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Research article

Improvements on overdetermined problems associated to the *p*-Laplacian†

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Abstract: This work presents some improvements on related papers that investigate certain overdetermined problems associated to elliptic quasilinear operators. Our model operator is the p-Laplacian. Under suitable structural conditions, and assuming that a solution exists, we show that the domain of the problem is a ball centered at the origin. Furthermore we discuss a convenient form of comparison principle for this kind of problems.

Keywords: p-Laplacian; overdetermined problems; comparison principle; boundary-point lemma

1. Introduction

This paper deals with the equation

$$- \Delta_p u = f(|x|, u) \tag{1.1}$$

where Δ_p denotes the p-Laplace operator $\Delta_p u = \operatorname{div}(|Du|^{p-2} Du)$ with $p \in (1, +\infty)$. The function $f(r, y)$ is defined for all $(r, y) \in (0, +\infty) \times [0, +\infty)$, it is positive whenever $y > 0$, and satisfies the following conditions (labels correspond to [4]).

(H_1) For almost every $r \in (0, +\infty)$, the function $y \mapsto f(r, y)$ is continuous with respect to $y \in [0, +\infty)$. Furthermore, for every $y \in [0, +\infty)$ and $r_0 > 0$, the function $r \mapsto f(r, y)$ belongs to $L^\infty((0, r_0))$.

(H_2) For a.e. $r \in (0, +\infty)$ the function $y \mapsto f(r, y)/y^{p-1}$ is strictly decreasing with respect to $y \in (0, +\infty)$.

(H_3) For every bounded interval $(0, r_0)$ there exists a constant $C(r_0)$ such that $f(r, y) \leq C(r_0)\,(y^{p-1} + 1)$ for a.e. $r \in (0, r_0)$ and for all $y \in [0, +\infty)$.

Under assumptions (H_1), (H_2) and (H_3), J. I. Díaz and J. E. Saa in the fundamental paper [4] proved uniqueness of the bounded, weak solution $u \in W_0^{1,p}(\Omega)$ to the Dirichlet problem

$$\begin{cases} -\Delta_p u = f(|x|, u), \ u \geq 0, \ u \not\equiv 0 & \text{in } \Omega; \\ u(x) = 0 & \text{for } x \in \partial\Omega, \end{cases} \tag{1.2}$$

where Ω is a bounded, smooth, open subset of $\mathbb{R}^N$, $N \geq 2$. To be precise, the function f in [4] is allowed to depend on (x, u), and the assumptions stated there are less demanding because they are tailored on the specific set Ω. In the present paper, instead, the set Ω is an unknown of the problem: thus, conditions (H_1) and (H_3) above correspond to the requirement that [4, (H_1) and (H_3)] hold in every bounded Ω. For the same reason we require

$$\lim_{y \to 0^+} \frac{f(r, y)}{y^{p-1}} = +\infty \quad \text{and} \quad \lim_{y \to +\infty} \frac{f(r, y)}{y^{p-1}} = 0 \quad \text{for a.e. } r \in (0, +\infty). \tag{1.3}$$

The last assumptions, together with (H_1), (H_2) and (H_3), ensure the existence of a bounded weak solution $u \in W_0^{1,p}(\Omega)$ of problem (1.2) by [4, Théorème 2] (see also [1]). Such a solution is in fact positive in Ω and (if the domain is sufficiently smooth) belongs to the Hölder class $C^{1,\alpha}(\overline{\Omega})$ ([4], p. 522, last paragraph). Here, however, we start from the assumption that problem (1.6) is solvable in the class $C^1(\overline{\Omega})$, so that the boundary conditions are intended in the classical sense: this implies that Ω is a domain of class C^1 (see Lemma A.2). Concerning the regularity of f, we need two assumptions. First we require that

$$f \text{ is locally uniformly Hölder continuous} \tag{1.4}$$

i.e., $f(r, y)$ is uniformly Hölder continuous in every compact subset of $(0, +\infty) \times [0, +\infty)$. Clearly, the first part of condition (H_1) is an immediate consequence of (1.4). If the weak solution u of problem (1.2) has a non-vanishing gradient at some point $x \in \Omega$, then the operator $\Delta_p u$ is non-degenerate, and if, furthermore, $x \neq 0$, then by (1.4) u belongs to the Hölder class $C^{2,\alpha}$ in a neighborhood of x (see, for instance, [5, Theorem 15.9]). The interior smoothness of u is required by Lemma 2.2 (a boundary-point lemma). In order to apply the lemma, we also need that for every $R_1 > 0$ there exist $\delta \in (0, R_1)$ and $L \in \mathbb{R}$ such that

$$\frac{f(r, y_1) - f(r, y_2)}{y_1 - y_2} \geq L \tag{1.5}$$

for every $r \in (R_1 - \delta, R_1)$ and $0 < y_1 < y_2 < \delta$. Assuming that Ω contains the origin, and denoting by $q\colon (0, +\infty) \to (0, +\infty)$ a prescribed function, we consider the overdetermined problem

$$\begin{cases} -\Delta_p u = f(|x|, u), \ u > 0 & \text{in } \Omega; \\ u(x) = 0, \ |Du(x)| = q(|x|) & \text{for } x \in \partial\Omega. \end{cases} \tag{1.6}$$

Contrary to what one may expect, counterexamples show that problem (1.6) may well be solvable even though the domain Ω is not radially symmetric: see, for instance, [6, pp. 488–489] and [8, Section 5]. The purpose of the present paper is to find conditions on f, q such that (1.6) is solvable only if Ω is a ball centered at the origin. More precisely, we remove the restrictions that $f(r, y)$ is monotone non-increasing with respect to r or y, which were imposed in [6] and [7], respectively. In general, given f and q, we introduce the function $F(r, \rho, \lambda)$ by letting

$$F(r, \rho, \lambda) = \frac{(q(r))^{p-1}}{r}\, f\Big(\frac{\rho}{r},\ \frac{\lambda}{r\,q(r)}\Big)$$

for $r, \rho, \lambda > 0$, and we assume that for every ρ, λ

$$F(r, \rho, \lambda) \text{ is monotone non-decreasing with respect to } r. \tag{1.7}$$

