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Research article

Singular Kähler-Einstein metrics on $\mathbb{Q}$-Fano compactifications of Lie groups[†]

Yan Li[1], Gang Tian[2] and Xiaohua Zhu[2,*]

[1] School of Mathematics and Statistics, Beijing Institute of Technology, Beijing, 100081, China
[2] Beijing International Centre of Mathematical Research and School of Mathematical Science, Peking University, Beijing 100871, China

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* **Correspondence:** Email: xhzhu@math.pku.edu.cn; Tel: +8601062759408;
Fax: +8601062759408.

Abstract: In this paper, we prove an existence result for Kähler-Einstein metrics on $\mathbb{Q}$-Fano compactifications of Lie groups by the variational method, provided their moment polytopes satisfy a *fine* condition. As an application, we prove that there is no $\mathbb{Q}$-Fano $SO_4(\mathbb{C})$-compactification which admits a Kähler-Einstein metric with the same volume as that of a smooth K-unstable Fano $SO_4(\mathbb{C})$-compactification.

Keywords: Kähler-Einstein metrics; $\mathbb{Q}$-Fano compactifications of Lie groups; variation method; reduced Ding functional; K-stability

1. Introduction

Let G be an n-dimensional connected, linear complex reductive Lie group which is the complexification of a compact Lie group K. Let T be a maximal torus of K, which has dimension r and Lie algebra $\mathfrak{t}$. Then $T^{\mathbb{C}}$ is a maximal complex torus of G. Denote by Φ_+ a chosen positive roots system associated to $T^{\mathbb{C}}$. Put

$$\rho = \frac{1}{2}\sum_{\alpha\in\Phi_+}\alpha. \tag{1.1}$$

It can be regarded as a character in $\mathfrak{a}^*$, where $\mathfrak{a}^*$ is the dual space of the non-compact part $\mathfrak{a} = \sqrt{-1}\mathfrak{t}$ of $\mathfrak{t}^{\mathbb{C}}$. Let π be a function on $\mathfrak{a}^*$ defined by

$$\pi(y) = \prod_{\alpha\in\Phi_+}\langle\alpha, y\rangle^2,\ y \in \mathfrak{a}^*,$$

where $\langle\cdot,\cdot\rangle$ * denotes the Cartan-Killing inner product on $\mathfrak{a}^*$.

Let M be a $\mathbb{Q}$-Fano G-compactification (cf. [4]). Denote by Z the closure of $T^{\mathbb{C}}$ in M. Then $(Z, -K_M|_Z)$ is a polarized toric variety. Hence there is an associated moment polytope P of $(Z, -K_M|_Z)$ induced by $(M, -K_M)$ [3, 4]. Let P_+ be the positive part of P defined by $P_+ = P \cap \overline{\mathfrak{a}^*_+}$, where

$$\mathfrak{a}_+ = \{y \in \mathfrak{a}^* | \langle\alpha, y\rangle > 0,\ \forall\ \alpha \in \Phi_+\}$$

is the positive Weyl chamber in $\mathfrak{a}^*$ defined by Φ_+. Denote by $2P_+$ a dilation of P_+ at rate 2. We define the barycenter of $2P_+$ with respect to the weighted measure $\pi(y)dy$ by

$$\mathrm{bar}(2P_+) = \frac{\int_{2P_+} y\pi(y)\,dy}{\int_{2P_+}\pi(y)\,dy}.$$

In [20], Delcroix proved the following existence theorem for Kähler-Einstein metrics on smooth Fano G-compactifications.

Theorem 1.1. *Let M be a smooth Fano G-compactification. Then M admits a Kähler-Einstein metric if and only if*

$$\mathrm{bar}(2P_+) \in 4\rho + \Xi, \tag{1.2}$$

where Ξ is the relative interior of the cone generated by Φ_+.

There is an alternative proof of Theorem 1.1 given by Li, Zhou and Zhu via the variation method [36]. They also showed that (1.2) is actually equivalent to the K-stability condition in terms of [44] and [24] by constructing $\mathbb{C}^*$-action through piecewisely rationally linear function which is invariant under the Weyl group action. In particular, it implies that M is K-unstable if $\mathrm{bar}(2P_+) \notin 4\rho + \Xi$. A more general construction of $\mathbb{C}^*$-action was also discussed in [21].

In the present paper, we extend the above theorem to $\mathbb{Q}$-Fano compactifications of G which may be singular. It is well known that any $\mathbb{Q}$-Fano G-compactification has klt-singularities [4, Section 5]. For a $\mathbb{Q}$-Fano variety M with klt-singularities, there is naturally a class of admissible Kähler metrics induced by the Fubini-Study metric (cf. [23]). In [10], Berman, Boucksom, Eyssidieux, Guedj and Zeriahi introduce a class of Kähler potentials associated to admissible Kähler metrics and refer it as the $\mathcal{E}^1(M, -K_M)$ space. Then they define the singular Kähler-Einstein metric on M with the Kähler potential in $\mathcal{E}^1(M, -K_M)$ via the complex Monge-Ampère equation, which is the usual Kähler-Einstein metric on the smooth part of M. It is a natural problem to establish an extension of the Yau-Tian-Donaldson conjecture we have solved for smooth Fano manifolds [44, 46], that is, an equivalence relation between the existence of such singular Kähler-Einstein metrics and the K-stability on a $\mathbb{Q}$-Fano variety M with klt-singularities. Actually, there are many recent works on this fundamental problem in terms of uniform K-stability. We refer the reader to [10, 11, 32–34], etc..

We assume that the associated polytope P of $(Z, K_M^{-1}|_Z)$ is fine in sense of [25], namely, each vertex of P is the intersection of precisely r facets. Then we prove

*Without of confusion, we also write it as $\alpha(y)$ for simplicity.

Theorem 1.2. *Let M be a $\mathbb{Q}$-Fano G-compactification such that the associated polytope P is* fine. *Then M admits a singular Kähler-Einstein metric if and only if (1.2) holds.*

By a result of Abreu [1], the assumption that the polytope P being *fine* is equivalent to that the metric induced by the Guillemin function can be extended to a Kähler orbifold metric on Z.[†] It follows that the Guillemin function of $2P$ in Theorem 1.2 induces a $K \times K$-invariant singular metric ω_{2P} in $\mathcal{E}^1(M, -K_M)$ (cf. Lemma 3.4). Moreover, we can prove that the Ricci potential of ω_{2P} on M is uniformly bounded above. We note that P is always fine when $\mathrm{rank}(G) = 2$. Thus for a $\mathbb{Q}$-Fano compactification of G with $\mathrm{rank}(G) = 2$, M admits a singular Kähler-Einstein metric if and only if (1.2) holds. As an application of Theorem 1.2, we show that there is only one example of non-smooth Gorenstein Fano $\mathrm{SO}_4(\mathbb{C})$-compactifications which admits a singular Kähler-Einstein metric (cf. Section 7.1).

On the other hand, it has been shown in [41] that there are only three smooth Fano compactifications of $\mathrm{SO}_4(\mathbb{C})$, i.e., *Case-1.1.2*, *Case-1.2.1* and *Case-2* in Section 7.1. The first two manifolds do not admit any Kähler-Einstein metric [20, 36]. By Theorem 1.2, we further prove

Theorem 1.3. *There is no $\mathbb{Q}$-Fano compactification of* $\mathrm{SO}_4(\mathbb{C})$ *which admits a singular Kähler-Einstein metric with the same volume as Case-1.1.2 or Case-1.2.1 in Section 7.1.*

Theorem 1.3 gives a partial answer to a question proposed in [38] about limit of Kähler-Ricci flow on either *Case-1.1.2* or *Case-1.2.1*. It has been proved there that the flow has type II singularities on each of *Case-1.1.2* and *Case-1.2.1*. By the Hamilton-Tian conjecture [7, 16, 44, 48], the limit should be a singular Kähler-Ricci soliton on a $\mathbb{Q}$-Fano variety with the same volume as that of initial metric. However, by Theorem 1.3, the limit can not be a $\mathbb{Q}$-Fano compactification of $\mathrm{SO}_4(\mathbb{C})$ with a singular Kähler-Einstein metric. This implies that the limiting soliton of flow on either *Case-1.1.2* or *Case-1.2.1* will have less *symmetric* than the initial one, which is totally different from the situation of smooth convergence of $K \times K$-invariant metrics on a smooth compactification of Lie group [38].

As in [9, 36, 49], we use the variational method to prove Theorem 1.2. More precisely, we will prove that a modified version of the Ding functional $\mathcal{D}(\cdot)$ is proper under the condition (1.2). This functional is defined for a class of convex functions $\mathcal{E}^1_{K\times K}(2P)$ associated to $K \times K$-invariant metrics on the orbit of G (cf. Section 4, 6). The key point is that the Ricci potential h_0 of the Guillemin metric ω_{2P} is bounded from above when P is *fine* (cf. Proposition 5.1). This enables us to control the nonlinear part $\mathcal{F}(\cdot)$ of $\mathcal{D}(\cdot)$ by modifying $\mathcal{D}(\cdot)$ as done in [24, 37] (cf. Section 6.1). We shall note that it is in general impossible to get a lower bound of h_0 if the compactification is a singular variety (cf. Remark 5.2). However, by a recent deep result of Li in [32], the additional *fine* condition in Theorem 1.2 can be actually dropped. For the completeness, we will also improve the theorem at the end of paper, Appendix 2.

The minimizer of $\mathcal{D}(\cdot)$ corresponds to a singular Kähler-Einstein metric. We will prove the semi-continuity of $\mathcal{D}(\cdot)$ and derive the Kähler-Einstein equation for the minimizer (cf. Proposition 6.6). Our proof is similar with what Berman and Berndtsson studied on toric varieties in [9].

The proof of the necessity part of Theorem 1.2 is the same as one in Theorem 1.1. In fact, a $\mathbb{Q}$-Fano compactification of G is not K-polystable if (1.2) is not satisfied [36, Proposition 3.4] (also see [21]). This will be a contradiction to the K-polystability of $\mathbb{Q}$-Fano variety with a singular Kähler-Einstein metric (cf. [8, 33]). We omit this part.

[†]It can not be guaranteed that the G–compactification is smooth even if Z is smooth, see an examples for $G = SO_{2n+1}(\mathbb{C})$ in [47, Section 11].

The organization of paper is as follows. In Section 2, we recall some notations in [10] for singular Kähler-Einstein metrics on $\mathbb{Q}$-Fano varieties. In Section 3, we introduce a subspace $\mathcal{E}^1_{K\times K}(M,-K_M)$ of $\mathcal{E}^1(M,-K_M)$ and prove that the Guillemin function lies in this space (cf. Lemma 3.4). In Section 4, we prove that $\mathcal{E}^1_{K\times K}(M,-K_M)$ is equivalent to a dual space $\mathcal{E}^1_{K\times K}(2P)$ of Legendre functions (cf. Theorem 4.2). In Section 5, we compute the Ricci potential h_0 of ω_{2P} and show that it is bounded from above (cf. Proposition 5.1). The sufficient part of Theorem 1.2 will be proved in Section 6. In Section 7, we construct many $\mathbb{Q}$-Fano compactifications of $SO_4(\mathbb{C})$ and in particular, we will prove Theorem 1.3.

2. Preliminary on $\mathbb{Q}$-Fano varieties

For a $\mathbb{Q}$-Fano variety M, by Kodaira's embedding, there is an integer $\ell > 0$ such that we can embed M into a projective space $\mathbb{CP}^N$ by a basis of $H^0(M, K_M^{-\ell})$, for simplicity, we assume $M \subset \mathbb{CP}^N$ and $K_M^{-\ell} = \mathcal{O}_{\mathbb{CP}^N}(1)$. Then we have a metric

$$\omega_0 = \frac{1}{\ell}\omega_{FS}|_M \in 2\pi c_1(M),$$

where ω_{FS} is the Fubini-Study metric of $\mathbb{CP}^N$. Moreover, there is a Ricci potential h_0 of ω_0 such that

$$\mathrm{Ric}(\omega_0) - \omega_0 = \sqrt{-1}\partial\bar{\partial}h_0, \text{ on } M_{\mathrm{reg}}.$$

In the case that M has only klt-singularities, e^{h_0} is L^p-integrable for some $p > 1$ (cf. [10, 23]). We call ω an admissible Kähler metric on M if there are an embedding $M \subset \mathbb{CP}^N$ as above and a smooth function ϕ on $\mathbb{CP}^N$ such that

$$\omega = \omega_\phi = \omega_0 + \sqrt{-1}\,\partial\bar{\partial}\phi|_M.$$

In particular, ϕ is a function on M with $\phi \in L^\infty(M) \cap C^\infty(M_{\mathrm{reg}})$, called an admissible Kähler potential associated to ω_0. ‡

For a general (possibly unbounded) Kähler potential ϕ, we define its complex Monge-Ampère measure ω_ϕ^n by

$$\omega_\phi^n = \lim_{j\to\infty}\omega_{\phi_j}^n,$$

where $\phi_j = \max\{\phi, -j\}$. According to [10], we say that ϕ (or ω_ϕ^n) has full Monge-Ampère (MA) mass if

$$\int_M \omega_\phi^n = \int_M \omega_0^n.$$

The MA-measure ω_ϕ^n with a full MA-mass has no mass on the pluripolar set of ϕ in M. Thus we need to consider the measure on M_{reg}. Moreover, $e^{-\phi}$ is L^p-integrable for any $p > 0$ associated to ω_0^n.

Definition 2.1. *We call ω_ϕ a (singular) Kähler-Einstein metric on M with full MA-mass if ϕ satisfies the following complex Monge-Ampère equation,*

$$\omega_\phi^n = e^{h_0-\phi}\omega_0^n. \tag{2.1}$$

It has been shown in [10] that ϕ is C^∞ on M_{reg} if it is a solution of (2.1). Thus ω_ϕ satisfies the Kähler-Einstein equation on M_{reg},

$$\mathrm{Ric}(\omega_\phi) = \omega_\phi.$$

‡For simplicity, we will denote a Kähler metric by its Kähler form thereafter.

2.1. The space $\mathcal{E}^1(M, -K_M)$ and the Ding functional

On a smooth Fano manifold, there is a well-known Euler-Lagrange functional for Kähler potentials associated to (2.1), often referred as the Ding functional or F-functional, defined by (cf. [22, 43]),

$$F(\phi) = -\frac{1}{(n+1)V}\sum_{k=0}^{n}\int_M \phi\omega_\phi^k \wedge \omega_0^{n-k} - \log\left(\frac{1}{V}\int_M e^{h_0-\phi}\omega_0^n\right). \quad (2.2)$$

In case of $\mathbb{Q}$-Fano manifold with klt-singularities, Berman-Boucksom-Eyssidieux-Guedj-Zeriahi [10] extended $F(\cdot)$ to the space $\mathcal{E}^1(M, -K_M)$ defined by

$$\mathcal{E}^1(M, -K_M) = \{\phi | \, \phi \text{ has full MA mass and} \\ \sup_M \phi = 0, \ I(\phi) = \int_M -\phi\omega_\phi^n < \infty\}.$$

They showed that $\mathcal{E}^1(M, -K_M)$ is compact in certain weak topology. By a result of Darvas [18], $\mathcal{E}^1(M, -K_M)$ is in fact compact in the topology of L^1-distance. It provides a variational approach to study (2.1).

Definition 2.2. *[10, 44] The functional $F(\cdot)$ is called proper if there is a continuous function $p(t)$ on $\mathbb{R}$ with the property $\lim_{t\to+\infty} p(t) = +\infty$, such that*

$$F(\phi) \geq p(I(\phi)), \ \forall \phi \in \mathcal{E}^1(M, -K_M). \quad (2.3)$$

In [10], Berman-Boucksom-Eyssidieux-Guedj-Zeriahi proved the existence of solutions for (2.1) under the properness assumption (2.3) of $F(\cdot)$. However, this assumption does not hold in the case of existence of non-zero holomorphic vector fields such as in our case of $\mathbb{Q}$-Fano G-compactifications. So we need to consider a modified version of properness of the Ding functional instead to overcome this new difficulty as done on toric varieties [9, 37].

