\lambda)}} dxdy \geqslant C\Big(\int\limits_\Omega \frac{v_n^{\frac{(\alpha+1)\widehat{\sigma}}{2}}}{|x|^\beta} dx\Big)^{\frac{2}{\widehat{\sigma}}}.$$

Now by using Hölder's inequality, it holds that

$$\int\limits_\Omega \widetilde{f}_n v_n^\alpha(x) dx \leqslant \Big(\int\limits_\Omega \widetilde{f}_n^q(x)\,|x|^{\beta(q-1)} dx\Big)^{\frac{1}{q}} \Big(\int\limits_\Omega \frac{v_n^{\alpha q'}(x)}{|x|^\beta} dx\Big)^{\frac{1}{q'}}. \tag{27}$$

Hence

$$C\Big(\int\limits_\Omega \frac{(v_n(x))^{\frac{(\alpha+1)\widehat{\sigma}}{2}}}{|x|^\beta} dx\Big)^{\frac{2}{\widehat{\sigma}}} \leqslant \Big(\int\limits_\Omega \widetilde{f}_n^q(x)|x|^{\beta(q-1)}(x) dx\Big)^{\frac{1}{q}} \Big(\int\limits_\Omega \frac{v_n^{\alpha q'}(x)}{|x|^\beta} dx\Big)^{\frac{1}{q'}}. \tag{28}$$

• If $\beta < 2(\mu(\lambda) + s)$ and $\frac{\widehat{\sigma}}{2} > q'$, namely $q > \frac{N-\beta}{2(\mu(\lambda)+s)-\beta}$, in this case we can prove that $v \in L^\infty(\Omega)$. The proof follows using the classical Stampacchia argument as in [44]. Let us give some details. Using $G_k(v_n)$ as a test function (26), it follows that

$$\frac{1}{2}\iint_{D_\Omega} \frac{(v_n(x) - v_n(y))(G_k(v_n(x)) - G_k(v_n(y)))}{|x-y|^{N+2s}|x|^{\mu(\lambda)}|y|^{\mu(\lambda)}} dxdy = \int\limits_\Omega \widetilde{f}_n(x) G_k(v_n(x))\, dx.$$

Since $\frac{\widehat{\sigma}}{2} > q'$, then $\frac{1}{\widehat{\sigma}} + \frac{1}{q} < 1 - \frac{1}{2q'}$. Thus Using the Hölder inequality, we get

$$\begin{aligned} & C\iint_{D_\Omega} \frac{(G_k(v_n(x)) - G_k(v_n(y)))^2}{|x-y|^{N+2s}|x|^{\mu(\lambda)}|y|^{\mu(\lambda)}} dxdy \\ \leqslant\ & \Big(\int\limits_\Omega \widetilde{f}^q(x)\,|x|^{\beta(q-1)} dx\Big)^{\frac{1}{q}} \Big(\int\limits_\Omega \frac{(G_k(v_n(x)))^{\widehat{\sigma}}}{|x|^\beta} dx\Big)^{\frac{1}{\widehat{\sigma}}} \Big|\{x \in \Omega : G_k(v_n(x)) > 0\}\Big|_{|x|^{-\beta}dx}^{1-\frac{1}{\widehat{\sigma}}-\frac{1}{q}}. \end{aligned}$$

Now, by the Caffarelli-Kohn-Nirenberg inequality in (25), we deduce that

$$\Big(\int\limits_\Omega \frac{(G_k(v_n(x)))^{\widehat{\sigma}}}{|x|^\beta} dx\Big)^{\frac{1}{\widehat{\sigma}}} \leqslant \Big(\int\limits_\Omega \widetilde{f}_n^q(x)\,|x|^{\beta(q-1)} dx\Big)^{\frac{1}{q}} \Big|\{x \in \Omega : G_k(v_n(x)) > 0\}\Big|_{|x|^{-\beta}dx}^{1-\frac{1}{\widehat{\sigma}}-\frac{1}{q}}.$$

Hence

$$\Big|\{x \in \Omega : v_n(x) > k\}\Big|_{|x|^{-\beta}dx}^{\frac{1}{\widehat{\sigma}}} \leqslant C\Big|\{x \in \Omega : v_n(x) > k\}\Big|_{|x|^{-\beta}dx}^{1-\frac{1}{\widehat{\sigma}}-\frac{1}{q}}.$$

Thus using the standard Stampacchia argument, see [44], we get the existence of $k_0 > 0$, independents of n such that

$$\Big|\{x \in \Omega : v_n(x) > k_0\}\Big| = 0 \text{ for all } n.$$

Hence $\Big|\{x \in \Omega : v(x) > k_0\}\Big|$ and then $v \in L^\infty(\Omega)$.

• If $\beta < 2(\mu(\lambda) + s)$ and $\frac{\widehat{\sigma}}{2} = q'$, since (28) holds for all $\alpha \geqslant 1$, then using Hölder's inequality, we reach that for all $n \geqslant 1$, $v_n^r|x|^{-\beta} \in L^1(\Omega)$, for all $r < \infty$ and

$$\left(\int_\Omega v_n^r\,|x|^{-\beta}dx\right)^{\frac{1}{r}} \leqslant C\|\widetilde{f_n}\|_{L^q(\Omega,|x|^{\beta(q-1)}dx)}, \quad \text{for all } 1 \leqslant r < +\infty.$$

Now using Fatou's Lemma we deduce that

$$\left(\int_\Omega v^r\,|x|^{-\beta}dx\right)^{\frac{1}{r}} \leqslant C\|\widetilde{f}\|_{L^q(\Omega,|x|^{\beta(q-1)}dx)}, \quad \text{for all } 1 \leqslant r < +\infty$$

as requested.

• Now, if $\beta < 2(\mu(\lambda) + s)$ and $\frac{\widehat{\sigma}}{2} < q'$, that is $q < \frac{N-\beta}{2(\mu+s)-\beta}$, and choosing $\alpha = \frac{\widehat{\sigma}}{2q'-\widehat{\sigma}}$, then $\frac{(\alpha+1)\widehat{\sigma}}{2} = \frac{q\widehat{\sigma}}{2q-(q-1)\widehat{\sigma}} = \frac{q(N-\beta)}{N-\beta-q(2(s+\mu(\lambda))-\beta)} := r^*$. Going back to (28), it follows that, for all $n \geqslant 1$, $v_n^{r^*}|x|^{-\beta} \in L^1(\Omega)$ and

$$\left(\int_\Omega v_n^{r^*}\,|x|^{-\beta}dx\right)^{\frac{1}{r^*}} \leqslant C\|\widetilde{f_n}\|_{L^q(\Omega,|x|^{\beta(q-1)}dx)}.$$

As above, using Fatou's lemma, we get

$$\left(\int_\Omega v^{r^*}\,|x|^{-\beta}dx\right)^{\frac{1}{r^*}} \leqslant C\|\widetilde{f}\|_{L^q(\Omega,|x|^{\beta(q-1)}dx)}.$$

• If $\beta = 2(\mu(\lambda) + s)$, then $\widehat{\sigma} = 2$. Again from (28) and choosing $\alpha = \frac{1}{q'-1}$, it follows that $r^* = q$ and $v_n^{r^*}|x|^{-\beta} \in L^1(\Omega)$ for all $n \geqslant 1$ with

$$\left(\int_\Omega v_n^q\,|x|^{-\beta}dx\right)^{\frac{1}{q}} \leqslant C\|\widetilde{f_n}\|_{L^q(\Omega,|x|^{\beta(q-1)}dx)}.$$

Thus

$$\left(\int_\Omega v^q\,|x|^{-\beta}dx\right)^{\frac{1}{q}} \leqslant C\|\widetilde{f}\|_{L^q(\Omega,|x|^{\beta(q-1)}dx)}.$$

■

As a consequence, we get the next corollary that improves the regularity results obtained in [8].

Corollary 3.5. *Let $s \in (0,1)$, $0 < \lambda < \lambda_{N,s}$ and u be the unique weak solution to problem* (18) *with $f|x|^{-\mu(\lambda)} \in L^1(\Omega)$. Suppose in addition that $f|x|^{\beta-\mu(\lambda)} \in L^q(\Omega, |x|^{-\beta}dx)$ where $q > 1$ and $\frac{2N\mu(\lambda)}{N-2s} \leqslant \beta \leqslant 2(\mu(\lambda) + s)$. Then*

1) *If $\beta < 2(\mu(\lambda) + s)$ and $q > \frac{(N-\beta)}{2(\mu(\lambda)+s)-\beta}$, then $u|x|^{\mu(\lambda)} \in L^\infty(\Omega)$. Moreover, there exists a positive constant $C := C(N, \beta, \lambda, s, q, \Omega)$ such that*

$$\|u|x|^{\mu(\lambda)}\|_{L^\infty(\Omega)} \leqslant C\|f|x|^{\beta-\mu(\lambda)}\|_{L^q(\Omega,|x|^{-\beta}dx)}.$$

2) *If $\beta < 2(\mu(\lambda)+s)$ and $q = \frac{(N-\beta)}{2(\mu(\lambda)+s)-\beta}$, then $u|x|^{\mu(\lambda)} \in L^r(\Omega, |x|^{-\beta}dx)$, for all $1 \leqslant r < +\infty$. Moreover, there exists a positive constant $C := C(N, \beta, \lambda, s, q, r, \Omega)$ such that*

$$\left(\int\limits_{\Omega} u^r |x|^{r\mu(\lambda)-\beta} dx\right)^{\frac{1}{r}} \leqslant C\|f|x|^{\beta-\mu(\lambda)}\|_{L^q(\Omega,|x|^{-\beta}dx)}.$$

3) *If either $\beta = 2(\mu(\lambda)+s)$ or $\beta < 2(\mu(\lambda)+s)$ and $1 < q < \frac{N-\beta}{2(\mu(\lambda)+s)-\beta}$, then $u|x|^{\mu(\lambda)} \in L^{r^*}(\Omega, |x|^{-\beta}dx)$ with $r^* = \frac{(N-\beta)q}{N-\beta-q(2(\mu(\lambda)+s)-\beta)}$. Moreover, there exists a positive constant $C := C(N, \beta, \lambda, s, q, \Omega)$ such that*

$$\left(\int\limits_{\Omega} u^{r^*} |x|^{r^*\mu(\lambda)-\beta} dx\right)^{\frac{1}{r^*}} \leqslant C\|f|x|^{\beta-\mu(\lambda)}\|_{L^q(\Omega,|x|^{-\beta}dx)}.$$

As a consequence we get the next fractional regularity.

