http://www.aimspress.com/journal/mine

Mathematics in Engineering, 2(4): 734–771.
DOI:10.3934/mine.2020035
Received: 25 February 2020
Accepted: 06 July 2020
Published: 10 July 2020

Research article

Fundamental solutions for Kolmogorov-Fokker-Planck operators with time-depending measurable coefficients†

Marco Bramanti[1,*] **and Sergio Polidoro**[2]

[1] Dipartimento di Matematica, Politecnico di Milano, Via Bonardi 9, I-20133 Milano, Italy

[2] Dipartimento di Scienze Fisiche, Informatiche e Matematiche, Università degli Studi di Modena e Reggio Emilia, Via G. Campi 213/b, I-41125 Modena, Italy

† **This contribution is part of the Special Issue:** Contemporary PDEs between theory and modeling—Dedicated to Sandro Salsa, on the occasion of his 70th birthday
Guest Editor: Gianmaria Verzini
Link: www.aimspress.com/mine/article/5753/special-articles

* **Correspondence:** Email: marco.bramanti@polimi.it; Tel: +390223994567; Fax: +390223994629.

Abstract: We consider a Kolmogorov-Fokker-Planck operator of the kind:

$$\mathcal{L}u = \sum_{i,j=1}^{q} a_{ij}(t)\,\partial^2_{x_i x_j} u + \sum_{k,j=1}^{N} b_{jk} x_k \partial_{x_j} u - \partial_t u, \qquad (x,t) \in \mathbb{R}^{N+1}$$

where $\left\{a_{ij}(t)\right\}_{i,j=1}^{q}$ is a symmetric uniformly positive matrix on $\mathbb{R}^q$, $q \leq N$, of bounded measurable coefficients defined for $t \in \mathbb{R}$ and the matrix $B = \left\{b_{ij}\right\}_{i,j=1}^{N}$ satisfies a structural assumption which makes the corresponding operator with constant a_{ij} hypoelliptic. We construct an explicit fundamental solution Γ for $\mathcal{L}$, study its properties, show a comparison result between Γ and the fundamental solution of some model operators with constant a_{ij}, and show the unique solvability of the Cauchy problem for $\mathcal{L}$ under various assumptions on the initial datum.

Keywords: degenerate Kolmogorov-Fokker-Planck operators; fundamental solution; Cauchy problem; hypoellipticity

We wish to dedicate this paper to Sandro Salsa, in occasion of his 70th birthday.

1. Introduction

We consider a Kolmogorov-Fokker-Planck (from now on KFP) operator of the kind:

$$\mathcal{L}u = \sum_{i,j=1}^{q} a_{ij}(t)\,\partial^2_{x_i x_j} u + \sum_{k,j=1}^{N} b_{jk} x_k \partial_{x_j} u - \partial_t u, \qquad (x,t) \in \mathbb{R}^{N+1} \tag{1.1}$$

where:

(H1) $A_0(t) = \left\{a_{ij}(t)\right\}_{i,j=1}^{q}$ is a symmetric uniformly positive matrix on $\mathbb{R}^q$, $q \le N$, of bounded measurable coefficients defined for $t \in \mathbb{R}$, so that

$$\nu\,|\xi|^2 \le \sum_{i,j=1}^{q} a_{ij}(t)\,\xi_i \xi_j \le \nu^{-1}\,|\xi|^2 \tag{1.2}$$

for some constant $\nu > 0$, every $\xi \in \mathbb{R}^q$, a.e. $t \in \mathbb{R}$.

Lanconelli-Polidoro in [13] have studied the operators (1.1) with constant a_{ij}, proving that they are hypoelliptic if and only if the matrix $B = \left\{b_{ij}\right\}_{i,j=1}^{N}$ satisfies the following condition. There exists a basis of $\mathbb{R}^N$ such that B assumes the following form:

(H2) For $m_0 = q$ and suitable positive integers $m_1, \dots, m_\kappa$ such that

$$m_0 \ge m_1 \ge \ldots \ge m_\kappa \ge 1, \quad \text{and} \quad m_0 + m_1 + \ldots + m_\kappa = N, \tag{1.3}$$

we have

$$B = \begin{bmatrix} * & * & \ldots & * & * \\ B_1 & * & \ldots & * & * \\ \mathbb{O} & B_2 & \ldots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbb{O} & \mathbb{O} & \ldots & B_\kappa & * \end{bmatrix} \tag{1.4}$$

where every block B_j is a $m_j \times m_{j-1}$ matrix of rank m_j with $j = 1, 2, \dots, \kappa$, while the entries of the blocks denoted by $*$ are arbitrary.

