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Research article

On an asymptotically log-periodic solution to the graphical curve shortening flow equation†

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Abstract: With the help of heat equation, we first construct an example of a graphical solution to the curve shortening flow. This solution $y(x,t)$ has the interesting property that it converges to a log-periodic function of the form

$$A\sin(\log t)+B\cos(\log t)$$

as $t\to\infty$, where $A,\ B$ are constants. Moreover, for any two numbers $\alpha<\beta$, we are also able to construct a solution satisfying the oscillation limits

$$\liminf_{t\to\infty} y(x,t)=\alpha,\quad \limsup_{t\to\infty} y(x,t)=\beta,\quad x\in K$$

on any compact subset $K\subset\mathbb{R}$.

Keywords: curve shortening flow; heat equation; geometric heat equation; log-periodic function; prescribing oscillation values

1. Introduction

The goal of this paper is to make a comparison between the geometric heat equation (graphical curve shortening flow equation) and the Euclidean heat equation (the usual heat equation on $\mathbb{R}$). Based on a nice result due to Nara-Taniguchi [9] (see Theorem 1.1), which says that under some suitable assumption on the initial condition $u_0(x)$, $x\in\mathbb{R}$, the solutions of both equations give rise to the same asymptotic behavior, we can construct a solution $y(x,t)$ of the graphical curve shortening flow equation so that it is asymptotically log-periodic (i.e., functions which are periodic in $\log t$) as $t \to \infty$. Moreover, for any two numbers $\alpha < \beta$, we can construct another solution $y(x,t)$ prescribing the oscillation behavior $\liminf_{t\to\infty} y(x,t) = \alpha$, $\limsup_{t\to\infty} y(x,t) = \beta$ on any compact subset $x \in K \subset \mathbb{R}$ (see Theorem 2.1, Theorem 2.7, Corollary 2.9).

1.1. The geometric heat equation and the Euclidean heat equation

The curve shortening flow (CSF) is a geometric heat equation which can be used to deform curves in the plane. This flow arises naturally in phase transitions and plays an important role in the thermomechanics of evolving phase boundaries in the plane; see [8]. Mathematically, CSF is also an interesting topic on its own with fundamental importance. Since 1986, it has been studied thoroughly by Gage-Hamilton [7], Grayson [6], Angenent [1] and many others. See the bibliography in the book by Chou-Zhu [3] for literature.

A family of smooth embedded curves $\gamma(\varphi,t) : I \times [0,T) \to \mathbb{R}^2$ ($I \subset \mathbb{R}$ is some interval) is said to evolve under CSF if it satisfies the equation

$$\frac{\partial \gamma}{\partial t}(\varphi,t) = \kappa(\varphi,t)\,\mathbf{N}(\varphi,t), \quad (\varphi,t) \in I \times [0,T), \quad (\mathbf{CSF}), \tag{1.1}$$

where $\kappa(\varphi,t)$ is the curvature of $\gamma(\varphi,t)$ and $\mathbf{N}(\varphi,t)$ is its unit normal vector. We use the convention that for a parametrized curve $\gamma \subset \mathbb{R}^2$ its Frenet frame $\{\mathbf{T},\mathbf{N}\}$, $\mathbf{T} = d\gamma/ds$, has positive orientation and its curvature κ is defined according to the identity $\kappa = \langle d\mathbf{T}/ds, \mathbf{N}\rangle$, where s is the arc length parameter of γ. By the identity $d\mathbf{T}/ds = \kappa\mathbf{N}$, one can write (1.1) as

$$\frac{\partial \gamma}{\partial t} = \kappa\mathbf{N} = \frac{\partial \mathbf{T}}{\partial s} = \frac{\partial^2 \gamma}{\partial s^2}. \tag{1.2}$$

Since Eq (1.2) resembles the familiar classical heat equation $u_t(x,t) = u_{xx}(x,t)$ and the second derivative operator $\partial^2/\partial s^2$ involves the geometry of the evolving curve $\gamma(\cdot,t)$, we call it a "geometric heat equation". As a contrast, the classical heat equation $u_t = u_{xx}$ is called an "Euclidean heat equation".

Denote the tangent angle of the curve $\gamma(\varphi,t)$ as $\theta(\varphi,t)$. Then we have

$$\mathbf{T}(\varphi,t) = (\cos\theta(\varphi,t), \sin\theta(\varphi,t)), \quad \mathbf{N}(\varphi,t) = (-\sin\theta(\varphi,t), \cos\theta(\varphi,t)) \tag{1.3}$$

and Eq (1.1) has the form (in the following, "angle" means "tangent angle")

$$\frac{\partial \gamma}{\partial t} = \kappa\mathbf{N} = \frac{\partial \theta}{\partial s}\mathbf{N} = \frac{\partial\,(\text{angle})}{\partial s}\mathbf{N}, \quad \kappa = \frac{\partial \theta}{\partial s}, \quad (\mathbf{CSF}). \tag{1.4}$$

It is also known that if $\gamma(\varphi,t) \subset \mathbb{R}^2$ is a family of graphical curves* evolving under CSF and is represented by the graphs of some function $y(x,t)$ over some interval J of the x-axis, then on its domain (in general the interval J on which $y(x,t)$ is defined may also depend on time) the function $y(x,t)$ will satisfy the equation

$$y_t(x,t) = \frac{y_{xx}(x,t)}{1 + y_x^2(x,t)} = \frac{\partial}{\partial x}\left(\tan^{-1}(y_x(x,t))\right) = \frac{\partial\,(\text{angle})}{\partial x}, \quad x \in J, \quad t > 0, \quad (\mathbf{CSF}), \tag{1.5}$$

*Our convention on the orientation of a graphical curve γ is that its tangent vector is pointing in the positive direction of x-axis.

where $y_x^2(x,t)$ means $(y_x(x,t))^2$. Note that the right hand side of (1.5) is the derivative of the tangent angle $\theta(x,t)$ of the graph (we have $\tan\theta(x,t) = y_x(x,t)$). In terms of the curvature $\kappa = y_{xx}\left(1+y_x^2\right)^{-3/2}$ for a graph, (1.5) can also be written as

$$y_t(x,t) = \sqrt{1+y_x^2(x,t)}\kappa(x,t), \quad (\mathbf{CSF}), \tag{1.6}$$

where $\kappa(x,t)$ is the curvature of the graph at the point $(x, y(x,t))$. By (1.5), if $y(x,t)$ is a solution of (1.5), so are $-y(x,t)$, $y(-x,t)$ and $\lambda y\left(\lambda^{-1}x, \lambda^{-2}t\right)$ (for any constant $\lambda > 0$).