Theorem 3.6. *Suppose that f satisfies the same condition as in Corollary 3.5. Let $u \in L^1(\Omega)$ be the unique weak solution to* (18) *with $\lambda < \Lambda_{N,s}$. Then*

1) *If $\beta < 2(\mu(\lambda)+s)$ and $q > \frac{(N-\beta)}{2(\mu(\lambda)+s)-\beta}$, then $\left|(-\Delta)^{\frac{s}{2}}u\right| \in L^p(\mathbf{R}^N)$ for all $1 \leqslant p < \frac{N}{\mu(\lambda)+s}$. In particular, there exists a positive constant $C := C(N, \beta, \lambda, s, q, p, \Omega)$ such that*

$$\|(-\Delta)^{\frac{s}{2}}u\|_{L^p(\mathbf{R}^N)} \leqslant C\|f|x|^{\beta-\mu(\lambda)}\|_{L^q(\Omega,|x|^{-\beta}dx)}.$$

2) *If $\beta < 2(\mu(\lambda)+s)$ and $q = \frac{(N-\beta)}{2(\mu(\lambda)+s)-\beta}$, then $\left|(-\Delta)^{\frac{s}{2}}u\right| \in L^p(\mathbf{R}^N)$ for all $1 \leqslant p < \frac{N}{\mu(\lambda)+s}$. In particular, there exists a positive constant $C := C(N, \beta, \lambda, s, q, p, \Omega)$ such that*

$$\|(-\Delta)^{\frac{s}{2}}u\|_{L^p(\mathbf{R}^N)} \leqslant C\|f|x|^{\beta-\mu(\lambda)}\|_{L^q(\Omega,|x|^{-\beta}dx)}.$$

3) *If either $\beta = 2(\mu(\lambda)+s)$ or $\beta < 2(\mu(\lambda)+s)$ and $\frac{\beta}{\beta-\mu(\lambda)} \leqslant q < \frac{N-\beta}{2(\mu(\lambda)+s)-\beta}$, then $\left|(-\Delta)^{\frac{s}{2}}u\right| \in L^p(\mathbf{R}^N)$ for all $1 \leqslant p < \frac{qN}{N-\beta-q(\mu+s-\beta)}$. In particular, there exists a positive constant $C := C(N, \beta, \lambda, s, q, p, \Omega)$ such that*

$$\|(-\Delta)^{\frac{s}{2}}u\|_{L^p(\mathbf{R}^N)} \leqslant C\|f|x|^{\beta-\mu(\lambda)}\|_{L^q(\Omega,|x|^{-\beta}dx)}.$$

4) *If either $\beta = 2(\mu(\lambda)+s)$ or $\beta < 2(\mu(\lambda)+s)$ and $1 < q \leqslant \frac{\beta}{\beta-\mu(\lambda)}$, then $\left|(-\Delta)^{\frac{s}{2}}u\right| \in L^p(\mathbf{R}^N)$ for all $1 \leqslant p < \frac{qN}{N-qs}$. In particular, there exists a positive constant $C := C(N, \beta, \lambda, s, q, p, \Omega)$ such that*

$$\|(-\Delta)^{\frac{s}{2}}u\|_{L^p(\mathbf{R}^N)} \leqslant C\|f|x|^{\beta-\mu(\lambda)}\|_{L^q(\Omega,|x|^{-\beta}dx)}.$$

Proof. We start with the first case. Since $\beta < 2(\mu(\lambda)+s)$ and $q > \frac{(N-\beta)}{2(\mu(\lambda)+s)-\beta}$, then by Corollary 3.5, we obtain that $u(x) \leqslant C|x|^{-\mu(\lambda)}$.

Hence $\frac{u}{|x|^{2s}} \leqslant C|x|^{-\mu(\lambda)-2s} \in L^\sigma(\Omega)$ for all $1 \leqslant \sigma < \frac{N}{\mu(\lambda)+2s}$. Since $q > \frac{(N-\beta)}{2(\mu(\lambda)+s)-\beta}$, then using Hölder inequality we can show the existence of $a > \frac{N}{\mu(\lambda)+2s}$ such that $f \in L^a(\Omega)$. Thus $g := \dfrac{u}{|x|^{2s}} + f \in L^\sigma(\Omega)$ for all $1 \leqslant \sigma < \frac{N}{\mu(\lambda)+2s}$. Using now the regularity result in Theorem 2.10, it holds that $\left|(-\Delta)^{\frac{s}{2}}u\right| \in L^p(\mathbf{R}^N)$ for all $1 \leqslant p < \frac{N}{\mu(\lambda)+s}$ and

$$\|(-\Delta)^{\frac{s}{2}}u\|_{L^p(\mathbf{R}^N)} \leqslant C\|f|x|^{\beta-\mu(\lambda)}\|_{L^q(\Omega,|x|^{-\beta}dx)}.$$

The second case follows as the first case using the fact that $u|x|^{\mu(\lambda)} \in L^r(\Omega, |x|^{-\beta}dx)$, for all $1 \leqslant r < +\infty$.

We consider the third case which is more involved. Assume that $\beta < 2(\mu(\lambda) + s)$ and $1 \leqslant q < \frac{N-\beta}{2(\mu(\lambda)+s)-\beta}$, then by Corollary 3.5, we reach that $u|x|^{\mu(\lambda)} \in L^{r^*}(\Omega, |x|^{-\beta}dx)$ with $r^* = \frac{(N-\beta)q}{N-\beta-q(2(\mu(\lambda)+s)-\beta)}$. We claim that $\dfrac{u}{|x|^{2s}} \in L^\theta(\Omega)$ for all $1 \leqslant \theta < \frac{qN}{N+q(\beta-\mu(\lambda))-\beta}$. To see this we will use Hölder's inequality. More precisely, for $1 \leqslant \theta < r^*$, we have

$$\begin{aligned}
\int\limits_\Omega \left(\frac{u}{|x|^{2s}}\right)^\theta dx &= \int\limits_\Omega (u|x|^{\mu(\lambda)})^\theta (|x|^{\beta-\theta(\mu(\lambda)+2s)})|x|^{-\beta}dx \\
&\leqslant \Big(\int\limits_\Omega (u|x|^{\mu(\lambda)})^{r^*}|x|^{-\beta}dx\Big)^{\frac{\theta}{r^*}} \Big(\int\limits_\Omega (|x|^{\beta-\theta(\mu(\lambda)+2s)})^{\frac{r^*}{r^*-\theta}}|x|^{-\beta}dx\Big)^{\frac{r^*-\theta}{r^*}} \\
&\leqslant C(\Omega)\|f|x|^{\beta-\mu(\lambda)}\|^\theta_{L^q(\Omega,|x|^{-\beta}dx)} \Big(\int\limits_\Omega (|x|^{\beta-\theta(\mu(\lambda)+2s)})^{\frac{r^*}{r^*-\theta}}|x|^{-\beta}dx\Big)^{\frac{r^*-\theta}{r^*}}.
\end{aligned}$$

The last integral is finite if and only if $(\beta - \theta(\mu(\lambda) + 2s))\frac{r^*}{r^*-\theta} - \beta > -N$. This is equivalent to the fact that $\theta < \frac{qN}{N+q(\beta-\mu(\lambda))-\beta}$. Notice that in this case we have $\frac{qN}{N+q(\beta-\mu(\lambda))-\beta} < r^*$. Then the claim follows.

In the same way and taking into consideration that $\frac{qN}{N+q(\beta-\mu(\lambda))-\beta} < q$, we can prove that $f \in L^\theta(\Omega)$ for all $1 \leqslant \theta < \frac{qN}{N+q(\beta-\mu(\lambda))-\beta}$. As in the previous cases, setting $g := \dfrac{u}{|x|^{2s}} + f$, then $g \in L^\theta(\Omega)$ for all $1 \leqslant \theta < \frac{qN}{N+q(\beta-\mu(\lambda))-\beta}$. Thus by the regularity result in Theorem 2.10, we obtain that $\left|(-\Delta)^{\frac{s}{2}}u\right| \in L^p(\boldsymbol{R}^N)$ for all $1 \leqslant p < \frac{\theta N}{N-\theta s}$. Hence $\left|(-\Delta)^{\frac{s}{2}}u\right| \in L^p(\boldsymbol{R}^N)$ for all $1 \leqslant p < \frac{qN}{N-\beta-q(\mu+s-\beta)}$ and

$$\|(-\Delta)^{\frac{s}{2}}u\|_{L^p(\boldsymbol{R}^N)} \leqslant C\|f|x|^{\beta-\mu(\lambda)}\|_{L^q(\Omega,|x|^{-\beta}dx)}.$$

Finally, the fourth case follows easily, using the approach of the previous case. □

To end this section we give the next weighted estimate for the fractional gradient if additional assumptions on f are satisfied. This will be used in order to show the existence of a solution to problem (1).

Suppose that $f \in L^1(|x|^{-\mu(\lambda)-a_0}dx, \Omega)$ for some $a_0 > 0$. Hence there exists $\lambda_1 \in (\lambda, \Lambda_{N,s})$ such that $\mu(\lambda_1) = \mu(\lambda) + a_0$. Define ψ to be the unique solution to problem

$$\left\{\begin{array}{rcll} (-\Delta)^s\psi &=& \lambda_1 \dfrac{\psi}{|x|^{2s}} + 1 & \text{in } \Omega, \\ \psi &=& 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{array}\right. \tag{29}$$

then $\psi \simeq |x|^{-\mu(\lambda)-a_0}$ near the origin. It is clear also that $\psi \in L^\infty(\Omega \backslash B_r(0))$.

Using ψ as a test function in problem (18), it holds that

$$(\lambda_1 - \lambda)\int\limits_\Omega \frac{u\psi}{|x|^{2s}}dx \leqslant \int\limits_\Omega f\psi dx.$$

Hence

$$\int\limits_\Omega \frac{u}{|x|^{2s+\mu(\lambda)}}dx \leqslant C(\Omega\,\lambda, a_0)\|f\|_{L^1(|x|^{-\mu(\lambda)-a_0}dx,\Omega)}. \tag{30}$$

The next proposition will be the crucial key in order to show a priori estimates when dealing with problem (1) with general datum f.

Proposition 3.7. *Assume that* $f \in L^1(|x|^{-\mu(\lambda)-a_0}dx, \Omega)$ *for some* $a_0 > 0$. *Let* v *be the unique weak solution to problem* (18)*, then*

$$\left\|(-\Delta)^{\frac{s}{2}} u\right\|_{L^\alpha(|x|^{-\mu(\lambda)}dx,\Omega)} \leqslant C(\Omega,\lambda,a_0)\|f\|_{L^1(|x|^{-\mu(\lambda)-a_0}dx,\Omega)} \text{ for all } 1 \leqslant \alpha < \frac{N}{N-s}. \tag{31}$$

To prove Proposition 3.7, we need the following lemma proved in [27].