In Section 4 we demonstrate some special cases when condition (1.7) holds true. Our main result is the following:

Theorem 1.1. *Let Ω be a bounded, connected open set in $\mathbb{R}^N$, $N \geq 2$, containing the origin. Suppose that the functions f and q are such that* (H_1)*,* (H_2) *and* (H_3) *hold, together with* (1.3)*,* (1.4)*,* (1.5) *and* (1.7)*. If there exists a weak solution $u \in C^1(\overline{\Omega})$ of the overdetermined problem* (1.6)*, then Ω is a ball centered at the origin.*

Similarly to [6–8], the result is achieved by means of a comparison with two radial functions. However, the supersolution is chosen following an idea in [9]: see Section 5 for the proof of the theorem. The result is new even in the case when $p = 2$, i.e., $\Delta_p = \Delta$: an example is given by the sublinear Hénon problem

$$\begin{cases} -\Delta u = |x|^{M_r}\, u^{M_y},\ u > 0 & \text{in } \Omega; \\ u = 0,\ |Du(x)| = |x|^{1+\mu} & \text{on } \partial\Omega; \end{cases} \tag{1.8}$$

with constants $\mu, M_r, M_y \geq 0$ satisfying

$$\frac{\mu - M_r}{2 + \mu} \geq M_y. \tag{1.9}$$

See Theorem 4.3 for details. A further advancement lies in the fact that the strict monotonicity of $F(r, \rho, \lambda)$ in r is not required. For instance, we have:

Corollary 1.2. *Let Ω be a bounded, connected open set in $\mathbb{R}^N$, $N \geq 2$, containing the origin. Suppose that* (H_1) *and* (H_3) *hold, together with* (1.3)*,* (1.4) *and* (1.5)*. Suppose, further, that $f(r, y)$ is non-increasing in $r > 0$, and there exists a constant $\varepsilon_0 \in (0,\ p - 1]$ such that for every $r \in (0, +\infty)$ the function $y \mapsto f(r, y)/y^{p-1-\varepsilon_0}$ is monotone non-increasing with respect to $y \in (0, +\infty)$. Let q be such that*

$$\textit{the ratio } \frac{q(r)}{r^{\frac{p}{\varepsilon_0}-1}} \textit{ is non-decreasing in } r. \tag{1.10}$$

If there exists a weak solution $u \in C^1(\overline{\Omega})$ of the overdetermined problem (1.6)*, then Ω is a ball centered at the origin.*

Corollary 1.2 improves [6, Theorem 1.2] because the monotonicity in (1.10) is intended in the broad sense. To achieve this, we develop in the next section a convenient boundary-point lemma. The lemma yields a sharper comparison between the solution u and the radial solution used in the proof of Theorem 1.1, which are regular enough by virtue of assumption (1.4).

2. Comparison principle and Hopf's lemma

Roughly speaking, the (weak) comparison principle asserts that if a subsolution u and a supersolution v of (1.1) satisfy $u \leq v$ on $\partial\Omega$, then the same inequality holds a.e. in Ω. The strong comparison principle, instead, under the same assumption asserts that either $u < v$ a.e. in Ω, or u and v are the same function. Note that the term *strong comparison principle* is sometimes used to refer to a class of theorems ensuring that $u < v$ in Ω provided that the weaker inequality $u \leq v$ holds in the whole domain Ω (and $u \not\equiv v$): see for instance [3, Theorem 1.4] and the results in [16, 17].

Due to the nonlinearity of the p-Laplacian, the difference $u - v$ is not a subsolution, in general (unless $p = 2$), and therefore the comparison principle, far from being a straightforward consequence of the maximum principle, is a self-standing and interesting task.

Several comparison principles for the p-Laplacian are found in the literature: for instance, the case when f vanishes identically is considered in [10, 11]. For $f = f(x)$ see [12]. For f depending on (x, y) and monotone in y let us mention [2, Proposition 2.3 (b)], as well as [15, Proposition 2.1], where $f(x, y)$ is required to be non-increasing in y. The case when $f(x, y) = a(x)\, y^{p-1}$ is considered in [8, Theorem 7.1]. More general nonlinearities (still non-decreasing in y) are admitted in [2, Theorem 2.1 and Proposition 2.3 (a)] provided that $u = v = 0$ on the boundary.

We make use of a comparison principle that holds regardless of the monotonicity of f in y, and does not require that $u = v = 0$ on $\partial\Omega$.

Theorem 2.1 (Comparison principle). *Let Ω be an open set in $\mathbb{R}^N$, $N \geq 2$, possibly non-smooth and unbounded. Let $u, v \in W^{1,p}(\Omega)$ be a weak subsolution and a weak supersolution, respectively, of equation* (1.1) *in Ω, where f satisfies* (H$_1$), (H$_2$) *and* (H$_3$). *Assume that $u, v > 0$ a.e. in Ω, the ratio u/v belongs to $L^\infty(\Omega)$, and $(u - v)^+ \in W_0^{1,p}(\Omega)$. Then $u \leq v$ a.e. in Ω.*

Proof. The claim is an application of [6, Theorem 1.1 (2)], which holds for $f = f(x, y)$, to the special case when $f = f(|x|, y)$. □

Note that if Ω is bounded, and if u, v are smooth up to the boundary, positive in Ω, satisfy $u \leq v$ on $\partial\Omega$, and have non-vanishing gradients Du, Dv on $\partial\Omega$, then the ratio u/v is bounded as required in Theorem 2.1: see Lemma A.1 for details. In order to prove Theorem 1.1 we also need a boundary-point lemma involving the outward derivatives of a smooth subsolution u and a smooth supersolution v with non-vanishing gradients:

Lemma 2.2 (Boundary-point lemma). *Let G be a connected, bounded open set in $\mathbb{R}^N$, $N \geq 2$, satisfying the interior sphere condition at $z_1 \in \partial G$, i.e., there exists a ball $B \subset G$ such that $z_1 \in \partial B$. Let $u, v \in C^2(G) \cap C^1(\overline{G})$ be such that $u \leq v$ in $\overline{G}$ and $Du(x), Dv(x) \neq 0$ in $\overline{G}$, as well as $u(z_1) = v(z_1)$. Assume*

$$\Delta_p v + f(x, v(x)) \leq \Delta_p u + f(x, u(x)) \text{ pointwise in } G, \tag{2.1}$$

where $f\colon G \times (a, b) \to \mathbb{R}$ satisfies a Lipschitz condition in y from below, i.e. there exists a constant L such that

$$\frac{f(x, y_1) - f(x, y_2)}{y_1 - y_2} \geq L \tag{2.2}$$

for all $x \in G$ and $a < y_1 < y_2 < b$. Here (a,b) is an interval such that $u(x), u(y) \in (a,b)$ for all $x \in G$. Suppose, further, that either u or v belongs to the class $C^2(\overline{G})$. Then either $u = v$ in G or

$$\frac{\partial u}{\partial \nu}(z_1) > \frac{\partial v}{\partial \nu}(z_1),$$

where ν denotes the outward derivative to the ball B at z_1.

Proof. Following [5, Theorem 10.1], we derive an inequality satisfied by the difference $w = u - v \in C^2(G) \cap C^1(\overline{G})$. However, in the present case assumption (iii) of [5] (monotonicity of f with respect to y) is not in effect: this difficulty is overcome because $w \le 0$ in the whole of $\overline{G}$ by assumption. To be more specific, for $\xi = (\xi_1, \dots, \xi_N) \in \mathbb{R}^N \setminus \{0\}$ and for $i, j = 1, \dots, N$ define

$$a^{ij}(\xi) = |\xi|^{p-2}\,\delta^{ij} + (p-2)\,|\xi|^{p-4}\,\xi_i\,\xi_j,$$

where δ^{ij} is Kronecker's delta, and notice that $a^{ij}(\xi)$ is continuously differentiable in the punctured space $\mathbb{R}^N \setminus \{0\}$. Since $u, v \in C^2(G)$ and $Du, Dv \neq 0$, the p-Laplacian may be rewritten as $\Delta_p u = a^{ij}(Du(x))\,u_{ij}(x)$, where u_{ij} denotes the second derivative of u with respect to $x_i\,x_j$ and the summation over repeated indices is understood. Similarly, we may write $\Delta_p v = a^{ij}(Dv(x))\,v_{ij}(x)$, and by (2.1) we obtain

$$\begin{aligned} a^{ij}(Dv(x))\,w_{ij}(x) + \big(a^{hk}(Du(x)) - a^{hk}(Dv(x))\big)\,u_{hk}(x) \\ + f(x,u(x)) - f(x,v(x)) \ge 0, \end{aligned}$$

where the summation over i, j and $h, k = 1, \dots, N$ is understood. Letting $w_i = \partial w/\partial x_i$, we have $(u_i(x) - v_i(x))\,w_i(x) = |Du(x) - Dv(x)|^2$, and therefore the inequality above may be rewritten as

$$a^{ij}(Dv(x))\,w_{ij}(x) + b^i(x)\,w_i(x) + c(x)\,w(x) \ge 0 \text{ in } G, \tag{2.3}$$

where the coefficients b^i and c are defined as follows:

$$b^i(x) = \begin{cases} \dfrac{a^{hk}(Du(x)) - a^{hk}(Dv(x))}{|Du(x) - Dv(x)|}\;\dfrac{u_i(x) - v_i(x)}{|Du(x) - Dv(x)|}\,u_{hk}(x), & Du(x) \neq Dv(x); \\ 0, & Du(x) = Dv(x), \end{cases}$$

$$c(x) = \begin{cases} \dfrac{f(x,u(x)) - f(x,v(x))}{u(x) - v(x)}, & u(x) \neq v(x); \\ 0, & u(x) = v(x). \end{cases}$$

The coefficients $a^{ij}(Dv(x))$ in (2.3) are bounded in $\overline{G}$ because $Dv(x)$ is continuous in $\overline{G}$ by assumption, and keeps away from zero. Assuming that $u \in C^2(\overline{G})$, let us check that the coefficients b^i are also bounded in $\overline{G}$. Since the derivatives $w_i = u_i - v_i$ satisfy $|w_i(x)| \le |Dw(x)|$, the ratio

$$\frac{u_i(x) - v_i(x)}{|Du(x) - Dv(x)|}$$

is clearly bounded. Hence suppose, contrary to the claim, that there exists a sequence of points $x_n \in \overline{G}$ converging to some $z_1 \in \overline{G}$ and such that $Du(x_n) \neq Dv(x_n)$ for every $n \geq 1$ and

$$\lim_{n\to+\infty} \frac{|a^{hk}(Du(x_n)) - a^{hk}(Dv(x_n))|}{|Du(x_n) - Dv(x_n)|} = +\infty. \tag{2.4}$$

Since $Du(x), Dv(x)$ are bounded in $\overline{G}$ and keep away from zero, the numerator keeps bounded, and therefore we must have $Du(z_1) = Dv(z_1) = \xi_0 \neq 0$. Letting $R = \frac{1}{2}|\xi_0|$, we have $Du(x_n), Dv(x_n) \in B_R(\xi_0)$ for n large, and therefore the whole segment joining $Du(x_n)$ and $Dv(x_n)$ is included in $B_R(\xi_0)$: hence it does not intersect the origin. We note in passing that the last assertion does not follow from the only fact that $Du(x), Dv(x)$ keep far from zero (think to the case when $Du = -Dv$). Letting

$$M^{hk} = \max_{\xi\in\overline{B}_R(\xi_0)} |Da^{hk}(\xi)|,$$

by the mean value theorem we have

$$|a^{hk}(Du(x_n)) - a^{hk}(Dv(x_n))| \leq M^{hk}\, |Du(x_n) - Dv(x_n)|,$$

contradicting (2.4). Thus, in order to conclude that the coefficients $b^i(x)$ are bounded in $\overline{G}$, it suffices to recall that $u_{hk}(x)$ is continuous in $\overline{G}$ by assumption. In the case when $v \in C^2(\overline{G})$, instead, a similar argument leads to the linear inequality