On the other hand, for the Euclidean heat equation $u_t = u_{xx}$, $x \in J$, we can interpret it geometrically as

$$u_t(x,t) = u_{xx}(x,t) = \frac{\partial}{\partial x}(u_x(x,t)) = \frac{\partial \tan\theta}{\partial x} = \frac{\partial(\text{slope})}{\partial x}, \quad x \in J, \quad t > 0 \tag{1.7}$$

and similar to (1.6), we have

$$u_t(x,t) = u_{xx}(x,t) = \left(\sqrt{1+u_x^2(x,t)}\right)^3 \kappa = \sqrt{1+u_x^2(x,t)}\left(\sec^2\theta\right)\kappa. \tag{1.8}$$

Similar to the equivalent relation between the flow Eq (1.1) and the graph Eq (1.6), one can verify that if $\gamma(\varphi,t) \subset \mathbb{R}^2$ is a family of graphical curves (represented by the graph of some function $u(x,t)$ over some interval J of the x-axis) evolving under the anisotropic curve shortening flow (ACSF) of the form

$$\frac{\partial\gamma}{\partial t}(\varphi,t) = \left(\sec^2\theta(\varphi,t)\right)\kappa(\varphi,t)\mathbf{N}(\varphi,t), \quad (\varphi,t) \in I \times [0,T), \quad (\mathbf{ACSF}), \tag{1.9}$$

where $\theta(\varphi,t)$ is the tangent angle of $\gamma(\varphi,t)$, then the graph function $u(x,t)$, $x \in J$, will satisfy the equation (1.7). Moreover, the converse is also true.

Note that the flow (1.9) is defined only for graphical curves with angle $\theta \in (-\pi/2, \pi/2)$. On the other hand, the classical curve shortening flow (1.1) can be defined for rather general curves, say simple closed curves in the plane. However, here we are confining the curve shortening flow to graphs and Eq (1.5) is not defined at points where the gradient blows up or equivalently the angle is vertical. Similar to (1.4), one can rewrite (1.9) as

$$\frac{\partial\gamma}{\partial t} = \frac{\partial(\tan\theta)}{\partial s}\mathbf{N} = \frac{\partial(\text{slope})}{\partial s}\mathbf{N}, \quad (\mathbf{ACSF}). \tag{1.10}$$

1.2. The result of Nara-Taniguchi

A major important connection between the two Eqs (1.5) and (1.7) (or between the two flows (1.1) and (1.9)) is the following equivalence result due to Nara-Taniguchi [9]:

Theorem 1.1. *([9]) Let $\gamma \in (0,1)$ be a constant. Assume that $y_0(x)$, $x \in \mathbb{R}$, is a bounded function lying in the space $C^{2+\gamma}(\mathbb{R})$. Then there exists a classical solution $y(x,t)$ to Eq (1.5) (with $y(x,0) = y_0(x)$, $x \in \mathbb{R}$) on the domain $\mathbb{R} \times [0,\infty)$. Moreover the solution $y(x,t)$ is smooth on $\mathbb{R} \times (0,\infty)$, continuous on $\mathbb{R} \times [0,\infty)$, and satisfies*

$$\sup_{x\in\mathbb{R}}\left|y(x,t) - \int_{\mathbb{R}}\Gamma(x-\xi,t)y_0(\xi)\,d\xi\right| \le \frac{C}{\sqrt{t}}, \quad t > 0 \tag{1.11}$$

for some constant $C > 0$ depending only on y_0. Here $\Gamma(\xi,t)$ is the heat kernel given by $\Gamma(\xi,t) = \left(1/\sqrt{4\pi t}\right)\exp\left(-\xi^2/(4t)\right)$.

In the following we give an interesting example of bounded $y_0(x)$, lying in the space $C^{2+\gamma}(\mathbb{R})$, which gives explicit $y(x,t)$. Therefore, we can check property (1.11) directly.

Example 1.2. *Let*

$$y_0(x) = \sinh^{-1}(\sin x) = \log\left(\sin x + \sqrt{\sin^2 x + 1}\right), \quad x \in \mathbb{R}, \tag{1.12}$$

which is an odd 2π-periodic smooth function on $\mathbb{R}$. Since there exists a constant $M > 0$ such that

$$\left|y_0(x),\ y_0'(x),\ y_0''(x),\ y_0'''(x)\right| \le M, \quad \forall\, x \in \mathbb{R}, \tag{1.13}$$

we have $y_0(x) \in C^{2+\gamma}(\mathbb{R})$ for any $\gamma \in (0,1)$. Under the evolution of Eq (1.5), one can verify that the solution $y(x,t)$ is given explicitly by

$$y(x,t) = \sinh^{-1}\left(e^{-t}\sin x\right) = \log\left[\left(e^{-t}\sin x\right) + \sqrt{\left(e^{-t}\sin x\right)^2 + 1}\right], \quad x \in \mathbb{R}, \quad t \in [0,\infty), \tag{1.14}$$

which decays to 0 uniformly on $x \in \mathbb{R}$ with exponential rate e^{-t} as $t \to \infty$. On the other hand, since $y_0(x)$ is an odd 2π-periodic smooth function on $\mathbb{R}$, Fourier series theory also implies that the convolution solution $\int_{\mathbb{R}} \Gamma(x-\xi,t)\, y_0(\xi)\, d\xi$ converges uniformly to the constant $\frac{1}{2\pi}\int_{-\pi}^{\pi} y_0(x)\, dx = 0$ uniformly on $x \in \mathbb{R}$ with exponential rate e^{-t} as $t \to \infty$. Hence (1.11) is verified.

Remark 1.3. *The solution (1.14) is known as the hairclip solution of the curve shortening flow. It is a graphical entire solution defined on $x \in \mathbb{R}$. It has been derived in Broadbridge-Vassiliou [2] and Doyle-Vassiliou [5] using symmetry and separation method. Also see Tsai-Wang [10] for a simple ODE method to derive it.*

2. The main result of the paper: an example converging to log-periodic function asymptotically

With the help of Theorem 1.1, one can find some interesting solution $y(x,t)$ of the graphical curve shortening flow. The upshot is that, as $t \to \infty$, the solution $y(x,t)$ approaches a non-constant bounded function $Y(t)$ which satisfies $Y(\lambda t) = Y(t)$ for some constant $\lambda > 1$ for all $t \in (0,\infty)$. The function $Y(t)$ will be periodic in $\log t$.