It is also proved in [13] that the operator $\mathcal{L}$ (corresponding to constant a_{ij}) is left invariant with respect to a suitable (noncommutative) Lie group of translations in $\mathbb{R}^N$. If, in addition, all the blocks $*$ in (1.4) vanish, then $\mathcal{L}$ is also 2-homogeneous with respect to a family of dilations. In this very special case, the operator $\mathcal{L}$ fits into the rich theory of left invariant, 2-homogeneus, Hörmander operators on homoegeneous groups.

Coming back to the family of hypoelliptic and left invariant operators with constant a_{ij} (and possibly nonzero blocks $*$ in (1.4)), an explicit fundamental solution is known, after [11] and [13].

A first result of this paper consists in showing that if, under the same structural assumptions considered in [13], the coefficients a_{ij} are allowed to depend on t, even just in an L^∞-way, then an explicit fundamental solution Γ can still be costructed. It is worth noting that, under our assumptions (H1)–(H2), $\mathcal{L}$ is hypoelliptic if and only if the coefficients $a_{i,j}$'s are C^∞ functions, which also means that Γ is smooth outside the pole. In our more general context, Γ will be smooth in x and only locally Lipschitz continuous in t, outside the pole. Our fundamental solution also allows to solve a Cauchy problem for $\mathcal{L}$ under various assumptions on the initial datum, and to prove its uniqueness. Moreover, we show that the fundamental solution of $\mathcal{L}$ satisfies two-sided bounds in terms of the fundamental solutions of model operators of the kind:

$$\mathcal{L}_\alpha u = \alpha \sum_{i=1}^{q} \partial^2_{x_i x_i} u + \sum_{k,j=1}^{N} b_{jk} x_k \partial_{x_j} u - \partial_t u, \tag{1.5}$$

whose explicit expression is more easily handled. This fact has other interesting consequences when combined with the results of [13], which allow to compare the fundamental solution of (1.5) with that of the corresponding "principal part operator", which is obtained from (1.5) by annihilating all the blocks $*$ in (1.4). The fundamental solution of the latter operator has an even simpler explicit form, since it possesses both translation invariance and homogeneity.

To put our results into context, let us now make some historical remarks. Already in 1934, Kolmogorov in [10] exhibited an explicit fundamental solution, smooth outside the pole, for the ultraparabolic operator

$$\partial^2_{xx} + x\partial_y - \partial_t \text{ in } \mathbb{R}^3.$$

For more general classes of ultraparabolic KFP operators, Weber [20], 1951, Il'in [9], 1964, Sonin [19], 1967, proved the existence of a fundamental solution smooth outside the pole, by the Levi method, starting with an approximate fundamental solution which was inspired by the one found by Kolmogorov. Hörmander, in the introduction of [8], 1967, sketches a procedure to compute explicitly (by Fourier transform and the method of characteristics) a fundamental solution for a class of KFP operators of type (1.1) (with constant a_{ij}). In all the aforementioned papers the focus is to prove that the operator, despite of its degenerate character, is hypoelliptic. This is accomplished by showing the existence of a fundamental solution smooth outside the pole, without explicitly computing it.

Kupcov in [11], 1972, computes the fundamental solution for a class of KFP operators of the kind (1.1) (with constant a_{ij}). This procedure is generalized by the same author in [12], 1982, to a class of operators (1.1) with time-dependent coefficients a_{ij}, which however are assumed of class C^κ for some positive integer κ related to the structure of the matrix B. Our procedure to compute the fundamental solution follows the technique by Hörmander (different from that of Kupcov) and works also for nonsmooth $a_{ij}(t)$.

Based on the explicit expression of the fundamental solution, existence, uniqueness and regularity issues for the Cauchy problem have been studied in the framework of the semigroup setting. We refer here to the article by Lunardi [14], and to Farkas and Lorenzi [7]. The parametrix method introduced in [9, 19, 20] was used by Polidoro in [18] and by Di Francesco and Pascucci in [5] for more general families of Kolmogorov equations with Hölder continuous coefficients. We also refer to the article [4] by Delaure and Menozzi, where a Lipschitz continuous drift term is considered in the framework of the stochastic theory. For a recent survey on the theory of KFP operators we refer to the paper [1] by Anceschi-Polidoro, while a discussion on several motivations to study this class of operators can be found for instance in the survey book [2, §2.1].