Lemma 3.8. *Let* $N \geqslant 1$, $R > 0$ *and* $\alpha, \beta \in (-\infty, N)$. *There exists* $C := C(N, R, \alpha, \beta) > 0$ *such that:*

- *If* $(N-\alpha-\beta) \neq 0$*, then*

$$\int_{B_R(0)} \frac{dz}{|x-z|^\alpha |y-z|^\beta} \leqslant C\Big(1 + |x-y|^{N-\alpha-\beta}\Big), \quad \text{for all } x, y \in B_R(0) \text{ with } x \neq y.$$

- *If* $(N-\alpha-\beta) = 0$*, then*

$$\int_{B_R(0)} \frac{dz}{|x-z|^\alpha |y-z|^\beta} \leqslant C\Big(1 + \big|\ln|x-y|\big|\Big), \quad \text{for all } x, y \in B_R(0) \text{ with } x \neq y.$$

Proof of Proposition 3.7. Since $\int_\Omega f|x|^{-\mu(\lambda)}dx < \infty$, then by Theorem 2.10, we know that

$$\left\|(-\Delta)^{\frac{s}{2}} u\right\|_{L^\alpha(\Omega)} \leqslant C(\Omega,\lambda,a_0)\|f\|_{L^1(|x|^{-\mu(\lambda)-a_0}dx,\Omega)} \text{ for all } 1 \leqslant \alpha < \frac{N}{N-s}.$$

Thus, to prove the claim we just need to show that

$$\int_{B_r(0)} |(-\Delta)^{\frac{s}{2}} u|^\alpha |x|^{-\mu(\lambda)} dx \leqslant C(\Omega,\lambda,a_0)\|f\|^\alpha_{L^1(|x|^{-\mu(\lambda)-a_0}dx,\Omega)} \text{ for all } 1 \leqslant \alpha < \frac{N}{N-s},$$

where $B_r(0) \subset\subset \Omega$.
We set $g(x) := \lambda \frac{u}{|x|^{2s}} + \mu f$, then $u(x) = \int_\Omega \mathcal{G}_s(x,y)g(y)dy$. Hence, for a.e. $x \in B_r(0)$,

$$|(-\Delta)^{\frac{s}{2}} u(x)| \leqslant \int_\Omega |(-\Delta)^{\frac{s}{2}} \mathcal{G}_s(x,y)|g(y)dy. \tag{32}$$

Notice that from [6], we know that

$$\left|(-\Delta)_x^{\frac{s}{2}} G_s(x,y)\right| \leqslant \frac{C}{|x-y|^{N-s}} \left(\left|\ln\frac{1}{|x-y|}\right| + \ln\Big(\frac{1}{\delta(x)}\Big)\right), \quad \text{for a.e. } x, y \in \Omega. \tag{33}$$

Since $B_r(0) \subset\subset \Omega$, one has

$$\left|(-\Delta)_x^{\frac{s}{2}} G_s(x,y)\right| \leqslant \frac{C}{|x-y|^{N-s}} \ln\left(\frac{C}{|x-y|}\right), \text{ for a.e. } (x,y) \in B_r(0) \times \Omega. \tag{34}$$

For the remaining part of this proof, we will use systematically this estimate for a.e. $(x, y) \in B_r(0) \times \Omega$.

Thus we conclude that

$$\left|(-\Delta)_x^{\frac{s}{2}} G_s(x,y)\right| \leqslant G_s(x,y)h(x,y), \qquad \text{for a.e. } (x,y) \in B_r(0) \times \Omega, \tag{35}$$

with

$$h(x,y) = \begin{cases} \frac{1}{|x-y|^s} \ln\left(\frac{C}{|x-y|}\right) \text{ if } |x-y| < \delta(y) \\ \\ \frac{1}{\delta^s(y)} \ln\left(\frac{C}{|x-y|}\right) \text{ if } |x-y| \geqslant \delta(y). \end{cases}$$

Fix $1 < \alpha < p_* = \frac{N}{N-s}$. Going back to (32), we deduce that, for a.e. $x \in B_r(0)$, we have

$$|(-\Delta)^{\frac{s}{2}} u(x)| \leqslant \int\limits_\Omega |(-\Delta)^{\frac{s}{2}} \mathcal{G}_s(x,y)| g(y) dy \leqslant \lambda \int\limits_\Omega |(-\Delta)^{\frac{s}{2}} \mathcal{G}_s(x,y)| \frac{u(y)}{|y|^{2s}} dy + \int\limits_\Omega |(-\Delta)^{\frac{s}{2}} \mathcal{G}_s(x,y)| f(y) dy.$$

Hence

$$|(-\Delta)^{\frac{s}{2}} u(x)||x|^{-\frac{\mu(\lambda)}{\alpha}} \leqslant \lambda |x|^{-\frac{\mu(\lambda)}{\alpha}} \int\limits_\Omega |(-\Delta)^{\frac{s}{2}} \mathcal{G}_s(x,y)| \frac{u(y)}{|y|^{2s}} dy + |x|^{-\frac{\mu(\lambda)}{\alpha}} \int\limits_\Omega |(-\Delta)^{\frac{s}{2}} \mathcal{G}_s(x,y)| f(y) dy.$$

We set

$$K_1(x) = \int\limits_\Omega |(-\Delta)^{\frac{s}{2}} \mathcal{G}_s(x,y)| \frac{u(y)}{|y|^{2s}} dy$$

and

$$K_2(x) = \int\limits_\Omega |(-\Delta)^{\frac{s}{2}} \mathcal{G}_s(x,y)| f(y) dy.$$

We begin estimating K_1. We have

$$\begin{aligned} K_1^\alpha(x) &\leqslant \left(\int\limits_\Omega |(-\Delta)^{\frac{s}{2}} \mathcal{G}_s(x,y)| \frac{u(y)}{|y|^{2s}} dy \right)^\alpha \leqslant \left(\int\limits_\Omega h(x,y) \mathcal{G}_s(x,y) \frac{u(y)}{|y|^{2s}} dy \right)^\alpha \\ &\leqslant \left(\int\limits_\Omega (h(x,y))^\alpha \mathcal{G}_s(x,y) \frac{u(y)}{|y|^{2s}} dy \right) \left(\int\limits_\Omega \mathcal{G}_s(x,y) g(y) dy \right)^{\alpha-1} \\ &\leqslant \int\limits_\Omega \left(h^\alpha(x,y) \mathcal{G}_s(x,y) \frac{u(y)}{|y|^{2s}} dy \right) u^{\alpha-1}(x). \end{aligned}$$

Thus, using Fubini's theorem, it holds that

$$\int_{B_r(0)} K_1^\alpha(x)|x|^{-\mu(\lambda)} dx \leqslant \lambda \int\limits_\Omega \frac{u(y)}{|y|^{2s}} \left(\int_{B_r(0)} h^\alpha(x,y) \mathcal{G}_s(x,y) u^{\alpha-1}(x)|x|^{-\mu(\lambda)} dx \right) dy.$$

Recall that, by (30), we have

$$\int\limits_\Omega \frac{u(y)}{|y|^{2s+\mu(\lambda)}} dy \leqslant C \int\limits_\Omega \frac{f(y)}{|y|^{\mu(\lambda)+a_0}} dy.$$

Therefore we obtain that

$$\begin{aligned}\int_{B_r(0)} K_1^{\alpha}(x)|x|^{-\mu(\lambda)}dx &\leqslant C_2 \int\limits_{\Omega} \frac{u(y)}{|y|^{2s}}\left(\int_{B_r(0)\cap\{|x-y|<\delta(y)\}} \frac{u^{\alpha-1}(x)\mathcal{G}_s(x,y)}{|x|^{\mu(\lambda)}}\frac{1}{|x-y|^{s\alpha}}\ln\left(\frac{C}{|x-y|}\right)^{\alpha}dx\right)dy\\ &+ C_2 \int\limits_{\Omega} \frac{u(y)}{|y|^{2s}\delta^{s\alpha}(y)}\left(\int_{B_r(0)\cap\{|x-y|\geqslant\delta(y)\}} \frac{u^{\alpha-1}(x)\mathcal{G}_s(x,y)}{|x|^{\mu(\lambda)}}\ln\left(\frac{C}{|x-y|}\right)^{\alpha}dx\right)dy\\ &= J_1 + J_2.\end{aligned}$$

Respect to J_1, using the fact that for all $\eta > 0$,

$$\frac{1}{|x-y|^{s\alpha}}\ln\left(\frac{C}{|x-y|}\right)^{\alpha} \leqslant \frac{\alpha}{\eta}\frac{C^{\eta}}{|x-y|^{s\alpha+\eta}},$$

and by Proposition 2.8, we reach that

$$\begin{aligned}J_1 &\leqslant C_2 \int\limits_{\Omega} \frac{u(y)}{|y|^{2s}}\left(\int_{B_r(0)} \frac{u^{\alpha-1}(x)}{|x|^{\mu(\lambda)}|x-y|^{N-(2s-s\alpha-\eta)}}dx\right)dy\\ &\leqslant C \int\limits_{\Omega} \frac{u(y)}{|y|^{2s}}\left(\int_{B_r(0)\cap\{|x|\geqslant\frac{1}{2}|y|\}} \frac{u^{\alpha-1}(x)}{|x|^{\mu(\lambda)}|x-y|^{N-(2s-s\alpha-\eta)}}dx\right)dy\\ &+ C \int\limits_{\Omega} \frac{u(y)}{|y|^{2s}}\left(\int_{B_r(0)\cap\{|x|<\frac{1}{2}|y|\}} \frac{u^{\alpha-1}(x)}{|x|^{\mu(\lambda)}|x-y|^{N-(2s-s\alpha-\eta)}}dx\right)dy\\ &\leqslant I_1 + I_2.\end{aligned}$$

To estimate I_1, we observe that

$$I_1 \leqslant C \int\limits_{\Omega} \frac{u(y)}{|y|^{2s+\mu(\lambda)}}\left(\int_{B_r(0)} \frac{u^{\alpha-1}(x)}{|x-y|^{N-(2s-s\alpha-\eta)}}dx\right)dy.$$

Recall that $u \in L^{\sigma}(\Omega)$ for all $\sigma < \frac{N}{N-2s}$. Since $\alpha < \frac{N}{N-s}$, fixing $\sigma_0 < \frac{N}{N-2s}$ and using Hölder inequality, we get

$$\int_{B_r(0)} \frac{u^{\alpha-1}(x)}{|x-y|^{N-(2s-s\alpha-\eta)}}dx \leqslant \left(\int_{B_r(0)} u^{\sigma_0}dx\right)^{\frac{\alpha-1}{\sigma_0}}\left(\int_{B_r(0)} \frac{1}{|x-y|^{\frac{(N-(2s-s\alpha-\eta))\sigma_0}{\sigma_0-(\alpha-1)}}}dx\right)^{\frac{\sigma_0-(\alpha-1)}{\sigma_0}}.$$