3. Associated polytopes and $K \times K$-invariant metrics

Let M be a $\mathbb{Q}$-Fano compactification of G with Z being the closure of a maximal complex torus $T^{\mathbb{C}}$-orbit. We first characterize the polytope P of Z associated to $(M, -K_M)$. Denote by $\mathfrak{M}$ the lattice of G-weights and W the Weyl group of $(G, T^{\mathbb{C}})$. Then P is an r-dimensional W-invariant, convex, rational polytope in $\mathfrak{a}^* = \mathfrak{M}_{\mathbb{R}}$. Let $\{F_A\}_{A=1,...,d_0}$ be the facets of P and $\{F_A\}_{A=1,...,d_+}$ be those whose relative interior intersects $\mathfrak{a}_+^*$. Suppose that

$$P = \cap_{A=1}^{d_0}\{l_A^o := \lambda_A - u_A(y) \geq 0\} \quad (3.1)$$

for some primitive vector $u_A \in \mathfrak{N}$ and the facet

$$F_A \subseteq \{l_A^o = 0\}, A = 1, ..., d_0.$$

By the W-invariance, for each $A \in \{1, ..., d_0\}$, there is some $w_A \in W$ such that $w_A(F_A) \in \{F_B\}_{B=1,...,d_+}$. Denote by $\rho_A = w_A^{-1}(\rho)$, where $\rho \in \mathfrak{a}_+^*$ is given by (1.1). Then $\rho_A(u_A)$ is independent of the choice of $w_A \in W$ and hence it is well-defined.

The following Lemma can be derived from a general result [26, Theorem 1.9] on polytopes of $\mathbb{Q}$-Fano spherical varieties. For readers' convenience, we sketch a direct proof in cases of group compactifications below by using [15, Section 4].

Lemma 3.1. *Let M be a $\mathbb{Q}$-Fano compactification of G with P being the associated moment polytope. Then for each $A = 1, ..., d_0$, it holds*

$$\lambda_A \;=\; 1 + 2\rho_A(u_A). \tag{3.2}$$

Conversely, a W-invariant convex polytope P given by (3.1) *is the associated polytope of some $\mathbb{Q}$-Fano G-compactification if* (3.2) *holds.*

Sketch of proof. Suppose that $-mK_M$ is a Cartier divisor for some $m \in \mathbb{N}_+$. Up to a dilation of the polytope, it suffices to consider the case when $m = 1$. Denote by B^+ the (positive) Borel subgroup of G corresponding to $(T^{\mathbb{C}}, \Phi_+)$ and B^- be the opposite one. Then by [15, Section 4], there exists a $B^+ \times B^-$-semiinvariant section of $-K_M$ whose divisor is

$$\mathfrak{d} = \sum_{A=1}^{d_+} X_A + 2 \sum_{\alpha_i \in \Phi_{+,s}} Y_{\alpha_i}, \tag{3.3}$$

where $\{X_{A'}\}$ is the set of $G \times G$-invariant prime divisors and Y_{α_i} is the prime $B^+ \times B^-$-invariant divisor corresponds to the weight α_i in $\Phi_{+,s}$, the set of simple roots in Φ_+. Note that the corresponding $B^+\times B^-$-weight of this divisor is 2ρ (cf. [21, Section 3.2.4]). Thus by adding the divisor of a $B^+ \times B^-$-semi-invariant rational function f_o with weight -2ρ, we have

$$\mathfrak{d} + \mathrm{div}(f_o) = \sum_{A=1}^{d_+} (1 + 2\rho(u_A))X_A \tag{3.4}$$

is a $G \times G$-invariant divisor. On the other hand, by [5, Theorem 2.4 (3)], the prime $G \times G$-invariant divisors of M are in bijections with W-orbits of prime toric divisors of Z. Restricting the above divisor to Z, we get (3.2).

Conversely, suppose that there is a W-invariant polytope P given by (3.1) and (3.2). Then P is rational and there is an $m \in \mathbb{N}_+$ so that mP is integral. The support function of mP is W-invariant and strictly convex. Hence it corresponds to an ample Cartier divisor $m\mathfrak{d}'$, where (cf. [14, Section 3.3])

$$\mathfrak{d}' = \sum_{A=1}^{d_+} (1 + 2\rho(u_A))X_A.$$

Obviously $\mathfrak{d}' - \mathrm{div}(f_o)$ equals to the divisor $\mathfrak{d}$ defined by (3.3), which is a divisor of $-K_M$ by [15, Section 4]. We conclude the Lemma. □

3.1. $K \times K$-invariant metrics

On a $\mathbb{Q}$-Fano compactification of G, we may regard the $G \times G$-action as a subgroup of $PGL_{N+1}(\mathbb{C})$ which acts holomorphically on the hyperplane bundle $L = O_{\mathbb{CP}^N}(1)$. Then any admissible $K \times K$-invariant Kähler metric $\omega_\phi \in \frac{2\pi}{\ell} c_1(L)$ can be regarded as a restriction of $K \times K$-invariant Kähler metric of $\mathbb{CP}^N$. Thus the moment polytope P associated to $(Z, L|_Z)$ is a W-invariant rational polytope in $\mathfrak{a}^*$. By the $K \times K$-invariance, the restriction of ω_ϕ on $T^{\mathbb{C}}$ is an open toric Kähler metric. Hence, it induces a strictly convex, W-invariant function ψ_ϕ on $\mathfrak{a}$ [6] (also see Lemma 3.3 below) such that

$$\omega_\phi \;=\; \sqrt{-1}\partial\bar{\partial}\psi_\phi, \text{ on } T^{\mathbb{C}}. \tag{3.5}$$

By the KAK-decomposition ([31, Theorem 7.39]), for any $g \in G$, there are k_1, $k_2 \in K$ and $x \in \mathfrak{a}$ such that $g = k_1 \exp(x) k_2$. Here x is uniquely determined up to a W-action. This means that x is unique in $\bar{\mathfrak{a}}_+$. Thus there is a bijection between $K \times K$-invariant functions Ψ on G and W-invariant functions ψ on $\mathfrak{a}$ which is given by

$$\Psi(\exp(\cdot)) = \psi(\cdot) : \mathfrak{a} \to \mathbb{R}.$$

Without of confusion, we will not distinguish ψ and Ψ, and call Ψ (or ψ) convex on G if ψ is convex on $\mathfrak{a}$.

The following KAK-integration formula can be found in [31, Proposition 5.28].

Proposition 3.2. *Let dV_G be a Haar measure on G and dx the Lebesgue measure on $\mathfrak{a}$. Then there exists a constant $C_H > 0$ such that for any $K \times K$-invariant, dV_G-integrable function ψ on G,*

$$\int_G \psi(g)\, dV_G = C_H \int_{\mathfrak{a}_+} \psi(x) \mathbf{J}(x)\, dx,$$

where

$$\mathbf{J}(x) = \prod_{\alpha \in \Phi_+} \sinh^2 \alpha(x).$$

Without loss of generality, we may normalize $C_H = 1$ for simplicity.

Next we recall a local holomorphic coordinate system on G used in [20]. By the standard Cartan decomposition, we can decompose $\mathfrak{g}$ as

$$\mathfrak{g} = (\mathfrak{t} \oplus \mathfrak{a}) \oplus (\oplus_{\alpha \in \Phi} V_\alpha),$$

where $\mathfrak{t}$ is the Lie algebra of T and

$$V_\alpha = \{X \in \mathfrak{g} |\ ad_H(X) = \alpha(H)X,\ \forall H \in \mathfrak{t} \oplus \mathfrak{a}\}$$

is the root space of complex dimension 1 with respect to α. By [28], one can choose $X_\alpha \in V_\alpha$ such that $X_{-\alpha} = -\iota(X_\alpha)$ and $[X_\alpha, X_{-\alpha}] = \alpha^\vee$, where ι is the Cartan involution and $\alpha^\vee$ is the dual of α by the Killing form. Let $E_\alpha = X_\alpha - X_{-\alpha}$ and $E_{-\alpha} = J(X_\alpha + X_{-\alpha})$. Denoted by $\mathfrak{k}_\alpha$, $\mathfrak{k}_{-\alpha}$ the real line spanned by E_α, $E_{-\alpha}$, respectively. Then we get the Cartan decomposition of Lie algebra $\mathfrak{k}$ of K as follows,

$$\mathfrak{k} = \mathfrak{t} \oplus (\oplus_{\alpha \in \Phi_+} (\mathfrak{k}_\alpha \oplus \mathfrak{k}_{-\alpha})).$$

Choose a real basis $\{E^0_1, ..., E^0_r\}$ of $\mathfrak{t}$, where r is the dimension of T. Then $\{E^0_1, ..., E^0_r\}$ together with $\{E_\alpha, E_{-\alpha}\}_{\alpha \in \Phi_+}$ forms a real basis of $\mathfrak{k}$, which is indexed by $\{E_1, ..., E_n\}$. We can also regard $\{E_1, ..., E_n\}$ as a complex basis of $\mathfrak{g}$. For any $g \in G$, we define local coordinates $\{z^i_{(g)}\}_{i=1,...,n}$ on a neighborhood of g by

$$(z^i_{(g)}) \to \exp(z^i_{(g)} E_i) g.$$

It is easy to see that $\theta^i|_g = dz^i_{(g)}|_g$, where the dual θ^i of E_i is a right-invariant holomorphic 1-form. Thus

$$dV_G|_g := \wedge_{i=1}^n \left(dz^i_{(g)} \wedge d\bar{z}^i_{(g)} \right)|_g,\ \forall g \in G \tag{3.6}$$

is also a right-invariant (n, n)-form, which defines a Haar measure.

For a $K \times K$-invariant function ψ, Delcroix computed the Hessian of ψ in the above local coordinates as follows [20, Theorem 1.2].

Lemma 3.3. *Let ψ be a $K \times K$ invariant function on G. Then for any $x \in \mathfrak{a}_+$, the complex Hessian matrix of ψ in the above coordinates is diagonal by blocks, and equals to*

$$\mathrm{Hess}_{\mathbb{C}}(\Psi)(\exp(x)) = \begin{pmatrix} \frac{1}{4}\mathrm{Hess}_{\mathbb{R}}(\psi)(x) & 0 & & 0 \\ 0 & M_{\alpha_{(1)}}(x) & & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & & M_{\alpha_{(\frac{n-r}{2})}}(x) \end{pmatrix}, \tag{3.7}$$

where $\Phi_+ = \{\alpha_{(1)}, ..., \alpha_{(\frac{n-r}{2})}\}$ is the set of positive roots and

$$M_{\alpha_{(i)}}(x) = \frac{1}{2}\alpha_{(i)}(\partial\psi(x)) \begin{pmatrix} \coth \alpha_{(i)}(x) & \sqrt{-1} \\ -\sqrt{-1} & \coth \alpha_{(i)}(x) \end{pmatrix}.$$

By (3.7) in Lemma 3.3, we see that ψ_ϕ induced by an admissible $K \times K$-invariant Kähler form ω_ϕ is convex on $\mathfrak{a}$. The complex Monge-Ampère measure is given by

$$\omega_\phi^n =: (\sqrt{-1}\partial\bar{\partial}\psi_\phi)^n = \mathrm{MA}_{\mathbb{C}}(\psi_\phi)\, dV_G. \tag{3.8}$$

By (3.6), for any $x \in \mathfrak{a}_+$ we have

$$\mathrm{MA}_{\mathbb{C}}(\psi_\phi)(\exp(x)) = \frac{1}{2^{r+n}}\mathrm{MA}_{\mathbb{R}}(\psi_\phi)(x)\frac{1}{\mathbf{J}(x)}\prod_{\alpha\in\Phi_+}\alpha^2(\partial\psi_\phi(x)) \tag{3.9}$$

in (3.8). In particular, by Proposition 3.2,

$$\mathrm{Vol}(M) = \int_M \omega_\phi^n = \int_{2P_+} \pi\, dy = V. \tag{3.10}$$

Clearly, (3.9) also holds for any Kähler potential in $\mathcal{E}^1(M, -K_M)$, which is smooth and $K \times K$-invariant on G. For the completeness, we introduce a subspace of $\mathcal{E}^1(M, -K_M)$ by

$$\mathcal{E}^1_{K\times K}(M, -K_M) = \{\phi \in \mathcal{E}^1(M, -K_M) |\ \psi_0 + \phi \text{ is } K \times K\text{-invariant and convex on } G\}. \tag{3.11}$$

Here ψ_0 is a convex function on $\mathfrak{a}$ associated to a background admissible $K \times K$-invariant metric ω_0 as in (3.5). $\mathcal{E}^1_{K\times K}(M, -K_M)$ is locally precompact in terms of convex functions on $\mathfrak{a}$. In Sections 4 and 6, we will prove its completeness by using the Legendre dual.

3.2. Fine polytope P

In this subsection, we show that the Legendre dual of Guillemin function u_{2P} on $2P$ lies in $\mathcal{E}^1_{K\times K}(M, -K_M)$ when P is fine.

Recall (3.1). For convenience, we set

$$l_A(y) = 2\lambda_A - u_A(y).$$

Then

$$2P = \cap_{A=1}^{d_0}\{l_A(y) \geq 0\}.$$

Thus, u_{2P} is given by (cf. [1])

$$u_{2P} \;=\; \frac{1}{2}\sum_{A=1}^{d_0} l_A \log l_A(y).$$

Clearly, it is W-invariant, so its Legendre function ψ_{2P} is also W-invariant, where

$$\psi_{2P}(x) \;=\; \sup_{y\in 2P}(\langle x,y\rangle - u_{2P}(y)),\;\; \forall\, x\in \mathfrak{a}. \tag{3.12}$$

Hence, by [1, Theorem 2] and [6] (also see Lemma 3.3), [§] we can extend

$$\omega_{2P} \;=\; \sqrt{-1}\partial\bar\partial\psi_{2P},\;\text{ on } \mathfrak{a},$$

to a $K\times K$-invariant metric on G.

Lemma 3.4. *Let ψ_0 be the background $K\times K$-invariant Kähler potential in (3.11). Assume that P is fine. Then the Kähler potential $(\psi_{2P}-\psi_0)$ of ω_{2P} lies in $\phi\in L^\infty(M)\cap C^\infty(M_{\rm reg})$. In particular, $(\psi_{2P}-\psi_0)\in \mathcal{E}^1_{K\times K}(M,-K_M)$.*

Proof. Fix an $m_0\in\mathbb{Z}_+$ such that $-m_0K_X$ is very ample. We consider the projective embedding

$$\iota: M\to \mathbb{CP}^N$$

given by $|-m_0K_M|$, where $N=h^0(M,-m_0K_M)-1$. By [39, Section 2.3], the pull back of the Fubini-Study metric on $\mathbb{CP}^N$ gives a $K\times K$-invariant, Hermitian metric h on $L=\mathcal{O}_{\mathbb{CP}^N}(-1)|_M$. Moreover, we have

$$h|_{T^{\mathbb{C}}}(x)=\sum_{\lambda\in mP\cap\mathfrak{M}}\bar n(\lambda)e^{2\lambda(x)}, \tag{3.13}$$

where $\bar n(\lambda)\in\mathbb{Z}_+$. Thus we have a Kähler potential on $T^{\mathbb{C}}$ by

$$\psi_{FS} \;=\; \frac{1}{m}\log h|_{T^{\mathbb{C}}}.$$

Since P is fine, one can show directly that

$$\psi_{FS}\in\mathcal{V}(2P) \;=\; \{\psi\in C^0(\mathfrak{a})|\;\psi\text{ is convex, } W\text{-invariant and }\max_{\mathfrak{a}}|v_{2P}-\psi|<\infty\},$$

where $v_{2P}(\cdot)$ is the support function on $\mathfrak{a}$ defined by

$$v_{2P}(x) \;=\; \sup_{y\in 2P}\langle x,y\rangle. \tag{3.14}$$

Recall that the Legendre function u_ψ of ψ is defined as in (3.12) by

$$u_\psi(y) \;=\; \sup_{x\in\mathfrak{a}}(\langle x,y\rangle-\psi(x)),\;\; y\in 2P. \tag{3.15}$$

[§]The corresponding moment map is given by $\frac{1}{2}\nabla\psi_{2P}$, whose image is P.