The interest in studying KFP operators with a possibly rough time-dependence of the coefficients comes from the theory of stochastic processes. Indeed, let $\sigma = \sigma(t)$ be a $N \times q$ matrix, with zero entries under the q-th row, let B as in (1.4), and let $(W_t)_{t \geq t_0}$ be a q-dimensional Wiener process. Denote by

$(X_t)_{t\geq t_0}$ the solution to the following N-dimensional stochastic differential equation

$$\begin{cases} dX_t = -BX_t\,dt + \sigma(t)\,dW_t \\ X_{t_0} = x_0. \end{cases} \tag{1.6}$$

Then the *forward Kolmogorov operator* $\mathcal{K}_f$ of $(X_t)_{t\geq t_0}$ agrees with $\mathcal{L}$ up to a constant zero order term:

$$\mathcal{K}_f v(x,t) = \mathcal{L}v(x,t) + \mathrm{tr}(B)v(x,t),$$

where

$$a_{ij}(t) = \tfrac{1}{2}\sum_{k=1}^{q}\sigma_{ik}(t)\sigma_{jk}(t) \quad i,j = 1,...,q. \tag{1.7}$$

Moreover, the *backward Kolmogorov operator* $\mathcal{K}_b$ of $(X_t)_{t\geq t_0}$ acts as follows

$$\mathcal{K}_b u(y,s) = \partial_s u(y,s) + \sum_{i,j=1}^{q} a_{ij}(s)\partial^2_{y_i y_j}u(y,s) - \sum_{i,j=1}^{N} b_{ij}y_j\partial_{y_i}u(y,s).$$

Note that $\mathcal{K}_f$ is the transposed operator of $\mathcal{K}_b$. In general, given a differential operator $\mathcal{K}$, its transposed operator $\mathcal{K}^*$ is the one which satisfies the relation

$$\int_{\mathbb{R}^{N+1}} \phi(x,t)\,\mathcal{K}^*\psi(x,t)\,dxdt = \int_{\mathbb{R}^{N+1}} \mathcal{K}\phi(x,t)\,\psi(x,t)\,dxdt$$

for every $\phi,\psi \in C_0^\infty\left(\mathbb{R}^{N+1}\right)$.

A further motivation for our study is the following one. A regularity theory for the operator $\mathcal{L}$ with Hölder continuous coefficients has been developed by several authors (see e.g., [6, 14, 15]). However, as Pascucci and Pesce show in the Example 1.3 of [16], the requirement of Hölder continuity in (x,t) with respect to the control distance may be very restrictive, due to the interaction of time and space variable in the drift term of $\mathcal{L}$. In view of this, a regularity requirement with respect to x-variables alone, for t fixed, with a possible rough dependence on t, seems a more natural assumption. This paper can be seen as a first step to study KFP operators with coefficients measurable in time and Hölder continuous or VMO in space, to overcome the objection pointed out in [16]. For these operators the fundamental solution of (1.1) could be used as a parametrix, as done in [17], to build a fundamental solution.

Notation 1.1. *Throughout the paper we will regard vectors $x \in \mathbb{R}^N$ as columns, and, we will write x^T, M^T to denote the transpose of a vector x or a matrix M. We also define the (symmetric, nonnegative) $N \times N$ matrix*

$$A(t) = \begin{bmatrix} A_0(t) & \mathbb{O} \\ \mathbb{O} & \mathbb{O} \end{bmatrix}. \tag{1.8}$$

Before stating our results, let us fix precise definitions of solution to the equation $\mathcal{L}u = 0$ and to a Cauchy problem for $\mathcal{L}$.

Definition 1.2. *We say that $u(x,t)$ is a solution to the equation $\mathcal{L}u = 0$ in $\mathbb{R}^N \times I$, for some open interval I, if:*

u is jointly continuous in $\mathbb{R}^N \times I$;

for every $t \in I$, $u(\cdot, t) \in C^2\left(\mathbb{R}^N\right)$;

for every $x \in \mathbb{R}^N$, $u(x, \cdot)$ *is absolutely continuous on* I, *and* $\frac{\partial u}{\partial t}$ *(defined for a.e.* t*) is essentially bounded for* t *ranging in every compact subinterval of* I;

for a.e. $t \in I$ *and every* $x \in \mathbb{R}^N$, $\mathcal{L}u(x, t) = 0$.

