http://www.aimspress.com/journal/mine

Mathematics in Engineering, 5(1): 1–16.
DOI:10.3934/mine.2023017
Received: 16 November 2021
Revised: 01 February 2022
Accepted: 09 February 2022
Published: 01 March 2022

Research article

Qualitative properties of solutions to the Dirichlet problem for a Laplace equation involving the Hardy potential with possibly boundary singularity†

Luigi Montoro* **and Berardino Sciunzi**

Dipartimento di Matematica e Informatica, UNICAL, Ponte Pietro Bucci 31B, 87036 Arcavacata di Rende, Cosenza, Italy

† **This contribution is part of the Special Issue:** The interplay between local and nonlocal equations
Guest Editors: Begona Barrios; Leandro Del Pezzo; Julio D. Rossi
Link: `www.aimspress.com/mine/article/6029/special-articles`

* **Correspondence:** Email: montoro@mat.unical.it.

Abstract: We consider positive solutions to semilinear elliptic problems with Hardy potential and a first order term in bounded smooth domain Ω with $0 \in \overline{\Omega}$. We deduce symmetry and monotonicity properties of the solutions via the moving plane procedure under suitable assumptions on the nonlinearity.

Keywords: semilinear elliptic equations; Hardy potential; qualitative properties

A Ireneo: Con cariño y afecto por todas tus enseñanzas y tu eterno entusiasmo.

1. Introduction

The aim of the paper is to investigate symmetry and monotonicity properties of weak solutions to semilinear elliptic equations concerning the Hardy term and a locally Lipschitz continuous from above (see below) nonlinearity in a bounded smooth domain $\Omega \subset \mathbb{R}^N$, $N \geq 3$ with $0 \in \overline{\Omega}$. More precisely let us consider the problem

$$\begin{cases} -\Delta u + k|\nabla u|^q = \vartheta \frac{u^p}{|x|^2} + f(u) & \text{in} \quad \Omega \\ u > 0 & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{$\mathcal{P}$}$$

where $p > 0$, $1 \leq q \leq 2$ and $k, \vartheta \geq 0$. Our results will be obtained by means of the moving plane technique, see [1, 11, 26, 34]. Such a technique can be performed in general domains providing partial monotonicity results near the boundary and symmetry when the domain is convex and symmetric with respect some direction. In particular, along this paper, we say that a domain is strictly convex with respect a direction, say for example the x_1-direction, if and only if

For any pairs of points $P_a, P_b \in \overline{\Omega}$ with

$$P_a = (x_1^a, x_2, \dots, x_N) \quad \text{and} \quad P_b = (x_1^b, x_2, \dots, x_N),$$

every point on the line segment connecting P_a, P_b other than the end points P_a and P_b is contained in the interior of $\overline{\Omega}$.

For simplicity of exposition we assume directly in all the paper that Ω is a strictly convex with respect the x_1-direction (or convex when it will be specified, see Theorem 1.5 below) domain which is symmetric with respect to the hyperplane $\{x_1 = 0\}$.

Moreover in all the paper the nonlinearity f will be assumed to be locally Lipschitz continuous from above. More precisely we assume that f satisfies the following condition denoted from now on by (h_f), namely

(h_f) $f : (0, +\infty) \to \mathbb{R}_0^+$ is a continuous function such that for $0 < t \le s \le M$ it holds

$$f(s) - f(t) \le C(M)(s - t),$$

where $C(M)$ is a positive constant depending on M.

A typical example is provided by positive solutions to equations involving nonlinearities given by $f(u) = g(u) + 1/u^\alpha$, where g is a locally Lipschitz continuous function and $\alpha > 0$. We recall here the following

Proposition 1.1. *(Hardy-Sobolev Inequality) Suppose $N \ge 3$ and $u \in H^1(\mathbb{R}^N)$. Then we have*

$$\int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2}\, dx \le C_N \int_{\mathbb{R}^N} |\nabla u|^2\, dx, \tag{1.1}$$

with $C_N = (2/N-2))^2$ optimal and not achieved constant.

Singular semilinear elliptic equations with Hardy potential have been intensely studied. The problem of the existence of solutions to $(\mathcal{P})$ exhibits a different behavior depending on the position of the pole on the domain. This acutually is strongly related to the Hardy-Sobolev inequality stated in Proposition 1.1. Let us consider, as a particular case of the problem $(\mathcal{P})$ the following

$$\begin{cases} -\Delta u = \frac{u^p}{|x|^2} & \text{in} \quad \Omega \\ u > 0 & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.2}$$

We analyze the two different cases:

- **Case 1:** $0 \in \partial\Omega$**.** The problem of the existence of solutions of problem shows a different behaviour depending on the exponent p. In the case $0 \in \partial\Omega$, the existence of a solutions to (1.2) generally depends on the geometry of the domain.

(i) If $0 < p < 1$ the existence of a solution to (1.2) is independent of the location of the origin. Indeed using Hardy inequality (Proposition 1.1) the functional satisfies

$$u \to \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \frac{1}{p+1}\int_\Omega \frac{u^{p+1}}{|x|^2}\,dx \geq \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - C\left(\int_\Omega |\nabla u|^2\,dx\right)^{\frac{p+1}{2}},$$

for some positive constant C and therefore the existence of a solution in $H_0^1(\Omega)$ follows by minimization, see [3].

(ii) In the linear case $p = 1$ the problem of the existence of (1.2) is related to the attainability of some constant less then C_N in (1.1). In [25], the authors give sufficient conditions to get the existence of solutions to (1.2). In this case the geometry of Ω at the origin plays a fundamental role. See also [16, 17, 20, 27, 28] for related problems.

(iii) The situation for $p > 1$ is also involved. Of course, if $0 \notin \overline{\Omega}$ the solution follows using the mountain pass theorem [9]. On the contrary, if $0 \in \Omega$ there is no solutions to (1.2). Actually in [13] was shoved that (1.2) has no weak supersolutions since this would be a contradiction with the Hardy inequality (Proposition 1.1). In the case $0 \in \partial\Omega$ the existence of solutions to (1.2) depends strongly on the geometry of the domain Ω. For example in starshaped (with respect to the origin) domains there are no solutions since a Pohozaev's identity is in force in this case. On the other hand in some suitable non-starshaped domains, e.g., dumbell domains, there exits a weak solution to (1.2) in the range $1 < p < (N+2)/(N-2)$, see [18,29]. Moreover we point out the if we perturb the problem (1.2) adding some sublinear term u^r with $0 < r < 1$, we get the existence of a weak solution without any restriction on the shape of the domain Ω and on the size of the exponent $p > 1$.

- **Case 2:** $0 \in \Omega$. As in the previous case, the existence of solutions to problem (1.2) is related to the exponent p and to the Hardy inequality (1.1). In particular

(i) If $0 < p < 1$, (as in the case $0 \in \partial\Omega$) the existence of a solution to (1.2) is independent of the location of the origin and follows by using a minimization procedure, see [3].

(ii) In the linear and superlinear cases $p \geq 1$ the problem (1.2) does not admit solutions (even in the weakest possible sense) because Proposition 1.1, see [2, 4–8, 12, 18, 29]. On the contrary, if the problem (1.2) is perturbed adding a first order term in the right hand side (that is, adding a first order term as an absorption term), then the existence of positive solutions of (1.2) can be proved by means of approximation and variational methods, see [2, 4–8, 12, 18, 29]. The absorption term $k|\nabla u|^q$ in ($\mathcal{P}$), despite of Proposition 1.1, is sufficient to break down the effect of the obstruction to the existence of solutions due to the presence of the Hardy potential in the problem ($\mathcal{P}$).