The theorem below is motivated by the initial condition (1.12) in the hairclip solution in Example 1.2. Roughly speaking, we switch the role of sine function and logarithmic function, and in order for $\log x$ to be defined on $(-\infty,\infty)$, we change it as $\log\left(x^2+1\right)$. More precisely, we have:

Theorem 2.1. *Let $y_0(x)$ be given by*

$$y_0(x) = \sin\left(\log\left(x^2+1\right)\right), \quad x \in (-\infty,\infty) \tag{2.1}$$

and consider Eq (1.5) with the above initial condition. Then the solution $y(x,t)$ of this Cauchy problem is defined on $\mathbb{R}\times[0,\infty)$, which is smooth on $\mathbb{R}\times(0,\infty)$ and continuous up to $t = 0$. Moreover, $y(x,t)$ satisfies the following two properties:

(1). (Asymptotic behavior as $t \to \infty$.) There exist two positive constants A, B such that

$$\lim_{t\to\infty}\sup_{x\in K}\left|y(x,t) - \left(A\sin(\log t) + B\cos(\log t)\right)\right| = 0 \tag{2.2}$$

for any compact set $K \subset \mathbb{R}$.

(2). (Asymptotic behavior as $|x| \to \infty$.) For fixed $t > 0$ we have

$$\limsup_{|x|\to\infty} \left| y(x,t) - \sin\left(\log\left(x^2+1\right)\right)\right| \le \frac{C}{\sqrt{t}} \tag{2.3}$$

for some constant $C > 0$ depending only on y_0.

Remark 2.2. *The limit function $Y(t) := A\sin(\log t) + B\cos(\log t)$, $t \in (0,\infty)$, is not periodic in time $t \in (0,\infty)$. Instead, it satisfies $Y\left(e^{2\pi}t\right) = Y(t)$ for all $t \in (0,\infty)$. One can see that a function $Y(t)$ defined on $(0,\infty)$ satisfying the identity $Y\left(e^{2\pi}t\right) = Y(t)$ for all $t \in (0,\infty)$ if and only if the function $F(s) = Y(e^s)$ is defined on $(-\infty,\infty)$ and satisfies the identity $F(s+2\pi) = F(s)$ for all $s \in (-\infty,\infty)$. Since we have $Y(t) = F(\log t)$, $t \in (0,\infty)$, we can say that $Y(t)$ is 2π-periodic in $\log t$, or for simplicity, just log-periodic.*

Proof. (1). We first note that $y_0(x) \in C^{2+\gamma}(\mathbb{R})$ for any $\gamma \in (0,1)$ since there exists a constant $M > 0$ such that (1.13) holds for $y_0(x)$. Hence Theorem 1.1 is applicable and to look at the asymptotic behavior of $y(x,t)$, it suffices to look at that of $y_0(x)$ under the heat equation $u_t = u_{xx}$. We have for $t > 0$ that

$$\begin{aligned} u(0,t) &= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{4t}} y_0(x)\,dx = \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{4}} y_0\left(\sqrt{t}z\right)dz \\ &= \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{4}} \sin\left[\log t + \log\left(z^2 + \frac{1}{t}\right)\right] dz = A(t)\sin(\log t) + B(t)\cos(\log t), \end{aligned} \tag{2.4}$$

where

$$\begin{cases} A(t) = \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{4}} \cos\left[\log\left(z^2 + \frac{1}{t}\right)\right] dz, \\ B(t) = \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{4}} \sin\left[\log\left(z^2 + \frac{1}{t}\right)\right] dz, \end{cases} \tag{2.5}$$

and we note that

$$\begin{cases} \lim_{t\to\infty} A(t) = \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{4}} \cos\left(\log\left(z^2\right)\right) dz = A \approx 0.26682 \\ \lim_{t\to\infty} B(t) = \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{4}} \sin\left(\log\left(z^2\right)\right) dz = B \approx 0.12278. \end{cases} \tag{2.6}$$

Thus we conclude

$$\lim_{t\to\infty} \left| u(0,t) - (A\sin(\log t) + B\cos(\log t))\right| = 0, \tag{2.7}$$

where the constants A, B are from (2.6).

Since $|y_0(x)| \le 1$ for all $x \in \mathbb{R}$, we have the following gradient estimate for the convolution solution of the heat equation with initial data $y_0(x)$:

$$|u_x(x,t)| \le \frac{1}{\sqrt{\pi t}}, \quad \forall\ (x,t) \in \mathbb{R} \times (0,\infty), \tag{2.8}$$

which, for each fixed x, implies

$$\lim_{t\to\infty} \left| u(x,t) - (A\sin(\log t) + B\cos(\log t))\right|$$

$$\leq \lim_{t\to\infty}\left\{|u(x,t)-u(0,t)|+\left|u(0,t)-(A\sin(\log t)+B\cos(\log t))\right|\right\}=0. \tag{2.9}$$

Thus for each fixed x, $u(x,t)$ also approaches to the same function as $t \to \infty$. Moreover, the above convergence is uniform in $x \in K$ for any compact set $K \subset \mathbb{R}$. This fact, together with Theorem 1.1, gives the proof of (2.2).

(2). By (2.12) below and letting $x = \sqrt{t}y$, we have

$$u(x,t) = u\left(\sqrt{t}y, t\right) = A(y,t)\sin(\log t) + B(y,t)\cos(\log t), \tag{2.10}$$

where now

$$\begin{cases} A(y,t) = \frac{1}{\sqrt{4\pi}}\int_{-\infty}^{\infty} e^{-\frac{z^2}{4}}\cos\left(\log\left((z+y)^2+\frac{1}{t}\right)\right)dz \\ B(y,t) = \frac{1}{\sqrt{4\pi}}\int_{-\infty}^{\infty} e^{-\frac{z^2}{4}}\sin\left(\log\left((z+y)^2+\frac{1}{t}\right)\right)dz, \quad y = \frac{x}{\sqrt{t}}. \end{cases}$$