$$a^{ij}(Du(x))\, w_{ij}(x) + \tilde{b}^i(x)\, w_i(x) + c(x)\, w(x) \geq 0 \text{ in } G,$$

with bounded coefficients $\tilde{b}^i$. In both cases, the boundedness of $c(x)$ from below in G follows from (2.2). The sign of $c(x)$ is not prescribed by assumption. However, since $w(x) \leq 0$, we have $-c^-(x)\, w(x) \geq c(x)\, w(x)$, where $-c^-(x) = \min\{\, c(x),\, 0\,\} \leq 0$. Hence, from (2.3) we deduce

$$a^{ij}(Dv(x))\, w_{ij}(x) + b^i(x)\, w_i(x) - c^-(x)\, w(x) \geq 0 \text{ in } G$$

and the conclusion follows from the Hopf boundary-point lemma: see, for instance, [14, Corollary 2.8.5] (the Hopf lemma is also found in [13, Theorem 8, p. 67], but the assumptions on the boundedness of the coefficients need to be recovered from Theorem 6 and the subsequent remarks). □

3. Radial solutions

In the following lemma we summarize the properties of the radial solutions of problem (1.2) that are needed in the proof of Theorem 1.1.

Lemma 3.1. *Let* $\Omega = B_R(0)$ *for some* $R > 0$*, and let* f *satisfy* (H$_1$), (H$_2$), (H$_3$), (1.3) *and* (1.4). *Then the unique solution* $u = u_R$ *of problem* (1.2) *possesses the following properties:*

1) *u_R is radially symmetric;*
2) *for every $\varepsilon \in (0, R)$ there exists $\alpha \in (0, 1)$ such that u_R belongs to the Hölder class $C^{1,\alpha}(\overline{B}_R(0)) \cap C^{2,\alpha}(\overline{B}_R(0) \setminus B_\varepsilon(0))$;*
3) *$Du_R \neq 0$ on $\partial B_R(0)$.*

Proof. As mentioned in the Introduction, uniqueness follows from (H_1), (H_2) and (H_3). Using (1.3), we also have the existence of a (positive) weak solution $u = u_R \in C^{1,\alpha}(\overline{B}_R(0))$ for each $R > 0$. Since problem (1.2) is invariant under rotations about the origin, uniqueness implies that u_R is a radial function (Claim 1), and we may write $u_R(x) = v_R(|x|)$ for a convenient function $v_R(r)$. Using the divergence theorem in the ball $B_r(0)$ for any $r \in (0, R]$, we obtain

$$-\,|v_R'(r)|^{p-2}\, v_R'(r) = \int_0^r f(\rho, v_R(\rho))\, d\rho, \tag{3.1}$$

which immediately implies the last claim: in fact, it turns out that $Du_R(x) = 0$ if and only if $x = 0$. Claim 2 follows from (1.4) and (3.1). □

4. Sufficient conditions

Before proving Theorem 1.1 in the next section, we show some special cases satisfying (1.7). In order to compare Theorem 1.1 with Theorem 1.2 of [6], we prove

Proposition 4.1. *Assume that $f(r,y)$ is non-negative and non-increasing in $r > 0$. Suppose, further, that there exists $\varepsilon_0 \in (0, p-1]$ such that* (1.10) *holds, and the ratio $f(r,y)/y^{p-1-\varepsilon_0}$ is non-increasing in y. Then* (1.7) *is satisfied.*

Proof. Fix $\rho, \lambda > 0$ and denote by $\zeta(r)$ the function $\zeta(r) = q(r)\, r^{1-p/\varepsilon_0}$, which is non-decreasing by (1.10). Since $r\, q(r) = \zeta(r)\, r^{p/\varepsilon_0}$, the variables $t = \rho/r$ and $y = \lambda/(r\, q(r))$ decrease as r increases. Keeping this in mind, we write

$$f\Big(\frac{\rho}{r}, \frac{\lambda}{r\, q(r)}\Big) = \frac{\lambda^{p-1-\varepsilon_0}}{(r\, q(r))^{p-1-\varepsilon_0}}\, \frac{1}{y^{p-1-\varepsilon_0}}\, f(t, y)$$

and we observe that $(q(r))^{p-1}/r = (\zeta(r))^{\varepsilon_0}\, (r\, q(r))^{p-1-\varepsilon_0}$. Hence

$$\frac{(q(r))^{p-1}}{r}\, f\Big(\frac{\rho}{r}, \frac{\lambda}{r\, q(r)}\Big) = \lambda^{p-1-\varepsilon_0}\, (\zeta(r))^{\varepsilon_0} f(t, y)/y^{p-1-\varepsilon_0}$$

and (1.7) follows. □

Now let us take [7, Theorem 1.1] into consideration. To this purpose we consider $f(r,y)$ non-negative and non-increasing in $y > 0$, and we assume that there exists $\mu \in [-p, +\infty)$ such that both

$$\frac{(q(r))^{p-1}}{r^{1+\mu}} \text{ and } r^{\mu}\, f\Big(\frac{1}{r},\, y\Big) \text{ are non-decreasing with respect to } r > 0. \tag{4.1}$$

Note that (4.1) follows from (1.4) and (1.5) in [7]: more precisely, letting $\mu = -1 - (p-1)\,\sigma$, condition (1.4) in [7] is equivalent to the *strict* monotonicity of $(q(r))^{p-1}/r^{1+\mu}$. Note, further, that if (H_1) is in effect, then μ must belong to the interval $[0, +\infty)$ in order that the second expression in (4.1) is non-trivial and non-decreasing in r: indeed, if we take $\mu < 0$ and let $r \to +\infty$, then $f(1/r,\, y)$ keeps bounded by (H_1) and therefore $r^{\mu}\, f(1/r,\, y) \to 0$. This and the monotonicity imply that the non-negative function f must vanish identically, but then problem (1.6) is unsolvable. Thus, the only case compatible with (H_1) is when $\mu \geq 0$, which corresponds to $\sigma \leq -1/(p-1)$: this was not mentioned in [7].