It is known that $\psi \in \mathcal{V}(2P)$ if and only if u_ψ is uniformly bounded on $2P$ since the Legendre function of v_{2P} is zero (cf. [42]). Thus the Legendre function u_h of $h|_{T^{\mathbb{C}}}(x)$ is uniformly bounded on $2P$. It follows that

$$|u_h - v_{2P}| \le C.$$

Hence, we get

$$\max_{\mathfrak{a}} |\psi_{FS} - \psi_{2P}| < +\infty.$$

Consequently,

$$\max_{\mathfrak{a}} |\psi_{2P} - \psi_0| < +\infty.$$

By (3.10), $(\psi_{2P} - \psi_0)$ has full MA-mass, so we have completed the proof. □

4. The space $\mathcal{E}^1_{K\times K}(2P)$

In this section, we describe the space $\mathcal{E}^1_{K\times K}(M, -K_M)$ in (3.11) via Legendre functions as in [17] for $\mathbb{Q}$-Fano toric varieties. Recall the background $K\times K$-invariant Kähler potential ψ_0 in Lemma 3.4. Then we can normalize ψ_0 up to an action of the centre $Z(G)$ of G as follows (cf. [38]),

$$\inf_{\mathfrak{a}} \psi_0 = \psi_0(O) = 0, \tag{4.1}$$

where O is the origin of $\mathfrak{a}$. Similarly, for any $\phi \in \mathcal{E}^1_{K\times K}(M, -K_M)$, $\psi_\phi = \psi_0 + \phi$ can be also normalized as in (4.1).

The following lemma is elementary.

Lemma 4.1. *For any $K \times K$-invariant potential ϕ normalized as in* (4.1)*, it holds*

$$\partial(\psi_\phi) \subseteq 2P, \text{ and } \psi_\phi \le v_{2P},$$

where $\partial(\psi_\phi)(\cdot)$ is the normal mapping of ψ_ϕ.

Proof. We choose a sequence of decreasing and uniformly bounded $K \times K$-invariant potential ϕ_i normalized as in (4.1) such that

$$\omega_0 + \sqrt{-1}\partial\bar\partial\phi_i > 0, \text{ on } M_{\mathrm{reg}}$$

and

$$\phi_i \to \phi, \text{ as } i \to +\infty.$$

Then

$$\sqrt{-1}\partial\bar\partial\psi_{\phi_i} > 0 \text{ in } G.$$

It follows that

$$\partial\psi_{\phi_i} \subseteq 2P.$$

This implies that $\partial\psi_\phi \subseteq 2P$. By the convexity, we also get $\psi_\phi \le v_{2P}$.

□

It is easy to see that the Legendre function u_ϕ of ψ_ϕ with $\phi \in \mathcal{E}^1_{K\times K}(M, -K_M)$ satisfies

$$\inf_{2P} u_\phi = u_\phi(O) = 0. \tag{4.2}$$

We set a class of W-invariant convex functions on $2P$ by

$$\mathcal{E}^1_{K\times K}(2P) = \{u|\ u \text{ is convex, } W\text{-invariant on } 2P \text{ which satisfies (4.2) and } \int_{2P_+} u\pi\, dy < +\infty\}.$$

Our main goal in this section is to prove

Theorem 4.2. *A Kähler potential $\phi \in \mathcal{E}^1_{K\times K}(M, -K_M)$ with normalized ψ_ϕ satisfying* (4.1) *if and only if the Legendre function u_ϕ of ψ_ϕ lies in $\mathcal{E}^1_{K\times K}(2P)$. In particular, u_ϕ is locally bounded in* Int$(2P)$ *if $\phi \in \mathcal{E}^1_{K\times K}(M, -K_M)$.*

As in [17], we need to establish a comparison principle for the complex Monge-Ampère measure in $\mathcal{E}^1_{K\times K}(M, -K_M)$. For our purpose, we will introduce a weighted Monge-Ampère measure on $\mathfrak{a}$ in the following.

4.1. Weighted Monge-Ampère measure

Definition 4.3. *Let $\Omega \subseteq \mathfrak{a}$ be a W-invariant domain and ψ any W-invariant convex function on Ω. Define a weighted Monge-Ampère measure on Ω by*

$$\int_{\Omega'} MA_{\mathbb{R};\pi}(\psi)dx = \int_{\partial\psi(\Omega')} \pi\, dy,\ \forall\ \Omega' \Subset \Omega,$$

where $\partial\psi(\cdot)$ is the normal mapping of ψ.

Remark 4.4. *Let $\{\psi_k\}$ be a sequence of convex functions which converges locally uniformly to ψ on Ω, then $MA_{\mathbb{R};\pi}(\psi_k)$ converges to $MA_{\mathbb{R};\pi}(\psi)$ (cf. [2, Section 15]).*

We have the following KAK-integration for the measure ω^n_ϕ with $\phi \in \mathcal{E}^1_{K\times K}(M, -K_M)$.

Lemma 4.5. *Let $\omega_\phi = \sqrt{-1}\partial\bar\partial\psi_\phi$ with $\phi \in \mathcal{E}^1_{K\times K}(M, -K_M)$. Then for any $K \times K$-invariant continuous uniformly bounded function f on G, it holds*

$$\int_M f\omega^n_\phi = \int_{\mathfrak{a}_+} fMA_{\mathbb{R};\pi}(\psi_\phi)dx. \tag{4.3}$$

Proof. First we assume that f is a $K\times K$-invariant continuous function with compact support on $\mathfrak{a}$. We take a sequence of smooth W-invariant convex functions $\psi_k \searrow \psi$ and let $\omega_k = \sqrt{-1}\partial\bar\partial\psi_k$. Then for any W-invariant $\Omega' \Subset \mathfrak{a}$, it holds

$$\int_{\Omega'} \mathrm{MA}_{\mathbb{R};\pi}(\psi_k)dx := \int_{\Omega'} \det(\nabla^2\psi_k)\pi(\nabla\psi_k)\, dy.$$

By the standard KAK-integration formula in Proposition 3.2, it follows that

$$\int_M f\omega^n_k = \int_{\mathfrak{a}_+} f \det(\nabla^2\psi_k)\pi(\nabla\psi_k)dx = \int_{\mathfrak{a}_+} f\mathrm{MA}_{\mathbb{R};\pi}(\psi_k)dx.$$

Since

$$\int_M f\omega_k^n \to \int_M f\omega^n,$$

we get (4.3) by Remark 4.4.

Next we choose a sequence of exhausting W-invariant convex domains Ω_k in $\mathfrak{a}$ and a sequence of W-invariant convex functions with the support on Ω_{k+1} such that $f_k = f|_{\Omega_k}$. Since ω^n has full MA-mass, we get

$$\begin{aligned}\int_M f\omega^n &= \lim_k \int_M f_k\omega^n \\ &= \lim_k \int_{\mathfrak{a}_+} f_k \mathrm{MA}_{\mathbb{R};\pi}(\psi_\phi)dx \\ &= \int_{\mathfrak{a}_+} f\mathrm{MA}_{\mathbb{R};\pi}(\psi_\phi)dx.\end{aligned}$$

□

4.2. Comparison principles

In this subsection, we establish an ordinary comparison principle for the weighted Monge-Ampère measure $\mathrm{MA}_{\mathbb{R};\pi}(\psi)$. As showed in [17], this will then lead a global comparison principle (see Proposition 4.7 below) which can be used to estimate the MA mass of Kähler potential. We will only prove Proposition 4.6 and omit other proofs, since the others follow directly from Proposition 4.6 by corresponding arguments in [17].

Proposition 4.6. *Let $\Omega \subseteq \mathfrak{a}$ be a W-invariant domain and φ, ψ be two convex functions on Ω such that*

$$\varphi \geq \psi \text{ and } (\varphi - \psi)|_{\partial\Omega} = 0. \tag{4.4}$$

Then

$$\int_\Omega MA_{\mathbb{R};\pi}(\varphi)dx \leq \int_\Omega MA_{\mathbb{R};\pi}(\psi)dx. \tag{4.5}$$

Proof. It is sufficient to prove (4.5) when φ and ψ are smooth, since we can approximate general φ and ψ by smooth W-invariant convex functions by Lemma 4.5. Let

$$\varphi_t = t\varphi + (1-t)\psi.$$

Then

$$\mathrm{MA}_{\mathbb{R};\pi}(\varphi_t) = \det(\nabla^2\varphi_t)\prod_{\alpha\in\Phi_+}\alpha^2(\nabla\varphi_t)$$

and

$$\begin{aligned}&\frac{d}{dt}\int_\Omega \det(\nabla^2\varphi_t)\prod_{\alpha\in\Phi_+}\alpha^2(\nabla\varphi_t)dx \\ = &\int_\Omega (\nabla^2\varphi_t)^{-1,ij}\nabla^2\dot{\varphi}_{t,ij}\det(\nabla^2\varphi_t)\prod_{\alpha\in\Phi_+}\alpha^2(\nabla\varphi_t)dx\end{aligned}$$

$$+\int_{\Omega}\left(\sum_{\alpha\in\Phi_+}\frac{2\alpha(\nabla\dot{\varphi}_t)}{\alpha(\nabla\varphi_t)}\right)\det(\nabla^2\varphi_t)\prod_{\alpha\in\Phi_+}\alpha^2(\nabla\varphi_t)dx. \tag{4.6}$$

Using the fact that

$$\left(\det(\nabla^2\varphi_t)(\nabla^2\varphi_t)^{-1,ij}\right)_{,j}=0$$

and integration by parts, we have

$$\begin{aligned}
&\int_{\Omega}(\nabla^2\varphi_t)^{-1,ij}\nabla^2\dot{\varphi}_{t,ij}\det(\nabla^2\varphi_t)\prod_{\alpha\in\Phi_+}\alpha^2(\nabla\varphi_t)dx\\
=&\int_{\partial\Omega}(\nabla^2\varphi_t)^{-1,ij}\nabla\dot{\varphi}_{t,i}\nu_j\det(\nabla^2\varphi_t)\prod_{\alpha\in\Phi_+}\alpha^2(\nabla\varphi_t)d\sigma\\
&-\int_{\partial\Omega}[(\nabla^2\varphi_t)^{-1,ij}\det(\nabla^2\varphi_t)\prod_{\alpha\in\Phi_+}\alpha^2(\nabla\varphi_t)]_{,j}\nu_i\dot{\varphi}_t d\sigma\\
&+\int_{\Omega}(\nabla^2\varphi_t)^{-1,ij}\dot{\varphi}_t\det(\nabla^2\varphi_t)\left(\prod_{\alpha\in\Phi_+}\alpha^2(\nabla\varphi_t)\right)_{,ij}dx.
\end{aligned}\tag{4.7}$$

Also

$$\begin{aligned}
&\int_{\Omega}\left(\sum_{\alpha\in\Phi_+}\frac{2\alpha(\nabla\dot{\varphi}_t)}{\alpha(\nabla\varphi_t)}\right)\det(\nabla^2\varphi_t)\prod_{\alpha\in\Phi_+}\alpha^2(\nabla\varphi_t)dx\\
=&2\int_{\partial\Omega}\sum_{\alpha\in\Phi_+}\frac{\alpha^i\nu_i}{\alpha(\nabla\varphi_t)}\det(\nabla^2\varphi_t)\prod_{\alpha\in\Phi_+}\alpha^2(\nabla\varphi_t)\dot{\varphi}_t d\sigma\\
=&-2\int_{\Omega}\left(\det(\nabla^2\varphi)\prod_{\alpha\in\Phi_+}\alpha^2(\nabla\varphi_t)\sum_{\alpha\in\Phi_+}\frac{\alpha^i}{\alpha(\nabla\varphi)}\right)_{,i}\dot{\varphi}_t dx.
\end{aligned}\tag{4.8}$$

Note that

$$\begin{aligned}
\frac{\left(\prod_{\alpha\in\Phi_+}\alpha^2(\nabla\varphi_t)\right)_{,ij}}{\prod_{\alpha\in\Phi_+}\alpha^2(\nabla\varphi_t)}=&-2\sum_{\alpha\in\Phi_+}\frac{\alpha^k\alpha^l\varphi_{t,ik}\varphi_{t,jl}}{\alpha^2(\nabla\varphi_t)}+2\sum_{\alpha\in\Phi_+}\frac{\alpha^k\varphi_{t,ijk}}{\alpha(\nabla\varphi_t)}+4\sum_{\alpha,\beta\in\Phi_+}\frac{\alpha^k\beta^l\varphi_{t,ik}\varphi_{t,jl}}{\alpha(\nabla\varphi_t)\beta(\nabla\varphi_t)}\\
=&2\frac{\left(\det(\nabla^2\varphi)\prod_{\alpha\in\Phi_+}\alpha^2(\nabla\varphi_t)\sum_{\alpha\in\Phi_+}\frac{\alpha^i}{\alpha(\nabla\varphi)}\right)_{,i}}{\det(\nabla^2\varphi)\prod_{\alpha\in\Phi_+}\alpha^2(\nabla\varphi_t)}.
\end{aligned}\tag{4.9}$$

Plugging (4.7)–(4.9) into (4.6) and using the boundary condition (4.4), we have

$$\frac{d}{dt}\int_{\Omega}\det(\nabla^2\varphi_t)\prod_{\alpha\in\Phi_+}\alpha^2(\nabla\varphi_t)dx=\int_{\partial\Omega}\nabla\dot{\varphi}_{t,i}\nu_i\det(\nabla^2\varphi_t)\prod_{\alpha\in\Phi_+}\alpha^2(\nabla\varphi_t)d\sigma\le 0.$$

Hence we get (4.5). □

By the above Proposition, we get the following analogue of [17, Lemma 2.3], which gives a global comparison principle for the weighted Monge-Ampère measures.

Proposition 4.7. *Let φ, ψ be two W-invariant convex functions on $\mathfrak{a}$ so that*

$$\varphi \geq \psi$$

and

$$\lim_{|x|\to+\infty} \varphi(x) = +\infty.$$

Then

$$\int_{\mathfrak{a}_+} MA_{\mathbb{R};\pi}(\varphi)dx \geq \int_{\mathfrak{a}_+} MA_{\mathbb{R};\pi}(\psi)dx.$$

As an application of Proposition 4.7, we get by the argument of [17, Lemma 2.7],

Lemma 4.8. *Let ψ be a W-invariant convex function on $\mathfrak{a}$ and u its Legendre function. Suppose that for some constant C,*

$$\psi \leq v_{2P} + C, \tag{4.10}$$

where v_{2P} is the support function of $2P$. Then

$$\int_{\mathfrak{a}_+} MA_{\mathbb{R};\pi}(\psi)dx = \int_{2P} \pi\, dy, \tag{4.11}$$

if $u < +\infty$ everywhere in the interior of $2P$.

The inverse of Lemma 4.8 is also true as an analogue of [17, Theorem 3.6]. In fact, we have

Proposition 4.9. *Let ϕ be a $K \times K$-invariant potential. Then ψ_ϕ satisfies* (4.11) *if and only if u_ϕ is finite everywhere in* $\mathrm{Int}(2P)$.

Proposition 4.9 will be used in the proof of Theorem 4.2 in next subsection.

4.3. Proof of Theorem 4.2

It is easy to see that (4.1) is equivalent to (4.2). Thus, to prove Theorem 4.2, we only need to show that

$$\phi \in \mathcal{E}^1_{K\times K}(M, -K_M) \Longleftrightarrow \int_{2P_+} |u_\phi|\pi\, dy < +\infty.$$

The following lemma can be found in [9, Lemma 2.7] (proved in [9, Appendix]).

Lemma 4.10. *Let ψ be a convex function on $\mathfrak{a}$ and u_ψ its Legendre dual on P.*

(1) u_ψ is differentiable at p if and only if the sup defining u_ψ is attained at a unique point $x_p \in \mathfrak{a}$ and $x_p = \nabla u_\psi(p)$;
(2) Suppose that $(\psi - \psi_0) \in \mathcal{E}^1_{K\times K}(M, -K_M)$. Let $p \in P$ at which u_ψ is differentiable. Then for any continuous uniformly bounded function v on $\mathfrak{a}$, it holds

$$\left.\frac{d}{dt}\right|_{t=0} u_{\psi+tv}(p) = -v(\nabla u_\psi(p)), \tag{4.12}$$

where $u_{\psi+tv}$ is the Legendre function of $\psi + tv$ as in (3.15) which is well-defined since v is continuous and uniformly bounded on $\mathfrak{a}$.