Therefore, taking into account these considerations on the existence of solutions to ($\mathcal{P}$), taking into account the presence of the Hardy potential and the presence of a Hölder nonlinearity in the right hand side and standard elliptic regularity theory results, in all the paper we assume that

$$u \in H_0^1(\Omega) \cap C^1(\overline{\Omega} \setminus \{0\}). \tag{1.3}$$

Thus the equation is understood in the following sense

Definition 1.2. *$u \in H_0^1(\Omega) \cap C^1(\overline{\Omega} \setminus \{0\})$ is a weak solution to $(\mathcal{P})$ if*

$$\frac{u^p}{|x|^2}, f(u) \in L^1(\Omega)$$

and

$$\int_\Omega \nabla u \nabla \varphi \, dx + k \int_\Omega |\nabla u|^q \varphi \, dx = \vartheta \int_\Omega \frac{u^p}{|x|^2} \varphi \, dx + \int_\Omega f(u) \varphi \, dx \qquad \forall \varphi \in C_c^1(\Omega), \tag{1.4}$$

Let us now state our main results.

Theorem 1.3. *Let Ω be a strictly convex domain with respect to the x_1-direction, which is symmetric with respect to the hyperplane $\{x_1 = 0\}$ and let*

$$u \in H_0^1(\Omega) \cap C^1(\overline{\Omega} \setminus \{0\})$$

be a solution to $(\mathcal{P})$. Assume that

$$p > 0 \quad \text{and} \quad f \text{ fulfills } (h_f).$$

Then, it follows that u is symmetric with respect to the hyperplane $\{x_1 = 0\}$ and increasing in the x_1-direction in $\Omega \cap \{x_1 < 0\}$. Furthermore

$$u_{x_1} \geq 0 \quad \text{in} \quad \Omega \cap \{x_1 < 0\}. \tag{1.5}$$

As an immediate consequence of the previous result we get the following

Corollary 1.4. *Let $\Omega = B_R(0)$, $R > 0$ and let $u \in H_0^1(\Omega) \cap C^1(\overline{\Omega} \setminus \{0\})$ be a solution to $(\mathcal{P})$. Assume that*

$$p > 0 \quad \text{and} \quad f \text{ fulfills } (h_f).$$

Then, it follows that u is radially symmetric with

$$\frac{\partial u}{\partial r}(r) < 0, \quad \text{for} \quad r \neq 0.$$

If we assume more regularity on the data of problem $(\mathcal{P})$, we can only assume that Ω is convex (not strictly) in the x_1-direction. In this case our result holds also for domains with a flat part on the boundary, as for example the case of a N-dimensional cube. We have the following

Theorem 1.5. *Let Ω be a convex domain with respect to the x_1-direction, which is symmetric with respect to the hyperplane $\{x_1 = 0\}$ and let*

$$u \in H_0^1(\Omega) \cap C^1(\overline{\Omega} \setminus \{0\})$$

be a solution to $(\mathcal{P})$. Assume that $p \geq 1$ and $f : [0, +\infty) \to \mathbb{R}$ is locally Lipschitz continuous in $[0, \infty)$. Then, it follows that u is symmetric with respect to the hyperplane $\{x_1 = 0\}$ and increasing in the x_1-direction in $\Omega \cap \{x_1 < 0\}$. Furthermore

$$u_{x_1} > 0 \quad \text{in} \quad \Omega \cap \{x_1 < 0\}. \tag{1.6}$$

Symmetry and monotonicity properties of solutions to quasilinear and semilinear elliptic problems involving the Hardy potential (and $0 \in \Omega$) or more general singular critical sets where the solution may be not regular, have been studied in [14, 15, 21–24, 30, 31, 33] for the local case and in [10, 19, 32] for the nonlocal case. In this direction our result is new and more general. Indeed in this paper we also deal with the case $0 \in \partial\Omega$ and we consider nonlinearities that are sum of a Hölder continuous term (the case $0 < p < 1$) and of a term f that is locally Lipschitz continuous (only) from above in $(0, +\infty)$.

Actually, all the non negative nonlinearities of the form

$$f(s) := f_1(s) + f_2(s),$$

where f_1 is a decreasing continuous function in $[0, \infty)$, $f_2(\cdot)$, is locally Lipschitz continuous in $[0, \infty)$, satisfy our assumptions (h_f).

The remaining part of the paper is devoted to the proofs of our results.

2. Proof of Theorem 1.3 and Theorem 1.5

Notation. Generic fixed numerical constants will be denoted by C (with subscript in some case) and will be allowed to vary within a single line or formula. Moreover f^+ and f^- will stand for the positive and negative part of a function, i.e., $f^+ = \max\{f, 0\}$ and $f^- = \min\{f, 0\}$. We also denote $|A|$ the Lebesgue measure of the set A.

For a real number λ we set

$$\Omega_\lambda = \{x \in \Omega : x_1 < \lambda\}$$
$$x_\lambda = R_\lambda(x) = (2\lambda - x_1, x_2, \ldots, x_n)$$

which is the reflection through the hyperplane $T_\lambda := \{x_1 = \lambda\}$and

$$u_\lambda(x) = u(x_\lambda). \tag{2.1}$$

Also let

$$a = \inf_{x\in\Omega} x_1. \tag{2.2}$$

In the following we will exploit the fact that u_λ is a solution to:

$$\int_{R_\lambda(\Omega)} \nabla u_\lambda \nabla\varphi \, dx + k \int_{R_\lambda(\Omega)} |\nabla u_\lambda|^q \varphi \, dx = \vartheta \int_{R_\lambda(\Omega)} \frac{u_\lambda^p}{|x_\lambda|^2}\varphi \, dx + \int_{R_\lambda(\Omega)} f(u_\lambda)\varphi \, dx, \tag{2.3}$$

for all $\varphi \in C^1_c(R_\lambda(\Omega))$ and we also observe that, for any $a < \lambda < 0$, the function $w_\lambda := u - u_\lambda$ satisfies

$$0 \le w_\lambda^+ \le u \text{ on } \Omega_\lambda$$

and so $w_\lambda^+ \in L^2(\Omega_\lambda)$, since $u \in C^0(\overline{\Omega_\lambda})$. Since in the range $0 < p < 1$ the right hand side of $(\mathcal{P})$

$$\vartheta \frac{u^p}{|x|^2} + f(u) \tag{2.4}$$

is the sum of a Hölder continuous term (with respect to the variable u) and of a Lipschitz continuous term from above in $(0, +\infty)$, first we need to prove the following weak comparison principle that holds in subdomain of Ω that lies far from the boundary of Ω where the right hand side (2.4) is more regular. Then we have to take into account this fact in the proof of Theorem 1.3, by exploiting the Hopf's boundary lemma and the strictly convexity (in the x_1-direction) of Ω. We have the following