Proposition 4.2. *If $f(r, y)$ is non-negative and non-increasing in $y > 0$, and if $q(r)$ and $f(r, y)$ satisfy* (4.1) *for some $\mu \geq -p$, then condition* (1.7) *holds true.*

Proof. Denoting by $\psi(r)$ the monotone non-decreasing function given by $\psi(r) = (q(r))^{p-1}/r^{1+\mu}$, we may write

$$\frac{(q(r))^{p-1}}{r} = \psi(r)\, r^{\mu}. \tag{4.2}$$

Hence, for every $\rho > 0$ we have

$$\begin{aligned}\frac{(q(r))^{p-1}}{r}\, f\Big(\frac{\rho}{r},\, y\Big) &= \psi(r)\, r^{\mu}\, f\Big(\frac{\rho}{r},\, y\Big)\\ &= \psi(r)\, \rho^{\mu}\, s^{\mu}\, f\Big(\frac{1}{s}, y\Big),\end{aligned}$$

where $s = r/\rho$. Since the right-hand side is non-decreasing by (4.1), we deduce that

$$\frac{(q(r))^{p-1}}{r}\, f\Big(\frac{\rho}{r},\, y\Big) \text{ is non-decreasing in } r > 0. \tag{4.3}$$

To conclude the proof, we rewrite (4.2) as $(r\, q(r))^{p-1} = \psi(r)\, r^{\mu+p}$, which implies that $r\, q(r)$ is non-decreasing (because $\mu + p \geq 0$): recalling that $f(r, y)$ is non-increasing in y, this and (4.3) imply (1.7). □

When the functions $q(r)$ and $f(r, y)$ are differentiable, a sufficient condition in order that (1.7) holds is given by the next theorem, which is applicable to the example in (1.8) and (1.9). Before proceeding further, recall that for every $M \in \mathbb{R}$ and for every positive, differentiable function $g(r)$ the condition $(g(r)/r^M)' \leq 0$ for $r > 0$ is equivalent to

$$\frac{r\, g'(r)}{g(r)} \leq M.$$

We will use several expressions like that in the sequel.

Theorem 4.3. *Let $q(r)$ be a positive, differentiable function of $r > 0$ and define $\phi(r) = (q(r))^{p-1}/r$. Suppose there exists a constant μ such that*

$$\inf_{r>0} \frac{r\, \phi'(r)}{\phi(r)} \geq \mu \in (-p, +\infty). \tag{4.4}$$

Furthermore, let $f(r, y)$ be positive and differentiable for every $r, y > 0$, and denote $f_r = \partial f/\partial r$ and $f_y = \partial f/\partial y$, for shortness. Suppose there exist constants M_r, M_y such that

$$\sup_{r,y>0} \frac{r\, f_r(r, y)}{f(r, y)} \leq M_r \in (-p, +\infty), \tag{4.5}$$

$$\sup_{r,y>0} \frac{y\, f_y(r, y)}{f(r, y)} \leq M_y \in (-\infty, +\infty). \tag{4.6}$$

If

$$\frac{\mu - M_r}{p + \mu} \geq \frac{M_y}{p - 1} \tag{4.7}$$

then (1.7) *holds true.*

Proof. Let us check that

$$\frac{d}{dr}\left\{\phi(r)\, f\Big(\frac{\rho}{r},\ \frac{\lambda}{r\, q(r)}\Big)\right\} \geq 0$$

for every $\rho, \lambda > 0$. Letting $t = \rho/r$ and $y = \lambda/(r\, q(r))$, and since f is differentiable, the inequality above may be rewritten as

$$\frac{r\,\phi'(r)}{\phi(r)} \geq \frac{t\, f_r(t,y)}{f(t,y)} + \frac{y\, f_y(t,y)}{f(t,y)}\Big(1 + \frac{r\, q'(r)}{q(r)}\Big). \tag{4.8}$$

Using the identity

$$1 + \frac{r\,\phi'(r)}{\phi(r)} = (p-1)\,\frac{r\, q'(r)}{q(r)} \tag{4.9}$$

together with assumption (4.4), we see

$$1 + \frac{r\, q'(r)}{q(r)} = \frac{1}{p-1}\Big(p + \frac{r\,\phi'(r)}{\phi(r)}\Big) > 0.$$

Hence, in view of assumptions (4.5) and (4.6), it is enough to ensure

$$\frac{r\,\phi'(r)}{\phi(r)} \geq M_r + \frac{M_y}{p-1}\Big(p + \frac{r\,\phi'(r)}{\phi(r)}\Big)$$

in order that (4.8) holds. Of course, the last inequality is equivalent to

$$\frac{\frac{r\,\phi'(r)}{\phi(r)} - M_r}{p + \frac{r\,\phi'(r)}{\phi(r)}} \geq \frac{M_y}{p-1}.$$

To complete the proof, observe that the rational function $(x - M_r)/(p + x)$ is strictly increasing in the variable $x > -p$ (because $M_r + p > 0$), hence assumption (4.7) implies the claim. □

Remark 4.4. By virtue of (4.9), we may define $\omega = (1 + \mu)/(p-1)$ and rewrite assumptions (4.4) and (4.7), respectively, as

$$\inf_{r>0} \frac{r\, q'(r)}{q(r)} \geq \omega \in (-1, +\infty),$$

$$\frac{(p-1)\,\omega - 1 - M_r}{\omega + 1} \geq M_y.$$

Remark 4.5. Assumption (H_1) implies $M_r, M_y \geq 0$. Indeed, if $M_r < 0$ then the ratio $\vartheta(r,y) = f(r,y)/r^{M_r}$ is strictly decreasing in r, and therefore

$$\lim_{r\to 0^+} f(r,y) = \lim_{r\to 0^+} r^{M_r}\,\vartheta(r,y) = +\infty,$$

which is in contrast with (H_1). A similar argument shows that $M_y \geq 0$. Assumption (H_2) immediately implies $M_y \leq p-1$. Finally, if $M_r, M_y \geq 0$ and (4.7) is in effect, then obviously $\mu \geq 0$.