Remark 4.11. *By Lemma 4.5 and Part (1) in Lemma 4.10, we can prove the following: Let* $\phi \in \mathcal{E}^1_{K\times K}(M, -K_M)$*, then for any* $K \times K$*-invariant continuous uniformly bounded function* f *on* G*, it holds*

$$\int_M f\omega_\phi^n = \int_{2P} f(\partial u_\phi)\pi dy. \tag{4.13}$$

Proof of Theorem 4.2. We will follow the arguments in [17, Proposition 3.9] to prove the theorem.

Necessary part. First we show that ϕ has full MA-mass by Proposition 4.9. In fact, by a result in [36, Lemma 4.5], we see that for any W-invariant convex polytope $2P' \subseteq 2P$, there is a constant $C = C(P')$ such that for any W-invariant convex $u_\phi \geq 0$,

$$\int_{2P'} u_\phi \, dy \leq C \int_{2P} u_\phi \pi \, dy < +\infty.$$

This implies that u_ϕ is finite everywhere in $\mathrm{Int}(2P)$ by the convexity of u_ϕ. Thus we get what we want from Proposition 4.9.

Next we prove that ϕ is L^1-integrate associated to the MA-measure ω_ϕ^n. Let $\psi_1 = \psi_0 + \phi$ (ϕ may be different to a constant) be satisfying (4.1). We define a distance between ψ_0 and ψ_1 for $p \geq 1$,

$$d_p(\psi_0, \psi_1) = \inf_{\phi_t} \int_0^1 \left(\int_M |\dot{\phi}_t|^p \omega_{\phi_t}^n \right)^{\frac{1}{p}} dt,$$

where $\phi_t \in \mathcal{E}^1(M, -K_M)$ ($t \in [0, 1]$) runs over all curves joining 0 and ϕ with $\omega_{\phi_t} \geq 0$. Choose a special path ϕ_t such that the corresponding Legendre functions of $\psi_t = \psi_0 + \phi_t$ are given by

$$u_t = tu_1 + (1 - t)u_0, \tag{4.14}$$

where u_1 and u_0 are the Legendre functions of ψ_1 and ψ_0, respectively. Note that by Lemma 4.10,

$$\dot{\psi}_t = -\dot{u}_t = u_0 - u_1, \text{ almost everywhere.}$$

Then by Lemma 4.5 (or Remark 4.11), we get

$$\begin{aligned} d_p(\psi_0, \psi_1) &\leq \int_0^1 \left(\int_{2P_+} |\dot{u}_t|^p \pi \, dy \right)^{\frac{1}{p}} dt \\ &\leq C(p) \left(\int_{2P_+} |u_1|^p \pi \, dy \right)^{\frac{1}{p}} + C'(p.\psi_0). \end{aligned} \tag{4.15}$$

On the other hand, by a result of Darvas-Rubinstein [19], there are uniform constant C_0 and C_1 such that for any Kähler potential ϕ with full MA-measure it holds,

$$-\int_M \phi\omega_\phi^n \leq C_0 d_1(\psi_0, \psi_1) + C_1.$$

Thus we obtain

$$-\int_M \phi\omega_\phi^n \leq C.$$

Hence, $\phi \in \mathcal{E}^1_{K\times K}(M, -K_M)$.

Sufficient part. Assume that $\phi \in \mathcal{E}^1_{K\times K}(M, -K_M)$. We first deal with the case of $\phi \in L^\infty(M)\cap C^\infty(G)$. Then

$$v_{2P} - C \ \le\ \psi_\phi \ \le\ v_{2P} \ \le\ \psi_0 + C, \tag{4.16}$$

and

$$\nabla\psi_\phi : \mathfrak{a} \to 2P$$

is a bijection. Thus

$$\begin{aligned} -\phi &= (\psi_0 - \psi_\phi)(\nabla u_\phi) \\ &\ge v_{2P}(\nabla u_\phi) - \psi_\phi(\nabla u_\phi) - C_2 \ge -C_2. \end{aligned}$$

Moreover,

$$\begin{aligned} (\psi_0 - \psi_\phi)(\nabla u_\phi) \ge& v_{2P}(\nabla u_\phi) - \psi_\phi(\nabla u_\phi) - C \\ =& \sup_{y'\in 2P}\langle\nabla u_\phi, y'\rangle - \psi(\nabla u_\phi) - C \\ \ge& \langle\nabla u_\phi, y\rangle - \psi(\nabla u_\phi) - C \\ =& u_\phi(y) - C. \end{aligned}$$

Hence,

$$\begin{aligned} \int_{2P_+} u_\phi \pi\, dy &\le \int_{2P_+} (\psi_0 - \psi_\phi)(\nabla u_\phi)\pi\, dy + C \\ &= \int_M |\phi|\omega_\phi^n + C < +\infty. \end{aligned} \tag{4.17}$$

Next for an arbitrary $\phi \in \mathcal{E}^1_{K\times K}(M, -K_M)$, we choose a sequence of smooth $K\times K$-invariant functions $\{\phi_j\}$ decreasing to ϕ such that $\phi_j \in C^\infty(G)$ and

$$\sqrt{-1}\partial\bar\partial(\psi_0 + \phi_j) > 0, \text{ in } G.$$

Then as in (4.17), we have

$$\begin{aligned} \int_{2P_+} u_j \pi\, dy &\le \int_{2P_+} (\psi_0 - \psi_j)(\nabla u_j)\pi\, dy \\ &= \int_M |\phi_j|\omega_j^n + C, \end{aligned}$$

where u_j is the Legendre function of $\psi_j = \psi_0 + \phi_j$. Note that

$$\int_M |\phi_j|\omega_j^n \to \int_M |\phi|\omega_\phi^n$$

and $u_j \nearrow u_\phi$. Thus by taking the above limit as $j \to +\infty$, we also get (4.17).

□

5. Computation of the Ricci potential

In this section, we assume that the associated polytope P is fine. Then by Lemma 3.4, $(\psi_{2P} - \psi_0) \in \mathcal{E}^1_{K\times K}(M, -K_M)$ is a smooth $K \times K$-invariant Kähler potential on G. It follows that

$$-\log\det(\partial\bar{\partial}\psi_{2P}) - \psi_{2P} = h_0 \tag{5.1}$$

gives a Ricci potential h_0 of ω_{2P}, which is smooth and $K \times K$-invariant on G.

The following proposition gives an upper bound of h_0.

Proposition 5.1. *The Ricci potential h_0 of ω_{2P} is uniformly bounded from above on G. In particular, e^{h_0} is uniformly bounded on G.*

Proof. As in [36, Sections 3.2 and 4.3], the proof is based on a direct computation of asymptotic behavior of h_0 near every point of $\partial(2P_+)$. Recall that

$$\mathbf{J}(x) = \prod_{\alpha\in\Phi_+} \sinh^2\alpha(x), x \in \mathfrak{a} \text{ and } \pi(y) = \prod_{\alpha\in\Phi_+} \alpha^2(y),\ y \in 2P.$$

Since the Ricci potential of h_0 is also $K \times K$-invariant, by (5.1) and (3.9),

$$\begin{aligned} h_0 &= -\log\det(\psi_{2P,ij}) - \psi_{2P} + \log\mathbf{J}(x) - \log\prod_{\alpha\in\Phi_+}\alpha^2(\nabla\psi_{2P}) \\ &= \log\det(u_{2P,ij}) - y_i u_{2P,i} + u_{2P} + \log\mathbf{J}(\nabla u_{2P}) - \log\pi(y). \end{aligned} \tag{5.2}$$

Here in the second line we take the Legendre transformation

$$u_{2P}(y(x)) = y_i(x)x^i - \psi_{2P}(x) \text{ and } y(x) = \nabla\psi_{2P}(x).$$

Note that

$$\begin{aligned} u_{2P,i} &= \frac{1}{2}\sum_{A=1}^{d_0}(-u_A^i)(1 + \log l_A), \\ u_{2P,ij} &= \frac{1}{2}\sum_{A=1}^{d_0}\frac{u_A^i u_A^j}{l_A} \end{aligned}$$

and

$$\log\mathbf{J}(t) = 2\sum_{\alpha\in\Phi_+}\log\sinh(t).$$

Thus we have

$$\begin{aligned} h_0 = &-\sum_{A=1}^{d_0}\log l_A + \frac{1}{2}\sum_{A=1}^{d_0}(u_A^i y_i)\log l_A \\ &+ 2\sum_{\alpha\in\Phi_+}\log\sinh(-\frac{1}{2}\sum_{A=1}^{d_0}\alpha(u_A)\log l_A) - 2\sum_{\alpha\in\Phi_+}\log\alpha(y) + O(1). \end{aligned} \tag{5.3}$$

By (5.3), h_0 is locally bounded in the interior of $2P_+$. Thus we need to prove that h_0 is bounded above near each $y_0 \in \partial(2P_+)$. There will be three cases as follows.

Case-1. $y_0 \in \partial(2P_+)$ and is away from any Weyl wall (see Figure 1).

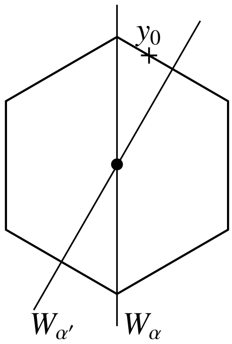

Figure 1. *Case-1*: y_0 is away from any Weyl wall.

Note that

$$\log\sinh(t) = \begin{cases} t + O(1), \ t \to +\infty, \\ \log t + O(1), \ t \to 0^+. \end{cases} \tag{5.4}$$

Then we get as $y \to y_0$,

$$\sum_{\alpha\in\Phi_+} \log\sinh(-\frac{1}{2}\sum_{A=1}^{d_0}\alpha(u_A)\log l_A) = -\sum_A \rho(u_A)\log l_A + O(1).$$

By (5.3), it follows that

$$h_0 = -\sum_{\{A|l_A(y_0)=0\}}\left(1 - \frac{1}{2}y_i u_A^i + 2\rho_i u_A^i\right)\log l_A(y) + O(1).$$

However, by Lemma 3.1, we have

$$h_0 = -\frac{1}{2}\sum_{\{A|l_A(y_0)=0\}} l_A(y)\log l_A(y) + O(1).$$

Hence h_0 is bounded near y_0.

Case-2. y_0 lies on some Weyl walls but away from any facet of $2P$ (see Figure 2).

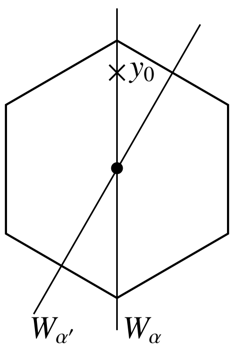

Figure 2. *Case-2*: y_0 lies on a Weyl wall but away from the facets.

In this case it is direct to see that h_0 is bounded near y_0 since

$$\log\det(u_{2P,ij}),\ y_i u_{2P,i},\ \frac{\mathbf{J}(\nabla u_{2P})}{\pi(y)}$$

are all bounded.

Case-3. y_0 lies on the intersection of $\partial(2P)$ with some Weyl walls. In this case, by (3.1), we rewrite (5.3) as

$$\begin{aligned} h_0 =& 2\sum_{A=1}^{d_0}\rho_A(u_A)\log l_A + 2\sum_{\alpha\in\Phi_+}\log\sinh(-\frac{1}{2}\sum_{A=1}^{d_0}\alpha(u_A)\log l_A) \\ & - 2\sum_{\alpha\in\Phi_+}\log\alpha(y) + O(1) \\ =& \sum_{\alpha\in\Phi_+}\left[\sum_{A=1}^{d_0}|\alpha(u_A)|\log l_A + 2\log\sinh(-\frac{1}{2}\sum_{A=1}^{d_0}\alpha(u_A)\log l_A)\right. \\ & \left. -2\log\alpha(y)\right] + O(1),\ y\to y_0. \end{aligned}$$

Here we used a fact that

$$2\rho_A(u_A) = \sum_{\alpha\in\Phi_+}|\alpha(u_A)|.$$

Set

$$I_\alpha(y) = \sum_{A=1}^{d_0}|\alpha(u_A)|\log l_A + 2\log\sinh(-\frac{1}{2}\sum_{A=1}^{d_0}\alpha(u_A)\log l_A) - 2\log\alpha(y)$$

for each $\alpha\in\Phi_+$. Then

$$h_0(y) = \sum_{\alpha\in\Phi_+} I_\alpha(y) + O(1),\ y\to y_0. \tag{5.5}$$

Note that each $I_\alpha(y)$ involves only one root α. Thus, without loss of generality, we may assume that y_0 lies on only one Weyl wall.

Assume that $y_0\in\partial(2P)\cap W_{\alpha_0}$ for some simple Weyl wall W_{α_0}, $\alpha_0\in\Phi_+$ and it is away from other Weyl walls. Now we estimate each $I_\alpha(y)$ in (5.5). When $\beta\neq\alpha_0$, it is easy to see that

$$\beta(y)\to c_\beta > 0,\ \text{as}\ y\to y_0.$$

Then, by (5.4), we have

$$\log\sinh(-\frac{1}{2}\sum_{A=1}^{d_0}\beta(u_A)\log l_A) = -\frac{1}{2}\sum_{\{A|l_A(y_0)=0\}}\beta(u_A)\log l_A + O(1), \forall\beta\neq\alpha_0.$$

Note that $y_0\in\{\beta(y)>0\}$. Thus any facet F_A passing through y_0 lies in $\{\beta(y)>0\}$ or is orthogonal to W_β. Since $2P$ is convex and s_β-invariant, where s_β is the reflection with respect to W_β, these facets must satisfy

$$\beta(u_A)\geq 0.$$

Hence, for any $\beta \neq \alpha_0$, we get

$$
\begin{aligned}
I_\beta(y) = &\sum_{A=1}^{d_0} |\beta(u_A)| \log l_A - 2\sum_{A=1}^{d_0} |\beta(u_A)| \log l_A - 2\log\beta(y) \\
=&O(1), \text{ as } y \to y_0. \qquad (5.6)
\end{aligned}
$$

It remains to estimate the second term in $I_{\alpha_0}(y)$,

$$\log\sinh(-\frac{1}{2}\sum_A \alpha_0(u_A)\log l_A). \qquad (5.7)$$

We first consider a simple case that y_0 lies on the intersection of W_{α_0} with at most two facets of $2P$. Then there will be two subcases: $y_0 \in W_{\alpha_0} \cap F_1$ or $y_0 \in W_{\alpha_0} \cap F_1 \cap F_2$, where F_1, F_2 are two facets of P.

Case-3.1. $y_0 \in W_{\alpha_0} \cap F_1$ is away from other facets of $2P$. Then F_1 is orthogonal to W_{α_0} (see Figure 3).

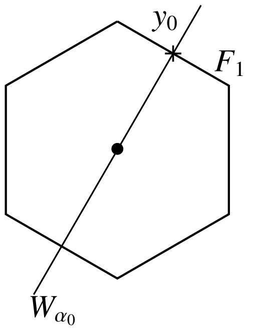

Figure 3. *Case-3.1*: $y_0 \in F_1 \cap W_{\alpha_0}$ and $F_1 \perp W_{\alpha_0}$.

It follows that $l_A(y_0) \neq 0$ for any $A \neq 1$. Thus

$$\langle\alpha_0, y\rangle = o(l_A(y)), y \to y_0, A \neq 1. \qquad (5.8)$$

Let $\{F_1, ..., F_{d_1}\}$ be all facets of P such that $\alpha_0(u_A) \geq 0, A = 1, ..., d_1$. Let s_{α_0} be the reflection with respect to W_{α_0}. Then by s_{α_0}-invariance of P, for each $A' \notin \{1, ..., d_1\}$ there is some $A \in \{1, ..., d_1\}$ such that

$$l_{A'} = l_A + 2\frac{\alpha_0(u_A)\langle\alpha_0, y\rangle}{|\alpha_0|^2}.$$

It follows that

$$
\begin{aligned}
\alpha_0(\nabla u_{2P}) &= -\frac{1}{2}\sum_{A=1}^{d_0} \alpha_0(u_A)\log l_A \\
&= \frac{1}{2}\sum_{A=2}^{d_1} \alpha_0(u_A)\log\left(1 + 2\frac{\alpha_0(u_A)\langle\alpha_0, y\rangle}{|\alpha_0|^2 l_A(y)}\right).
\end{aligned}
$$

Thus, by (5.8) and the fact that $\alpha_0(u_1) = 0$, we obtain

$$\begin{aligned}
&\log\sinh(-\frac{1}{2}\sum_{A=1}^{d_0}\alpha_0(u_A)\log l_A)\\
&= \log\sinh\sum_{A=2}^{d_1}\alpha_0(u_A)\log\left(1+2\frac{\alpha_0(u_A)\langle\alpha_0, y\rangle}{|\alpha_0|^2 l_A(y)}\right)\\
&= \log\langle\alpha_0, y\rangle + O(1).
\end{aligned}$$

Hence

$$I_{\alpha_0}(y) = O(1), \text{ as } y \to y_0.$$

Together with (5.6), we see that h_0 is bounded near y_0.