Proposition 2.1 (Weak Comparison Principle 1)**.** *Assume that*

$$p > 0 \quad \textit{and} \quad f \textit{ fulfills } (h_f).$$

Let $\lambda \le \hat{\lambda} < 0$ and $\tilde{\Omega}$ be a bounded domain such that $\tilde{\Omega} \subset\subset \Omega_\lambda$. Assume that u is a solution to $(\mathcal{P})$ such that $u \le u_\lambda$ on $\partial\tilde{\Omega}$. Then there exists a positive constant

$$\hat{\delta} = \hat{\delta}(k, p, q, f, \hat{\lambda}, \vartheta, \operatorname{dist}(\tilde{\Omega}, \partial\Omega), \|u\|_{L^\infty(\Omega_{\hat{\lambda}})}, \|\nabla u\|_{L^\infty(\Omega_{\hat{\lambda}})})$$

such that if we assume $|\tilde{\Omega}| \le \hat{\delta}$, then it holds

$$u \le u_\lambda \quad \textit{in } \tilde{\Omega}. \tag{2.5}$$

Proof. We have (in the weak sense, see (1.4))

$$-\quad \Delta u + k|\nabla u|^q = \vartheta \frac{u^p}{|x|^2} + f(u) \quad \text{in } \Omega, \tag{2.6}$$

$$-\quad \Delta u_\lambda + k|\nabla u_\lambda|^q = \vartheta \frac{u_\lambda^p}{|x_\lambda|^2} + f(u_\lambda) \quad \text{in } R_\lambda(\Omega), \tag{2.7}$$

By contradiction, we assume the (2.5) is false.

First of all we start proving that

$$(u - u_\lambda)^+ \in H_0^1(\tilde{\Omega}) \cap L^\infty(\tilde{\Omega}). \tag{2.8}$$

It is immediate to show that $(u-u_\lambda)^+ \in L^\infty(\tilde{\Omega})$ because $0 \le (u-u_\lambda)^+ \le u \in C^0(\overline{\Omega_{\hat{\lambda}}})$. On the other hand, the fact that $(u - u_\lambda)^+ \in H_0^1(\tilde{\Omega})$ is not a priori obvious since it can happen that $\partial\tilde{\Omega} \cap 0_\lambda \ne \emptyset$ and there the reflected function u_λ is not defined. For the reader's convenience we give some details.

Let us define $\varphi_\varepsilon(x) \in C_c^\infty(\Omega)$, $\varphi_\varepsilon \ge 0$ such that

$$\begin{cases} \varphi_\varepsilon \equiv 1 & \text{in } \Omega \setminus B_{2\varepsilon} \\ \varphi_\varepsilon \equiv 0 & \text{in } B_\varepsilon \\ |\nabla \varphi_\varepsilon| \le \frac{C}{\varepsilon} & \text{in } B_{2\varepsilon} \setminus B_\varepsilon, \end{cases} \tag{2.9}$$

where $B_\varepsilon = B_\varepsilon(0)$ denotes the open ball with center 0 and radius $\varepsilon > 0$. For $x \in \Omega_\lambda$, we consider

$$\hat{\varphi}_\varepsilon(x) = \varphi_\varepsilon(x_\lambda)\ ,$$

with φ_ε defined in (2.9). Let us set

$$\phi_\varepsilon := (u - u_\lambda)^+ \hat{\varphi}_\varepsilon \qquad \text{in } \Omega_\lambda.$$

Since by hypothesis $u \in H_0^1(\Omega)$ and $u \le u_\lambda$ on $\partial\tilde{\Omega}$, we readily have that $\phi_\varepsilon \in H_0^1(\Omega_\lambda)$, for all $\varepsilon > 0$ and that

$$\phi_\varepsilon \to (u - u_\lambda)^+ \text{ a.e. in } \Omega_\lambda, \tag{2.10}$$

if $\varepsilon \to 0$. Setting $w_\lambda = (u - u_\lambda)$, we deduce

$$\int_{\Omega_\lambda} |\nabla \phi_\varepsilon|^2\, dx \le C \int_{\Omega_\lambda} |\hat{\varphi}_\varepsilon|^2 |\nabla w_\lambda^+|^2\, dx + C \int_{\Omega_\lambda} (w_\lambda^+)^2 |\nabla \hat{\varphi}_\varepsilon|^2\, dx$$

$$
\begin{aligned}
&\leq C\int_{\Omega_\lambda} |\hat{\varphi}_\varepsilon|^2 |\nabla w_\lambda^+|^2\,dx + C(\|u\|_{L^\infty(\Omega_{\hat\lambda})})\varepsilon^{N-2}\\
&\leq C_1(\|u\|_{H_0^1(\Omega)}) + C_2(\|u\|_{L^\infty(\Omega_{\hat\lambda})})\varepsilon^{N-2},
\end{aligned}
$$

where we used (2.9). Therefore (recall that $N \geq 3$)

$$
\phi_\varepsilon \rightharpoonup \hat{\phi} \text{ in } H_0^1(\Omega_\lambda), \qquad \text{if } \varepsilon \to 0. \tag{2.11}
$$

Finally, by Sobolev embedding, from (2.10) and (2.11), we deduce (2.8).

Therefore, because (2.8), by a density argument we consider $(u-u_\lambda)^+ \in H_0^1(\tilde\Omega) \cap L^\infty(\tilde\Omega)$ as a test function in both (2.6) and (2.7). We first consider the

Case: $1 \leq q < 2$. Subtracting in the weak formulation of (2.6) and (2.7), we get

$$
\begin{aligned}
&\int_{\tilde\Omega} |\nabla(u-u_\lambda)^+|^2\,dx + k\int_{\tilde\Omega}(|\nabla u|^q - |\nabla u_\lambda|^q)(u-u_\lambda)^+\,dx \qquad (2.12)\\
&= \vartheta\int_{\tilde\Omega}\frac{u^p}{|x|^2}(u-u_\lambda)^+\,dx - \vartheta\int_{\tilde\Omega}\frac{u_\lambda^p}{|x_\lambda|^2}(u-u_\lambda)^+\,dx\\
&+ \int_{\tilde\Omega} f(u)(u-u_\lambda)^+\,dx - \int_{\tilde\Omega} f(u_\lambda)(u-u_\lambda)^+\,dx\\
&= \vartheta\int_{\tilde\Omega}\left(\frac{u^p}{|x|^2}(u-u_\lambda)^+ - \frac{u_\lambda^p}{|x_\lambda|^2}(u-u_\lambda)^+\right)dx + \int_{\tilde\Omega}(f(u)-f(u_\lambda))(u-u_\lambda)^+\,dx.
\end{aligned}
$$

We note that in the set $\tilde\Omega \subset\subset \Omega_\lambda$ we have that $|x| \geq |x_\lambda|$. Therefore (2.12) becomes

$$
\begin{aligned}
&\int_{\tilde\Omega} |\nabla(u-u_\lambda)^+|^2\,dx \leq k\left|\int_{\tilde\Omega}(|\nabla u|^q - |\nabla u_\lambda|^q)(u-u_\lambda)^+\,dx\right|\\
&+\vartheta\int_{\tilde\Omega}\frac{u^p-u_\lambda^p}{|x|^2}(u-u_\lambda)^+\,dx + \int_{\tilde\Omega}(f(u)-f(u_\lambda))(u-u_\lambda)^+\,dx\\
&\leq k\int_{\tilde\Omega}|(|\nabla u|^q - |\nabla u_\lambda|^q)|\,(u-u_\lambda)^+\,dx + \vartheta\int_{\tilde\Omega}\frac{u^p-u_\lambda^p}{|x|^2}(u-u_\lambda)^+\,dx\\
&+C(f,\|u\|_{L^\infty(\Omega_{\hat\lambda})})\int_{\tilde\Omega}[(u-u_\lambda)^+]^2\,dx,
\end{aligned} \tag{2.13}
$$

where in the last inequality of (2.13) we used the assumption (h_f) (recall that we are working where $u \geq u_\lambda$). Since $q \geq 1$, for every $0_{\hat\lambda} \neq x \in \tilde\Omega_{\hat\lambda}$ by the mean value's theorem we get

$$
(|\nabla u|^q - |\nabla u_\lambda|^q) \leq q(|\nabla u| + |\nabla u_\lambda|)^{q-1}|\nabla(u-u_\lambda)^+|.
$$