Since $\alpha < \frac{N}{N-s}$, then we can chose σ_0 close to $\frac{N}{N-2s}$ and η small enough such that $\frac{(N-(2s-s\alpha-\eta))\sigma_0}{\sigma_0-(\alpha-1)} < N$. Thus

$$\int_{B_r(0)} \frac{1}{|x-y|^{\frac{(N-(2s-s\alpha-\eta))\sigma_0}{\sigma_0-(\alpha-1)}}}dx \leqslant C(r,\Omega),$$

and then

$$I_1 \leqslant C\left(\int_{B_r(0)} u^{\sigma_0}dx\right)^{\frac{\alpha-1}{\sigma_0}}\left(\int\limits_{\Omega} \frac{u(y)}{|y|^{2s+\mu(\lambda)}}dy\right) \leqslant C\left(\int\limits_{\Omega} \frac{f(y)}{|y|^{\mu(\lambda)+a_0}}dy\right)^{\alpha}. \tag{36}$$

We deal now with I_2. Notice that $\{|x| \leqslant \frac{1}{2}|y|\} \subset \{|x-y| \geqslant \frac{1}{2}|y|\}$. Thus

$$I_2 \leqslant C \int\limits_{\Omega} \frac{u(y)}{|y|^{\mu(\lambda)+2s}} \left(\int_{B_r(0)} \frac{u^{\alpha-1}(x)}{|x|^{\mu(\lambda)}|x-y|^{N-(2s+\mu(\lambda)-s\alpha-\eta)}} dx \right) dy.$$

As in the estimate of I_1, setting $\theta = \dfrac{\sigma_0}{\sigma_0 - (\alpha - 1)}$, we get

$$\begin{aligned} &\int_{B_r(0)} \frac{u^{\alpha-1}(x)}{|x|^{\mu(\lambda)}|x-y|^{N-(2s+\mu(\lambda)-s\alpha-\eta)}} dx \\ \leqslant & \left(\int_{B_r(0)} u^{\sigma_0} dx \right)^{\frac{\alpha-1}{\sigma_0}} \left(\int_{B_r(0)} \frac{1}{|x|^{\mu(\lambda)\theta}|x-y|^{(N-(2s+\mu(\lambda)-s\alpha-\eta))\theta}} dx \right)^{\frac{1}{\theta}}. \end{aligned}$$

For $\alpha < \frac{N}{N-s}$ fixed, we can chose η small enough and σ_0 close to p_2 such that

$$N - (N - (2s + \mu(\lambda) - s\alpha - \eta))\theta - \mu(\lambda)\theta < N.$$

Hence using Lemma 3.8, it holds that

$$\int_{B_r(0)} \frac{1}{|x|^{\mu(\lambda)\theta}|x-y|^{(N-(2s+\mu(\lambda)-s\alpha-\eta))\theta}} dx \leqslant C(r_0).$$

Therefore we conclude that

$$I_2 \leqslant C \int\limits_{\Omega} \frac{u(y)}{|y|^{\mu(\lambda)+2s}} \Big(\int_{B_r(0)} u^{\sigma_0} dx \Big)^{\frac{\alpha-1}{\sigma_0}} \leqslant C \Big(\int\limits_{\Omega} \frac{f(y)}{|y|^{\mu(\lambda)+a_0}} dy \Big)^{\alpha}. \tag{37}$$

As a consequence, we have

$$J_1 \leqslant C \Big(\int\limits_{\Omega} \frac{f(y)}{|y|^{\mu(\lambda)+a_0}} dy \Big)^{\alpha}. \tag{38}$$

We deal now with J_2. Let $c_1 > 0$ be a positive constant to be chosen later, then

$$\begin{aligned} J_2 \leqslant & \int\limits_{\Omega} \frac{u(y)}{|y|^{2s}\delta^{s\alpha}(y)} \left(\int_{B_r(0)\cap\{|x-y|\geqslant\delta(y)\}} \frac{u^{\alpha-1}(x)\mathcal{G}_s(x,y)}{|x|^{\mu(\lambda)}} \ln\left(\frac{C}{|x-y|}\right)^{\alpha} dx \right) dy \\ \leqslant & \int_{\Omega\cap\{\delta(y)>c_1\}} \frac{u(y)}{|y|^{2s}\delta^{s\alpha}(y)} \left(\int_{B_r(0)\cap\{|x-y|\geqslant\delta(y)\}} \frac{u^{\alpha-1}(x)\mathcal{G}_s(x,y)}{|x|^{\mu(\lambda)}} \ln\left(\frac{C}{|x-y|}\right)^{\alpha} dx \right) dy \\ + & \int_{\Omega\cap\{\delta(y)<c_1\}} \frac{u(y)}{|y|^{2s}\delta^{s\alpha}(y)} \left(\int_{B_r(0)\cap\{|x-y|\geqslant\delta(y)\}} \frac{u^{\alpha-1}(x)\mathcal{G}_s(x,y)}{|x|^{\mu(\lambda)}} \ln\left(\frac{C}{|x-y|}\right)^{\alpha} dx \right) dy \\ \leqslant & C_1 J_1 + \int_{\Omega\cap\{\delta(y)<c_1\}} \frac{u(y)}{|y|^{2s}\delta^{s\alpha}(y)} \left(\int_{B_r(0)\cap\{|x-y|\geqslant\delta(y)\}} \frac{u^{\alpha-1}(x)\mathcal{G}_s(x,y)}{|x|^{\mu(\lambda)}} \ln\left(\frac{C}{|x-y|}\right)^{\alpha} dx \right) dy. \end{aligned}$$

We set

$$A = \int_{\Omega\cap\{\delta(y)<c_1\}} \frac{u(y)}{|y|^{2s}\delta^{s\alpha}(y)} \left(\int_{B_r(0)\cap\{|x-y|\geqslant\delta(y)\}} \frac{u^{\alpha-1}(x)\mathcal{G}_s(x,y)}{|x|^{\mu(\lambda)}} \ln\left(\frac{C}{|x-y|}\right)^{\alpha} dx \right) dy.$$

Choosing c_1 small, we get the existence of a positive constant c_2 such that for $\delta(y) < c_1$ and $x \in B_r(0)$, we have $|x-y| > c_2 > 0$. Hence using again Proposition 2.8, we deduce that

$$A \leqslant \int_{\Omega\cap\{\delta(y)<c_1\}} \frac{u(y)}{\delta^{s(\alpha-1)}(y)} \left(\int_{B_r(0)\cap\{|x-y|\geqslant\delta(y)\}} \frac{u^{\alpha-1}(x)}{|x|^{\mu(\lambda)}} dx \right) dy.$$

As above, for $\alpha_0 < \frac{N}{N-2s}$, we have

$$\begin{aligned}
\int_{B_r(0)} \frac{u^{\alpha-1}(x)}{|x|^{\mu(\lambda)}} dx &\leqslant \left(\int_{B_r(0)} u^{\sigma_0} dx \right)^{\frac{\alpha-1}{\sigma_0}} \left(\int_{B_r(0)} \frac{1}{|x|^{\mu(\lambda)\theta}} dx \right)^{\frac{1}{\theta}} \\
&\leqslant C\left(\int_{B_r(0)} u^{\sigma_0} dx \right)^{\frac{\alpha-1}{\sigma_0}} \leqslant C\left(\int_{\Omega} \frac{f(y)}{|y|^{\mu(\lambda)+a_0}} dy \right)^{\alpha-1}.
\end{aligned}$$

On the other hand, we have

$$\int_{\Omega\cap\{\delta(y)<c_2\}} \frac{u(y)}{\delta^{s(\alpha-1)}(y)} dy \leqslant C \int_{\Omega} \frac{u(y)}{\delta^{s}(y)} dy \leqslant C \int_{\Omega} \frac{f(y)}{|y|^{\mu(\lambda)+a_0}} dy.$$

Hence

$$J_2 \leqslant C\left(\int_{\Omega} \frac{f(y)}{|y|^{\mu(\lambda)+a_0}} dy \right)^{\alpha}.$$

As a consequence we deduce that

$$\int_{B_r(0)} K_1^{\alpha}(x)|x|^{-\mu(\lambda)} dx \leqslant C\left(\int_{\Omega} \frac{f(y)}{|y|^{\mu(\lambda)+a_0}} dy \right)^{\alpha}.$$

We treat now the term K_2. Recall that

$$K_2(x) = \int_{\Omega} |(-\Delta)^{\frac{s}{2}} \mathcal{G}_s(x,y)| f(y) dy.$$

Notice that for $\eta > 0$, small enough, to be chosen later, we have

$$|(-\Delta)^{\frac{s}{2}} \mathcal{G}_s(x,y)| \leqslant \frac{C}{|x-y|^{N-s}} \ln\left(\frac{C}{|x-y|}\right) \leqslant \frac{C}{|x-y|^{N-(s-\eta)}} \text{ for a.e. } (x,y) \in B_r(0)\times\Omega.$$

Thus

$$\begin{aligned}
K_2(x)|x|^{-\frac{\mu(\lambda)}{\alpha}} &\leqslant \frac{C}{|x|^{\frac{\mu(\lambda)}{\alpha}}} \int_{\Omega} \frac{f(y)}{|x-y|^{N-(s-\eta)}} dy \\
&\leqslant \frac{C}{|x|^{\frac{\mu(\lambda)}{\alpha}}} \int_{\Omega\cap\{|y|\leqslant 4|x|\}} \frac{f(y)}{|x-y|^{N-(s-\eta)}} dy + \frac{C}{|x|^{\frac{\mu(\lambda)}{\alpha}}} \int_{\Omega\cap\{|y|\geqslant 4|x|\}} \frac{f(y)}{|x-y|^{N-(s-\eta)}} dy \\
&\leqslant C \int_{\Omega\cap\{|y|\leqslant 4|x|\}} \frac{f(y)}{|y|^{\mu(\lambda)}} \frac{1}{|x-y|^{N-(s-\eta)}} dy + \frac{C}{|x|^{\frac{\mu(\lambda)}{\alpha}}} \int_{\Omega\cap\{|y|\geqslant 4|x|\}} \frac{f(y)}{|x-y|^{N-(s-\eta)}} dy \\
&= L_1(x) + L_2(x).
\end{aligned}$$

We start with the estimate of term L_1. Since $\frac{f(y)}{|y|^{\mu(\lambda)}} \in L^1(\Omega)$, then by Theorem 2.4, we deduce that $L_1 \in L^\sigma(B_r(0))$ for all $1 \leqslant \sigma < \frac{N}{N-(s-\eta)}$. Thus $L_1 \in L^\alpha(\Omega)$ and

$$\int_{B_r(0)} L_1^\alpha(x)dx \leqslant C\Big(\int\limits_\Omega \frac{f(y)}{|y|^{\mu(\lambda)+a_0}}dy\Big)^\alpha.$$