For fixed $z \in (-\infty,\infty)$ and fixed $t > 0$, we have

$$\begin{cases} \lim_{|y|\to\infty}\left|e^{-\frac{z^2}{4}}\cos\left(\log\left((z+y)^2+\frac{1}{t}\right)\right) - e^{-\frac{z^2}{4}}\cos\left(\log\left(y^2+\frac{1}{t}\right)\right)\right| = 0 \\ \lim_{|y|\to\infty}\left|e^{-\frac{z^2}{4}}\sin\left(\log\left((z+y)^2+\frac{1}{t}\right)\right) - e^{-\frac{z^2}{4}}\sin\left(\log\left(y^2+\frac{1}{t}\right)\right)\right| = 0 \end{cases}$$

and the Lebesgue Dominated Convergence Theorem implies, for fixed $t > 0$, the limits

$$\begin{cases} \lim_{|y|\to\infty}\left|A(y,t) - \frac{1}{\sqrt{4\pi}}\int_{-\infty}^{\infty} e^{-\frac{z^2}{4}}\cos\left(\log\left(y^2+\frac{1}{t}\right)\right)dz\right| = 0 \\ \lim_{|y|\to\infty}\left|B(y,t) - \frac{1}{\sqrt{4\pi}}\int_{-\infty}^{\infty} e^{-\frac{z^2}{4}}\sin\left(\log\left(y^2+\frac{1}{t}\right)\right)dz\right| = 0. \end{cases}$$

Since $\left(\sqrt{4\pi}\right)^{-1}\int_{-\infty}^{\infty} e^{-z^2/4}dz = 1$, we conclude

$$\begin{cases} \lim_{|y|\to\infty}\left|A(y,t) - \cos\left(\log\left(y^2+\frac{1}{t}\right)\right)\right| = 0 \\ \lim_{|y|\to\infty}\left|B(y,t) - \sin\left(\log\left(y^2+\frac{1}{t}\right)\right)\right| = 0 \end{cases}$$

and so

$$\begin{aligned} &\lim_{|y|\to\infty}\left|u\left(\sqrt{t}y,t\right) - \left[\cos\left(\log\left(y^2+\frac{1}{t}\right)\right)\sin(\log t) + \sin\left(\log\left(y^2+\frac{1}{t}\right)\right)\cos(\log t)\right]\right| \\ &= \lim_{|y|\to\infty}\left|u\left(\sqrt{t}y,t\right) - \sin\left(\log\left(ty^2+1\right)\right)\right| = \lim_{|x|\to\infty}\left|u(x,t) - \sin\left(\log\left(x^2+1\right)\right)\right| = 0, \end{aligned} \tag{2.11}$$

which, together with (1.11), implies

$$\begin{aligned} &\limsup_{|x|\to\infty}\left|y(x,t) - \sin\left(\log\left(x^2+1\right)\right)\right| \\ &\leq \limsup_{|x|\to\infty}\left\{|y(x,t)-u(x,t)| + \left|u(x,t) - \sin\left(\log\left(x^2+1\right)\right)\right|\right\} \leq \frac{C}{\sqrt{t}}. \end{aligned}$$

Hence (2.3) follows. □

Remark 2.3. *Note that the convergence in (2.2) cannot be uniform in the whole space $x \in (-\infty, \infty)$. If we choose $x = \sqrt{t},\ t > 0$, in the identity*

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{4}} y_0\left(x + \sqrt{t}z\right) dz \\ &= \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{4}} \sin\left[\log t + \log\left(\frac{x^2}{t} + \frac{2xz}{\sqrt{t}} + z^2 + \frac{1}{t}\right)\right] dz, \end{aligned} \tag{2.12}$$

then the function $u\left(\sqrt{t}, t\right)$ will converge to $\tilde{A}\sin(\log t) + \tilde{B}\cos(\log t)$ as $t \to \infty$, where $\tilde{A},\ \tilde{B}$ are constants different from the $A,\ B$ in (2.6), given by

$$\begin{cases} \tilde{A} = \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{4}} \cos\left(\log(z+1)^2\right) dz \approx 0.2003 \\ \tilde{B} = \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{4}} \sin\left(\log(z+1)^2\right) dz \approx 0.24081. \end{cases}$$

The following says that it is impossible to find an initial data $y_0(x) \in C^{2+\gamma}(\mathbb{R})$ for some $\gamma \in (0,1)$ so that we have convergence to a periodic function.

Lemma 2.4. *There does not exist $y_0(x) \in C^{2+\gamma}(\mathbb{R})$ for some $\gamma \in (0,1)$ such that under the graphical curve shortening flow Eq (1.5) we have*

$$\lim_{t\to\infty} |y(0,t) - P(t)| = 0, \tag{2.13}$$

where $P(t)$ is a non-constant periodic function.

Proof. Assume (2.13) holds for some $P(t)$. Then by Theorem 1.1 we will have

$$\lim_{t\to\infty} \left| \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{4}} y_0\left(\sqrt{t}z\right) dz - P(t) \right| = 0, \tag{2.14}$$

which will give a contradiction due to

$$\begin{aligned} &\frac{d}{dt}\left(\frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{4}} y_0\left(\sqrt{t}z\right) dz\right) \\ &= \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{4}} y_0'\left(\sqrt{t}z\right) \frac{z}{2\sqrt{t}} dz = O\left(\frac{1}{\sqrt{t}}\right) \to 0 \quad \text{as} \quad t \to \infty. \end{aligned} \tag{2.15}$$

□

Remark 2.5. *In view of Lemma 2.4, it seems reasonable to see that for $y_0(x) \in C^{2+\gamma}(\mathbb{R})$, instead of converging to $A\sin t + B\cos t$, we get convergence to $A\sin(\log t) + B\cos(\log t)$, which is the result of Theorem 2.1. The proof of Lemma 2.4 also says that for heat equation with initial data in $C^{2+\gamma}(\mathbb{R})$, the solution $u(x,t)$, for fixed x, cannot converge to a periodic function as $t \to \infty$. On the other hand, it is possible if we allow the initial data to be unbounded. Let*

$$u_0(x) = \lambda + Ae^{\frac{x}{\sqrt{2}}} \cos\left(\frac{x}{\sqrt{2}}\right) + Be^{\frac{x}{\sqrt{2}}} \sin\left(\frac{x}{\sqrt{2}}\right), \quad x \in \mathbb{R},$$

where $\lambda,\ A,\ B$ are arbitrary constants. Then the function

$$u(x,t)=\lambda+Ae^{\frac{x}{\sqrt{2}}}\cos\left(t+\frac{x}{\sqrt{2}}\right)+Be^{\frac{x}{\sqrt{2}}}\sin\left(t+\frac{x}{\sqrt{2}}\right),\quad (x,t)\in\mathbb{R}^2,$$

is a solution of the heat equation with $u(x,0)=u_0(x),\ x\in\mathbb{R}$. For each fixed x, $u(x,t)$ is a non-constant 2π-periodic function in time. However, as seen in Section 3, for unbounded initial data, solutions to equation (1.5) and solutions to the heat equation (1.7) may not have the same asymptotic behavior as $t\to\infty$.