Hence from (2.13) we deduce that

$$
\begin{aligned}
&\int_{\tilde\Omega} |\nabla(u-u_\lambda)^+|^2\,dx\\
&\leq qk\int_{\tilde\Omega}(|\nabla u| + |\nabla u_\lambda|)^{q-1}|\nabla(u-u_\lambda)^+|(u-u_\lambda)^+\,dx + \vartheta\int_{\tilde\Omega}\frac{u^p-u_\lambda^p}{|x|^2}(u-u_\lambda)^+\,dx
\end{aligned} \tag{2.14}
$$

$$
\begin{aligned}
&+C(f,\|u\|_{L^\infty(\Omega_{\tilde\lambda})})\int_{\tilde\Omega}[(u-u_\lambda)^+]^2\,dx\\
&\le qk\int_{\{x\in\tilde\Omega\,:\,|\nabla u_\lambda|\le 2|\nabla u|\}}(|\nabla u|+|\nabla u_\lambda|)^{q-1}|\nabla(u-u_\lambda)^+|(u-u_\lambda)^+\,dx\\
&+qk\int_{\{x\in\tilde\Omega\,:\,|\nabla u_\lambda|> 2|\nabla u|\}}(|\nabla u|+|\nabla u_\lambda|)^{q-1}|\nabla(u-u_\lambda)^+|(u-u_\lambda)^+\,dx\\
&+\vartheta\int_{\tilde\Omega}\frac{u^p-u_\lambda^p}{|x|^2}(u-u_\lambda)^+\,dx+C(f,\|u\|_{L^\infty(\Omega_{\tilde\lambda})})\int_{\tilde\Omega}[(u-u_\lambda)^+]^2\,dx\\
&\le C(k,q,\|\nabla u\|_{L^\infty(\Omega_{\tilde\lambda})})\int_{\tilde\Omega}|\nabla(u-u_\lambda)^+|(u-u_\lambda)^+\,dx\\
&+qk\int_{\{x\in\tilde\Omega\,:\,|\nabla u_\lambda|> 2|\nabla u|\}}(|\nabla u|+|\nabla u_\lambda|)^{q-1}|\nabla(u-u_\lambda)^+|(u-u_\lambda)^+\,dx\\
&+\vartheta\int_{\tilde\Omega}\frac{u^p-u_\lambda^p}{|x|^2}(u-u_\lambda)^+\,dx+C(f,\|u\|_{L^\infty(\Omega_{\tilde\lambda})})\int_{\tilde\Omega}[(u-u_\lambda)^+]^2\,dx.
\end{aligned}
$$

In the set $\{x\in\tilde\Omega\,:\,|\nabla u_\lambda|>2|\nabla u|\}$ using standard triangular inequalities we can deduce that

$$
\frac{1}{2}|\nabla u_\lambda|\le|\nabla u_\lambda|-|\nabla u|\le|\nabla(u-u_\lambda)|\le|\nabla u_\lambda|+|\nabla u|\le\frac{3}{2}|\nabla u_\lambda|. \tag{2.15}
$$

Note that, here below, we shall exploit (2.15) in the support of $(u-u_\lambda)^+$ since otherwise the functions involved vanish. By (2.15) we therefore obtain

$$
\begin{aligned}
&\int_{\{x\in\tilde\Omega\,:\,|\nabla u_\lambda|> 2|\nabla u|\}}(|\nabla u|+|\nabla u_\lambda|)^{q-1}|\nabla(u-u_\lambda)^+|(u-u_\lambda)^+\,dx\\
&\le C(q)\int_{\{x\in\tilde\Omega\,:\,|\nabla u_\lambda|> 2|\nabla u|\}}|\nabla u_\lambda|^q(u-u_\lambda)^+\,dx\\
&\le \varepsilon C(q)\int_{\{x\in\tilde\Omega\,:\,|\nabla u_\lambda|> 2|\nabla u|\}}|\nabla u_\lambda|^2\,dx+C(q,\varepsilon)\int_{\{x\in\tilde\Omega\,:\,|\nabla u_\lambda|> 2|\nabla u|\}}[(u-u_\lambda)^+]^{\frac{2}{2-q}}\,dx\\
&\le \varepsilon C(q)\int_{\{x\in\tilde\Omega\,:\,|\nabla u_\lambda|> 2|\nabla u|\}}|\nabla u_\lambda|^2\,dx\\
&+C(q,\varepsilon,\|u\|_{L^\infty(\Omega_{\tilde\lambda})})\int_{\{x\in\tilde\Omega\,:\,|\nabla u_\lambda|> 2|\nabla u|\}}[(u-u_\lambda)^+]^2\,dx,
\end{aligned}
$$

where we have used weighted Young's inequality and the fact that $2/(2-q)\ge 2$ for $1\le q<2$. Therefore from (2.14) we deduce

$$
\begin{aligned}
&\int_{\tilde\Omega}|\nabla(u-u_\lambda)^+|^2\,dx\\
&\le C(k,q,\|\nabla u\|_{L^\infty(\Omega_{\tilde\lambda})})\int_{\tilde\Omega}|\nabla(u-u_\lambda)^+|(u-u_\lambda)^+\,dx\\
&+\varepsilon C(k,q)\int_{\{x\in\tilde\Omega\,:\,|\nabla u_\lambda|> 2|\nabla u|\}}|\nabla u_\lambda|^2\,dx\\
&+C(k,q,\varepsilon,\|u\|_{L^\infty(\Omega_{\tilde\lambda})})\int_{\{x\in\tilde\Omega\,:\,|\nabla u_\lambda|> 2|\nabla u|\}}[(u-u_\lambda)^+]^2\,dx
\end{aligned} \tag{2.16}
$$

$$
\begin{aligned}
&+\vartheta\int_{\tilde\Omega}\frac{u^p-u_\lambda^p}{|x|^2}(u-u_\lambda)^+\,dx+C(f,\|u\|_{L^\infty(\Omega_{\hat\lambda})})\int_{\tilde\Omega}[(u-u_\lambda)^+]^2\,dx\\
&\le C(k,q,\|\nabla u\|_{L^\infty(\Omega_{\hat\lambda})})\int_{\tilde\Omega}|\nabla(u-u_\lambda)^+|(u-u_\lambda)^+\,dx\\
&+\varepsilon C(k,q)\int_{\{x\in\tilde\Omega\,:\,|\nabla u_\lambda|>2|\nabla u|\}}|\nabla(u-u_\lambda)^+|^2\,dx\\
&+C(k,q,\varepsilon,\|u\|_{L^\infty(\Omega_{\hat\lambda})})\int_{\{x\in\tilde\Omega\,:\,|\nabla u_\lambda|>2|\nabla u|\}}[(u-u_\lambda)^+]^2\,dx\\
&+\vartheta\int_{\tilde\Omega}\frac{u^p-u_\lambda^p}{|x|^2}(u-u_\lambda)^+\,dx+C(f,\|u\|_{L^\infty(\Omega_{\hat\lambda})})\int_{\tilde\Omega}[(u-u_\lambda)^+]^2\,dx,
\end{aligned}
$$