We consider now L_2. Since $|y| \geqslant 4|x|$, then $|x-y| \geqslant \frac{3}{4}|y|$ and $|x-y| \geqslant 3|x|$. Hence

$$L_2(x) \leqslant \frac{C}{|x|^{\frac{\mu(\lambda)}{\alpha}+N-(\mu(\lambda)+s+a_0-\eta)}} \int\limits_\Omega \frac{f(y)}{|y|^{\mu(\lambda)+a_0}}dy.$$

Since $(\frac{\mu(\lambda)}{\alpha} + N - (\mu(\lambda) + s + a_0 - \eta))\alpha < N$, then we conclude that $I_2 \in L^\alpha(B_r(0))$ and

$$\int_{B_r(0)} L_2^\alpha(x)dx \leqslant C\Big(\int\limits_\Omega \frac{f(y)}{|y|^{\mu(\lambda)+a_0}}dy\Big)^\alpha.$$

As a consequence, we have proved that

$$\int_{B_r(0)} K_2^\alpha(x)|x|^{-\mu(\lambda)}dx \leqslant C\Big(\int\limits_\Omega \frac{f(y)}{|y|^{\mu(\lambda)+a_0}}dy\Big)^\alpha.$$

Therefore we conclude that

$$\int_{B_r(0)} |(-\Delta^{\frac{s}{2}})u|^\alpha|x|^{-\mu(\lambda)}dx \leqslant C\Big(\int\limits_\Omega \frac{f(y)}{|y|^{\mu(\lambda)+a_0}}dy\Big)^\alpha.$$

Hence the main estimate follows and this finishes the proof of our proposition. □

4. Existence and non existence results for the KPZ problem: the case $\mathfrak{F}(u) \equiv |(-\Delta)^{\frac{s}{2}}u|$

In this section we consider the question of existence and non existence of a positive solution to problem (1) with $\mathfrak{F}(u) \equiv |(-\Delta)^{\frac{s}{2}}u|$. Namely we will treat the problem

$$\left\{\begin{array}{rcll} (-\Delta)^s u &=& \lambda\dfrac{u}{|x|^{2s}} + |(-\Delta)^{\frac{s}{2}}u|^p + \rho f & \text{in } \Omega,\\ u &>& 0 & \text{in } \Omega,\\ u &=& 0 & \text{in } (\boldsymbol{R}^N \setminus \Omega), \end{array}\right. \tag{39}$$

where $\Omega \subset \boldsymbol{R}^N$ is a bounded regular domain containing the origin, $s \in (0,1)$, $\lambda \leqslant \Lambda_{N,s}$, $\rho > 0$, $1 < p < \infty$ and f is positive measurable function satisfies some hypothesis that will be precised later.

Let us begin with the next definition.

Definition 4.1. Assume that $f \in L^1(\Omega)$ is a nonnegative function. We say that u is a weak solution to problem (39) if $|(-\Delta)^{\frac{s}{2}}u|^p \in L^1(\Omega)$, $\frac{u}{|x|^{2s}} \in L^1(\Omega)$ and, setting $g \equiv \lambda\frac{u}{|x|^{2s}} + |(-\Delta)^{\frac{s}{2}}u|^p + \rho f$, then u is a weak solution to problem (8) in the sense of Definition 2.6.

The existence of a solution in the case $\lambda = 0$ was proved in [7] without any limitation on p under suitable hypotheses on f. However, if $\lambda > 0$, taking into consideration the singularity generated by the Hardy potential, it is possible to show a non existence result for p large. In the next computation we will find the exact critical exponent for the non existence.

4.1. Non existence result: the existence of the critical exponent

Recall that we are considering the case $\mathfrak{F}(u)(x) = |(-\Delta)^{\frac{s}{2}} u(x)|$. We begin by analyzing the radial case in the whole space as in [8]. Consider the equation

$$(-\Delta)^s w - \lambda \frac{w}{|x|^{2s}} = |(-\Delta)^{\frac{s}{2}} w|^p \text{ in } \boldsymbol{R}^N, \tag{40}$$

then we search radial positive solution in the form $w = A|x|^{\beta - \frac{N-2s}{2}}$, with $A > 0$. By a direct computation, it follows that

$$A\gamma_{\beta,s}|x|^{-2s-\frac{N-2s}{2}+\beta} - \lambda A|x|^{\beta-2s-\frac{N-2s}{2}} = \frac{A^p|\gamma_{\beta,\frac{s}{2}}|^p}{|x|^{(\frac{N-2s}{2}-\beta+s)p}},$$

with

$$\gamma_{\beta,t} := \gamma_{-\beta,t} := \frac{2^{2t}\Gamma(\frac{N+2t+2\beta}{4})\Gamma(\frac{N+2t-2\beta}{4})}{\Gamma(\frac{N-2t-2\beta}{4})\Gamma(\frac{N-2t+2\beta}{4})} \tag{41}$$

and $t \in \{\frac{s}{2},\ s\}$.
Hence, by homogeneity, we need to have

$$p = \frac{\frac{N-2s}{2} - \beta + 2s}{\frac{N-2s}{2} - \beta + s},$$

which means that $\beta = \frac{N-2s}{2} + \frac{ps}{p-1} - \frac{2s}{p-1}$. Hence the constant A satisfies

$$\gamma_{\beta,s} - \lambda = A^{p-1}|\gamma_{\beta,\frac{s}{2}}|^p.$$

Using the fact that $A > 0$, it holds that $\gamma_\beta - \lambda > 0$. Define the application

$$\begin{array}{rcl} \Upsilon : (-\frac{N-2s}{2}, \frac{N-2s}{2}) & \mapsto & (0, \Lambda_{N,s}) \\ \beta & \mapsto & \gamma_\beta \end{array}$$

Then Υ is even and the restriction of Υ to the set $[0, \frac{N-2s}{2})$ is decreasing, see [24] and [26]. So there exists a unique $\alpha_\lambda \in (0, \Lambda_{N,s}]$ such that $\gamma_{\alpha_\lambda} = \gamma_{-\alpha_\lambda} = \lambda$.

Let $\beta_0 = -\beta_1 = \alpha_\lambda$. Setting

$$p_+(\lambda, s) := \frac{\frac{N-2s}{2} - \beta_0 + 2s}{\frac{N-2s}{2} - \beta_0 + s} = \frac{N + 2s - 2\alpha_\lambda}{N - 2\alpha_\lambda}, \tag{42}$$

and

$$p_-(\lambda, s) := \frac{\frac{N-2s}{2} - \beta_1 + 2s}{\frac{N-2s}{2} - \beta_1 + s} = \frac{N + 2s + 2\alpha_\lambda}{N + 2\alpha_\lambda}, \tag{43}$$

it holds that $p_-(\lambda, s) < p_+(\lambda, s)$ and $\gamma_\beta - \lambda > 0$ if and only if

$$p_-(\lambda, s) < p < p_+(\lambda, s).$$

It is easy to check that $p_+(\lambda, s)$ and $p_-(\lambda, s)$ are respectively an increasing and a decreasing function in α_λ and, therefore, are respectively a decreasing and an increasing function in the variable λ. Thus

$$\frac{N}{N-s} < p_-(\lambda, s) < \frac{N+2s}{N} < p_+(\lambda, s) < 2, \text{ for } 0 < \lambda < \Lambda_{N,s}.$$

Notice that

$$p_+(\lambda, s) = \frac{\mu(\lambda) + 2s}{\mu(\lambda) + s} \text{ and } p_-(\lambda, s) = \frac{\bar{\mu}(\lambda) + 2s}{\bar{\mu}(\lambda) + s},$$

where $\mu(\lambda)$ and $\bar{\mu}(\lambda)$ are defined by (21).

Hence, for $p_-(\lambda, s) < p < p_+(\lambda, s)$ fixed, using the fact that $\Omega \subset B_R(0)$ for R large, we get the existence of a positive constant $C_1 > 0$ such that $w(x) = C_1|x|^{\beta - \frac{N-2s}{2}}$ is a radial supersolution for the Dirichlet problem (39) if $f(x) \leqslant \frac{1}{|x|^{\frac{N-2s}{2}+2s-\beta}}$ with ρ small.

To show that $p_+(\lambda, s)$ is critical, we prove the next non existence result.

Theorem 4.2. *Let $s \in (0, 1)$ and suppose that $p > p_+(\lambda, s)$. Then for all $\rho \geqslant 0$, problem* (39) *has no positive weak solution u in the sense of Definition 4.1.*

Proof. We argue by contradiction. Assume that problem (39) has a positive solution u in the sense of Definition 4.1, then $(-\Delta)^{\frac{s}{2}} u \in L^p(\Omega)$ and $\frac{u}{|x|^{2s}} \in L^1(\Omega)$. By Lemma 3.1, it follows that

$$u(x) \geqslant C|x|^{-\mu(\lambda)} \text{ in } B_r(0) \subset\subset \Omega.$$

Since $|(-\Delta)^{\frac{s}{2}} u|^p + \lambda \frac{u}{|x|^{2s}} + \rho f \in L^1(\Omega)$, then from the regularity result in Theorem 2.10, we deduce that $(-\Delta)^{\frac{s}{2}} u \in L^t(\mathbf{R}^N)$ for all $1 \leqslant t < \frac{N}{N-s}$.

Let $\theta \in L^\infty(\Omega)$ be a nonnegative function such that $\operatorname{Supp}\theta \subset\subset B_{\frac{r}{2}}(0) \subset B_r(0) \subset\subset \Omega$ and define $\phi_\theta \in H^s_0(\Omega) \cap L^\infty(\Omega)$ to be the unique solution of the problem

$$\begin{cases} (-\Delta)^{\frac{s}{2}} \phi_\theta = \theta, & \text{in } \Omega, \\ \phi_\theta = 0, & \text{in } \mathbf{R}^N \setminus \Omega. \end{cases} \tag{44}$$

From [40], it holds that $\phi_\theta \simeq \delta^{\frac{s}{2}}$ near the boundary of Ω. Using ϕ_θ as test function in (39), we get

$$\begin{aligned} \lambda \int_\Omega \frac{u\phi_\theta}{|x|^{2s}} dx + \int_\Omega |(-\Delta)^{\frac{s}{2}} u|^p \phi_\theta \, dx + \rho \int\limits_\Omega f\phi_\theta &= \int_\Omega u(-\Delta)^s \phi_\theta \, dx = \int_{\mathbf{R}^N} (-\Delta)^{\frac{s}{2}} u \, (-\Delta)^{\frac{s}{2}} \phi_\theta \, dx \\ = \int_\Omega (-\Delta)^{\frac{s}{2}} u \, (-\Delta)^{\frac{s}{2}} \phi_\theta \, dx &+ \int_{\mathbf{R}^N \setminus \Omega} (-\Delta)^{\frac{s}{2}} u \, (-\Delta)^{\frac{s}{2}} \phi_\theta \, dx. \end{aligned} \tag{45}$$

We treat separately each term in the right hand of the above identity.