Definition 1.3. *We say that* $u(x, t)$ *is a solution to the Cauchy problem*

$$\begin{cases} \mathcal{L}u = 0 \text{ in } \mathbb{R}^N \times (t_0, T) \\ u(\cdot, t_0) = f \end{cases} \tag{1.9}$$

for some $T \in (-\infty, +\infty]$, $t_0 \in (-\infty, T)$, *where* f *is continuous in* $\mathbb{R}^N$ *or belongs to* $L^p\left(\mathbb{R}^N\right)$ *for some* $p \in [1, \infty)$ *if:*

(a) u *is a solution to the equation* $\mathcal{L}u = 0$ *in* $\mathbb{R}^N \times (t_0, T)$ *(in the sense of the above definition);*

(b_1*) if* $f \in C^0\left(\mathbb{R}^N\right)$ *then* $u(x, t) \to f(x_0)$ *as* $(x, t) \to \left(x_0, t_0^+\right)$, *for every* $x_0 \in \mathbb{R}^N$;

(b_2*) if* $f \in L^p\left(\mathbb{R}^N\right)$ *for some* $p \in [1, \infty)$ *then* $u(\cdot, t) \in L^p\left(\mathbb{R}^N\right)$ *for every* $t \in (t_0, T)$, *and* $\|u(\cdot, t) - f\|_{L^p(\mathbb{R}^N)} \to 0$ *as* $t \to t_0^+$.

In the following, we will also need the *transposed operator* of $\mathcal{L}$, defined by

$$\mathcal{L}^* u = \sum_{i,j=1}^{q} a_{ij}(s)\, \partial^2_{y_i y_j} u - \sum_{k,j=1}^{N} b_{jk} y_k \partial_{y_j} u - u \operatorname{Tr} B + \partial_s u. \tag{1.10}$$

The definition of solution to the equation $\mathcal{L}^* u = 0$ is perfectly analogous to Definition 1.2.

We can now state precisely the main results of the paper.

Theorem 1.4. *Under the assumptions (H1)–(H2) above, denote by* $E(s)$ *and* $C(t, t_0)$ *the following* $N \times N$ *matrices*

$$E(s) = \exp(-sB), \qquad C(t, t_0) = \int_{t_0}^{t} E(t - \sigma) A(\sigma) E(t - \sigma)^T \, d\sigma \tag{1.11}$$

for $s, t, t_0 \in \mathbb{R}$ *and* $t > t_0$. *Then the matrix* $C(t, t_0)$ *is symmetric and positive for every* $t > t_0$. *Let*

$$\Gamma(x, t; x_0, t_0) = \frac{1}{(4\pi)^{N/2} \sqrt{\det C(t, t_0)}} e^{-\left(\frac{1}{4}(x - E(t-t_0)x_0)^T C(t,t_0)^{-1}(x - E(t-t_0)x_0) + (t-t_0)\operatorname{Tr} B\right)} \tag{1.12}$$

for $t > t_0$, $\Gamma = 0$ *for* $t \le t_0$. *Then* Γ *has the following properties (so that* Γ *is a fundamental solution for* $\mathcal{L}$ *with pole* (x_0, t_0)*).*

(i) In the region

$$\mathbb{R}^{2N+2}_* = \left\{(x, t, x_0, t_0) \in \mathbb{R}^{2N+2} : (x, t) \neq (x_0, t_0)\right\} \tag{1.13}$$

the function Γ *is jointly continuous in* (x, t, x_0, t_0) *and is* C^∞ *with respect to* x, x_0. *The functions* $\frac{\partial^{\alpha+\beta}\Gamma}{\partial x^\alpha \partial x_0^\beta}$ *(for every multiindices* α, β*) are jointly continuous in* $(x, t, x_0, t_0) \in \mathbb{R}^{2N+2}_*$. *Moreover* Γ *and and* $\frac{\partial^{\alpha+\beta}\Gamma}{\partial x^\alpha \partial x_0^\beta}$ *are Lipschitz continuous with respect to* t *and with respect to* t_0 *in any region* $H \le t_0 + \delta \le t \le K$ *for fixed* $H, K \in \mathbb{R}$ *and* $\delta > 0$.

$\lim_{|x| \to +\infty} \Gamma(x, t; x_0, t_0) = 0$ *for every* $t > t_0$ *and every* $x_0 \in \mathbb{R}^N$.

$\lim_{|x_0|\to+\infty} \Gamma(x,t;x_0,t_0) = 0$ *for every* $t > t_0$ *and every* $x \in \mathbb{R}^N$.

(ii) For every fixed $(x_0,t_0) \in \mathbb{R}^{N+1}$, *the function* $\Gamma(\cdot,\cdot;x_0,t_0)$ *is a solution to* $\mathcal{L}u = 0$ *in* $\mathbb{R}^N \times (t_0,+\infty)$ *(in the sense of Definition 1.2);*

(iii) For every fixed $(x,t) \in \mathbb{R}^{N+1}$, *the function* $\Gamma(x,t;\cdot,\cdot)$ *is a solution to* $\mathcal{L}^* u = 0$ *in* $\mathbb{R}^N \times (-\infty,t)$*;*