where in the last inequality we used (2.15). Applying one more time weighted Young's inequality in the r.h.s of (2.16) we obtain

$$
\begin{aligned}
&\int_{\tilde\Omega}|\nabla(u-u_\lambda)^+|^2\,dx\\
&\le \varepsilon C(q,k,\|\nabla u\|_{L^\infty(\Omega_{\hat\lambda})})\int_{\tilde\Omega}|\nabla(u-u_\lambda)^+|^2\,dx+\vartheta\int_{\tilde\Omega}\frac{u^p-u_\lambda^p}{|x|^2}(u-u_\lambda)^+\,dx\\
&+C(k,f,q,\varepsilon,\|u\|_{L^\infty(\Omega_{\hat\lambda})},\|\nabla u\|_{L^\infty(\Omega_{\hat\lambda})})\int_{\tilde\Omega}[(u-u_\lambda)^+]^2\,dx,
\end{aligned}
$$

and for ε small we deduce

$$
\int_{\tilde\Omega}|\nabla(u-u_\lambda)^+|^2\,dx\le C\int_{\tilde\Omega}\frac{u^p-u_\lambda^p}{|x|^2}(u-u_\lambda)^+\,dx+C\int_{\tilde\Omega}[(u-u_\lambda)^+]^2\,dx, \tag{2.17}
$$

where $C=C(k,f,q,\vartheta,\|u\|_{L^\infty(\Omega_{\hat\lambda})},\|\nabla u\|_{L^\infty(\Omega_{\hat\lambda})})$ is a positive constant. Taking into account that for $\lambda\le\hat\lambda<0$ one has $|x|\ge C$ in Ω_λ for some positive constant C depending only on $\hat\lambda$ (but not on λ), from (2.17) we obtain

$$
\begin{aligned}
&\int_{\tilde\Omega}|\nabla(u-u_\lambda)^+|^2\,dx\le C\int_{\tilde\Omega}(u^p-u_\lambda^p)(u-u_\lambda)^+\,dx+C\int_{\tilde\Omega}[(u-u_\lambda)^+]^2\,dx,\\
&\le C\int_{\tilde\Omega}[(u-u_\lambda)^+]^2\,dx,
\end{aligned}
\tag{2.18}
$$

where $C=C(k,f,p,q,\hat\lambda,\vartheta,\operatorname{dist}(\tilde\Omega,\partial\Omega),\|u\|_{L^\infty(\Omega_{\hat\lambda})},\|\nabla u\|_{L^\infty(\Omega_{\hat\lambda})})$ is a positive constant. We note that, in the last inequality, we have used the fact that the term $u^p-u_\lambda^p$ is locally Liptschitz continuous in $(0,+\infty)$ and that the solution u of $(\mathcal{P})$ is strictly positive in Ω.

Case: $q=2$. In this case we consider we consider $e^{-ku}(u-u_\lambda)^+\in H_0^1(\tilde\Omega)\cap L^\infty(\tilde\Omega)$, as a test function in (2.6) and $e^{-ku_\lambda}(u-u_\lambda)^+\in H_0^1(\tilde\Omega)\cap L^\infty(\tilde\Omega)$, as a test function in (2.7). Subtracting in the weak formulation of (2.6) and (2.7), we get

$$
\int_{\tilde\Omega}e^{-u_\lambda}|\nabla(u-u_\lambda)^+|^2\,dx \tag{2.19}
$$

$$\leq \int_{\tilde{\Omega}} |e^{-u} - e^{-u_\lambda}||\nabla u||\nabla(u-u_\lambda)^+|\,dx + \vartheta \int_{\tilde{\Omega}} e^{-u}\frac{u^p}{|x|^2}(u-u_\lambda)^+\,dx$$
$$-\vartheta \int_{\tilde{\Omega}} e^{-u_\lambda}\frac{u_\lambda^p}{|x_\lambda|^2}(u-u_\lambda)^+\,dx + \int_{\tilde{\Omega}} e^{-u} f(u)(u-u_\lambda)^+\,dx - \int_{\tilde{\Omega}} e^{-u_\lambda} f(u_\lambda)(u-u_\lambda)^+\,dx.$$

Notice that we are considering the set $\tilde{\Omega} \cap \{u \geq u_\lambda\}$ and there $|x| \geq |x_\lambda|$. Then (2.19) becomes

$$\int_{\tilde{\Omega}} e^{-u_\lambda}|\nabla(u-u_\lambda)^+|^2\,dx$$
$$\leq \int_{\tilde{\Omega}} |e^{-u} - e^{-u_\lambda}||\nabla u||\nabla(u-u_\lambda)^+|\,dx + \vartheta \int_{\tilde{\Omega}} e^{-u}\left(\frac{u^p - u_\lambda^p}{|x|^2}\right)(u-u_\lambda)^+\,dx$$
$$+ \int_{\tilde{\Omega}} e^{-u}\left(f(u) - f(u_\lambda)\right)(u-u_\lambda)^+\,dx.$$

As in the previous case, taking into account that for $\lambda < 0$ one has $|x| \geq C$ in $\Omega_{\hat{\lambda}}$ for some positive constant C, that the term $u^p - u_\lambda^p$ is locally Liptschitz continuous in $(0, +\infty)$ and that u is positive in Ω, we obtain

$$\int_{\tilde{\Omega}} |\nabla(u-u_\lambda)^+|^2\,dx$$
$$\leq C\int_{\tilde{\Omega}} |\nabla(u-u_\lambda)^+|(u-u_\lambda)^+\,dx + C\int_{\tilde{\Omega}} \left(u^p - u_\lambda^p\right)(u-u_\lambda)^+\,dx$$
$$+C\int_{\tilde{\Omega}} [(u-u_\lambda)^+]^2\,dx$$
$$\leq C\int_{\tilde{\Omega}} |\nabla(u-u_\lambda)^+|(u-u_\lambda)^+\,dx + C\int_{\tilde{\Omega}} [(u-u_\lambda)^+]^2\,dx,$$

where $C = C(k, f, p, \hat{\lambda}, \vartheta, \mathrm{dist}\,(\tilde{\Omega}, \partial\Omega), \|u\|_{L^\infty(\Omega_{\hat{\lambda}})}, \|\nabla u\|_{L^\infty(\Omega_{\hat{\lambda}})})$ is a positive constant. Using weighted Young inequality finally we obtain

$$\int_{\tilde{\Omega}} |\nabla(u-u_\lambda)^+|^2\,dx \leq C\int_{\tilde{\Omega}} [(u-u_\lambda)^+]^2\,dx. \tag{2.20}$$

We get a similar estimate as the one in (2.18). The conclusion follows using classical Poincaré inequality in (2.18) and in (2.20). Indeed we deduce

$$\int_{\tilde{\Omega}} |\nabla(u-u_\lambda)^+|^2\,dx \leq CC_P^2(\tilde{\Omega})\int_{\tilde{\Omega}} \nabla(u-u_\lambda)^+|^2\,dx,$$

where C_P we denotes the Poincaré constant. By choosing

$$\hat{\delta} = \hat{\delta}(k, f, p, q, \hat{\lambda}, \vartheta, \mathrm{dist}\,(\tilde{\Omega}, \partial\Omega), \|u\|_{L^\infty(\Omega_{\hat{\lambda}})}, \|\nabla u\|_{L^\infty(\Omega_{\hat{\lambda}})})$$

small such that $CC_p^2(\tilde{\Omega}) < 1$, we get $(u-u_\lambda)^+ = 0$ in $\tilde{\Omega}$ since by hypothesis we have $u \leq u_\lambda$ on $\partial\tilde{\Omega}$. This concludes the proof. □