Remark 2.6. *This is for comparison. Let $M>0$ be a constant and $y_0(x)\in C^{2+\gamma}(\mathbb{R})$ be such that for $|x|\geq M$ it is equal to $\sin(|x|^\alpha)$ for some constant $\alpha\in(0,1]$ (if $\alpha>1$, the function will not lie in the space $C^{2+\gamma}(\mathbb{R})$). Then as $|x|\to\infty$, $\sin(|x|^\alpha)$ oscillates faster than $\sin\left(\log\left(x^2+1\right)\right)$. Evolving $y_0(x)$ under the heat equation, we have*

$$\begin{aligned}u(0,t)&=\frac{1}{\sqrt{4\pi t}}\int_{|x|<M}e^{-\frac{x^2}{4t}}y_0(x)\,dx+\frac{1}{\sqrt{4\pi}}\int_{|z|\geq M/\sqrt{t}}e^{-\frac{z^2}{4}}\sin\left(\left(\sqrt{t}\right)^\alpha|z|^\alpha\right)dz\\&=\frac{1}{\sqrt{4\pi t}}\int_{|x|<M}e^{-\frac{x^2}{4t}}y_0(x)\,dx+\frac{1}{\alpha\sqrt{\pi}}\int_{\left(M/\sqrt{t}\right)^\alpha}^{\infty}\theta^{\frac{1}{\alpha}-1}e^{-\frac{\theta^{2/\alpha}}{4}}\sin\left(\left(\sqrt{t}\right)^\alpha\theta\right)d\theta,\end{aligned}$$

where, similar to the Riemann-Lebesgue lemma, we can prove that $\lim_{t\to\infty}u(0,t)=0$. Hence by Theorem 1.1, under Eq (1.5), we have $\lim_{t\to\infty}y(x,t)=0$ uniformly on compact set $K\subset\mathbb{R}$ if we choose the above initial data. Thus the moral is: a bounded slow-oscillation function, under either the heat Eq (1.7) or (1.5), has more chance to preserve its profile as $t\to\infty$. In the extreme case that when $y_0(x)\equiv c$ is a constant (which has no oscillation at all), we have $y(x,t)\equiv c$ and the profile is unchanged at all. See Remark 2.8 also.

Finally, we note that if we take the initial data of the heat equation as

$$p\sin\left(\lambda\log\left(x^2+1\right)\right)+q\cos\left(\lambda\log\left(x^2+1\right)\right),\quad x\in(-\infty,\infty),\tag{2.16}$$

where $p,\ q,\ \lambda>0$ are constants, then in the limit we have

$$\lim_{t\to\infty}\left|u(x,t)-\left[(pA_\lambda-qB_\lambda)\sin(\lambda\log t)+(pB_\lambda+qA_\lambda)\cos(\lambda\log t)\right]\right|=0,\ \ \forall\ x\in(-\infty,\infty),\tag{2.17}$$

where

$$A_\lambda=\frac{1}{\sqrt{4\pi}}\int_{-\infty}^{\infty}e^{-\frac{z^2}{4}}\cos\left(\lambda\log\left(z^2\right)\right)dz,\quad B_\lambda=\frac{1}{\sqrt{4\pi}}\int_{-\infty}^{\infty}e^{-\frac{z^2}{4}}\sin\left(\lambda\log\left(z^2\right)\right)dz\tag{2.18}$$

with

$$\lim_{\lambda\to0^+}A_\lambda=1,\quad \lim_{\lambda\to0^+}B_\lambda=0,\quad \lim_{\lambda\to\infty}A_\lambda=\lim_{\lambda\to\infty}B_\lambda=0.\tag{2.19}$$

Thus in the limit $\lambda\to\infty$, the solution $u(x,t)$ will converge to zero since the two functions in (2.16) are no longer of slow-oscillation. This matches with the moral stated in Remark 2.6. By Theorem 1.1, the solution $y(x,t)$ of Eq (1.5) with initial data (2.16) also satisfies (2.17), uniformly on any compact set $K\subset\mathbb{R}$ of x.

Another result similar to Theorem 2.1 is the following:

Theorem 2.7. *Let $r_0(x)$ be given by*

$$r_0(x) = \sin\left(\log\left(\log\left(x^2+2\right)\right)\right), \quad x \in (-\infty, \infty) \tag{2.20}$$

and consider Eq (1.5) with the above initial condition. Then the solution $y(x,t)$ of this Cauchy problem is defined on $\mathbb{R}\times[0,\infty)$, which is smooth on $\mathbb{R}\times(0,\infty)$ and continuous up to $t=0$. Moreover, $y(x,t)$ satisfies the following two properties:

(1). (Asymptotic behavior as $t\to\infty$.) On any compact set $K\subset\mathbb{R}$ we have

$$\lim_{t\to\infty}\sup_{x\in K}\left|y(x,t)-\sin(\log(\log t))\right| = 0. \tag{2.21}$$

(2). (Asymptotic behavior as $|x|\to\infty$.) For fixed $t>0$ we have

$$\limsup_{|x|\to\infty}\left|y(x,t)-\sin\left(\log\left(\log\left(x^2+2\right)\right)\right)\right| \le \frac{C}{\sqrt{t}}, \tag{2.22}$$

for some constant $C>0$ depending only on r_0.