Case-3.2. $y_0 \in W_{\alpha_0} \cap F_1 \cap F_2$ and is away from other facets of $2P$ (see Figure 4).

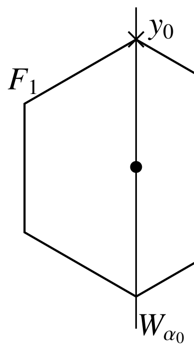

Figure 4. *Case-3.2*: $y_0 \in W_{\alpha_0} \cap F_1 \cap F_2$ and is away from other facets.

By the W-invariance of $2P$, it must hold $F_1 = s_{\alpha_0}(F_2)$. We may assume that $F_2 \subseteq \overline{\mathfrak{a}_+}$ and then

$$l_1 = l_2 + \frac{2\alpha_0(u_2)\langle\alpha_0, y\rangle}{|\alpha_0|^2}.$$

As $y \to y_0$ we have

$$\begin{aligned}
&\alpha_0(y), l_1(y), l_2(y) \to 0,\\
&l_A(y) \not\to 0, \ \forall A \neq 1, 2.
\end{aligned}$$

It follows that

$$\sum_{A=1}^{d_0}|\alpha_0(u_A)|\log l_A = \alpha_0(u_2)(\log l_1 + \log l_2) + O(1). \tag{5.9}$$

Then the second term in $I_{\alpha_0}(y)$ becomes

$$\log\sinh(-\frac{1}{2}\sum_{A=1}^{d_0}\alpha_0(u_A)\log l_A) = \log\sinh\frac{1}{2}\left[\alpha_0(u_2)\log\left(1+2\frac{\alpha_0(u_2)\langle\alpha_0, y\rangle}{|\alpha_0|^2 l_2(y)}\right)\right.$$

$$+\sum_{A\neq 2,\alpha_0(u_A)>0}^{d_1}\alpha_0(u_A)\log\left(1+2\frac{\alpha_0(u_A)\langle\alpha_0,y\rangle}{|\alpha_0|^2 l_A(y)}\right)\Bigg].$$

We will settle it down according to the different rate of $\frac{\alpha_0(y)}{l_2(y)}$ below.

Case-3.2.1. $\alpha_0(y)=o(l_2(y))$. Then

$$\log\sinh\alpha_0(\nabla u_{2P})=\log\alpha_0(y)-\log l_2(y). \tag{5.10}$$

Note that $s_{\alpha_0}(u_1)=u_2\in\overline{\mathfrak{a}_+}$, we have

$$\sum_{A=1,2}|\alpha_0(u_A)|\log l_A=\alpha_0(u_2)(\log l_1+\log l_2).$$

Using the above relation, (5.9) and (5.10), we get

$$\begin{aligned}I_{\alpha_0}(y)=&\alpha_0(u_2)\log l_1+(\alpha_0(u_2)-2)\log l_2+O(1)\\=&2(\alpha_0(u_2)-1)\log l_2+O(1).\end{aligned} \tag{5.11}$$

Here we used $l_1=l_2(1+o(1))$ in the last equality.

Note that by our assumption $\alpha_0(u_2)>0$. Then

$$\alpha_0(u_2)\geq 1,$$

since $\alpha_0(u_2)\in\mathbb{Z}$. Hence, as $l_1(y),l_2(y)\to 0^+$, by (5.5), (5.6) and (5.11), we see that h_0 is bounded from above in this case.

Case-3.2.2. $c\leq\frac{\alpha_0(y)}{l_2(y)}\leq C$ for some constants $C,c>0$. Then

$$\log\alpha_0(y)=\log l_2+O(1),\ \ \log\sinh\alpha_0(\nabla u_{2P})=O(1)$$

and the right hand side of (5.5) becomes

$$\begin{aligned}&\alpha_0(u_2)(\log l_1+\log l_2)-2\log\alpha_0(y)+O(1)\\=&2(\alpha_0(u_2)-1)\log l_2+O(1).\end{aligned} \tag{5.12}$$

Again h_0 is also bounded from above.

Case-3.2.3. $\frac{\alpha_0(y)}{l_2(y)}\to+\infty$. Then

$$\begin{aligned}\log\sinh\alpha_0(\nabla u_{2P})=&\frac{1}{2}\alpha_0(u_2)(\log\alpha_0(y)-\log l_2(y)),\\l_1(y)=&\alpha_0(y)(1+o(1))\end{aligned}$$

and the right hand side of (5.5) becomes

$$\begin{aligned}&\alpha_0(u_2)(\log l_1+\log l_2)+\alpha_0(u_2)(\log\alpha_0(y)-\log l_2(y))-2\log\alpha_0(y)+O(1)\\=&\alpha_0(u_2)\log l_1+[\alpha_0(u_2)-2]\log\alpha_0(y)+O(1)\end{aligned}$$

$$=2(\alpha_0(u_2)-1)\log\alpha_0(y)+O(1). \tag{5.13}$$

Hence h_0 is bounded from above as in *Case-3.2.1*.

Next we consider the case that there are facets $F_1..., F_s$ ($s \geq 3$) such that

$$y_0 \in W_{\alpha_0} \cap F_1 \cap ... \cap F_s$$

and it is away from any other facet of $2P$. We only need to control the term (5.7) as above. If $F_1, ..., F_s$ are all orthogonal to W_{α_0} as in *Case-3.1*, we see that $h_0(y)$ is uniformly bounded. Otherwise, for any y nearby y_0 there is a facet $F = F_{i'}$ for some $i' \in \{1, ..., s\}$ such that

$$l_{i'}(y) = \min\{l_i(y)|\ i = 1, ..., s \text{ such that } \alpha_0(u_i) \neq 0\}.$$

As $y \to y_0$, up to passing to a subsequence, we can fix this i'. Clearly, $y_0 \in W_{\alpha_0} \cap F_1 \cap F_2$ as in *Case-3.2*, where $F_2 = F \subseteq \overline{\mathfrak{a}_+}$ and $F_1 = s_{\alpha_0}(F)$ for the reflection s_{α_0}. Hence by following the argument in *Case-3.2*, we can also prove that $h_0(y)$ is uniformly bounded from above. Therefore, the proposition is true in *Case-3*. The proof of our proposition is completed. □

Remark 5.2. *We note that h_0 is always uniformly bounded in Case-1, Case-2 and Case-3.1. Furthermore, if rank$(G) = 2$, there are at most two facets F_1, F_2 intersecting at a same point y_0 of W_{α_0} as in Cases-3.2.1–3.2.3, thus, by the asymptotic expressions of h_0 in* (5.11), (5.12) *and* (5.13), *respectively, we see that h_0 is uniformly bounded if and only if the following relation holds,*

$$\alpha_0(u_2) = 1. \tag{5.14}$$

In other words, in Cases-3.2.1–3.2.3,

$$\lim_{y\to y_0} h_0 = -\infty,$$

if (5.14) *does not hold.*

Remark 5.3. (5.14) *always holds when M is a smooth G-compactification. This is because the Guillemin metric can be extended to a global one on M and so h_0 is uniformly bounded (cf. [5, Proposition 3.2]). We also note that in this case* (5.14) *can come from Lemma 6.1 in [39] directly.*

6. Reduced Ding functional and existence criterion

By Theorem 4.2, we see that for any $u \in \mathcal{E}^1_{K\times K}(2P)$, its Legendre function

$$\psi_u(x) = \sup_{y\in 2P}\{\langle x, y\rangle - u(y)\} \leq v_{2P}(x)$$

corresponds to a $K \times K$-invariant weak Kähler potential $\phi_u = \psi_u - \psi_0$ which belongs to $\mathcal{E}^1_{K\times K}(M, -K_M)$. Here we can choose ψ_0 to be the Legendre function ψ_{2P} of Guillemin function u_{2P} as in (3.12). As we know, $e^{-\phi_u} \in L^p(\omega_0)$ for any $p \geq 0$. Thus $\int_{\mathfrak{a}_+} e^{-\psi_u}\mathbf{J}(x)dx$ is well-defined.

We introduce the following functional on $\mathcal{E}^1_{K\times K}(2P)$ by

$$\mathcal{D}(u) = \mathcal{L}(u) + \mathcal{F}(u),$$

where

$$\mathcal{L}(u) \;=\; \frac{1}{V}\int_{2P_+} u\pi\, dy - u(4\rho)$$

and

$$\mathcal{F}(u) \;=\; -\log\left(\int_{\mathfrak{a}_+} e^{-\psi_u}\mathbf{J}(x)dx\right) + u(4\rho).$$

It is easy to see that on a smooth Fano compactification of G,

$$\mathcal{L}(u_\phi) + u_\phi(4\rho) \;=\; -\frac{1}{(n+1)V}\sum_{k=0}^{n}\int_M \phi\omega_\phi^k \wedge \omega_0^{n-k}$$

and $\mathcal{D}(u_\phi)$ is just the Ding functional $F(\phi)$. We note that a similar functional on such Fano manifolds has been studied for Mabuchi solitons in [37, Section 4]). Hence, for convenience, we call $\mathcal{D}(\cdot)$ the reduced Ding functional on a $\mathbb{Q}$-Fano compactifications of G.

In this section, we will use the variation method to prove Theorem 1.2 by verifying the properness of $\mathcal{D}(\cdot)$. We assume that the associated polytope P is fine so that the Ricci potential h_0 is uniformly bounded above by Proposition 5.1.

6.1. *A criterion for the properness of $\mathcal{D}(\cdot)$*

In this subsection, we establish a properness criterion for $\mathcal{D}(u_\phi)$, namely,

Proposition 6.1. *Let M be a $\mathbb{Q}$-Fano compactification of G. Suppose that the associated polytope P is fine and satisfies* (1.2). *Then there are constants δ and C_δ such that*

$$\mathcal{D}(u) \;\geq\; \delta\int_{2P_+} u\pi(y)\, dy + C_\delta,\;\; u \in \mathcal{E}^1_{K\times K}(2P). \tag{6.1}$$

The proof follows the line of argument in [37]. We note that u_ϕ satisfies the normalized condition $u \geq u(O) = 0$. Then we have the following estimate for the linear term $\mathcal{L}(\cdot)$ as in [36, Proposition 4.2].

Lemma 6.2. *Under the assumption (1.2), there exists a constant $\lambda > 0$ such that*

$$\mathcal{L}(u) \;\geq\; \lambda\int_{2P_+} u\pi(y)\, dy,\;\; \forall\; u \in \mathcal{E}^1_{K\times K}(2P). \tag{6.2}$$

For the non-linear term $\mathcal{F}(\cdot)$, we can also get an analogy of [37, Lemma 4.8] as follows.

Lemma 6.3. *For any $\phi \in \mathcal{E}^1_{K\times K}(M, -K_M)$, let*

$$\tilde{\psi}_\phi := \psi_\phi - 4\rho_i x^i,\;\; x \in \mathfrak{a}_+.$$

Then

$$\mathcal{F}(u_\phi) \;=\; -\log\left(\int_{\mathfrak{a}_+} e^{-(\tilde{\psi}_\phi - \inf_{\mathfrak{a}_+}\tilde{\psi}_\phi)}\prod_{\alpha\in\Phi_+}\left(\frac{1-e^{-2\alpha_i x^i}}{2}\right)^2 dx\right). \tag{6.3}$$

Consequently, for any $c > 0$,

$$\mathcal{F}(u_\phi) \;\geq\; \mathcal{F}\left(\frac{u_\phi}{1+c}\right) - n\cdot\log(1+c). \tag{6.4}$$

Let $\phi_0, \phi_1 \in \mathcal{E}^1_{K\times K}(M, -K_M)$ and u_0, u_1 be two Legendre functions of $\psi_0+\phi_0$ and $\psi_0+\phi_1$, respectively. Let u_t ($t \in [0,1]$) be a linear path connecting u_0 to u_1 as in (4.14). Then by Theorem 4.2, the corresponding Legendre functions ψ_t of u_t give a path in $\mathcal{E}^1_{K\times K}(M, -K_M)$. The following lemma shows that $\mathcal{F}(\psi_t)$ is convex in t.

Lemma 6.4. *Let*

$$\hat{\mathcal{F}}(t) = -\log \int_{\mathfrak{a}_+} e^{-\psi_t}\mathbf{J}(x)dx, \ t \in [0,1].$$

Then $\hat{\mathcal{F}}(t)$ is convex in t and so is $\mathcal{F}(\psi_t)$.

Proof. By definition, we have

$$\begin{aligned}\psi_t(tx_1 + (1-t)x_0) =& \sup_y \{\langle y, tx_1 + (1-t)x_0\rangle - (tu_1(y) + (1-t)u_0(y))\}\\ \leq& t\sup_y\{\langle y, x_1\rangle - u_1(y)\}\\ &+ (1-t)\sup_y\{\langle y, x_0\rangle - u_0(y))\}\\ \leq& t\psi_1(x_1) + (1-t)\psi_0(x_0), \ \forall\ x_0, x_1 \in \mathfrak{a}. \end{aligned} \tag{6.5}$$

On the other hand,

$$\log \mathbf{J}(tx_1 + (1-t)x_0) \geq t\log \mathbf{J}(x_1) + (1-t)\log \mathbf{J}(x_0), \ \forall x_0, x_1 \in \mathfrak{a}_+.$$

Combining these two inequalities, we get

$$(e^{-\psi_t}\mathbf{J})(tx_1 + (1-t)x_0) \geq (e^{-\psi_1}\mathbf{J})^t(x_1)(e^{-\psi_0}\mathbf{J})^{1-t}(x_0), \ \forall x_0, x_1 \in \mathfrak{a}_+.$$

Hence, by applying the Prekopa-Leindler inequality to three functions $e^{-\psi_t}\mathbf{J}, e^{-\psi_1}\mathbf{J}$ and $e^{-\psi_0}\mathbf{J}$ (cf. [27, Theorem 7.1]), we prove

$$-\log \int_{\mathfrak{a}_+} e^{-\psi_t}\mathbf{J}(x)dx \leq -t\log \int_{\mathfrak{a}_+} e^{-\psi_1}\mathbf{J}(x)dx - (1-t)\log \int_{\mathfrak{a}_+} e^{-\psi_0}\mathbf{J}(x)dx.$$

This means that $\hat{\mathcal{F}}(t)$ is convex. □

Proof of Proposition 6.1. By Proposition 5.1,

$$A(y) = \frac{V}{\int_{\mathfrak{a}_+} e^{-\psi_0}\mathbf{J}(x)dx} e^{h_0(\nabla u_0(y))}$$

is bounded, where $y(x) = \nabla\psi_0(x)$. Then the functional

$$\mathcal{D}_A(u) = \mathcal{L}^0_A(u) + \mathcal{F}(u),$$

is well-defined on $\mathcal{E}^1_{K\times K}(2P)$, where

$$\mathcal{L}^0_A(u) = \frac{1}{V}\int_{2P_+} uA(y)\pi(y)\,dy - u(4\rho).$$

It is easy to see that u_0 is a critical point of $\mathcal{D}_A(\cdot)$. On the other hand, by Lemma 6.4, $\mathcal{F}(\cdot)$ is convex along any path in $\mathcal{E}^1_{K\times K}(M, -K_M)$ determined by their Legendre functions as in (4.14). Note that $\mathcal{L}^0_A(\cdot)$ is convex in $\mathcal{E}^1_{K\times K}(2P)$. Hence

$$\mathcal{D}_A(u) \ \geq \ \mathcal{D}_A(u_0), \ \forall u \in \mathcal{E}^1_{K\times K}(2P).$$

Now together with Lemma 6.2 and Lemma 6.3, we can apply arguments in the proof of [37, Proposition 4.9] to proving that there is a constant $C > 0$ such that for any $u \in \mathcal{E}^1_{K\times K}(2P)$,

$$\mathcal{D}(u) \ \geq \ \frac{C\lambda}{1+C}\int_{2P_+} u\pi(y)\,dy + \mathcal{D}_A(u_0) - n\log(1+C).$$

Therefore, we get (6.1).