In the case $p \geq 1$, adapting straightforwardly the proof of Proposition 2.1 we are able to get the next

Proposition 2.2 (Weak Comparison Principle 2)**.** *Assume that*

$$p \geq 1 \quad \textit{and} \quad f \textit{ fulfills } (h_f).$$

Let $\lambda \leq \hat{\lambda} < 0$ and $\tilde{\Omega}$ be a bounded domain such that $\tilde{\Omega} \subseteq \Omega_\lambda$. Assume that u is a solution to $(\mathcal{P})$ such that $u \leq u_\lambda$ on $\partial\tilde{\Omega}$. Then there exists a positive constant

$$\hat{\delta} = \hat{\delta}(k, p, q, f, \hat{\lambda}, \vartheta, \|u\|_{L^\infty(\Omega_{\hat{\lambda}})}, \|\nabla u\|_{L^\infty(\Omega_{\hat{\lambda}})})$$

such that if we assume $|\tilde{\Omega}| \leq \hat{\delta}$, then it holds

$$u \leq u_\lambda \quad \textit{in } \tilde{\Omega}.$$

Now we are ready to prove our main results. We start with the

Proof of Theorem 1.3. To prove the theorem, we exploit the moving plane method. To start with the procedure, we take advantage of the application of Hopf's boundary lemma. We recall that in the case $0 < p < 1$ we do have to use Proposition 2.1 far from the boundary $\partial\Omega$, since the loss of regularity of the right hand side of $(\mathcal{P})$. Thus let $a < \lambda < 0$ with λ sufficiently close to a, see (2.2). By Hopf's boundary lemma, it follows that

$$u - u_\lambda \leq 0 \quad \text{in} \quad \Omega_\lambda.$$

We define

$$\Lambda_0 = \{\lambda > a : u \leq u_t \text{ in } \Omega_t \text{ for all } t \in (a, \lambda]\} \tag{2.21}$$

and

$$\lambda_0 = \sup \Lambda_0. \tag{2.22}$$

Notice that by the continuity of the solution u we obtain $u \leq u_{\lambda_0}$ in Ω_{λ_0}. To prove our theorem, we have to show that

$$\lambda_0 = 0.$$

Assume by contradiction $\lambda_0 < 0$. We can exploit the strong maximum (or comparison) principle for the laplacian operator, to get that

$$u < u_{\lambda_0} \qquad \text{or} \qquad u \equiv u_{\lambda_0}$$

in Ω_{λ_0}. It follows now that the case $u \equiv u_{\lambda_0}$ in Ω_{λ_0} is not possible, since the Dirichlet condition would imply the existence of some point $x \in \Omega$ such that $u(x) = 0$. This is a contradiction with $(\mathcal{P})$, in particular with the assumption $u > 0$. Thus $u < u_{\lambda_0}$ in $\Omega_{\lambda_0} \setminus \{0_{\lambda_0}\}$ (let us observe that u_{λ_0} may not defined in Ω_{λ_0}).

We point out that, because $0 \in \overline{\Omega}$, in general we have that the reflected point 0_{λ_0} may belong to Ω_{λ_0}. There, u_{λ_0} is not smooth. Since the domain is strictly convex in the x_1-direction, by Hopf's boundary lemma and the Dirichlet condition, we get that there exists a neighborhood $\mathcal{I}_{\lambda_0}$ of

$$(\partial\Omega_{\lambda_0} \setminus T_\lambda) \subseteq \partial\Omega$$

such that

$$u < u_{\lambda_0} \text{ in } \mathcal{I}_{\lambda_0} \setminus B_\delta(0_{\lambda_0}),$$

for some positive δ. In particular: for $x \in \mathcal{I}_{\lambda_0} \setminus B_\delta(0_{\lambda_0})$ far from $\partial\Omega_{\lambda_0} \cap T_{\lambda_0}$ we exploit the uniform continuity of the solution and the zero Dirichlet boundary condition; on the other hand, in a neighborhood of $\partial\Omega_{\lambda_0} \cap T_{\lambda_0}$ we exploit the Hopf's boundary lemma since by our assumption, the domain is smooth and strictly convex.

Therefore we deduce that there exists a compact set K in Ω_{λ_0} such that $K \cap B_\delta(0_{\lambda_0}) = \emptyset$ and

$$|\Omega_{\lambda_0} \setminus (K \cup (\mathcal{I}_{\lambda_0} \setminus B_\delta(0_{\lambda_0})))| \tag{2.23}$$

is sufficiently small (eventually reducing δ) so that $u_{\lambda_0} - u$ is positive in $K \cup (\mathcal{I}_{\lambda_0} \setminus B_\delta(0_{\lambda_0})))$ and Proposition 2.1 applies in the set $\Omega_{\lambda_0} \setminus (K \cup (\mathcal{I}_{\lambda_0} \setminus B_\delta(0_{\lambda_0})))$. We point out that, without loss of generality, we can suppose $0_{\lambda_0} \in \partial\Omega$. Actually in the case $0_{\lambda_0} \in \Omega$ we can choose the neighborhood $\mathcal{I}_{\lambda_0}$ and δ such that $\mathcal{I}_{\lambda_0} \cap B_\delta(0_{\lambda_0}) = \emptyset$ and (2.23) reduces to $|\Omega_{\lambda_0} \setminus (K \cup \mathcal{I}_{\lambda_0})|$.

Arguing by continuity, we also have $u_{\lambda_0} - u > 0$ on $\partial(K \cup (\mathcal{I}_{\lambda_0} \setminus B_\delta(0_{\lambda_0})))$. Hence it follows $u \le u_{\lambda_0+\varepsilon}$ on $\partial(K \cup (\mathcal{I}_{\lambda_0} \setminus B_\delta(0_{\lambda_0})))$, for sufficiently small ε. Using Proposition 2.1 we obtain that

$$u \le u_{\lambda_0+\varepsilon} \quad \text{in } \Omega_{\lambda_0+\varepsilon} \setminus (K \cup (\mathcal{I}_{\lambda_0+\varepsilon} \setminus B_\delta(0_{\lambda_0})))|. \tag{2.24}$$

To get the desired contradiction, it remains to show that

$$u \le u_{\lambda_0+\varepsilon} \quad \text{in } B_\delta(0_{\lambda_0}). \tag{2.25}$$

To do this let us consider the ball $B_r = B_r(0)$ for r small such that $B_r \subset\subset \Omega$. Since u is positive in Ω and (1.3), we infer that $m = m(r) := \min_{x \in \partial B_r} u(x) > 0$. We claim that there exist r such that

$$u(x) \ge m(r) > 0 \quad \text{in } \overline{B}_r. \tag{2.26}$$

Arguing by contradiction, let us define $\varphi = (u - m)^-$ if $x \in B_r$ and $\varphi = 0$ elsewhere. Clearly $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$. Therefore using the weak formulation of $(\mathcal{P})$ we have

$$\int_{B_r} |\nabla(u-m)^-|^2 \, dx \le -k \int_{B_r} |\nabla(u-m)^-|^q (u-m)^- \, dx$$