Proof. (1). We have $r_0(x)\in C^{2+\gamma}(\mathbb{R})$ for any $\gamma\in(0,1)$; hence Theorem 1.1 is applicable. The solution $u(x,t)$ of the heat equation with initial data $r_0(x)$ satisfies

$$u(0,t) = \frac{1}{\sqrt{4\pi}}\int_{-\infty}^{\infty} e^{-\frac{z^2}{4}}\sin\left\{\log\left[\log t+\log\left(z^2+\frac{2}{t}\right)\right]\right\}dz, \quad t\in(0,\infty).$$

For convenience, let β be the quantity $\beta = \log\left[1+\frac{1}{\log t}\log\left(z^2+\frac{2}{t}\right)\right]$. Then the above becomes

$$u(0,t) = \frac{1}{\sqrt{4\pi}}\int_{-\infty}^{\infty} e^{-\frac{z^2}{4}}\sin\left[\log(\log t)+\beta\right]dz = A(t)\sin(\log(\log t)) + B(t)\cos(\log(\log t)),$$

where

$$\begin{cases} A(t) = \frac{1}{\sqrt{4\pi}}\int_{-\infty}^{\infty} e^{-\frac{z^2}{4}}\cos\beta dz, & \lim_{t\to\infty}A(t)=1 \\ B(t) = \frac{1}{\sqrt{4\pi}}\int_{-\infty}^{\infty} e^{-\frac{z^2}{4}}\sin\beta dz, & \lim_{t\to\infty}B(t)=0. \end{cases} \tag{2.23}$$

Thus we have $\lim_{t\to\infty}\left|u(0,t)-\sin(\log(\log t))\right| = 0$ and (2.21) follows from the gradient estimate (2.8).

(2). Similar to (2.10), if we let $x=\sqrt{t}y$, we have

$$u(x,t) = u\left(\sqrt{t}y,t\right) = A(y,t)\sin(\log(\log t)) + B(y,t)\cos(\log(\log t)),$$

where now

$$A(y,t) = \frac{1}{\sqrt{4\pi}}\int_{-\infty}^{\infty} e^{-\frac{z^2}{4}}\cos\sigma dz, \quad B(y,t) = \frac{1}{\sqrt{4\pi}}\int_{-\infty}^{\infty} e^{-\frac{z^2}{4}}\sin\sigma dz,$$

with

$$\sigma = \log\left[1+\frac{1}{\log t}\log\left((y+z)^2+\frac{2}{t}\right)\right], \quad y = \frac{x}{\sqrt{t}}.$$

For fixed $z \in (-\infty, \infty)$ and fixed $t > 0$, by the limit

$$\lim_{|y|\to\infty} \left[\log\left((y+z)^2 + \frac{2}{t} \right) - \log\left(y^2 + \frac{2}{t} \right) \right] = 0,$$

we have

$$\lim_{|y|\to\infty} |\cos\sigma - \cos\rho| = 0, \quad \lim_{|y|\to\infty} |\sin\sigma - \sin\rho| = 0,$$

where

$$\rho = \log\left[1 + \frac{1}{\log t} \log\left(y^2 + \frac{2}{t} \right) \right], \quad y = \frac{x}{\sqrt{t}}.$$

The Lebesgue Dominated Convergence Theorem implies, for fixed $t > 0$, the limits

$$\lim_{|y|\to\infty} |A(y,t) - \cos\rho| = 0, \quad \lim_{|y|\to\infty} |B(y,t) - \sin\rho| = 0$$

and so

$$\lim_{|y|\to\infty} \left| u\left(\sqrt{t}y, t \right) - \sin\left[\log(\log t) + \rho \right] \right| = \lim_{|y|\to\infty} \left| u\left(\sqrt{t}y, t \right) - \sin\left[\log\left(\log\left(ty^2 + 2 \right) \right) \right] \right| = 0. \qquad (2.24)$$

Hence we have

$$\lim_{|x|\to\infty} \left| u(x,t) - \sin\left(\log\left(\log\left(x^2 + 2 \right) \right) \right) \right| = 0,$$

which, together with (1.11), implies the following estimate for fixed $t > 0$:

$$\begin{aligned} &\limsup_{|x|\to\infty} \left| y(x,t) - \sin\left(\log\left(\log\left(x^2 + 2 \right) \right) \right) \right| \\ &\le \limsup_{|x|\to\infty} \left\{ |y(x,t) - u(x,t)| + \left| u(x,t) - \sin\left(\log\left(\log\left(x^2 + 2 \right) \right) \right) \right| \right\} \le \frac{C}{\sqrt{t}}. \end{aligned}$$

Hence (2.22) follows. The proof is done.

Remark 2.8. *The function* $\sin\left(\log\left(\log\left(x^2 + 2 \right) \right) \right)$ *oscillates even more slowly than* $\sin\left(\log\left(x^2 + 1 \right) \right)$ *as* $|x| \to \infty$. *Under the heat equation, its profile is totally unchanged as* $t \to \infty$. *On the other hand, as* $t \to \infty$ *the profile of* $\sin\left(\log\left(x^2 + 1 \right) \right)$ *is slightly changed into a linear combination of* $\sin(\log t)$ *and* $\cos(\log t)$. *Similarly, if* $r_0(x)$ *is given by* $r_0(x) = \cos\left(\log\left(\log\left(x^2 + 2 \right) \right) \right)$, *we have*

$$\lim_{t\to\infty} \sup_{x\in K} \left| y(x,t) - \cos(\log(\log t)) \right| = 0 \qquad (2.25)$$

for any compact set $K \subset \mathbb{R}$. *By analogy, if we take the initial condition in the space* $C^{2+\gamma}(\mathbb{R})$ *as*

$$\sin\left\{ \log\left[\cdots \log\left(\log\left(x^2 + m \right) \right) \right] \right\} \quad (k \text{ copies of } \log,\ k \ge 3),$$

where $m > 0$ *is some constant so that the function is defined on* $x \in (-\infty, \infty)$, *then the limit of* $u(0,t)$, *as* $t \to \infty$, *is* $\sin\{\log[\cdots \log(\log t)]\}$. *The same result holds if we replace the sine function by cosine function.*

As a consequence of Theorem 1.1 and Theorem 2.7, we can prescribe the oscillation limits of $y(x,t)$ as $t \to \infty$.

Corollary 2.9. *For any two numbers $\alpha < \beta$, if we choose the initial data as*

$$r_0(x) = \frac{\beta-\alpha}{2}\sin\left[\log\left(\log\left(x^2+2\right)\right)\right] + \frac{\beta+\alpha}{2} \in C^{2+\gamma}(\mathbb{R}), \quad x \in (-\infty,\infty) \tag{2.26}$$

and consider Eq (1.5) with the above initial condition, then on any compact set $K \subset \mathbb{R}$ we have

$$\lim_{t\to\infty}\sup_{x\in K}\left|y(x,t) - \left(\frac{\beta-\alpha}{2}\sin(\log(\log t)) + \frac{\beta+\alpha}{2}\right)\right| = 0. \tag{2.27}$$

In particular, for any fixed $x_0 \in (-\infty,\infty)$, we have

$$\alpha = \liminf_{x\to\infty} r_0(x) = \liminf_{t\to\infty} y(x_0,t) < \limsup_{t\to\infty} y(x_0,t) = \limsup_{x\to\infty} r_0(x) = \beta. \tag{2.28}$$

Thus, in the limit, the oscillation of $y(x_0,t)$ can attain any two arbitrary numbers $\alpha < \beta$.