□

6.2. Semi-continuity

Write $\mathcal{E}^1_{K\times K}(2P)$ as

$$\mathcal{E}^1_{K\times K}(2P) \ = \ \bigcup_{\kappa\geq 0}\mathcal{E}^1_{K\times K}(2P;\kappa),$$

where

$$\mathcal{E}^1_{K\times K}(2P;\kappa) = \{u \in \mathcal{E}^1_{K\times K}(2P) | \int_{2P_+} u\pi\,dy \leq \kappa\}.$$

By [36, Lemma 6.1] and Fatou's lemma, it is easy to see that any sequence $\{u_n\} \subseteq \mathcal{E}^1_{K\times K}(2P;\kappa)$ has a subsequence which converges locally uniformly to some $u_\infty \in \mathcal{E}^1_{K\times K}(2P;\kappa)$. Thus each $\mathcal{E}^1_{K\times K}(2P;\kappa)$, and so $\mathcal{E}^1_{K\times K}(2P)$ is complete. Moreover, we have

Proposition 6.5. *The reduced Ding functional $\mathcal{D}(\cdot)$ is lower semi-continuous on the space $\mathcal{E}^1_{K\times K}(2P)$. Namely, for any sequence $\{u_n\} \subseteq \mathcal{E}^1_{K\times K}(2P)$, which converges locally uniformly to some u_∞, we have $u_\infty \in \mathcal{E}^1_{K\times K}(2P)$ and it holds*

$$\mathcal{D}(u_\infty) \ \leq \ \liminf_{n\to\infty}\mathcal{D}(u_n). \tag{6.6}$$

Proof. By Fatou's lemma, we have

$$\int_{2P_+} u_\infty\pi\,dy \ \leq \ \liminf_{n\to+\infty}\int_{2P_+} u_n\pi\,dy < +\infty. \tag{6.7}$$

Then $u_\infty \in \mathcal{E}^1_{K\times K}(2P)$ and

$$\mathcal{L}(u_\infty) \ \leq \ \liminf_{n\to+\infty}\mathcal{L}(u_n).$$

It remains to estimate $\mathcal{F}(u_\infty)$. Note that u_∞ is finite everywhere in $\mathrm{Int}(2P)$ by the locally uniform convergence and its Legendre function $\psi_\infty \leq v_{2P}$. Thus, for any $\epsilon_0 \in (0,1)$ there is a constant $M_{\epsilon_0} > 0$ such that (cf. [17, Lemma 2.3]),

$$\psi_\infty(x) \ \geq \ (1-\epsilon_0)v_{2P}(x) - M_{\epsilon_0}, \forall x \in \mathfrak{a}. \tag{6.8}$$

On the other hand, the Legendre function ψ_n of u_n also converges locally uniformly to ψ_∞. Then

$$\partial\psi_n \to \partial\psi_\infty$$

almost everywhere. Since

$$\psi_n(O) = \psi_\infty(O) = 0, \forall n \in \mathbb{N}_+,$$

we have

$$\psi_n(x) \geq (1-\epsilon_0)v_{2P}(x) - M_{\epsilon_0}, \forall x \in \mathfrak{a} \tag{6.9}$$

as long as $n \gg 1$. Note that

$$0 \leq \mathbf{J}(x) \leq e^{4\rho(x)}, \forall x \in \mathfrak{a}_+.$$

By choosing an ϵ_0 such that $4\rho \in (1-\epsilon_0)\mathrm{Int}(2P)$, we get

$$\int_{\mathfrak{a}_+} e^{M_{\epsilon_0}-(1-\epsilon_0)v_{2P}(x)}\mathbf{J}(x)dx < +\infty.$$

Hence, combining this with (6.8) and (6.9) and using Fatou's lemma, we derive

$$-\log\left(\int_{\mathfrak{a}_+} e^{-\psi_\infty}\mathbf{J}(x)dx\right) \leq \liminf_{n\to+\infty}\left[-\log\left(\int_{\mathfrak{a}_+} e^{-\psi_n}\mathbf{J}(x)dx\right)\right].$$

Therefore, we have proved (6.6) by (6.7). □

6.3. *Proof of Theorem 1.2*

Now we prove the sufficient part of Theorem 1.2. Suppose that (1.2) holds. Then by Propositions 6.1 and 6.5, there is a minimizing sequence $\{u_n\}$ of $\mathcal{D}(\cdot)$ on $\mathcal{E}^1_{K\times K}(2P)$, which converges locally uniformly to some $u_\star \in \mathcal{E}^1_{K\times K}(2P)$ such that

$$\mathcal{D}(u_\star) \leq \lim_{u\in\mathcal{E}^1_{K\times K}(2P)} \mathcal{D}(u). \tag{6.10}$$

Let $\psi_\star$ be the Legendre function of $u_\star$. Then by Theorem 4.2, we have

$$\phi_\star = \psi_\star - \psi_0 \in \mathcal{E}^1_{K\times K}(M, -K_M).$$

We need to show that $\phi_\star$ satisfies the Kähler-Einstein equation (2.1).

Proposition 6.6. *$\phi_\star$ satisfies the Kähler-Einstein equation* (2.1).

Proof. Let $\{u_t\}_{t\in[0,1]} \subseteq \mathcal{E}^1_{K\times K}(2P)$ be a family of convex functions with $u_0 = u_\star$ and ψ_t the corresponding Legendre functions of u_t. Then by Part (2) in Lemma 4.10,

$$\dot{\psi}_0 = -\dot{u}_0, \text{ almost everywhere.}$$

Note that

$$\int_{\mathfrak{a}_+} e^{-\psi_\star}\mathbf{J}(x)dx = V,$$

Thus by (4.11) in Lemma 4.8, we get

$$\frac{d}{dt}\Big|_{t=0}\mathcal{D}(u_t) = \frac{1}{V}\int_{2P_+}\dot{u}_0\pi\, dy + \frac{\int_{\mathfrak{a}_+}\dot{\psi}_0 e^{-\psi_\star}\mathbf{J}(x)dx}{V}$$
$$= \frac{1}{V}\int_{\mathfrak{a}_+}\dot{\psi}_0[e^{-\psi_\star}\mathbf{J}(x) - \mathrm{MA}_{\mathbb{R};\pi}(\psi_\star)]dx. \tag{6.11}$$

For any continuous, compactly supported W-invariant function $\eta \in C_0(\mathfrak{a})$, we consider a family of functions $u_\star + t\eta$. In general, it may not be convex for $t \neq 0$ since $u_\star$ is just weakly convex. In the following, we use a trick to modify the function $\mathcal{D}(u_t)$ as in [9, Section 2.6]. Define a family of W-invariant functions by

$$\hat{\psi}_t = \sup_{\phi\in\mathcal{E}^1_{K\times K}(M,-K_M)}\{\psi_\phi|\psi_\phi \le \psi_\star + t\eta\}.$$

Then it is easy to see that the Legendre function $\hat{u}_t$ of $\hat{\psi}_t$ satisfies

$$|\hat{u}_t - u_0| \le C, \ \forall |t| \ll 1.$$

By Theorem 4.2, we see that $(\hat{\psi}_t - \psi_0) \in \mathcal{E}^1_{K\times K}(M, -K_M)$. Without loss of generality, we may assume that $\hat{\psi}_t$ satisfies (4.1).

Let

$$\tilde{\mathcal{D}}(t) = \mathcal{L}(\hat{u}_t) + \mathcal{F}(\hat{u}_t).$$

Then

$$\tilde{\mathcal{D}}(0) = \mathcal{D}(u_\star) \tag{6.12}$$

and

$$\tilde{\mathcal{D}}(t) \ge \mathcal{D}(u_\star). \tag{6.13}$$

Claim 6.7. *$\mathcal{L}(\hat{u}_t) + \hat{u}_t(4\rho)$ is differentiable for t. Moreover,*

$$\frac{d}{dt}\Big|_{t=0}(\mathcal{L}(\hat{u}_t) + \hat{u}_t(4\rho)) = -\frac{1}{V}\int_M \eta\omega^n_{\phi_\star}. \tag{6.14}$$

To prove this claim, we let a convex function $g(t) = \hat{u}_t(p)$ for each fixed $p \in 2P$. Then it has left and right derivatives $g'_-(t;p), g'_+(t;p)$, respectively. Moreover, they are monotone and $g'_-(t;p) \le g'_+(t;p)$. Thus, $g'_-, g'_+ \in L^\infty_{\mathrm{loc}}$. It follows that

$$\frac{d}{dt}\Big|_{t=\tau^\pm}\int_{2P_+}\hat{u}_\tau\pi dy = \lim_{\tau'\to 0^\pm}\frac{1}{\tau'}\int_{2P_+}(\hat{u}_{\tau+\tau'} - \hat{u}_\tau)\pi dy$$

and by the Lebesgue monotone convergence theorem,

$$\frac{d}{dt}\Big|_{t=\tau^\pm}\int_{2P_+}\hat{u}_\tau\pi dy = \int_{2P_+} g'_\pm(\tau;p)\pi dy.$$

Recall that $g'_-(t;p) = g'_+(t;p)$ holds almost everywhere. Thus we see that

$$\mathcal{L}(\hat{u}_t) + u(4\rho) = \frac{1}{V}\int_{2P_+} \hat{u}_t \pi dy$$

is differentiable.

Note that

$$u_{\hat{\psi}_t} = u_{\psi_\star + t\eta},$$

where $u_{\psi_\star + t\eta}$ is the Legendre function of $\psi_\star + t\eta$. It follows from Part (2) in Lemma 4.10 that

$$\dot{\hat{\psi}}_0 = -\dot{u}_0 = \eta, \text{ almost everywhere.}$$

Hence by Lemma 4.5 (or Remark 4.11), we get

$$\begin{aligned}
\left.\frac{d}{dt}\right|_{t=0} (\mathcal{L}(\hat{u}_t) + \hat{u}_t(4\rho)) &= \frac{1}{V}\int_{2P_+} \dot{\hat{u}}_0 \pi dy \\
&= -\frac{1}{V}\int_{2P} \eta\pi dy = -\frac{1}{V}\int_{\mathfrak{a}_+} \eta \mathrm{MA}_{\mathbb{R};\pi}(\psi_0) dx \\
&= -\frac{1}{V}\int_M \eta\omega_{\phi_\star}^n,
\end{aligned}$$

where $\phi_\star = \psi_* - \psi_0$. The claim is proved.

Similar with Claim 6.7, we have

$$\begin{aligned}
\left.\frac{d}{dt}\right|_{t=0} (\mathcal{F}(\hat{u}_t) - \hat{u}_t(4\rho)) &= \frac{1}{V}\int_{\mathfrak{a}_+} \eta e^{-\psi_\star} \mathbf{J}(x) dx \\
&= \int_G \eta e^{-\phi_\star + h_0}\omega_0^n. \qquad (6.15)
\end{aligned}$$

Thus, by (6.12)–(6.15), we derive

$$0 = \left.\frac{d}{dt}\right|_{t=0} \tilde{\mathcal{D}}(t) = \frac{1}{V}\int_G \eta[e^{-\phi_\star + h_0}\omega_0^n - \omega_{\phi_\star}^n]dx. \qquad (6.16)$$

As a consequence,

$$\omega_{\phi_*}^n = e^{-\psi_\star + h_0}\omega_0^n, \text{ in } G.$$

Therefore, by Lemma 4.5 and KAK-integration formula, we prove that $\phi_\star$ satisfies (2.1) on G.

Next we show that $\omega_{\phi_\star}$ can be extended to a singular Kähler-Einstein metric on M. Choose an ϵ_0 such that $4\rho \in \mathrm{Int}(2(1-\epsilon_0)P)$. Since $u_\star$ is locally uniformly bounded on $2P$, there is a constant $C_\star > 0$ such that

$$\psi_\star \geq (1-\epsilon_0)v_{2P} - C_\star.$$

Thus

$$e^{-\psi_\star(x)}\mathbf{J}(x)$$

is bounded on $\mathfrak{a}_+$. Also $\pi(\partial\psi_\star)$ is bounded. Therefore, by (4.11), for any $\epsilon > 0$, we can find a neighborhood U_ϵ of $M \setminus G$ such that

$$\left|\int_{U_\epsilon} (\omega_{\phi_\star}^n - e^{h_0 - \psi_\star}\omega_0^n)\right| < \epsilon.$$

This implies that $\phi_\star$ can be extended to be a global solution of (2.1) on M. The proposition is proved. □

7. $\mathbb{Q}$-Fano compactifications of $SO_4(\mathbb{C})$

In this section, we will construct $\mathbb{Q}$-Fano compactifications of $SO_4(\mathbb{C})$ as examples and in particular, we will prove Theorem 1.3. Note that in this case $\mathrm{rank}(G) = 2$. Thus we can use Theorem 1.2 to verify whether there exists a Kähler-Einstein metric on a $\mathbb{Q}$-Fano $SO_4(\mathbb{C})$-compactification by computing the barycenter of their associated polytopes P_+. For convenience, we will work with P_+ instead of $2P_+$ throughout this section. It is easy to see that the condition (1.2) is equivalent to

$$\mathrm{bar}(P_+) \in 2\rho + \Xi. \tag{7.1}$$

Let

$$R(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}.$$

Then we can choose a maximal torus of $\mathrm{SO}_4(\mathbb{C})$ in $GL_4(\mathbb{C})$ as follows,

$$T^{\mathbb{C}} = \left\{ \begin{pmatrix} R(z^1) & O \\ O & R(z^2) \end{pmatrix} | z^1, z^2 \in \mathbb{C} \right\}.$$

Recall $\mathfrak{M}$ the lattice of $\mathrm{SO}_4(\mathbb{C})$-weights. Denote the basis of $\mathfrak{N} = \mathrm{Hom}_{\mathbb{Z}}(\mathfrak{M}, \mathbb{Z})$ by E_1, E_2 which generates the actions of $R(z^1)$ and $R(z^2)$. Thus we have the two positive roots in $\mathfrak{M}$,

$$\alpha_1 = (1, -1), \alpha_2 = (1, 1).$$

Also we get

$$\mathfrak{a}_+^* = \{(x, y) | -x < y < x\},\ 2\rho = (2, 0)$$

and

$$2\rho + \Xi = \{(x, y) | -x + 2 < y < x - 2\}. \tag{7.2}$$

7.1. Gorenstein Fano $\mathrm{SO}_4(\mathbb{C})$*-compactifications*

In this subsection, we use Lemma 3.1 to exhaust all polytopes associated to the Gorenstein Fano compactifications. Here by Gorenstein, we mean that $K_{M_{reg}}^{-1}$ can be extended to a holomorphic line bundle on M. In this case, the whole polytope P is a lattice polytope. Also, since $2\rho = (2, 0)$, each outer edge [¶] of P_+ must lies on some line

$$l_{p,q}(x, y) = (1 + 2p) - (px + qy) = 0 \tag{7.3}$$

for some coprime pair (p, q). Assume that $l_{p,q} \geq 0$ on P. By the convexity and W-invariance of P, (p, q) must satisfy

$$p \geq |q| \geq 0.$$

Let us start at the outer edge F_1 of P_+ which intersects the Weyl wall

$$W_1 = \{x - y = 0\}.$$

[¶] An edge of P_+ is called an *outer* one if it does not lie in any Weyl wall, cf. [36].

There are two cases: *Case-1.* F_1 is orthogonal to W_1; *Case-2.* F_1 is not orthogonal to W_1.

Case-1. F_1 is orthogonal to W_1. Then F_1 lies on

$$\{(x,y)|\ l_{1,1}(x,y) = 3 - x - y = 0\}.$$

Consider the vertex $A_1 = (x_1, 3 - x_1)$ of P_+ on this edge and suppose that the other edge F_2 at this point lies on

$$\{(x,y)|\ l_{p_2,q_2}(x,y) = 0\}.$$

Thus

$$2p_2 + 1 = x_1 p_2 + (3 - x_1) q_2, \tag{7.4}$$

and by the convexity of P,

$$p_2 > q_2 \geq 0.$$

We will have two subcases according to the possible choice $A_1 = (2,1)$ or $(3,0)$.