Since $\operatorname{Supp}\theta \subset\subset B_r(0) \subset\subset \Omega$, using the fact that $\phi_\theta > 0$ in Ω and then Hölder inequality, we reach that

$$\begin{aligned} \Big| \int_\Omega (-\Delta)^{\frac{s}{2}} u \, (-\Delta)^{\frac{s}{2}} \phi_\theta \, dx \Big| &= \Big| \int_\Omega (-\Delta)^{\frac{s}{2}} u \, \theta \, dx \Big| \\ &\leqslant \int_\Omega |(-\Delta)^{\frac{s}{2}} u| \phi_\theta \frac{\theta}{\phi_\theta} dx. \end{aligned} \tag{46}$$

Next, applying Young's inequality, it holds that

$$\Big| \int_\Omega (-\Delta)^{\frac{s}{2}} u \, (-\Delta)^{\frac{s}{2}} \phi_\theta \, dx \Big| \leqslant \varepsilon \int_\Omega |(-\Delta)^{\frac{s}{2}} u|^p \phi_\theta \, dx + C(\varepsilon) \int\limits_\Omega \frac{\theta^{p'}}{\phi_\theta^{p'-1}} dx, \tag{47}$$

where $\epsilon > 0$ will be chosen later.
Now we deal with the term $\int_{\mathbf{R}^N\setminus\Omega}(-\Delta)^{\frac{s}{2}}u\,(-\Delta)^{\frac{s}{2}}\phi_\theta\,dx$.

Since $x \in \mathbf{R}^N\setminus\Omega$, using again Hölder inequality, it follows that

$$|\int_{\mathbf{R}^N\setminus\Omega}(-\Delta)^{\frac{s}{2}}u\,(-\Delta)^{\frac{s}{2}}\phi_\theta\,dx| \leqslant ||(-\Delta)^{\frac{s}{2}}u||_{L^t(\mathbf{R}^N\setminus\Omega)}||(-\Delta)^{\frac{s}{2}}\phi_\theta||_{L^{t'}(\mathbf{R}^N\setminus\Omega)}$$

with $t < \frac{N}{N-s}$. By hypothesis $||(-\Delta)^{\frac{s}{2}}u||_{L^t(\mathbf{R}^N\setminus\Omega)} < \infty$. Now respect to $||(-\Delta)^{\frac{s}{2}}\phi_\theta||_{L^{t'}(\mathbf{R}^N\setminus\Omega)}$, we have

$$\begin{aligned}
||(-\Delta)^{\frac{s}{2}}\phi_\theta||^{t'}_{L^{t'}(\mathbf{R}^N\setminus\Omega)} &= \int_{\mathbf{R}^N\setminus\Omega}\Big(\int\limits_{\Omega}\frac{\phi_\theta(y)}{|y-x|^{N+s}}dy\Big)^{t'}dx\\
&= \int_{\mathbf{R}^N\setminus\Omega}\Big(\int_{B_r(0)}\frac{\phi_\theta(y)}{|y-x|^{N+s}}dy\Big)^{t'}dx + \int_{\mathbf{R}^N\setminus\Omega}\Big(\int_{\Omega\setminus B_r(0)}\frac{\phi_\theta(y)}{|y-x|^{N+s}}dy\Big)^{t'}dx.
\end{aligned}$$

Recalling that $\operatorname{Supp}\theta \subset B_{\frac{r}{2}}(0) \subset B_r(0) \subset\subset \Omega$, then we can prove that

$$||\phi_\theta||_{L^\infty(\Omega\setminus B_r(0))} \leqslant C||\theta||_{L^1(B_r(0))},$$

where C depends only on the N, s. Thus

$$\int_{\mathbf{R}^N\setminus\Omega}\Big(\int_{\Omega\setminus B_r(0)}\frac{\phi_\theta(y)}{|y-x|^{N+s}}dy\Big)^{t'}dx \leqslant C\int_{\mathbf{R}^N\setminus\Omega}\Big(\int_{\Omega\setminus B_r(0)}\frac{1}{|y-x|^{N+s}}dy\Big)^{t'}dx \leqslant C(r,\Omega,s,N,t).$$

Now, choosing r small enough, we obtain that, for $x \in \mathbf{R}^N\setminus\Omega$ and $y \in B_r(0)$, $|x-y| \geqslant c(|x|+1)$. Hence

$$\int_{\mathbf{R}^N\setminus\Omega}\Big(\int_{B_r(0)}\frac{\phi_\theta(y)}{|y-x|^{N+s}}dy\Big)^{t'}dx \leqslant \int_{\mathbf{R}^N\setminus\Omega}\frac{C}{(|x|+1)^{t'(N+s)}}\Big(\int_{B_r(0)}\phi_\theta(y)dy\Big)^{t'}dx \leqslant C||\phi_\theta||^{t'}_{L^1(\Omega)}.$$

Now, going back to (45), choosing $\varepsilon << 1$ in estimate (47), we obtain that

$$\lambda\int_\Omega\frac{u\phi_\theta}{|x|^{2s}}dx \leqslant C(\varepsilon)\int\limits_\Omega\frac{\theta^{p'}}{\phi_\theta^{p'-1}}dx + C||\phi_\theta||^{t'}_{L^1(\Omega)} + C||(-\Delta)^{\frac{s}{2}}u||_{L^t(\mathbf{R}^N)} + C. \tag{48}$$

Recall that $p > p_+(\lambda,s) = \frac{2s+\mu(\lambda)}{s+\mu(\lambda)}$, hence $p' < \frac{2s+\mu(\lambda)}{s}$. Using an approximating argument we can take $\theta = \frac{1}{|x|^\beta}\chi_{B_{\frac{r}{4}}(0)}$ with $N-(\mu(\lambda)+s) \leqslant \beta < N-(p'-1)s$. In this case $\phi_\theta \backsimeq \frac{1}{|x|^{\beta-s}}$ near the origin and $\phi_\theta \in L^\infty(B\setminus B_r(0)$. Therefore $\phi \in L^1(\Omega)$. From (48), it holds that

$$C\int_{B_{\frac{r}{4}}}\frac{1}{|x|^{\beta+\mu+s}}dx \leqslant C(\varepsilon)\int\limits_\Omega\frac{\theta^{p'}}{\phi_\theta^{p'-1}}dx + C||\phi_\theta||^{t'}_{L^1(\Omega)} + C||(-\Delta)^{\frac{s}{2}}u||_{L^t(\mathbf{R}^N)} + C.$$

Since $(\beta+\mu+s) \geqslant N$, then in order to conclude we have just to show that $\int\limits_\Omega\frac{\theta^{p'}}{\phi_\theta^{p'-1}}dx < \infty$. Notice that

$$\int\limits_\Omega\frac{\theta^{p'}}{\phi_\theta^{p'-1}}dx \leqslant \int_{B_{\frac{r}{4}}(0)}\frac{1}{|x|^{p'\beta-(p'-1)(\beta-s)}}dx = \int_{B_{\frac{r}{4}}(0)}\frac{1}{|x|^{(p'-1)s+\beta}}dx.$$

Taking into consideration that $p' < \frac{2s+\mu(\lambda)}{s}$, it follows that $(p'-1)s+\beta < N$ and so we are done. $\square$

Remarks 4.3. Following the same arguments as above, we can prove that problem (39) has no positive supersolutions u in the following sense: $u = 0$ *a.e* in $\mathbb{R}^N \setminus \Omega$, $\left|(-\Delta)^{\frac{s}{2}}u\right| \in L^r(\mathbb{R}^N)$ for some $r > 1$, $g := |(-\Delta)^{\frac{s}{2}}u|^p + \lambda\dfrac{u}{|x|^{2s}} + \rho f \in L^1(\Omega)$ and for all nonnegative $\phi \in \mathcal{T}$ (defined in (9)), we have

$$\int_{\mathbb{R}^N}(-\Delta)^{\frac{s}{2}}u(-\Delta)^{\frac{s}{2}}\phi\,dx \geqslant \int_\Omega g\phi\,dx.$$

For ρ large, we are also able to prove another non existence result.

Theorem 4.4. *Assume that $f \gneqq 0$ and $p > \frac{2s+2}{s+2}$, then there exists $\rho^* > 0$ such that problem* (39) *has non positive solution for $\rho > \rho^*$.*

Proof. Without loss of generality we assume that $f \in L^\infty(\Omega)$.
Assume that u is a positive solution to problem (39). For $\theta \in C_0^\infty(\Omega)$ with $\theta \gneqq 0$, we define ϕ_θ to be the unique solution to the problem

$$\begin{cases}(-\Delta)^s\phi_\theta = \theta, & \text{in } \Omega,\\ \phi_\theta = 0, & \text{in } \mathbb{R}^N \setminus \Omega.\end{cases}$$

Notice that $\phi_\theta \simeq \delta^s(x)$ where $\delta(x) \equiv dist(x, \partial\Omega)$, see for instance [41].
Using ϕ_θ as a test function in (39), it holds that

$$\int\limits_\Omega(-\Delta)^s\phi_\theta\,u dx \geqslant \int\limits_\Omega |(-\Delta)^{\frac{s}{2}}u|^p\phi_\theta dx + \rho\int\limits_\Omega f\phi_\theta dx.$$

Hence

$$\int\limits_\Omega \theta\,u\,dx \geqslant \int\limits_\Omega |(-\Delta)^{\frac{s}{2}}u|^p\phi_\theta dx + \rho\int\limits_\Omega f\phi_\theta\,dx. \tag{49}$$

Let ψ_θ to be the unique solution to the problem

$$\begin{cases}(-\Delta)^{\frac{s}{2}}\psi_\theta = \theta, & \text{in } \Omega,\\ \psi_\theta = 0, & \text{in } \mathbb{R}^N \setminus \Omega.\end{cases}$$

Thus

$$\int\limits_\Omega(-\Delta)^{\frac{s}{2}}\psi_\theta\,u\,dx \geqslant \int\limits_\Omega |(-\Delta)^{\frac{s}{2}}u|^p\phi_\theta\,dx + \rho\int\limits_\Omega f\phi_\theta\,dx.$$

Then

$$\int\limits_\Omega(-\Delta)^{\frac{s}{2}}u\,\psi_\theta\,dx \geqslant \int\limits_\Omega |(-\Delta)^{\frac{s}{2}}u|^p\phi_\theta\,dx + \rho\int\limits_\Omega f\phi_\theta\,dx.$$

Notice that

$$\int\limits_\Omega(-\Delta)^{\frac{s}{2}}u\,\psi_\theta\,dx \leqslant \int\limits_\Omega |(-\Delta)^{\frac{s}{2}}u|\phi_\theta\frac{\psi_\theta}{\phi_\theta}\,dx.$$

Hence, using Young's inequality, for any $\varepsilon > 0$, we get the existence of a positive constant $C(\varepsilon)$ such that

$$\int\limits_\Omega(-\Delta)^{\frac{s}{2}}u\,\psi_\theta\,dx \leqslant \varepsilon\int\limits_\Omega |(-\Delta)^{\frac{s}{2}}u|^p\phi_\theta\,dx + C(\varepsilon)\int\limits_\Omega \phi_\theta\Big(\frac{\psi_\theta}{\phi_\theta}\Big)^{p'}\,dx.$$

Since θ is bounded, according with [40], then $\psi_\theta \backsimeq \delta^{\frac{s}{2}}$ and $\phi_\theta \backsimeq \delta^s$, it follows that $\int\limits_\Omega \phi_\theta \Big(\frac{\psi_\theta}{\phi_\theta}\Big)^{p'} dx < \infty$ if, $p > \frac{2s+2}{s+2}$. Therefore, in this case, we deduce that

$$\rho \int\limits_\Omega f\,\phi\,dx \leqslant C(\varepsilon) \int\limits_\Omega \phi_\theta \Big(\frac{\psi_\theta}{\phi_\theta}\Big)^{p'} dx,$$

which implies that

$$\rho \leqslant \frac{C(\varepsilon) \displaystyle\int\limits_\Omega \phi_\theta \Big(\frac{\psi_\theta}{\phi_\theta}\Big)^{p'} dx}{\displaystyle\int\limits_\Omega f\,\phi_\theta\,dx} =: \rho^*.$$

Hence the result follows in this case.