Proof. This is because the solution of the heat equation with initial data (2.26) satisfies (2.27). The result follows due to Theorem 1.1. □

3. Theorem 1.1 fails if the initial data is not in the space $C^{2+\gamma}(\mathbb{R})$

Theorem 1.1 fails if the initial condition $y_0(x)$ is not in the space $C^{2+\gamma}(\mathbb{R})$. This is not discussed in the paper [9]. In the following we give one example of unbounded $y_0(x)$, $x \in (-\infty,\infty)$ to demonstrate this. This example is related to a travelling wave solution for the heat equation (1.7) along the y-direction.

We choose the initial data to be unbounded, with $y_0(x) = x^2/2$, $x \in (-\infty,\infty)$. It is not in the space $C^{2+\gamma}(\mathbb{R})$ for any $\gamma \in (0,1)$. Under the heat equation the solution is given by $u(x,t) = t + x^2/2$, which is a travelling wave solution translating in the positive y-direction. On the other hand, the curve $y_0(x) = x^2/2$ divides the plane into two regions, each with infinite area. By the main theorem in p. 472 of [4], if we evolve $y_0(x) = x^2/2$ under equation (1.5), the solution $y(x,t)$ is defined for all time $t > 0$, i.e., defined on $\mathbb{R}\times(0,\infty)$, which is smooth in $\mathbb{R}\times(0,\infty)$ and continuous on $\mathbb{R}\times[0,\infty)$.

It seems quite obvious that the initial curve $y_0(x) = x^2/2$, under the evolution of (1.7) and (1.5) respectively, will reveal different behavior as $t \to \infty$. Nonetheless, a simple analysis on the angle function $\theta(x,t)$ is useful in general and can provide us a rigorous proof of the above claim. Moreover, as soon as the evolution of $\theta(x,t)$ is known, the evolution of the curvature $\kappa(x,t)$ will come out immediately. See equations (3.7) and (3.8) below.

In the following we evolve $y_0(x) = x^2/2$ under Eq (1.5). The angle function $\theta(x,t) = \tan^{-1}(y_x(x,t)) \in (-\pi/2,\pi/2)$ of the evolving curve $y(x,t)$ satisfies the equation

$$\partial_t\theta(x,t) = \partial_t\left(\tan^{-1}(y_x(x,t))\right) = \frac{\partial_t y_x(x,t)}{1+y_x^2(x,t)} = \left(\cos^2\theta(x,t)\right)\theta_{xx}(x,t), \tag{3.1}$$

where we have used the identity $y_t(x,t) = \theta_x(x,t)$ from (1.5) in (3.1). As a consequence, letting $v(x,t) = \theta_x(x,t)$, we have

$$\partial_t v(x,t) = (\partial_t\theta)_x(x,t) = \left(\cos^2\theta(x,t)\right)v_{xx}(x,t) - (\sin 2\theta(x,t))\,v(x,t)\,v_x(x,t) \tag{3.2}$$

and for $w(x,t) = \theta_{xx}(x,t) = v_x(x,t)$ we have the equation

$$\partial_t w(x,t) = \begin{cases} \left(\cos^2 \theta(x,t)\right) w_{xx}(x,t) - 2(\sin 2\theta(x,t)) v(x,t) w_x(x,t) \\ -(\sin 2\theta(x,t)) w^2(x,t) - 2(\cos 2\theta(x,t)) v^2(x,t) w(x,t). \end{cases} \tag{3.3}$$

For the initial data $y(x,0) = x^2/2$, we have and $\theta(x,0) = \tan^{-1} y_x(x,0) = \tan^{-1} x$ and so $v(x,0) = \theta_x(x,0) = 1/\left(1+x^2\right) > 0$ on $\mathbb{R}$. We also have

$$w(x,0) = \theta_{xx}(x,0) = \frac{-2x}{(1+x^2)^2} = \begin{cases} <0, & x \in (0,\infty), \\ =0, & x = 0, \\ >0, & x \in (-\infty,0). \end{cases} \tag{3.4}$$

In particular, we see that $v(x,0)$ is an even positive function in $x \in (-\infty,\infty)$ and $w(x,0)$ is an odd function in $x \in (-\infty,\infty)$ with a simple zero at $x = 0$. By the symmetry of $y_0(x) = x^2/2$ and its curvature $\kappa_0(x)$ (both are even functions) and the geometry of equation (1.6), $\theta(x,t)$ and $w(x,t)$ must remain odd functions in $x \in (-\infty,\infty)$ as long as the solution exists. Also $v(x,t)$ must remain an even function in $x \in (-\infty,\infty)$.

Note that the initial data $w(x,0)$, over $x \in (-\infty,\infty)$, has only one simple zero at $x = 0$. Since we know that the number of zeros (counted with multiplicity) for solutions to Eq (3.3) cannot increase with time (see Angenent [1], p. 607), we must have

$$w(x,t) = \theta_{xx}(x,t) = \begin{cases} <0, & x \in (0,\infty), \quad t \in (0,\infty), \\ =0, & x = 0, \quad t \in (0,\infty), \\ >0, & x \in (-\infty,0), \quad t \in (0,\infty), \end{cases} \tag{3.5}$$

which implies

$$\partial_t \theta(x,t) = \left(\cos^2 \theta(x,t)\right) w(x,t) = \begin{cases} <0, & x \in (0,\infty), \quad t \in (0,\infty), \\ =0, & x = 0, \quad t \in (0,\infty), \\ >0, & x \in (-\infty,0), \quad t \in (0,\infty). \end{cases} \tag{3.6}$$

The above says that, if we evolve $y_0(x) = x^2/2$ under (1.5), the angle function $\theta(x,t)$ (with $\theta(x,0) = \tan^{-1} x$) is decreasing in time for $x \in (0,\infty)$ and increasing in time for $x \in (-\infty,0)$. Thus the asymptotic behavior for $y_0(x) = x^2/2$, evolving by (1.7) and (1.5) respectively, will be different as time goes on.