Case-1.1. $A_1 = (2,1)$. Then by (7.4),

$$2p_2 + 1 = 2p_2 + q_2.$$

Thus $q_2 = 1$ and $p_2 \geq 2$.

On the other hand, l_{p_2,q_2} must pass another lattice point $A_2 = (x_2, y_2)$ as the other endpoint of F_2. It is direct to see that there are only two possible choices $p_2 = 2, 4$ and three choices of $A_2 = (5,-5)$, $(3,-1)$ and $(3,-3)$.

Case-1.1.1. $A_2 = (5,-5)$ which lies on the other Weyl wall $W_2 = \{x + y = 0\}$. There can not be any other outer edges of P_+, and P_+ is given by the first case in Figure 5 (we denote it by $P_+^{(1)}$). By Theorem 1.2 (or equivalently (7.1)), this compactification admits no Kähler-Einstein metric.

Case-1.1.2. $A_2 = (3,-1)$. Then we exhaust the third edge F_3 which lies on

$$l_{p_3,q_3} = 2p_3 + 1 - p_3 x - q_3 y,$$

so that

$$2p_3 + 1 = 3p_3 - q_3,$$
$$p_3 > 2q_3 \geq 0.$$

Hence the only possible choice is $p_3 = 1, q_3 = 0$ and the other endpoint of F_3 is $A_3 = (3,-3)$. Then P_+ is given by the second case in Figure 5. Again, this compactification admits no Kähler-Einstein metric.

Case-1.1.3. $A_2 = (3,-3)$ which lies on the other Weyl wall $W_2 = \{x + y = 0\}$. There can not be any other outer edges of P_+, and P_+ is given by the third one in Figure 5. By Theorem 1.2, this compactification admits no Kähler-Einstein metric.

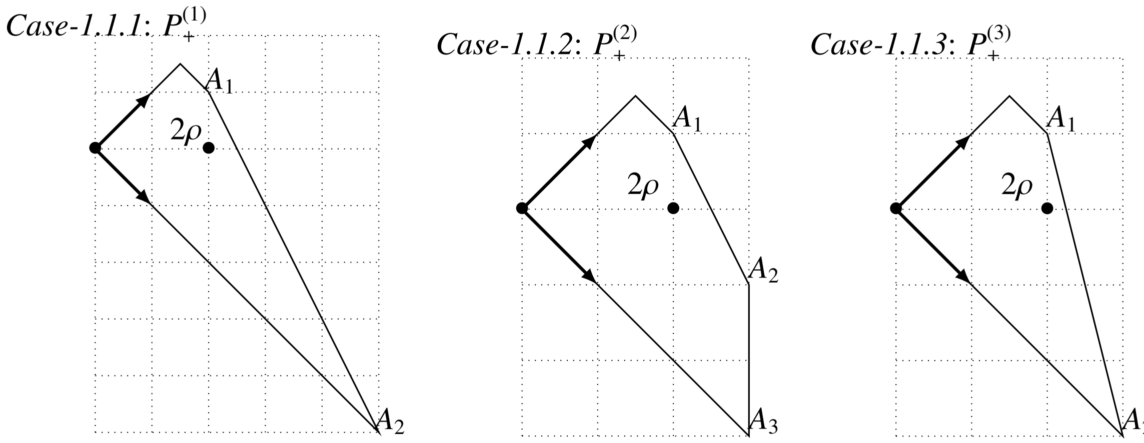

Figure 5. The three subcases of *Case-1.1*.

Case-1.2. $A_1 = (3, 0)$. By the same exhausting progress as in *Case-1.1.* There are two possible polytopes P_+, *Case-1.2.1* and *Case-1.2.2* (see the first two cases of Figure 6).

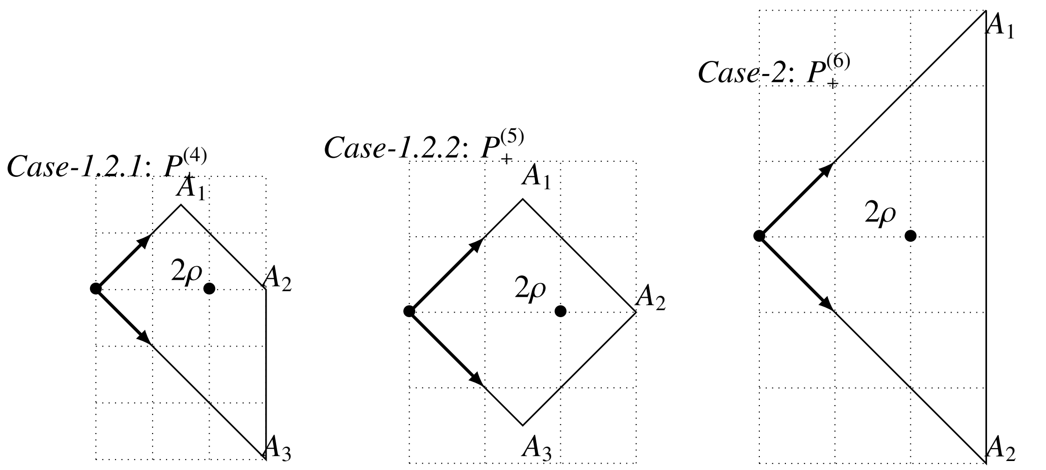

Figure 6. Subcases of *Case-1.2* and *Case-2*.

Case-1.2.1. This compactification admits no Kähler-Einstein metric.

Case-1.2.2. This compactification admits a Kähler-Einstein metric.

Case-2. F_1 is not orthogonal to W_1. Then its intersection $A_1 = (x_1, x_1)$ with W_1 is a vertex of P. We see that F_1 lies on l_{p_1,q_1} and

$$\begin{aligned} 2p_1 + 1 &= (p_1 + q_1)x_1, \\ p_1 &> q_1 \geq 0, \\ x_1 &= 2 + \frac{1 - 2q_1}{p_1 + q_1} \in \mathbb{N}_+. \end{aligned}$$

So the only choice is

$$p_1 = 1, q_1 = 0$$

and $A_1 = (3,3)$. The only new polytope P_+ is given by the last one of Figure 6, which admits a Kähler-Einstein metric.

It is known that *Case-1.1.2*, *Case-1.2.1* and *Case-2* are the only smooth $SO_4(\mathbb{C})$-compactifications as shown in [41]. We summarize results of this subsection in Table 1.

Table 1. Gorenstein Fano $SO_4(\mathbb{C})$-compactifications.

Cases.	Edges, except Weyl walls	Volume	KE?	Smoothness
Case-1.1.1	3-x-y=0; 5-2x-y=0	$\frac{411}{4}$	No	Singular
Case-1.1.2	3-x-y=0; 5-2x-y=0; 3-x=0	$\frac{10751}{180}$	No	Smooth
Case-1.1.3	3-x-y=0; 9-4x-y=0	$\frac{16349}{972}$	No	Singular
Case-1.2.1	3-x-y=0; 3-x=0	$\frac{1701}{20}$	No	Smooth
Case-1.2.2	3-x-y=0; 3-x+y=0	$\frac{81}{2}$	Yes	Singular
Case-2	3-x=0	$\frac{648}{5}$	Yes	Smooth

7.2. $\mathbb{Q}$-*Fano* $SO_4(\mathbb{C})$-*compactifications*

In general, for a fixed integer $m > 0$, it will be hard to give a classification of all $\mathbb{Q}$-Fano compactifications such that $-mK_X$ is Cartier. This is because when m is sufficiently divisible, there will be too many repeated polytopes according to Lemma 3.1. In the following, we give a way to exhaust all $\mathbb{Q}$-Fano polytopes according to the intersection point of ∂P_+ with x-axis.

We will adopt the notations from the previous subsection. We consider the intersection of P_+ with the positive part of the x-axis, namely $(x_0, 0)$. Then

$$x_0 = 2 + \frac{1}{p_0}$$

for some $p_0 \in \mathbb{N}_+$, and there is an edge which lies on some $\{l_{p_0,q_0} = 0\}$. Without loss of generality, we may also assume that $\{l_{p_0,q_0} = 0\} \cap \{y > 0\} \neq \emptyset$. Thus by symmetry, it suffices to consider the case

$$p_0 \geq q_0 \geq 0.$$

Indeed, by the prime condition, $q_0 \neq 0, \pm p_0$ if $p_0 \neq 1$. Hence, we may assume

$$p_0 > q_0 > 0, p_0 \geq 2. \tag{7.5}$$

We associate this number p_0 to determine each $\mathbb{Q}$-Fano polytope P (and hence $\mathbb{Q}$-Fano compactifications of $SO_4(\mathbb{C})$). By the convexity, other edges determined by $l_{p,q}$ must satisfy (see Figure 7 below)

$$p \leq p_0,$$

since we assume that

$$P_+ \subseteq (\{l_{p_0,q_0} \geq 0\} \cap \mathfrak{a}_+). \tag{7.6}$$

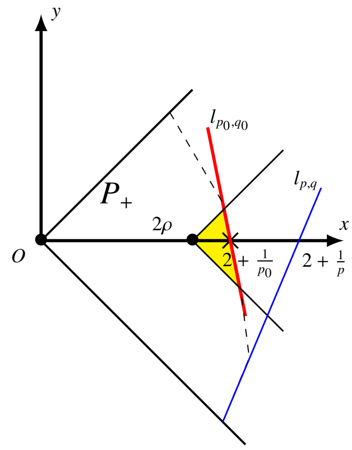

Figure 7. The relation $p \leq p_0$.

Thus, once p_0 is fixed, there are only finitely possible $\mathbb{Q}$-Fano compactifications of $SO_4(\mathbb{C})$ associated to it. In Table 2, we list all possible $\mathbb{Q}$-Fano compactifications with $p_0 \leq 2$, and test the existence of Kähler-Einstein metrics on these compactifications. In Appendix 1 we figure out the associated polytopes P of nine non-smooth examples in Table 2.

Table 2. $\mathbb{Q}$-Fano $SO_4(\mathbb{C})$-compactifications of cases $p_0 \leq 2$.

No.	p_0	(p,q) of edges, except Weyl walls	Volume	KE?	Smoothness/Multiple
(1)	**1**	$(1,0)$	$\frac{648}{5}$	Yes	Smooth
(2)		$(1,0),(1,1)$	$\frac{1701}{20}$	No	Smooth
(3)		$(1,-1),(1,1)$	$\frac{81}{2}$	Yes	Multiple=1
(4)	**2**	$(2,1)$	$\frac{25000}{243}$	No	Multiple=3
(5)		$(2,1),(1,1)$	$\frac{411}{4}$	No	Multiple=1
(6)		$(1,0),(2,1)$	$\frac{72728}{1215}$	No	Multiple=3
(7)		$(2,1),(1,-1)$	$\frac{947}{36}$	No	Multiple=3
(8)		$(2,-1),(2,1)$	$\frac{165625}{7776}$	No	Multiple=6
(9)		$(2,1),(1,0),(1,1)$	$\frac{10751}{180}$	No	Smooth
(10)		$(2,1),(1,-1),(1,1)$	$\frac{12721}{486}$	No	Multiple=1
(11)		$(2,1),(2,-1),(1,1)$	$\frac{164609}{7776}$	No	Multiple=6
(12)		$(2,1),(2,-1),(1,1),(1,-1)$	$\frac{6059}{288}$	No	Multiple=6

7.3. *Proof of Theorem 1.3*

Proof. We introduce some notations for convenience: For any domain $\Omega \subset \overline{\mathfrak{a}_+^*}$, define

$$\mathrm{Vol}(\Omega) := \int_\Omega \pi dx \wedge dy,$$
$$\bar{x}(\Omega) := \frac{1}{V(\Omega)} \int_\Omega x\pi dx \wedge dy,$$
$$\bar{y}(\Omega) := \frac{1}{V(\Omega)} \int_\Omega y\pi dx \wedge dy,$$

and

$$\bar{c}(\Omega) := \bar{x} + \bar{y}.$$

By Theorem 1.2 and (7.2), we have $\bar{c}(P_+) > 2$ whenever the $\mathbb{Q}$-Fano compactification of $SO_4(\mathbb{C})$ admits a Kähler-Einstein metric.

Recall the numbers p_0, q_0 introduced in Section 7.2. Consider the line segment I_t cut by P_+ on $\{y = x - 2t\}$ for $t \geq 0$ (see Figure 8 below). Set

$$\bar{C}(t) := \frac{\int_{I_t} \bar{c}(y)\pi(y)ds}{\int_{I_t} \pi(y)ds} \text{ and } l(t) := \int_{I_t} \pi(y)ds,$$

where ds is the standard Lebesgue measure on I_t. Then $\bar{C}(t), l(t)$ are the mean value of $\bar{c}(\cdot)$ and length of I_t against the weight π, respectively. Also, since P_+ is bounded, there is some $0 \leq t_0 < +\infty$ so that I_t is non-empty only on $[0, t_0]$. Thus we get

$$\bar{c}(P_+) = \frac{\int_0^{t_0} l(t)\bar{C}(t)dt}{\int_0^{t_0} l(t)dt}. \tag{7.7}$$

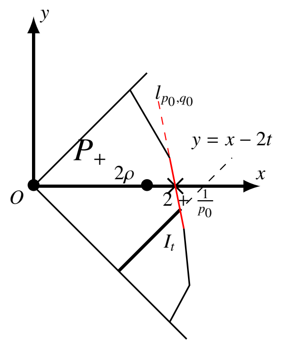

Figure 8. The line segment $I_t = P_+ \cap \{y = x - 2t\}$.

On the other hand, for each $0 \leq t \leq t_0$, the line segment $I_t = \{(x_0, -x_0) + s(1,1) | \, 0 \leq s \leq S_t\}$, where $(x_0, -x_0) \in P_+$ and $(x_0, -x_0) + S_t(1,1) \in \partial P$, satisfies

$$\bar{C}(t) = \frac{3}{2}S_t \leq \frac{3}{2}S_0 \leq \frac{6p_0 + 3}{2p_0 + 2q_0}.$$

Here in the first equality we use the relation (7.5), and the second follows from the fact that the endpoint of I_0 can not exceed the intersection point

$$l_{p_0,q_0} \cap \{t(1,1) | t > 0\} = \left(\frac{2p_0 + 1}{p_0 + q_0}, \frac{2p_0 + 1}{p_0 + q_0}\right).$$

Thus by (7.7), we get

$$\bar{c}(P_+) \leq \frac{6p_0 + 3}{2p_0 + 2q_0}.$$

By the fact $\bar{c}(P_+) > 2$, we derive from the above upper bound of $\bar{c}(P_+)$,

$$q_0 < \frac{1}{2}p_0 + \frac{3}{4}. \tag{7.8}$$

Then

$$\begin{aligned}\mathrm{Vol}(P_+) &\leq \mathrm{Vol}(\{l_{p_0,q_0} \geq 0, x \geq y \geq -x\}) \\ &= \frac{8(1+2p_0)^6}{45(p_0^2 - q_0^2)^3} \\ &\leq \frac{8(1+2p_0)^6}{45(p_0^2 - ((1/2)p_0 + (3/4))^2)^3}.\end{aligned} \tag{7.9}$$

It turns that for $p_0 \geq 9$,

$$\mathrm{Vol}(P_+) \leq \frac{224755712}{4100625}.$$

However,

$$\mathrm{Vol}(P_+^{(4)}) = \frac{1701}{20} > \mathrm{Vol}(P_+^{(2)}) = \frac{10751}{180} > \frac{224755712}{4100625},$$

where $\mathrm{Vol}(P_+^{(2)})$ and $\mathrm{Vol}(P_+^{(4)})$ are volumes of polytopes in *Case-1.1.2* and *Case-1.2.1*, respectively. Hence, there is no desired Kähler-Einstein polytope with its volume equals to $\mathrm{Vol}(P_+^{(2)})$ or $\mathrm{Vol}(P_+^{(4)})$ when $p_0 \geq 9$.