$\square$

Remarks 4.5. The condition $p > \frac{2s+2}{s+2}$ in Theorem 4.4 seems to be technical. We conjecture that the non existence result in Theorem 4.4 holds for all $p > 1$. However the above arguments does not hold if $p \leqslant \frac{2s+2}{s+2}$.

4.2. Existence result

To show the optimality of the exponent $p_+(\lambda, s)$, we show the existence of a supersolution to problem (39). Notice that, in some cases, under suitable conditions on the datum f and the exponent p, we are able to prove the existence of a weak solution to problem (39).

Fix $p_-(\lambda, s) < p < p_+(\lambda, s) < 2$ and let $w_1(x) = \dfrac{A}{|x|^{\theta_0}}$, with $\theta_0 = \frac{N-2s}{2} - \beta$, be the solution to the Eq (40) obtained in the previous section. Recall that

$$\begin{aligned} (-\Delta)^s w_1(x) - \lambda \frac{w_1}{|x|^{2s}} &= \frac{A(\gamma_{\beta,s} - \lambda)}{|x|^{\theta_0+2s}} \\ &= \frac{A^p |\gamma_{\beta,\frac{s}{2}}|^p}{|x|^{(\theta_0+s)p}} = |(-\Delta)^{\frac{s}{2}} w_1|^p. \end{aligned}$$

Taking into consideration the definition of $\gamma_{\beta,t}$ given in (41) (with $t \in \{\frac{s}{2}, s\}$), it holds that $(\gamma_{\beta,t} - \lambda) > 0$ if and only if $\theta_0 \in (\mu(\lambda), \bar{\mu}(\lambda))$. Now, if $f \leqslant \dfrac{1}{|x|^{2s+\theta_0}}$, using the fact that Ω is bounded, we can choose $C_1 > 0$ such that $\widehat{w}_1 = C_1 w$ is a supersolution to problem (39) for $\rho < \rho^*$. In this way we have obtained the following result.

Theorem 4.6. *Let Ω be a bounded domain containing the origin. Suppose that $p_-(\lambda, s) < p < p_+(\lambda, s)$. If $f \leqslant \dfrac{1}{|x|^{2s+\theta}}$, with θ given as above, then problem* (39) *has a supersolution w such that $w, \dfrac{w}{|x|^{2s}}, |(-\Delta)^{\frac{s}{2}} w_1|^p \in L^1(\Omega)$.*

Notice that in order to show the existence of a solution under the presence of a supersolution, we need a comparison principle in the spirit of the work of [14] for the fractional gradient. This is missing at the present time but will be investigated in a forthcoming paper. However, using the compactness approach developed in [7] we are able to show the existence of a solution in some particular cases. More precisely, we have:

Theorem 4.7. *Let $s \in (0,1)$, $0 < \lambda < \Lambda_{N,s}$ and $f \in L^1(\Omega)$ be a nonnegative function such that $\int\limits_{\Omega} f|x|^{-\mu(\lambda)-a_0}\,dx < \infty$, for some $a_0 > 0$. Assume that $1 < p < p_* = \frac{N}{N-s}$. Then, there exists $\rho^* := \rho^*(N,p,s,f,\lambda,\Omega) > 0$ such that if $\rho < \rho^*$, problem (39) has a solution $u \in L_0^{s,\sigma}(\Omega)$, for all $1 < \sigma < \frac{N}{N-s}$. Moreover $\int_{R^N} |(-\Delta)^{\frac{s}{2}} u|^p |x|^{-\mu(\lambda)} dx < \infty$.*

Proof. We follow again the arguments used in [11]. Fix $1 < p < p_*$ and let $f \in L^1(\Omega)$ be a nonnegative function with $\int\limits_{\Omega} f|x|^{-\mu(\lambda)-a_0}\,dx < \infty$.

Fix $r > 1$ be fixed such that $1 < p < r < p_*$. Then, we get the existence of $\rho^* > 0$ such that for some $l > 0$, we have

$$C_0(l + \rho^* \|f\|_{L^1(|x|^{-\mu(\lambda)-a_0}dx,\Omega)}) = l^{\frac{1}{p}},$$

where C_0 is a positive constant depending only on Ω, λ and the regularity constant in Theorems (2.10).

Let $\rho < \rho^*$ be fixed and define the set

$$E = \{v \in L_0^{s,1}(\Omega) : v \in L_0^{s,r}(|x|^{-\mu(\lambda)}dx, \Omega) \text{ and } \|(-\Delta)^{\frac{s}{2}} v\|_{L^r(|x|^{-\mu(\lambda)}dx,\Omega)} \leqslant l^{\frac{1}{p}}\}. \tag{50}$$

It is clear that E is a closed convex set of $L_0^{s,1}(\Omega)$. Consider the operator

$$\begin{array}{rcl} T : E & \to & L_0^{s,1}(\Omega) \\ v & \to & T(v) = u, \end{array}$$

where u is the unique solution to problem

$$\begin{cases} (-\Delta)^s u &=& \lambda \dfrac{u}{|x|^{2s}} + |(-\Delta)^{\frac{s}{2}} v|^p + \rho f & \text{in } \Omega, \\ u &=& 0 & \text{in } \mathbb{R}^N \setminus \Omega, \\ u &>& 0 & \text{in } \Omega. \end{cases} \tag{51}$$

Setting

$$g(x) = |(-\Delta)^{\frac{s}{2}} v|^p + \gamma f,$$

then taking into consideration the definition of E, it holds that $g \in L^1(|x|^{-\mu(\lambda)}dx, \Omega)$. Hence the existence and the uniqueness of u follows using the result of [8] with $u \in L_0^{s,\sigma}(\Omega)$ for all $1 < \sigma < \frac{N}{N-s}$. Thus T is well defined.

We claim that $T(E) \subset E$. Since $r > p$, using Hölder inequality we get the existence of $\hat{a}_0 > 0$ such that

$$\int\limits_{\Omega} |(-\Delta)^{\frac{s}{2}} v|^p |x|^{-\mu(\lambda)-\hat{a}_0} dx \leqslant C(\Omega) \Big(\int\limits_{\Omega} |(-\Delta)^{\frac{s}{2}} v|^r |x|^{-\mu(\lambda)} dx \Big)^{\frac{p}{r}} < \infty.$$

Setting $\bar{a}_0 = \min\{a_0, \hat{a}_0\}$, it holds that $g \in L^1(|x|^{-\mu(\lambda)-\bar{a}}dx, \Omega)$. Thus by Proposition 3.7, we reach that, for all $1 \leqslant \sigma < \frac{N}{N-s}$,

$$\Big(\int\limits_{\Omega} |(-\Delta)^{\frac{s}{2}} u|^{\sigma} |x|^{-\mu(\lambda)} dx\Big)^{\frac{1}{\sigma}} \leqslant C(N, p, \bar{a}) \Big\| |(-\Delta)^{\frac{s}{2}} v|^p + \rho f \Big\|_{L^1(|x|^{-\mu(\lambda)-\bar{a}}dx,\Omega)}.$$

Since $v \in E$, we conclude that

$$\begin{aligned}\Big(\int\limits_{\Omega} |(-\Delta)^{\frac{s}{2}} u|^{\sigma} |x|^{-\mu(\lambda)} dx\Big)^{\frac{1}{\sigma}} &\leqslant C(N, p, \bar{a})\Big(\Big(\int\limits_{\Omega} |(-\Delta)^{\frac{s}{2}} v|^{r} |x|^{-\mu(\lambda)} dx\Big)^{\frac{p}{r}} + \rho \|f\|_{L^1(|x|^{-\mu(\lambda)-a_0}dx,\Omega)}\Big) \\ &\leqslant C(l + \rho^* \|f\|_{L^1(|x|^{-\mu(\lambda)-a_0}dx,\Omega)}) \leqslant l.\end{aligned}$$

Choosing $\sigma = r$, it holds that $u \in E$.

The continuity and the compactness of T follow using closely the same arguments as in [11].

As a conclusion and using the Schauder Fixed Point Theorem as in [11], there exists $u \in E$ such that $T(u) = u$, $u \in L_0^{s,p}(\Omega)$ and

$$\|(-\Delta)^{\frac{s}{2}} u\|_{L^r(|x|^{-\mu(\lambda)}dx,\Omega)} \leqslant C.$$

Therefore, u solves (39). □

5. Some extensions and further results

Let us consider now the case where $\mathfrak{F}(u) \equiv (\mathbb{D}_s(u))$. Then problem (1) takes the form

$$\begin{cases} (-\Delta)^s u &= \lambda \dfrac{u}{|x|^{2s}} + (\mathbb{D}_s(u))^p + \rho f \quad \text{in } \Omega, \\ u &> 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{in } (\mathbb{R}^N \setminus \Omega). \end{cases} \tag{52}$$

Recall that $\mathbb{D}_s(u)(x) = \Big(\dfrac{a_{N,s}}{2} \displaystyle\int_{R^N} \dfrac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dy\Big)^{\frac{1}{2}}$. If we consider the equation

$$(-\Delta)^s w - \lambda \frac{w}{|x|^{2s}} = (\mathbb{D}_s(w))^p \text{ in } \boldsymbol{R}^N, \tag{53}$$

then, using the same radial computation as in the previous section, searching for a radial solution in the form $w = A|x|^{\beta-\frac{N-2s}{2}}$, one sees that we need $p = \dfrac{\frac{N-2s}{2} - \beta + 2s}{\frac{N-2s}{2} - \beta + s}$, which means that $\beta = \frac{N-2s}{2} + \frac{ps}{p-1} - \frac{2s}{p-1}$. Hence, as in the previous case, we obtain that w is a solution to (53) if $p_-(\lambda, s) < p < p_+(\lambda, s)$ where $p_-(\lambda, s)$, $p_+(\lambda, s)$ are defined by (42) and (43) respectively.