(iv) Let $f \in C_b^0\left(\mathbb{R}^N\right)$ *(bounded continuous), or* $f \in L^p\left(\mathbb{R}^N\right)$ *for some* $p \in [1,\infty)$. *Then there exists one and only one solution to the Cauchy problem (1.9) (in the sense of Definition 1.3, with* $T = \infty$*) such that* $u \in C_b^0\left(\mathbb{R}^N \times [t_0,\infty)\right)$ *or* $u(t,\cdot) \in L^p\left(\mathbb{R}^N\right)$ *for every* $t > t_0$, *respectively. The solution is given by*

$$u(x,t) = \int_{\mathbb{R}^N} \Gamma(x,t;y,t_0) f(y)\, dy \tag{1.14}$$

and is $C^\infty\left(\mathbb{R}^N\right)$ *with respect to* x *for every fixed* $t > t_0$. *If moreover* f *is continuous and vanishes at infinity, then* $u(\cdot,t) \to f$ *uniformly in* $\mathbb{R}^N$ *as* $t \to t_0^+$.

(v) Let f *be a (possibly unbounded) continuous function on* $\mathbb{R}^N$ *satisfying the condition*

$$\int_{\mathbb{R}^N} |f(x)|\, e^{-\alpha|x|^2} dx < \infty, \tag{1.15}$$

for some $\alpha > 0$. *Then there exists* $T > 0$ *such that there exists one and only one solution* u *to the Cauchy problem (1.9) satisfying condition*

$$\int_{t_0}^{T} \int_{\mathbb{R}^N} |u(x,t)|\; e^{-C|x|^2}\, dx\, dt < +\infty \tag{1.16}$$

for some $C > 0$. *The solution* $u(x,t)$ *is given by (1.14) for* $t \in (t_0,T)$. *It is* $C^\infty\left(\mathbb{R}^N\right)$ *with respect to* x *for every fixed* $t \in (t_0,T)$.

(vi) Γ *satisfies for every* $x_0 \in \mathbb{R}^N$, $t_0 < t$ *the integral identities*

$$\int_{\mathbb{R}^N} \Gamma(x_0,t;y,t_0)\, dy = 1$$
$$\int_{\mathbb{R}^N} \Gamma(x,t;x_0,t_0)\, dx = e^{-(t-t_0)\operatorname{Tr} B}.$$

(vii) Γ *satisfies the reproduction formula*

$$\Gamma(x,t;y,s) = \int_{\mathbb{R}^N} \Gamma(x,t;z,\tau)\, \Gamma(z,\tau;y,s)\, dz$$

for every $x,y \in \mathbb{R}^N$ *and* $s < \tau < t$.

Remark 1.5. *Our uniqueness results only require the condition (1.16). Indeed, as we will prove in Proposition 4.14 all the solutions to the Cauchy problem (1.9), in the sense of Definition 1.3, with* $f \in L^p\left(\mathbb{R}^N\right)$ *for some* $p \in [1,\infty)$, $f \in C_b^0\left(\mathbb{R}^N\right)$ *or* $f \in C^0\left(\mathbb{R}^N\right)$ *with* f *satisfying (1.15), do satisfy the condition (1.16).*

Remark 1.6. *All the statements in the above theorem still hold if the coefficients* $a_{ij}(t)$ *are defined only for* t *belonging to some interval* I. *In this case the above formulas need to be considered only for* $t,t_0 \in I$. *In order to simplify notation, throughout the paper we will only consider the case* $I = \mathbb{R}$.

The above theorem will be proved in section 4.

The second main result of this paper is a comparison between Γ and the fundamental solutions Γ_α of the model operators (1.5) corresponding to $\alpha = \nu, \alpha = \nu^{-1}$ (with ν as in (1.2)). Specializing (1.12) to the operators (1.5) we have

$$\Gamma_\alpha (x, t; x_0, t_0) = \Gamma_\alpha (x - E (t - t_0) x_0, t - t_0; 0, 0)$$

with

$$\Gamma_\alpha (x, t; 0, 0) = \frac{1}{(4\pi\alpha)^{N/2} \sqrt{\det C_0 (t)}} e^{-\left(\frac{1}{4\alpha} x^T C_0(t)^{-1} x + t \operatorname{Tr} B\right)} \tag{1.17}$$

where, here and in the following, $C_0 (t) = C (t, 0)$ with $A_0 (t) = I_q$ (identity $q \times q$ matrix). Explicitly:

$$C_0 (t) = \int_0^t E (t - \sigma) I_{q,N} E (t - \sigma)^T d\sigma, \tag{1.18}$$

where $I_{q,N}$ is the $N \times N$ matrix given by

$$I_{q,N} = \begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix}.$$

Then:

Theorem 1.7. *For every $t > t_0$ and $x, x_0 \in \mathbb{R}^N$ we have*

$$\nu^N \Gamma_\nu (x, t; x_0, t_0) \leq \Gamma (x, t; x_0, t_0) \leq \frac{1}{\nu^N} \Gamma_{\nu^{-1}} (x, t; x_0, t_0). \tag{1.19}$$

The above theorem will be proved in section 3. The following example illustrates the reason why our comparison result is useful.