To end this section, we point out that, in terms of the angle function $\theta(x,t)$ and its evolution equation (3.1), one can compute the evolution equation of the curvature κ as follows:

Lemma 3.1. *(The evolution equation of the curvature for CSF and ACSF.) Assume* $y(x,t) \in C^\infty(\mathbb{R} \times (0,T_{\max})) \bigcap C^0(\mathbb{R} \times [0,T_{\max}))$ *is a solution to Eq (1.5) on the domain* $\mathbb{R} \times [0,T_{\max})$. *Then the curvature* $\kappa(x,t)$ *of the graph* $y(\cdot,t)$ *satisfies the equation*

$$\partial_t \kappa = \left(\cos^2 \theta\right) \kappa_{xx} + \kappa^3, \quad \kappa = \kappa(x,t), \quad (x,t) \in \mathbb{R} \times (0,T_{\max}). \tag{3.7}$$

Similarly, if $u(x,t) \in C^{\infty}(\mathbb{R}\times(0,T_{\max}))\bigcap C^{0}(\mathbb{R}\times[0,T_{\max}))$ satisfies the heat Eq (1.7) on $\mathbb{R}\times[0,T_{\max})$, then its curvature $\kappa(x,t)$ satisfies

$$\partial_t\kappa = \kappa_{xx} + 6(\tan\theta)\theta_x\kappa_x + \left(\frac{6}{\cos^4\theta} - \frac{3}{\cos^2\theta}\right)\kappa^3, \quad \kappa = \kappa(x,t), \quad (x,t)\in\mathbb{R}\times(0,T_{\max}). \tag{3.8}$$

Remark 3.2. *Since Eq (1.5) is a geometric equation, we see that Eq (3.7) looks better than Eq (3.8).*

Proof. For (3.7), by (1.6) we can express the curvature as

$$\kappa(x,t) = \frac{\theta_x(x,t)}{\sqrt{1+y_x^2(x,t)}} = \frac{\theta_x}{\sqrt{1+\tan^2\theta}} = (\cos\theta)\theta_x = (\sin\theta)_x, \tag{3.9}$$

and get

$$\begin{cases} \kappa_x = (\cos\theta)\theta_{xx} - (\sin\theta)\theta_x^2 \\ \kappa_{xx} = (\cos\theta)\theta_{xxx} - 3(\sin\theta)\theta_x\theta_{xx} - (\cos\theta)\theta_x^3. \end{cases} \tag{3.10}$$

By the second identity in (3.10) and (3.1), we conclude

$$\begin{aligned} \partial_t\kappa &= -(\sin\theta)\theta_t\theta_x + (\cos\theta)(\theta_t)_x \\ &= -(\sin\theta)\left(\cos^2\theta\right)\theta_{xx}\theta_x + (\cos\theta)\left[\left(\cos^2\theta\right)\theta_{xxx} - (\sin 2\theta)\theta_x\theta_{xx}\right] \\ &= -(\sin\theta)\left(\cos^2\theta\right)\theta_x\theta_{xx} + \left(\cos^2\theta\right)\left[\kappa_{xx} + 3(\sin\theta)\theta_x\theta_{xx} + (\cos\theta)\theta_x^3\right] - (\cos\theta)(\sin 2\theta)\theta_x\theta_{xx} \\ &= \left(\cos^2\theta\right)\kappa_{xx} + \left(\cos^3\theta\right)\theta_x^3 = \left(\cos^2\theta\right)\kappa_{xx} + \kappa^3, \end{aligned} \tag{3.11}$$

which gives (3.7).

For (3.8), we first have $u_t = u_{xx} = (\tan\theta)_x$ and $\kappa = (\cos\theta)\theta_x$. Similar to the derivation of (3.11), we need to compute $\partial_t\theta$ and $\partial_t\theta_x$ first. We have

$$\begin{aligned} \partial_t\theta = \partial_t\left(\tan^{-1}u_x\right) = \frac{(\partial_t u)_x}{1+u_x^2} &= \left(\cos^2\theta\right)(\tan\theta)_{xx} \\ &= \left(\cos^2\theta\right)\left(\left(\sec^2\theta\right)\theta_x\right)_x = \theta_{xx} + 2(\tan\theta)\theta_x^2 \end{aligned} \tag{3.12}$$

and

$$\partial_t\theta_x = (\partial_t\theta)_x = \theta_{xxx} + 2\left(\sec^2\theta\right)\theta_x^3 + 4(\tan\theta)\theta_x\theta_{xx} \tag{3.13}$$

and similar to (3.11), together with (3.10), we get

$$\begin{aligned} \partial_t\kappa &= -(\sin\theta)\left[\theta_{xx} + 2(\tan\theta)(\theta_x)^2\right]\theta_x + (\cos\theta)\left[\theta_{xxx} + 2\left(\sec^2\theta\right)\theta_x^3 + 4(\tan\theta)\theta_x\theta_{xx}\right] \\ &= -(\sin\theta)\theta_x\left[\theta_{xx} + 2(\tan\theta)(\theta_x)^2\right] + (\cos\theta)\begin{pmatrix} \frac{\kappa_{xx}}{\cos\theta} + 3\frac{\sin\theta}{\cos\theta}\theta_x\theta_{xx} + \theta_x^3 \\ +2\left(\sec^2\theta\right)\theta_x^3 + 4(\tan\theta)\theta_x\theta_{xx} \end{pmatrix} \\ &= \kappa_{xx} + 6(\sin\theta)\theta_x\theta_{xx} + \left(-2\frac{\sin^2\theta}{\cos\theta} + \cos\theta + 2\sec\theta\right)\theta_x^3 \\ &= \kappa_{xx} + 6(\sin\theta)\theta_x\left(\frac{\kappa_x}{\cos\theta} + \frac{\sin\theta}{\cos\theta}\theta_x^2\right) + \left(-2\frac{\sin^2\theta}{\cos\theta} + \cos\theta + 2\sec\theta\right)\theta_x^3 \\ &= \kappa_{xx} + 6(\tan\theta)\theta_x\kappa_x + \left(\frac{6}{\cos^4\theta} - \frac{3}{\cos^2\theta}\right)\kappa^3, \end{aligned}$$

which gives (3.8). □

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Conflict of interest

The authors declare no conflict of interest.

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