Example 4.6. Theorem 4.3 is applicable, for instance, to the case when $p = 2$, $q(r) = r^{1+\mu}$, and $f(r,y) = r^{M_r}\, y^{M_y}$, with constants $\mu, M_r, M_y \geq 0$ satisfying (1.9). In this case, the assumptions in [6] are not satisfied (when $M_r > 0$), nor hold the assumptions in [7] (if $M_y > 0$).

5. Proofs of the main results

Proof of Theorem 1.1. Part I. Assume that there exists a weak solution $u \in C^1(\overline{\Omega})$ of the overdetermined problem (1.6). Then Ω is a domain of class C^1 by Lemma A.2. Define

$$R_1 = \min_{x \in \partial\Omega} |x|, \qquad R_2 = \max_{x \in \partial\Omega} |x|,$$

and let u_1 be the positive solution of problem (1.2) in the ball $B_1 = B_{R_1}(0)$. Since $u \ge 0$ in $\overline{\Omega}$, we have $0 = u_1 \le u$ on ∂B_1. In order to apply the comparison principle (Theorem 2.1) in the ball B_1 we need to check that $u_1/u \in L^\infty(B_1)$: this follows from Lemma A.1 because u vanishes at a boundary point $z_1 \in \partial B_1$ if and only if $z_1 \in \partial\Omega$: but then $|Du(z_1)| = q(|z_1|) > 0$, hence the assumptions of the lemma are satisfied. By the comparison principle we have $u_1 \le u$ in B_1, and therefore for each $z_1 \in \partial B_1 \cap \partial\Omega$ we may write

$$|Du_1(z_1)| \le |Du(z_1)|. \tag{5.1}$$

Part II. We claim that the equality holds in (5.1) if and only if $\Omega = B_1$. To prove this, suppose that $|Du_1(z_1)| = |Du(z_1)| > 0$ at some $z_1 \in \partial B_1 \cap \partial\Omega$. By continuity, there exists $\varepsilon \in (0, R_1)$ such that if we define $G = B_1 \cap B_\varepsilon(z_1)$ then we have $Du_1, Du \ne 0$ in $\overline{G}$. By (1.5), and by reducing ε if necessary, we may assume that for every $x \in G$ the pairs $(|x|, u_1(x))$ and $(|x|, u(x))$ belong to the set $(R_1-\delta,\ R_1)\times(0,\delta)$ where f satisfies a Lipschitz condition w.r.t. y from below. Hence by the boundary-point lemma (Lemma 2.2) we have $u_1 = u$ in G, which implies $u = 0$ on $\overline{G} \cap \partial B_1$. This shows that the set of all $z \in \partial B_1$ such that $u(z) = 0$ is a relatively open subset of ∂B_1. Since such a set is obviously closed, it follows that $u = 0$ on ∂B_1, hence $\partial B_1 \subset \partial\Omega$. Finally, since Ω is connected, we must have $\Omega = B_1$. Thus, the equality holds in (5.1) if and only if $\Omega = B_1$. To complete the proof of the theorem, it is enough to verify that $|Du_1(z_1)| = |Du(z_1)|$.

Part III. Let $B_2 = B_{R_2}(0)$, for shortness, and define

$$a = \frac{R_2\, q(R_2)}{R_1\, q(R_1)}.$$

With such a value of the parameter a, the function $v\colon B_2 \to \mathbb{R}$ given by $v(x) = a\, u_1(R_1\, x/R_2)$ satisfies

$$\frac{|Dv|_{\partial B_2}}{q(R_2)} = \frac{|Du_1|_{\partial B_1}}{q(R_1)} \le 1, \tag{5.2}$$

where the last inequality is a consequence of (5.1). Let us check that v is a supersolution of (1.1). By a straightforward computation we obtain

$$\begin{aligned} -\Delta_p\, v(x) &= -\Big(\frac{q(R_2)}{q(R_2)}\Big)^{p-1} \frac{R_1}{R_2}\, \Delta_p\, u_1(R_1\, x/R_2) \\ &= \Big(\frac{q(R_2)}{q(R_2)}\Big)^{p-1} \frac{R_1}{R_2}\, f\Big(\frac{\rho}{R_2},\ \frac{\lambda}{R_2\, q(R_2)}\Big), \end{aligned}$$

where we have put $\rho = R_1\, x$ and $\lambda = R_1\, q(R_1)\, v(x)$. Now, using assumption (1.7), it follows that $-\Delta_p\, v(x) \ge f(|x|, v(x))$, hence v is a supersolution of (1.1), as claimed.

Conclusion. Since $v \geq 0$ in $\overline{B}_2 \supset \partial\Omega$, and $u = 0$ on $\partial\Omega$, using Theorem 2.1 we find $u \leq v$ in Ω. Therefore at each $z_2 \in \partial B_2 \cap \partial\Omega$ we have $|Du(z_2)| \leq |Dv(z_2)|$, hence

$$1 \leq \frac{|Dv|_{\partial B_2}}{q(R_2)}.$$

By comparing the last inequality with (5.2) we see that $|Du_1|_{\partial B_1} = q(R_1)$, hence the equality holds in (5.1), and the theorem follows. □

Proof of Corollary 1.2. To prove the corollary it suffices to observe that its assumptions imply (1.7) by virtue of Proposition 4.1. Note that assumption (H_2), although not mentioned in the statement, is silently in effect as a consequence of the monotonicity of the ratio $f(r, y)/y^{p-1-\varepsilon_0}$. □

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Conflict of interest

The authors declare no conflict of interest.

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A. Appendix

In order to apply Theorem 2.1 (comparison principle) we need to know that $u/v \in L^\infty(\Omega)$. It was observed in [4, p. 522, last two lines] that the ratio u/v is bounded in a bounded domain Ω in the case when u and v are positive in Ω, vanish along the boundary, and have a positive inward derivative $\partial u/\partial \nu$ on $\partial\Omega$ (which is equivalent to having a non-vanishing gradient there).