Example 1.8. *Let us consider the operator*

$$\mathcal{L}u = a (t) u_{x_1 x_1} + x_1 u_{x_2} - u_t$$

with $x \in \mathbb{R}^2$, $a (t)$ measurable and satisfying

$$0 < \nu \leq a (t) \leq \nu^{-1} \textit{ for every } t \in \mathbb{R}.$$

Let us compute $\Gamma (x, t; 0, 0)$ in this case. We have:

$$A = \begin{bmatrix} a (t) & 0 \\ 0 & 0 \end{bmatrix}; B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}; E (s) = \begin{bmatrix} 1 & 0 \\ -s & 1 \end{bmatrix};$$

$$C (t) \equiv C (t, 0) = \int_0^t \begin{bmatrix} 1 & 0 \\ -s & 1 \end{bmatrix} \begin{bmatrix} a (t - s) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -s \\ 0 & 1 \end{bmatrix} ds = \int_0^t a (t - s) \begin{bmatrix} 1 & -s \\ -s & s^2 \end{bmatrix} ds$$

(*after two integrations by parts*)

$$= \begin{bmatrix} a^* (t) & -a^{**} (t) \\ -a^{**} (t) & 2a^{***} (t) \end{bmatrix}$$

where we have set:

$$a^*(t) = \int_0^t a(s)\,ds; a^{**}(t) = \int_0^t a^*(s)\,ds; a^{***}(t) = \int_0^t a^{**}(s)\,ds.$$

Therefore we find, for $t > 0$:

$$\Gamma(x,t;0,0) = \frac{1}{4\pi\sqrt{\det C(t)}} e^{-\left(\frac{1}{4}x^T C(t)^{-1}x\right)}$$

with

$$C(t)^{-1} = \frac{1}{\det C(t)} \begin{bmatrix} 2a^{***}(t) & a^{**}(t) \\ a^{**}(t) & a^*(t) \end{bmatrix}$$

so that, explicitly, we have

$$\Gamma(x,t;0,0) = \frac{1}{4\pi\sqrt{\det C(t)}} \exp\left(-\frac{\left(2a^{***}(t)\,x_1^2 + 2a^{**}(t)\,x_1x_2 + a^*(t)\,x_2^2\right)}{4\det C(t)}\right)$$
$$\text{with } \det C(t) = 2a^*(t)\,a^{***}(t) - a^{**}(t)^2.$$

On the other hand, when considering the model operator

$$L_\alpha u = \alpha u_{x_1x_1} + x_1 u_{x_2} - u_t$$

with constant $\alpha > 0$, we have

$$\Gamma_\alpha(x,t;0,0) = \frac{\sqrt{3}}{2\pi\alpha t^2} \exp\left(-\frac{1}{\alpha}\left(\frac{x_1^2}{t} + \frac{3x_1x_2}{t^2} + \frac{3x_2^2}{t^3}\right)\right).$$

The comparison result of Theorem 1.7 then reads as follows:

$$\nu^2\Gamma_\nu(x,t;0,0) \le \Gamma(x,t;0,0) \le \frac{1}{\nu^2}\Gamma_{\nu^{-1}}(x,t;0,0)$$

or, explicitly,

$$\nu\frac{\sqrt{3}}{2\pi t^2} \exp\left(-\frac{1}{\nu}\left(\frac{x_1^2}{t} + \frac{3x_1x_2}{t^2} + \frac{3x_2^2}{t^3}\right)\right) \le \Gamma(x,t;0,0) \le \frac{1}{\nu}\frac{\sqrt{3}}{2\pi t^2} \exp\left(-\nu\left(\frac{x_1^2}{t} + \frac{3x_1x_2}{t^2} + \frac{3x_2^2}{t^3}\right)\right).$$

Plan of the paper. In §2 we compute the explicit expression of the fundamental solution Γ of $\mathcal{L}$ by using the Fourier transform and the method of characteristics, showing how one arrives to the the explicit formula (1.12). This procedure is somehow formal as, due to the nonsmoothness of the coefficients $a_{ij}(t)$, we cannot plainly assume that the functional setting where the construction is done is the usual distributional one. Since all the properties of Γ which qualify it as a fundamental solution will be proved in the subsequent sections, on a purely logical basis one could say that §2 is superfluous. Nevertheless, we prefer to present this complete computation to show how this formula has been built. A further reason to do this is the following one. The unique article where the analogous computation in the constant coefficient case is written in detail seems to be [11], and it is written in Russian language.