For $1 \le q < 2$ (by weighted Young inequality) we obtain

$$\begin{aligned}
&\int_{B_r} |\nabla(u-m)^-|^2 \, dx \le \varepsilon C(k) \int_{B_r} |\nabla(u-m)^-|^2 \, dx + C(k,\varepsilon) \int_{B_r} [(u-m)^-]^{\frac{2}{2-q}} \, dx \\
&\le \varepsilon C(k) \int_{B_r} |\nabla(u-m)^-|^2 \, dx + m^{\frac{2q-2}{2-q}} C(k,\varepsilon) \int_{B_r} [(u-m)^-]^2 \, dx \\
&\le \varepsilon C(k) \int_{B_r} |\nabla(u-m)^-|^2 \, dx + C(k,q,\varepsilon) \int_{B_r} [(u-m)^-]^2 \, dx,
\end{aligned}$$

where $C(k,q,\varepsilon)$ is some constant that does not depend on r since $0 < \min\limits_{0<\varrho\le r} \le m(r)$ (recall also that $(2q-2)/(2-q) \ge 0$ since $1 \le q < 2$). For ε small enough we get

$$\int_{B_r} |\nabla(u-m)^-|^2 \, dx \le C(k,q) \int_{B_r} [(u-m)^-]^2 \, dx.$$

with $(2q-2)/(2-q) \geq 0$ since $1 \leq q < 2$. Using Poincaré inequality in the right hand side we obtain

$$\int_{B_r} |\nabla(u-m)^-|^2 \, dx \leq C_P^2(|B_r|)C(k,q) \int_{B_r} [\nabla(u-m)^-]^2 \, dx,$$

Therefore for r small we have that actually $(u-m)^- = 0$ in B_r. This proves (2.26) for the case $1 \leq q < 2$.

For $q = 2$ we take $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$ such that $\varphi = e^{-ku}(u-m)^-$ if $x \in B_r$ and $\varphi = 0$ elsewhere, as test function in the weak formulation of $(\mathcal{P})$. Then

$$\int_{B_r} e^{-ku} |\nabla(u-m)^-|^2 \, dx \leq 0,$$

that is (2.26) for the case $q = 2$.

Thanks to (2.26) (see also (2.1)), we deduce as well that $u_{\lambda_0}(x) \geq m > 0$ in $B_\delta(0_{\lambda_0})$, for $0 < \delta \leq r/2$. Hence using the boundary Dirichlet condition and the continuity of u in $B_r(0_{\lambda_0})$, reducing δ if it is necessary, we deduce (2.25) for ε small.

Consequently from (2.24) and (2.25) we have that $u \leq u_{\lambda_0+\varepsilon}$ in $\Omega_{\lambda_0+\varepsilon}$. This contradicts the assumption $\lambda_0 < 0$. Therefore, $\lambda_0 = 0$. We point out that we are exploiting Proposition 2.1 in the set $\Omega_{\lambda_0+\varepsilon} \setminus (K \cup (\mathcal{I}_{\lambda_0+\varepsilon} \setminus B_\delta(0_{\lambda_0}))$ which is bounded away from the boundary $\partial\Omega$ and then the constant $\hat{\delta}$ in the statement is uniformly bounded.

In the same way, performing the moving plane method in the opposite direction, namely $-x_1$, we obtain

$$u(x) \geq u_\lambda \quad \text{for } x \in \Omega_0,$$

that is, u is symmetric. Moreover, it is implicit in the moving plane procedure the fact that the solution is increasing in the x_1-direction in $\{x_1 < 0\}$. Since (see (1.3)) u is C^1 far away the origin $0 \in \partial\Omega$, using the monotonicity of the solution u wen readily get (1.6). □

Proof of Corollary 1.4. If Ω is a ball, applying Theorem 1.5 along any direction, it follows that u s radially symmetric. The fact that $u_r < 0$ for $r \neq 0$, follows by the Hopf's boundary lemma which works in this case since the level sets are balls and therefore fulfill the interior sphere condition. □

Proof of Theorem 1.5. Since the origin $0 \in \overline{\Omega}$ is contained in the hyperplane $\{x_1 = 0\}$, then the moving plane procedure can be started in the standard way and, for $a < \lambda < a + \sigma$ with $\sigma > 0$ small, we have that $u - u_\lambda \leq 0$ in Ω_λ, by Proposition 2.2. In this case the standard weak comparison principle holds since the right hand side (far away to zero) is locally Lipschitz continuous. Moreover note that u, u_λ are smooth far from zero. Therefore for λ close to a the singularity at zero coming from the Hardy potential does not play a role. To proceed further we define as we did above

$$\Lambda_0 = \{\lambda > a : u \leq u_t \text{ in } \Omega_t \text{ for all } t \in (a, \lambda],\}$$

that is not empty for λ close to a, and $\lambda_0 = \sup \Lambda_0$. Assuming by contradiction that $\lambda_0 < 0$, we have by the strong maximum principle that $u < u_{\lambda_0}$ in Ω_{λ_0}. Therefore exploiting the fact that $u < u_{\lambda_0}$ in $\Omega_{\lambda_0} \setminus \{0_{\lambda_0}\}$ and the fact that the solution u is continuous in $\overline{\Omega} \setminus \{0\}$ (resp. u_{λ_0} is continuous in $\overline{\Omega} \setminus \{0_{\lambda_0}\}$), we deduce that there exist a compact set K such that $\Omega_{\lambda_0} \setminus K$ is sufficiently small in order to apply the weak comparison principle, Proposition 2.2. The rest of the proof is standard (follow the proof of Theorem 1.3). Moreover, it is implicit in the moving plane procedure the fact that the solution is increasing in the x_1-direction in $\{x_1 < 0\}$. Since (see (1.3)) u is C^1 far from the origin, using the monotonicity of the solution u we get that $u_{x_1} \geq 0$ in $\Omega \cap \{x_1 < 0\}$. The fact that u_{x_1} is positive for $x_1 < 0$ (see (1.6)), follows by the maximum principle for u_{x_1} that applies in this case and by the Hopf's boundary lemma. □

Aknowledgments

The authors would like to thank the anonymous referee for his/her useful suggestions and comments. L. Montoro and B. Sciunzi are partially supported by PRIN project 2017JPCAPN (Italy): Qualitative and quantitative aspects of nonlinear PDEs, and L. Montoro by Agencia Estatal de Investigación (Spain), project PDI2019-110712GB-100.

Conflict of interest

The authors declare no conflict of interest.