The assumption $u = v = 0$ on $\partial\Omega$ can be turned into $0 \le u \le v$ on $\partial\Omega$ and the conclusion continues to hold: this is used in the proof of Theorem 1.1, and it was also used in [6] (see the three lines following (17) on p. 405). Let us give a precise statement and proof.

Recall that a domain $\Omega \subset \mathbb{R}^N$ with nonempty boundary is of class C^1 when $\partial\Omega$ is locally the graph of a continuously differentiable function. Following [5, p. 10], we write $u \in C^1(\overline{\Omega})$ if $u \in C^1(\Omega) \cap C^0(\overline{\Omega})$ and each derivative $u_i \colon \Omega \to \mathbb{R}$ is the restriction to Ω of a continuous function (still denoted by u_i) defined on $\overline{\Omega}$. In such a case, the gradient $Du = (u_1, \dots, u_N)$ is defined in the closure $\overline{\Omega}$. In this section we represent $x \in \mathbb{R}^N$ as $x = (x', x_N)$, where $x' = (x_1, \dots, x_{N-1}) \in \mathbb{R}^{N-1}$, and we denote by e_N the N-th element of the canonical base of $\mathbb{R}^N$. Furthermore we let $D_{x'} u = (u_1, \dots, u_{N-1})$. By a cylindrical neighborhood of the origin we mean the set $\mathcal{U}_r = \{ (x', x_N) : |x'|, |x_N| < r \}$ for some $r > 0$.

Lemma A.1. *Let Ω be a bounded domain of class C^1 in $\mathbb{R}^N$, $N \ge 2$, and let $u, v \in C^1(\overline{\Omega})$ be positive in Ω and satisfy $0 \le u \le v$ on $\partial\Omega$. Suppose that for any boundary point $z \in \partial\Omega$ where $v(z) = 0$, the gradient $Dv(z)$ does not vanish. Then the ratio u/v is bounded in Ω.*

Proof. Suppose, contrary to the claim, that there exists a sequence of points $x_k \in \Omega$ such that $u(x_k)/v(x_k) \to +\infty$. Without loss of generality we may assume that x_k converges to $z \in \overline{\Omega}$. Since the ratio $u(x)/v(x)$ is continuous (and finite) in Ω, we must have $z \in \partial\Omega$, and $u(z) = v(z) = 0$. Since Ω is a C^1-domain, after a convenient translation and rotation of the coordinate frame we may further assume that $z = 0$, and there exists a cylindrical neighborhood $\mathcal{U}_r$ of the origin such that the intersection $\mathcal{U}_r \cap \partial\Omega$ is the graph of a continuously differentiable function $x_N = f(x')$ satisfying $D_{x'} f(0) = 0$. We may also assume that the intersection $\mathcal{U}_r \cap \Omega$ lies above the graph of f, i.e., if $(x', x_N) \in \mathcal{U}_r \cap \Omega$ then $x_N > f(x)$. Since $v(0) = 0 \le v(x)$ for all $x \in \overline{\Omega}$, the directional derivative $v_\xi(0) = \xi \cdot Dv(0)$ satisfies $v_\xi(0) \ge 0$ for every direction ξ such that $\xi \cdot e_N \ge 0$: this implies $D_{x'} v(0) = 0$ and therefore we may let $\lambda_0 = v_N(0) > 0$ because $Dv(0) \ne 0$. By shrinking the neighborhood $\mathcal{U}_r$ if necessary, we may achieve that $v_N(x) > \frac{1}{2}\lambda_0$ in $\mathcal{U}_r \cap \Omega$. Furthermore we define $C = \sup_\Omega u_N(x) < +\infty$. Finally, recall that $u \le v$ on $\partial\Omega$. By the fundamental theorem of calculus, for every $x = (x', x_N) \in \mathcal{U}_r \cap \Omega$ we have

$$\begin{aligned}\frac{u(x)}{v(x)} &\le \frac{u(x', f(x')) + C\,(x_N - f(x'))}{v(x', f(x')) + \frac{1}{2}\lambda_0\,(x_N - f(x'))} \\ &\le \frac{v(x', f(x')) + C\,(x_N - f(x'))}{v(x', f(x')) + \frac{1}{2}\lambda_0\,(x_N - f(x'))} = \frac{\rho(x) + C}{\rho(x) + \frac{1}{2}\lambda_0} \le \max\Big\{1, \frac{2C}{\lambda_0}\Big\}\end{aligned}$$

where

$$\rho(x) = \frac{v(x', f(x'))}{x_N - f(x')} \ge 0.$$

This contradicts the assumption that u/v is unbounded, and the lemma follows. □

The domain Ω is required to belong to the smoothness class C^1 in order that the preceding lemma holds: let us prove that the regularity of the domain, which is not mentioned explicitly in the statement of Theorem 1.1, follows from the other assumptions.

Lemma A.2. *Consider an open, proper subset $\Omega \subset \mathbb{R}^N$, $N \ge 2$. If there exists $u \in C^1(\overline{\Omega})$, $u > 0$ in Ω, such that $u(z) = 0$ and $Du(z) \ne 0$ for every $z \in \partial\Omega$, then Ω is a domain of class C^1. Furthermore $Du(z)$ has the direction of the inward normal to $\partial\Omega$ at z.*