In §3 we prove Theorem 1.7, comparing Γ with the fundamental solutions of two model operators, which is easier to write explicitly and to study. In §4 we will prove Theorem 1.4, namely: point (i) in §4.1; points (ii), (iii), (vi) in §4.2; points (iv), (v), (vii) in §4.3.

2. Computation of the fundamental solution Γ

As explained at the end of the introduction, this section contains a formal computation of the fundamental solution Γ. To this aim, we choose any $(x_0, t_0) \in \mathbb{R}^{N+1}$, and we look for a solution to the Cauchy Problem

$$\begin{cases} \mathcal{L}u = 0 & \text{for } x \in \mathbb{R}^N, t > t_0 \\ u(\cdot, t_0) = \delta_{x_0} & \text{in } \mathcal{D}'\left(\mathbb{R}^N\right) \end{cases} \tag{2.1}$$

by applying the Fourier transform with respect to x, and using the notation

$$\widehat{u}(\xi, t) = \mathcal{F}(u(\cdot, t))(\xi) := \int_{\mathbb{R}^N} e^{-2\pi i x^T \xi} u(x, t) dx.$$

We have:

$$\sum_{i,j=1}^{q} a_{ij}(t)\left(-4\pi^2 \xi_i \xi_j\right)\widehat{u} + \sum_{k,j=1}^{N} b_{jk}\mathcal{F}\left(x_k \partial_{x_j} u\right) - \partial_t \widehat{u} = 0.$$

By the standard properties of the Fourier transform, it follows that

$$\mathcal{F}\left(x_k \partial_{x_j} u\right) = \frac{1}{-2\pi i}\partial_{\xi_k}\left(\mathcal{F}\left(\partial_{x_j} u\right)\right) = \frac{1}{-2\pi i}\partial_{\xi_k}\left(2\pi i \xi_j \widehat{u}\right) = -\left(\delta_{jk}\widehat{u} + \xi_j \partial_{\xi_k}\widehat{u}\right).$$

then the problem (2.1) is equivalent to the following Cauchy problem that we write in compact form (recalling the definition of the $A(t)$ given in (1.8)) as

$$\begin{cases} \left(\nabla_\xi \widehat{u}(\xi, t)\right)^T B^T \xi + \partial_t \widehat{u}(\xi, t) = -\left(4\pi^2 \xi^T A(t)\xi + \operatorname{Tr} B\right)\widehat{u}(\xi, t), \\ \widehat{u}(\xi, t_0) = e^{-2\pi i \xi^T x_0}. \end{cases} \tag{2.2}$$

Now we solve the problem (2.2) by the method of characteristics. Fix any initial condition $\eta \in \mathbb{R}^N$, and consider the system of ODEs:

$$\begin{cases} \frac{d\xi}{ds}(s) = B^T \xi(s), & \xi(0) = \eta, \\ \frac{dt}{ds}(s) = 1, & t(0) = t_0, \\ \frac{dz}{ds}(s) = -\left(4\pi^2 \xi^T(s) A(t(s))\xi(s) + \operatorname{Tr} B\right)z(s), & z(0) = e^{-2\pi i \eta^T x_0}. \end{cases} \tag{2.3}$$

We plainly find $t(s) = t_0 + s$ and $\xi(s) = \exp\left(sB^T\right)\eta$, so that the last equation becomes

$$\frac{dz}{ds}(s) = -\left(4\pi^2\left(\exp\left(sB^T\right)\eta\right)^T A(t_0 + s)\exp\left(sB^T\right)\eta + \operatorname{Tr} B\right)z(s),$$

whose solution, with initial condition $z(0) = e^{-2\pi i \eta^T x_0}$, is

$$z(s) = \exp\left(-4\pi^2 \int_0^s \eta^T\left[\exp(\sigma B) A(t_0 + \sigma)\exp\left(\sigma B^T\right)\right]\eta d\sigma - s \operatorname{Tr} B - 2\pi i \eta^T x_0\right).$$