References

1. A. D. Alexandrov, A characteristic property of the spheres, *Annali di Matematica*, **58** (1962), 303–315. http://dx.doi.org/10.1007/BF02413056
2. B. Abdellaoui, L. Boccardo, I. Peral, A. Primo, Quasilinear elliptic equations with natural growth, *Differ. Integral Equ.*, **20**, (2007), 1005–1020.
3. B. Abdellaoui, I. Peral, Some results for semilinear elliptic equations with critical potential, *Proc. Roy. Soc. Edinb. A*, **132** (2002), 2002, 1–24. http://dx.doi.org/10.1017/S0308210500001505
4. B. Abdellaoui, I. Peral, A. Primo, Elliptic problems with a Hardy potential and critical growth in the gradient: non-resonance and blow-up results, *J. Differ. Equations*, **239** (2007), 386–416. http://dx.doi.org/10.1016/j.jde.2007.05.010
5. B. Abdellaoui, I. Peral, A. Primo, Breaking of resonance and regularizing effect of a first order quasi-linear term in some elliptic equations, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **25** (2008), 969–985. http://dx.doi.org/10.1016/j.anihpc.2007.06.003
6. B. Abdellaoui, I. Peral, A. Primo, Influence of the Hardy potential in a semilinear heat equation, *Proc. Roy. Soc. Edinb. A*, **139** (2009), 897–926. http://dx.doi.org/10.1017/S0308210508000152
7. B. Abdellaoui, I. Peral, A. Primo, Strong regularizing effect of a gradient term in the heat equation with the Hardy potential, *J. Funct. Anal.*, **258** (2010), 1247–1272. http://dx.doi.org/10.1016/j.jfa.2009.11.008
8. B. Abdellaoui, I. Peral, A. Primo, Optimal results for parabolic problems arising in some physical models with critical growth in the gradient respect to a Hardy potential, *Adv. Math.*, **225** (2010), 2967–3021. http://dx.doi.org/10.1016/j.aim.2010.04.028
9. A. Ambrosetti, P. H. Rabinowitz, Dual variational methods in critical point theory and applications, *J. Funct. Anal.*, **14** (1973), 349–381. http://dx.doi.org/10.1016/0022-1236(73)90051-7

10. B. Barrios, L. Montoro, B. Sciunzi, On the moving plane method for nonlocal problems in bounded domains, *JAMA*, **135** (2018), 37–57. http://dx.doi.org/10.1007/s11854-018-0031-1

11. H. Berestycki, L. Nirenberg, On the method of moving planes and the sliding method, *Bol. Soc. Bras. Mat.*, **22** (1991), 1–37. http://dx.doi.org/10.1007/BF01244896

12. L. Boccardo, L. Orsina, I. Peral, A remark on existence and optimal summability of solutions of elliptic problems involving Hardy potential, *Discrete Contin. Dyn. Syst.*, **16**, (2006), 513–523. http://dx.doi.org/10.3934/dcds.2006.16.513

13. H. Brezis, X. Cabré, Some simple nonlinear PDE's without solutions, *Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. (8)*, **1** (1998), 223–262.

14. A. Canino, L. Montoro, B. Sciunzi, The moving plane method for singular semilinear elliptic problems, *Nonlinear Anal.*, **156** (2017), 61–69. http://dx.doi.org/10.1016/j.na.2017.02.009

15. A. Canino, F. Esposito, B. Sciunzi, On the Höpf boundary lemma for singular semilinear elliptic equations, *J. Differ. Equations*, **266** (2019), 5488–5499. http://dx.doi.org/10.1016/j.jde.2018.10.039

16. C. Cazacu, On Hardy inequalities with singularities on the boundary, *C. R. Math.*, **349** (2011), 273–277. http://dx.doi.org/10.1016/j.crma.2011.02.005

17. C. Cazacu, Schrödinger operators with boundary singularities: Hardy inequality, Pohozaev identity and controllability results, *J. Funct. Anal.*, **263** (2012), 3741–3783. http://dx.doi.org/10.1016/j.jfa.2012.09.006

18. J. Dávila, I. Peral, Nonlinear elliptic problems with a singular weight on the boundary, *Calc. Var.*, **41** (2011), 567–586. http://dx.doi.org/10.1007/s00526-010-0376-5

19. S. Dipierro, L. Montoro, I. Peral, B. Sciunzi, Qualitative properties of positive solutions to nonlocal critical problems involving the Hardy-Leray potential, *Calc. Var.*, **55** (2016), 99. http://dx.doi.org/10.1007/s00526-016-1032-5

20. H. Egnell, Positive solutions of semilinear equations in cones, *Trans. Amer. Math. Soc.*, **330** (1992), 191–201. http://dx.doi.org/10.1090/S0002-9947-1992-1034662-5

21. F. Esposito, Symmetry and monotonicity properties of singular solutions to some cooperative semilinear elliptic systems involving critical nonlinearity, *Discrete Contin. Dyn. Syst.*, **40** (2020), 549–577. http://dx.doi.org/10.3934/dcds.2020022

22. F. Esposito, A. Farina, B. Sciunzi, Qualitative properties of singular solutions to semilinear elliptic problems, *J. Differ. Equations*, **265** (2018), 1962–1983. http://dx.doi.org/10.1016/j.jde.2018.04.030

23. F. Esposito, L. Montoro, B. Sciunzi, Monotonicity and symmetry of singular solutions to quasilinear problems, *J. Math. Pure Appl.*, **126** (2019), 214–231. http://dx.doi.org/10.1016/j.matpur.2018.09.005

24. F. Esposito, B. Sciunzi, On the Höpf boundary lemma for quasilinear problems involving singular nonlinearities and applications, *J. Funct. Anal.*, **278** (2020), 108346. http://dx.doi.org/10.1016/j.jfa.2019.108346

25. M. M. Fall, R. Musina, Hardy-Poincaré inequalities with boundary singularities, *Proc. Roy. Soc. Edinb. A*, **142** (2012), 769–786. http://dx.doi.org/10.1017/S0308210510000740

26. B. Gidas, W. M. Ni, L. Nirenberg, Symmetry and related properties via the maximum principle, *Commun. Math. Phys.*, **68** (1979), 209–243. http://dx.doi.org/10.1007/BF01221125

27. N. Ghoussoub, X. S. Kang, Hardy-Sobolev critical elliptic equations with boundary singularities, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **21** (2004), 767–793. http://dx.doi.org/10.1016/j.anihpc.2003.07.002

28. N. Ghoussoub, F. Robert, The effect of curvature on the best constant in the Hardy-Sobolev inequalities, *Geom. Funct. Anal.*, **16** (2006), 1201–1245. http://dx.doi.org/10.1007/s00039-006-0579-2

29. S. Merchán, L. Montoro, Remarks on the existence of solutions to some quasilinear elliptic problems involving the Hardy-Leray potential, *Annali di Matematica*, **193** (2014), 609–632. http://dx.doi.org/10.1007/s10231-012-0293-7

30. L. Montoro, Harnack inequalities and qualitative properties for some quasilinear elliptic equations, *Nonlinear Differ. Equ. Appl.*, **26** (2019), 45. http://dx.doi.org/10.1007/s00030-019-0591-5

31. S. Merchán, L. Montoro, I. Peral, B. Sciunzi, Existence and qualitative properties of solutions to a quasilinear elliptic equation involving the Hardy-Leray potential, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **31** (2014), 1–22. http://dx.doi.org/10.1016/j.anihpc.2013.01.003

32. L. Montoro, F. Punzo, B. Sciunzi, Qualitative properties of singular solutions to nonlocal problems, *Annali di Matematica*, **197** (2018), 941–964. http://dx.doi.org/10.1007/s10231-017-0710-z

33. B. Sciunzi, On the moving Plane Method for singular solutions to semilinear elliptic equations, *J. Math. Pure. Appl.*, **108** (2017), 111–123. http://dx.doi.org/10.1016/j.matpur.2016.10.012

34. J. Serrin, A symmetry problem in potential theory, *Arch. Rational Mech. Anal.*, **43** (1971), 304–318. http://dx.doi.org/10.1007/BF00250468

AIMS Press