Notice that if $f \leqslant \dfrac{1}{|x|^{2s+\theta}}$ with $\theta = \frac{N-2s}{2} - \beta$, then we can chose $C_1 > 0$ such that $C_1 w$ is a supersolution to problem (52) for ρ small enough.

Let us show that $p_+(\lambda, s)$ is the critical exponent for the existence of a weak solution. More precisely we have the next non existence result.

Theorem 5.1. *Assume that $s \in (0,1)$ and $p > p_+(\lambda, s)$. For $\lambda > 0$, problem* (52) *has no positive solution u in the sense of Definition 4.1.*

Proof. We follow closely the arguments in [5]. Without loss of generality we assume that $f \in L^\infty(\Omega)$.

According to the value of p, we will divide the proof in two parts.

The case $p_+(\lambda, s) < p < 2_s^*$**.** In this case $p' > \frac{2N}{N+2s}$. Assume by contradiction that problem (52) has a weak positive u. Let $\phi \in C_0^\infty(\Omega)$ be a nonnegative function such that Supp $\subset B_{\frac{r}{2}}(0) \subset B_r(0) \subset\subset \Omega$ to be chosen later. Using $\phi^{p'}$ as test function in (52), it holds that

$$\int_\Omega (-\Delta)^s u\, \phi^{p'}(x)dx \geqslant \int_\Omega (\mathbb{D}_s(u)(x))^p \phi^{p'}(x)dx + \lambda \int\limits_\Omega \frac{u\phi^{p'}}{|x|^{2s}}dx + \rho \int_\Omega f(x)\phi^{q'}(x)dx. \tag{54}$$

Using the algebraic inequality, for $a, b \geqslant 0, m > 1$,

$$(a^m - b^m) \simeq (a-b)(a^{m-1} + b^{m-1}),$$

it holds that

$$\begin{aligned}
\int_\Omega (-\Delta)^s u\, \phi^{p'}(x)dx &= \iint_{D_\Omega} \frac{(u(x)-u(y))(\phi^{p'}(x) - \phi^{p'}(y))}{|x-y|^{N+2s}}dydx \\
&\leqslant C \iint_{D_\Omega} \frac{|u(x)-u(y)||\phi(x)-\phi(y)|(\phi^{p'-1}(x) + \phi^{p'-1}(y))}{|x-y|^{N+2s}}dydx \\
&\leqslant C \iint_{D_\Omega} \frac{|u(x)-u(y)||\phi(x)-\phi(y)|}{|x-y|^{N+2s}}\phi^{p'-1}(x)dydx \\
&+ C \iint_{D_\Omega} \frac{|u(x)-u(y)||\phi(x)-\phi(y)|}{|x-y|^{N+2s}}\phi^{p'-1}(y)dydx \\
&\leqslant 2C \iint_{D_\Omega} \frac{|u(x)-u(y)||\phi(x)-\phi(y)|}{|x-y|^{N+2s}}\phi^{p'-1}(x)dydx \\
&\leqslant C \int_{\mathbb{R}^N} \Big(\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dy\Big)^{\frac12} \Big(\int_{\mathbb{R}^N} \frac{|\phi(x)-\phi(y)|^2}{|x-y|^{N+2s}}dy\Big)^{\frac12} \phi^{p'-1}(x)dx \\
&\leqslant C \int\limits_\Omega \mathbb{D}_s(u)\mathbb{D}_s(\phi)\phi^{p'-1}(x)dx.
\end{aligned}$$

Therefore, using Young's inequality, we deduce that for any $\varepsilon > 0$, we get the existence of a positive constant $C(\varepsilon)$ such that

$$\int_\Omega (-\Delta)^s u\, \phi^{p'}(x)dx \;\leqslant\; \varepsilon \int\limits_\Omega (\mathbb{D}_s(u))^p \phi^{p'}(x)dx + C(\varepsilon) \int\limits_\Omega (\mathbb{D}_s(\phi))^{p'}(x)dx.$$

Choosing ε small enough and going back to (54), we get

$$(1-\varepsilon)\int_\Omega (\mathbb{D}_s(u)(x))^p \phi^{p'}(x)dx + \lambda \int\limits_\Omega \frac{u\phi^{p'}}{|x|^{2s}}dx + \rho \int_\Omega f(x)\phi^{q'}(x)dx \leqslant C(\varepsilon) \int\limits_\Omega (\mathbb{D}_s(\phi)(x))^{p'}dx. \tag{55}$$

Recall that $u(x) \geqslant C|x|^{-\mu(\lambda)}$ in $B_r(0) \subset\subset \Omega$. Hence fixed $\phi \in C_0^\infty(B_{\frac{r}{2}}(0))$, we have that

$$\lambda C \int_{B_{\frac{r}{2}}(0)} \frac{\phi^{p'}}{|x|^{2s+\mu(\lambda)}} dx \leqslant C(\varepsilon) \int_\Omega (\mathbb{D}_s(\phi)(x))^{p'} dx.$$

Replacing ϕ by $|\phi|$ in the above estimate, we deduce that

$$\lambda C \int_{B_{\frac{r}{2}}(0)} \frac{|\phi|^{p'}}{|x|^{2s+\mu(\lambda)}} dx \leqslant C(\varepsilon) \int_\Omega (\mathbb{D}_s(\phi)(x))^{p'} dx \leqslant C(\varepsilon) \int_{\mathbb{R}^N} (\mathbb{D}_s(\phi)(x))^{p'} dx. \tag{56}$$

Since $p > p_+(\lambda, s)$, then $sp' < 2s + \mu(\lambda)$. Recall that $p' > \frac{2N}{N+2s}$. Hence (56) is in contradiction with the Hardy inequality in Proposition 2.3. Thus we conclude.

The case $p > 2_s^* > p_+(\lambda, s)$. Notice that for all $\lambda < \Lambda_{N,s}$, we have $p_+(\lambda, s) < 2 < 2_s^*$. By a continuity argument we get the existence of $\lambda_1 < \lambda$ and $p_1 < 2_s^*$ such that $p_1 > p_+(\lambda_1, s)$. Assume that u is a weak solution to problem (52), then

$$(-\Delta)^s u \geqslant \lambda \frac{u}{|x|^{2s}} + (\mathbb{D}_s(u))^{p_1} - C(p_1) \text{ in } B_r(0).$$

Notice that $u(x) \geqslant C_1|x|^{-\mu(\lambda)}$ in $B_r(0)$. Hence

$$(-\Delta)^s u \geqslant \lambda_1 \frac{u}{|x|^{2s}} + (\mathbb{D}_s(u))^{p_1} + \frac{C}{|x|^{2s+\mu(\lambda)}} - C(p_1) \text{ in } B_r(0).$$

Choosing r small, it holds that

$$(-\Delta)^s u \geqslant \lambda_1 \frac{u}{|x|^{2s}} + (\mathbb{D}_s(u))^{p_1} \text{ in } B_r(0).$$

Since $p_+(\lambda_1, s) < p_1 < \frac{2N}{N+2s}$, repeating the same argument as in the first case, we reach the same contradiction. Hence we conclude. □

Taking advantage of the previous estimate, we can show that the problem (52) has no solution for large value of ρ.

Theorem 5.2. *Assume that $f \gneqq 0$ and $p > 1$, then there exists $\rho^* > 0$ such that problem* (52) *does not have a positive solution for $\rho > \rho^*$.*

Proof. Suppose that u is a nonnegative weak solution to problem (52). Let $\phi \in C_0^\infty(\Omega)$ be a nonnegative function such that

$$\int_\Omega f(x)\phi^{q'}(x) dx > 0.$$

From estimate (55), fixing $\varepsilon \in (0, 1)$, we obtain that

$$\rho \int_\Omega f(x)\phi^{q'}(x) dx \leqslant C(\varepsilon) \int_\Omega (\mathbb{D}_s(\phi)(x))^{p'} dx.$$

In particular

$$\rho \leqslant \inf_{\{\phi\in C^\infty(\Omega),\phi\gneqq 0\}} \frac{C(\varepsilon)\displaystyle\int_\Omega (\mathbb{D}_s(\phi))^{p'}(x)dx}{\displaystyle\int_\Omega f(x)\phi^{q'}(x)dx} := \rho^*,$$

and this is in contradiction with our initial assumption. □

Remarks 5.3. 1) As in Theorem 4.6, if $1 < p < p_+(\lambda, s)$ and $f \leqslant \dfrac{1}{|x|^{2s+\theta}}$, with θ given as above, then problem (39) has a supersolution w such that $w, \dfrac{w}{|x|^{2s}}, \mathbb{D}_s(u) \in L^p(\Omega)$.

2) Using the same compactness approach, we can also treat the case $(\mathfrak{F}(u)(x)) = |\nabla^s u(x)|$, where $\nabla^s u(x)$ is defined in (4).

5.1. Some new perspectives and an open problem

1) In the local case $s = 1$ or in the nonlocal case under the existence of a local gradient term, an interesting maximum principle is obtained in the sense that if $w \in W_0^{s,1}(\Omega)$ is a subsolution to the problem

$$\begin{cases} (-\Delta)^s w = a(x)|\nabla w| & \text{in } \Omega,\\ w = 0 & \text{in } (\mathbb{R}^N \setminus \Omega),\end{cases} \tag{57}$$

with $a \in L^\sigma(\Omega), \sigma > \frac{N}{s}$, then $w \leqslant 0$ in Ω (see for instance [14] and [11]). It would be very interesting to get a similar result replacing the gradient term $|\nabla w|$ by the nonlocal fractional gradient $|(-\Delta)^{\frac{s}{2}} w|$, namely for the problem

$$\begin{cases} (-\Delta)^s w = a(x)|(-\Delta)^{\frac{s}{2}} w| & \text{in } \Omega,\\ w = 0 & \text{in } (\mathbb{R}^N \setminus \Omega).\end{cases} \tag{58}$$

2) Since non comparison principle is known for problem (58), then to get general existence result to problem (1), under natural integrability conditions for f, it is necessary to prove a new class of weighted CKN inequalities as in [3], using the norm $\||(-\Delta)^{\frac{s}{2}} u||x|^\beta\|_{L^p(\mathbb{R}^N)}$. This will be considered in a forthcoming work.

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Conflict of interest

The authors declare no conflict of interest.

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