Since $q_0 \in \mathbb{N}$, we can improve (7.9) to

$$\mathrm{Vol}(P_+) \leq \frac{8(1+2p_0)^6}{45(p_0^2 - [(1/2)p_0 + (3/4)]^2)^3}.$$

Here $[x] = \max_{n\in\mathbb{Z}}\{n \leq x\}$. By the above estimation, when $p_0 = 4, 6, 7, 8$, we have

$$\mathrm{Vol}(P_+^{(4)}) \ > \ \mathrm{Vol}(P_+^{(2)}) \ > \ \mathrm{Vol}(P_+). \tag{7.10}$$

As a consequence, they are not Kähler-Einstein polytopes. Hence, it remains to deal with the cases when $p_0 = 3, 5$. In these two cases, we shall rule out polytopes that may not satisfy (7.10).

When $p_0 = 5$, there are three possible choices of q_0, i.e., $q_0 = 1, 2, 3$ by (7.8). It is easy to see that (7.10) still holds for the first two cases by the second relation in (7.9). Thus we only need to consider all possible polytopes when $q_0 = 3$. In this case, $\{l_{5,3} = 0\}$ is an edge of P_+.

Case-7.3.1. P_+ has only one outer face which lies on $\{l_{5,3} = 0\}$. Then

$$\mathrm{Vol}(P_+) \ = \ \frac{1771561}{23040}.$$

Case-7.3.2. P_+ has two outer edges. Assume that the second one lies on $\{l_{p_1,q_1} = 0\}$. Then

$$|q_1| \leq p_1 \leq 4 \text{ or } p_1 = 5, q_1 = -3.$$

By a direct computation, we see that (7.10) holds except the following two subcases:

Case-7.3.2.1. $p_1 = 4, q_1 = 3$,

$$\mathrm{Vol}(P_+) = \frac{383478671}{5000940}.$$

Case-7.3.2.2. $p_1 = 2, q_1 = 1$,

$$\mathrm{Vol}(P_+) = \frac{567779}{7680}.$$

Case-7.3.3. P_+ has three outer edges. Then P_+ is obtained by cutting one of polytopes in *Case-7.3.2* with adding new edge $\{l_{p_2,q_2} = 0\}$. In fact we only need to consider P_+ obtained by cutting *Case-7.3.2.1* and *Case-7.3.2.2* above, since it obviously satisfies (7.10) in the other cases. By our construction, we can assume that $|q_2| \leq p_2 \leq p_1$. The only possible P which does not satisfy (7.10) is the case that $p_1 = 4, q_1 = 3$ and $p_2 = 2, q_2 = 1$. However,

$$\mathrm{Vol}(P_+) = \frac{92167583}{1250235}.$$

Case-7.3.4. P_+ has four outer edges. We only need to consider P_+ which is obtained by cutting *Case-7.3.3* with adding new edge $\{l_{p_3,q_3} = 0\}$ with $|q_3| \leq p_3 \leq 2$. One can show that all of these possible P_+ satisfy (7.10). Thus we do not need to consider more polytopes with more than four outer edges in case of $p_0 = 5$. Hence we conclude that for all polytopes P with $p_0 = 5$,

$$\mathrm{Vol}(P_+) \neq \mathrm{Vol}(P_+^{(2)}) \text{ or } \mathrm{Vol}(P_+^{(4)}).$$

Theorem 1.3 is true when $p_0 = 5$.

The case $p_0 = 3$ can be ruled out in the same way. We only list the exceptional polytopes such that the volumes of P_+ do not satisfy (7.10):

Case-7.3.1'. P_+ has only one outer face $\{l_{3,2} = 0\}$. Then

$$\mathrm{Vol}(P_+) = \frac{941192}{5625}.$$

Case-7.3.2'. P_+ has two outer face $\{l_{3,2} = 0\}$ and $\{l_{2,1} = 0\}$. Then

$$\mathrm{Vol}(P_+) = \frac{177064}{1875}.$$

In summary, when $p_0 \geq 3$, the volume of P_+ is not equal to either $\mathrm{Vol}(P_+^{(2)})$ or $\mathrm{Vol}(P_+^{(3)})$. Finally by exhausting all possible compactifications for $p_0 = 1, 2$ (see Table-2), we finish the proof of Theorem 1.3.

□

Remark 7.1. *If P_+ is further symmetric under the reflection with respect to the x-axis, it is easy to see its barycenter is $(\bar{x}(P_+), 0)$ and*

$$\bar{x}(P_+) \leq \bar{x}(\{-x \leq y \leq x, 0 \leq x \leq (2 + \frac{1}{p_0})\}) = \frac{6}{7}(2 + \frac{1}{p_0}).$$

Thus a Kähler-Einstein polytope of this type must satisfy

$$p_0 \leq 3.$$

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Conflict of interest

The authors declare no conflict of interest.

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A. Appendix 1: Non-smooth $\mathbb{Q}$-Fano $SO_4(\mathbb{C})$-compactifications with $p_0 \leq 2$

In this appendix we list all polytopes P_+ of non-smooth $\mathbb{Q}$-Fano $SO_4(\mathbb{C})$-compactifications with $p_0 \leq 2$, namely, (3)–(8) and (10)–(12) labeled as in Table 2 (see Figure 9 below).

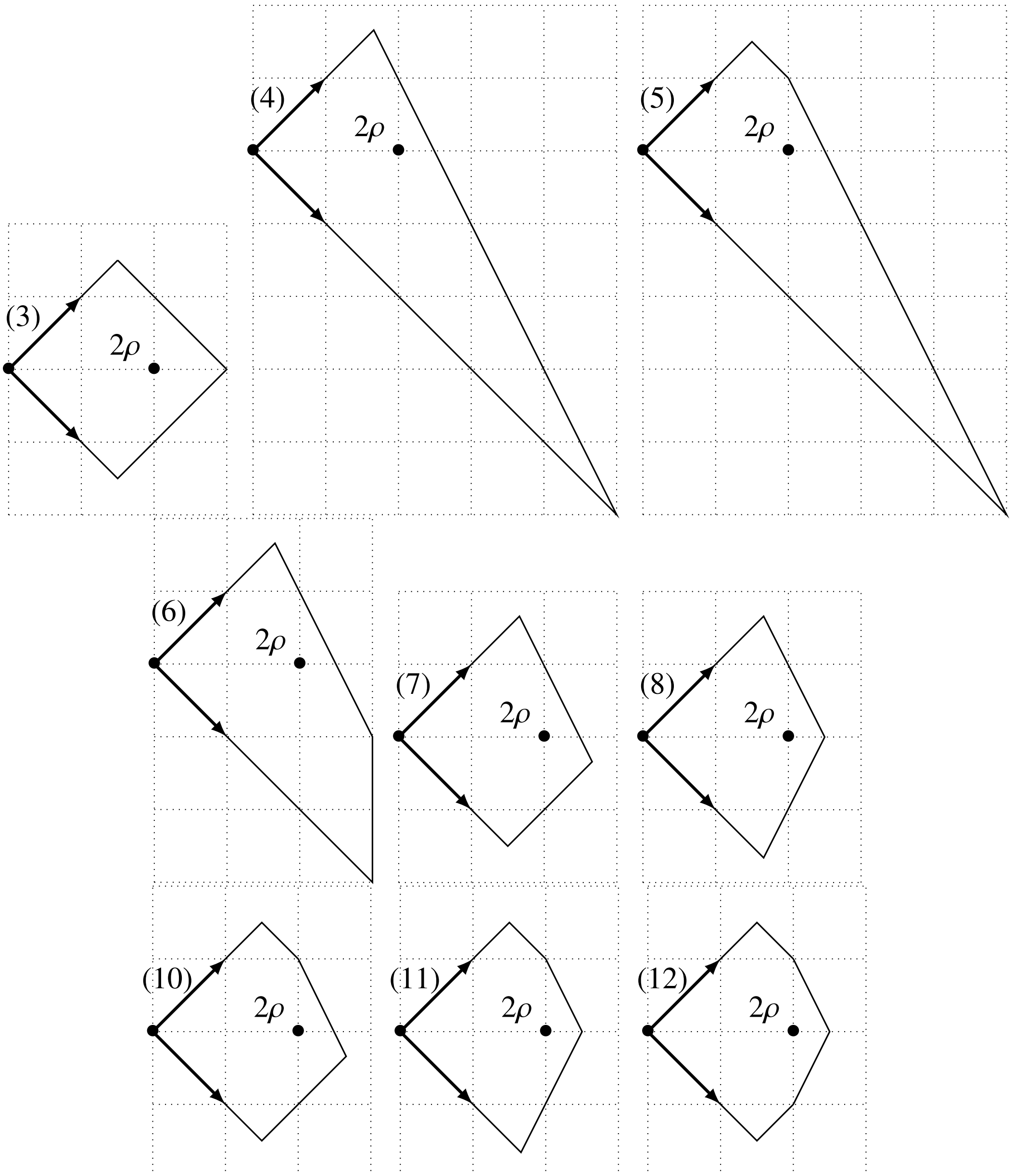

Figure 9. The Cases when $p_0 \leq 2$. The polytopes are numbered according to Table 2.

B. Appendix 2: An improvement of Theorem 1.2

In this appendix, we show that the *fine* condition in Theorem 1.2 can be dropped by a recent result of Li in [32, Theorem 1.2]. Namely, we can generalize Theorem 1.1 to a $\mathbb{Q}$-Fano G-compactification M.

Let $\mathbb{G}$ be the image of $G \times G$ embedded in $\mathrm{Aut}_0(M)$, the identity component of $\mathrm{Aut}(M)$. Then the image $\mathbb{T}'$ of $Z(G) \times \{e\}$ is a subtorus of $\mathbb{T} = Z(\mathbb{G})$. Let $\mathbf{M}^{\mathrm{NA}}(\cdot)$ and $\mathbf{J}_{\mathbb{T}}^{\mathrm{NA}}(\cdot)$ be the non-Archimedean Mabuchi K-energy and J-functional defined in [13, 32], respectively. By [32, Theorem 1.2], it suffices to check that M is $\mathbb{G}$-uniformly K-stable under the assumption (1.2). More precisely, by [36, Proposition 4.2] (an analogous version of Lemma 6.2), we prove

Theorem B.1. *Suppose that* (1.2) *holds. Then there is a constant* $c_0 > 0$ *such that for any* $\mathbb{G}$-*equivariant normal test configuration* $(\mathcal{X}, \mathcal{L})$ *of* $(M, -K_M)$*, it holds*

$$\mathbf{M}^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) \geq c_0 \cdot \mathbf{J}^{\mathrm{NA}}_{\mathbb{T}}(\mathcal{X}, \mathcal{L}). \tag{B.1}$$

Consequently, Proposition 6.1 holds and so M admits a (singular) Kähler-Einstein metric.

Proof. We need to compute the two functionals $\mathbf{M}^{\mathrm{NA}}(\cdot)$ and $\mathbf{J}^{\mathrm{NA}}_{\mathbb{T}}(\cdot)$. Note that these two functionals can be computed via the $\mathbb{C}^*$-weights on $(\mathcal{X}, \mathcal{L})$ (cf. [13, 40]). Moreover, the first one is also same with the CM-weight in [40, 44, 45] when the central fiber is reduced. In our case for a general $\mathbb{G}$-equivariant test configuration, it can be normalized with a reduced central fiber via a base change as follows.

Recall that the $\mathbb{G}$-equivariant normal test configurations are in one-one correspondence with W-invariant, convex, piecewise linear functions with rational coefficients on P (cf. [5, Section 2.4]). In particular, when M is a toric manifold with torus action $\mathbb{T}$, $\mathbb{T}$-equivariant normal test configurations are same with toric degenerations. Thus there is such a function f associated to $\pi : (\mathcal{X}, \mathcal{L}) \to \mathbb{C}$. By a base change $z \to z^d$ on $\mathbb{C}$ for sufficiently divisible $d \in \mathbb{N}_+$, the normalization

$$(\mathcal{X}^{(d)}, \mathcal{L}^{(d)}) := (\mathcal{X}, \mathcal{L})^{\text{normalization}}_{z \to z^d}$$

has a reduced central fibre (cf. [13, Proposition 7.16]). In fact, from the proof of [35, Theorem 4.1], $(\mathcal{X}^{(d)}, \mathcal{L}^{(d)})$ is still a $\mathbb{G}$-equivariant normal test configuration associated to df.

By [13, Proposition 2.8], we have

$$\mathbf{M}^{\mathrm{NA}}(\mathcal{X}^{(d)}, \mathcal{L}^{(d)}) = \mathrm{Fut}(\mathcal{X}^{(d)}, \mathcal{L}^{(d)}). \tag{B.2}$$

Note that $\mathbf{M}^{\mathrm{NA}}(\cdot)$ is linear under the base change. It follows

$$\mathbf{M}^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) = \frac{1}{d}\mathrm{Fut}(\mathcal{X}^{(d)}, \mathcal{L}^{(d)}).$$

On the other hand, by (3.11) and (3.13) in [36],

$$\mathrm{Fut}(\mathcal{X}^{(d)}, \mathcal{L}^{(d)}) = \frac{d}{V}\int_{2P_+} \langle y - 4\rho, \nabla f \rangle \pi dy.$$

This formula was proved by Donaldson for f with integral coefficients on a toric manifold [24, Proposition 4.2.1], but the arguments in his proof do not generalize to general cases.[‖] Thus

$$\mathbf{M}^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) = \frac{1}{V}\int_{2P_+} \langle y - 4\rho, \nabla f \rangle \pi dy \tag{B.3}$$

Also, by an analogous argument for any toric degeneration on a toric manifold in [29], we can get

$$\mathbf{J}^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) = \frac{1}{V}\int_{2P_+} (f - \min_{2P_+} f) \pi dy, \tag{B.4}$$

[‖]In fact, by using (B.3) and [13, 40, 44, 45], Proposition 4.2.1 in [24] is equivalent to that the central fiber of $(\mathcal{X}, \mathcal{L})$ is reduced. Clearly, this condition on the central fiber does not hold in general cases.

and for a twisted test configuration it holds

$$\mathbf{J}^{\mathrm{NA}}(\mathcal{X}_\xi, \mathcal{L}_\xi) = \frac{1}{V} \int_{2P_+} (f - \xi(y) - \min_{2P_+}\{f - \xi(y)\})\pi dy, \tag{B.5}$$

where ξ lies in the Lie algebra $\mathrm{Lie}(\mathbb{T}')$ of $\mathbb{T}'$ [29, 32].

By [36, Proposition 4.2], we see

$$\begin{aligned}&\int_{2P_+} \langle y - 4\rho, \nabla u \rangle \pi dy \\ &\geq c_0 \int_{\partial(2P_+)} u \langle y, \nu \rangle \pi d\sigma_0, \ \forall \text{ convex, } W\text{-invariant } u \text{ satisfying (4.2).}\end{aligned} \tag{B.6}$$

Since u is convex and W-invariant,

$$\nabla u(y) \in \overline{\mathfrak{a}_+}, \ \forall y \in P_+.$$

It implies that $u(ty)$ is non-decreasing for $t \geq 0$. Thus by (4.2),

$$\begin{aligned}\int_{2P_+} u\pi dy = &\int_0^1 t^{n-1} \left(\int_{\partial(2P_+)} u(ty)\langle y, \nu \rangle \pi d\sigma_0 \right) dt \\ \leq &\frac{1}{n} \int_{\partial(2P_+)} u(y) \langle y, \nu \rangle \pi d\sigma_0.\end{aligned}$$

Combining with (B.6), we get

$$\begin{aligned}&\int_{2P_+} \langle y - 4\rho, \nabla u \rangle \pi dy \\ &\geq c_0 \int_{2P_+} u \pi dy, \ \forall \text{ convex, } W\text{-invariant } u \text{ satisfying (4.2).}\end{aligned} \tag{B.7}$$

For the function f in (B.3), by the W-invariance, there is always an $\xi \in \mathrm{Lie}(\mathbb{T}')$ such that

$$f_\xi := f - \xi(y) - \min\{f - \xi(y)\}$$

satisfies (4.2). Thus applying (B.7) to f_ξ together with (B.3) and (B.5), we obtain

$$\mathbf{M}^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) \geq \frac{c_0}{V} \int_{2P_+} f_\xi \pi dy = c_0 \mathbf{J}^{\mathrm{NA}}(\mathcal{X}_\xi, \mathcal{L}_\xi) \geq c_0 \mathbf{J}^{\mathrm{NA}}_{\mathbb{T}}(\mathcal{X}, \mathcal{L}).$$

Hence (B.1) holds. □

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