Proof. Part I: Definitions. Let us fix a boundary point $z_0 \in \partial\Omega$. Without loss of generality, we may assume that $z_0 = 0$ and $Du(0) = \lambda_0\, e_N$ with some $\lambda_0 > 0$. By the continuity of Du at 0, for every positive $\varepsilon < \lambda_0/2$ there exists a cylindrical neighborhood $\mathcal{U}_r$ of the origin, with a convenient $r = r(\varepsilon) > 0$, such that $|D_{x'} u|, |u_N - \lambda_0| < \varepsilon$ in $\mathcal{U}_r \cap \overline{\Omega}$. In particular, $u_N > \lambda_0 - \varepsilon > 0$ in $\mathcal{U}_r \cap \overline{\Omega}$. Let ϑ be the unique solution of the equation $\tan\vartheta = (\lambda_0 - \varepsilon)/\varepsilon$ in the interval $(\frac{\pi}{4}, \frac{\pi}{2})$. For every $\overline{x} \in \mathbb{R}^N$ we define the upper cone with vertex in $\overline{x}$ and half-opening ϑ by $V^+_\vartheta(\overline{x}) = \{ (x', x_N) : |x' - \overline{x}'| < (x_N - \overline{x}_N)\tan\vartheta \}$. The corresponding lower cone is $V^-_\vartheta(\overline{x}) = \{ (x', x_N) : |x' - \overline{x}'| < (\overline{x}_N - x_N)\tan\vartheta \}$. Clearly, $x \in V^+_\vartheta(\overline{x})$ if and only if $\overline{x} \in V^-_\vartheta(x)$. Since $\vartheta > \frac{\pi}{4}$, for every $x' \in \mathbb{R}^{N-1}$ satisfying $|x'| < r$ the vertical line $\ell_{x'} = \{ (x', t) : t \in \mathbb{R} \}$ intersects both $\mathcal{U}_r \cap V^+_\vartheta(0)$ and $\mathcal{U}_r \cap V^-_\vartheta(0)$, hence

$$\ell_{x'} \cap \mathcal{U}_r \cap V^+_\vartheta(0) \ne \emptyset, \qquad \ell_{x'} \cap \mathcal{U}_r \cap V^-_\vartheta(0) \ne \emptyset. \tag{A.1}$$

Part II. Take any point $\overline{x} = (\overline{x}', \overline{x}_N) \in \mathcal{U}_r \cap \Omega$. We claim that $\mathcal{U}_r \cap V^+_\vartheta(\overline{x}) \subset \Omega$. Indeed, since the intersection $\mathcal{U}_r \cap \Omega$ is an open, nonempty subset of $\mathbb{R}^N$, then the segment S described by $(\overline{x}', x_N)$, when $\overline{x}'$ is kept fixed and $x_N > \overline{x}_N$ is let vary, is included in $\mathcal{U}_r \cap \Omega$ provided that $x_N - \overline{x}_N$ is small. Furthermore the value of $u(\overline{x}', x_N)$ is easily estimated by means of the fundamental theorem of calculus, and for $x_N > \overline{x}_N$ we may write

$$u(\overline{x}', x_N) = u(\overline{x}', \overline{x}_N) + \int_{\overline{x}_N}^{x_N} u_N(\overline{x}', t)\, dt$$

$$> u(\overline{x}', \overline{x}_N) + (\lambda_0 - \varepsilon)\,(x_N - \overline{x}_N).$$

Thus, $u(\overline{x}', x_N)$ stays positive along S and increases with x_N, and therefore the segment S can be extended by allowing x_N to range in the whole interval $(\overline{x}_N, r)$ without hitting the boundary $\partial\Omega$, where u vanishes.

Now, integrating along the orthogonal directions to e_N, we find that $\mathcal{U}_r \cap V^+_\vartheta(\overline{x}) \subset \Omega$, as claimed, and for every $x = (x', x_N) \in \mathcal{U}_r \cap V^+_\vartheta(\overline{x})$ we can estimate $u(x)$ in terms of $u(\overline{x})$ as follows:

$$\begin{aligned} u(x) - u(\overline{x}) &= u(x) - u(\overline{x}', x_N) + u(\overline{x}', x_N) - u(\overline{x}) \\ &= \int_0^1 (x' - \overline{x}') \cdot D_{x'} u(tx' + (1-t)\,\overline{x}')\,dt + \int_{\overline{x}_N}^{x_N} u_N(\overline{x}', t)\,dt \\ &> -\varepsilon\,|x' - \overline{x}'| + (\lambda_0 - \varepsilon)\,(x_N - \overline{x}_N) > 0. \end{aligned}$$

Part III. Observe that for every $z \in \mathcal{U}_r \cap \partial\Omega$ we have $\mathcal{U}_r \cap V^+_\vartheta(z) \subset \Omega$ because there exists a sequence of points $\overline{x}_k \in \mathcal{U}_r \cap \Omega$ converging to z, and we may apply Part II to each $\overline{x}_k$. Furthermore, no point $\overline{x} \in \mathcal{U}_r \cap V^-_\vartheta(z)$ can be in Ω, for otherwise we would have $z \in \mathcal{U}_r \cap V^+_\vartheta(\overline{x}) \subset \Omega$, a contradiction. Choosing $z = 0$, and by (A.1), we see that for every x' satisfying $|x'| < r$ there exists $x_N \in (-r, r)$ such that the point (x', x_N) belongs to $\mathcal{U}_r \cap \partial\Omega$. Choosing $z = (x', x_N)$ we see that the value of x_N is unique: in other terms, the intersection $\mathcal{U}_r \cap \partial\Omega$ is the graph of a function $x_N = f(x')$. The argument above also shows that f satisfies a Lipschitz condition with constant $L(\varepsilon) = \cot\vartheta = \varepsilon/(\lambda_0 - \varepsilon)$. Since ε is arbitrary (and $\mathcal{U}_r$ shrinks to the origin, in general, when $\varepsilon \to 0$), it follows that $f(x')$ is differentiable at $x' = 0$, and we have $D_{x'} f(0) = 0$. This implies that $\partial\Omega$ has an inward normal ν at 0, which has the same direction as $Du(0)$. Since the choice of the boundary point $z_0 \in \partial\Omega$ is arbitrary, $f(x')$ must be differentiable at every x' such that $|x'| < r = r(\varepsilon)$, and the gradient $D_{x'} f(x')$ satisfies $|D_{x'} f(x')| \le L(\varepsilon)$. Since $L(\varepsilon) \to 0$ when $\varepsilon \to 0$, we have that $D_{x'} f(x')$ is continuous at $x' = 0$, and the lemma follows. □