Hence, substituting $s = t - t_0, \eta = \exp\left((t_0 - t) B^T\right)\xi$, recalling the notation introduced in (1.11), we find

$$\begin{aligned}\widehat{u}(\xi, t) &= z(t - t_0)\\ &= \exp\Bigg(-4\pi^2 \int_0^{t-t_0} \xi^T \exp\left((t_0 - t + \sigma) B\right) A\left(t_0 + \sigma\right) \exp\left((t_0 - t + \sigma) B^T\right) \xi d\sigma\\ &- (t - t_0) \operatorname{Tr} B - 2\pi i \xi^T \exp\left((t_0 - t) B\right) x_0\Bigg)\end{aligned}$$

$$\begin{aligned}&= \exp\Bigg(-4\pi^2 \xi^T \left(\int_{t_0}^{t} E(\sigma - t) A(\sigma) E(\sigma - t)^T d\sigma\right)\xi\\ &\qquad -(t - t_0) \operatorname{Tr} B - 2\pi i \xi^T E(t - t_0) x_0\Bigg)\\ &= \exp\left(-4\pi^2 \xi^T C(t, t_0) \xi - (t - t_0) \operatorname{Tr} B - 2\pi i \xi^T E(t - t_0) x_0\right). \qquad (2.4)\end{aligned}$$

Let

$$\begin{aligned}G(\xi, t; x_0, t_0) &= \exp\left(-4\pi^2 \xi^T C(t, t_0) \xi - (t - t_0) \operatorname{Tr} B - 2\pi i \xi^T E(t - t_0) x_0\right)\\ G_0(\xi, t, t_0) &= \exp\left(-4\pi^2 \xi^T C(t, t_0) \xi\right) \qquad (2.5)\end{aligned}$$

and note that if

$$\mathcal{F}\left(k(\cdot, t, t_0)\right)(\xi) = G_0(\xi, t, t_0)$$

then

$$\mathcal{F}\left(k(\cdot - E(t - t_0) x_0, t, t_0) \exp\left(-(t - t_0) \operatorname{Tr} B\right)\right)(\xi) = G(\xi, t; x_0, t_0), \qquad (2.6)$$

hence it is enough to compute the antitransform of $G_0(\xi, t, t_0)$. In order to do that, the following will be useful:

Proposition 2.1. *Let A be an $N \times N$ real symmetric positive constant matrix. Then:*

$$\mathcal{F}\left(e^{-(x^T A x)}\right)(\xi) = \left(\frac{\pi^N}{\det A}\right)^{1/2} e^{-\pi^2 \xi^T A^{-1} \xi}.$$

The above formula is a standard known result in probability theory, being the characteristic function of a multivariate normal distribution (see for instance [3, Prop. 1.1.2]).

To apply the previous proposition, and antitransform the function $G_0(\xi, t, t_0)$, we still need to know that the matrix $C(t, t_0)$ is strictly positive. By [13] we know that the matrix $C_0(t)$ (see (1.18)) is positive, under the structure conditions on B expressed in (1.4). Exploiting this fact, let us show that the same is true for our $C(t, t_0)$:

Proposition 2.2. *For every $\xi \in \mathbb{R}^N$ and every $t > t_0$ we have*

$$\nu^{-1} \xi^T C_0(t - t_0) \xi \geq \xi^T C(t, t_0) \xi \geq \nu \xi^T C_0(t - t_0) \xi. \qquad (2.7)$$

In particular, the matrix $C(t, t_0)$ is positive for $t > t_0$.

Proof.

$$\xi^T C(t,t_0)\xi = \int_{t_0}^{t} \xi^T E(t-s) A(s) E(t-s)^T \xi ds.$$

Next, letting $E(s) = \left(e_{ij}(s)\right)_{i,j=1}^N$ and $\eta_h(s) = \sum_{k=1}^N \xi_k e_{kh}(s)$ we have

$$\xi^T E(t-s) A(s) E(t-s)^T \xi = \sum_{i,j,h,k=1}^{N} \xi_i e_{ij}(t-s) a_{jh}(s) e_{kh}(t-s)\xi_k$$

$$= \sum_{j,h=1}^{q} a_{jh}(s)\eta_j(t-s)\eta_h(t-s) \geq \nu \sum_{j=1}^{q} \eta_j(t-s)^2 = \nu\xi^T E(t-s) I_{q,N} E(t-s)^T \xi$$

where

$$I_{q,N} = \begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix}.$$

Integrating for $s \in (t_0, t)$ the previous inequality we get

$$\xi^T C(t,t_0)\xi \geq \nu\xi^T \int_{t_0}^{t} E(t-s) I_{q,N} E(t-s)^T ds\xi = \nu\xi^T C_0(t-t_0)\xi.$$

Analogously we get the other bound. □

By the previous proposition, the matrix $C(t,t_0)$ is positive definite for every $t > t_0$, since, under our assumptions, this is true for $C_0(t-t_0)$. Therefore we can invert $C(t,t_0)$ and antitransform the function $G_0(\xi,t,t_0)$ in (2.5). Namely, applying Proposition 2.1 to $C(t,t_0)^{-1}$ we get:

$$\mathcal{F}\left(e^{-\left(x^T C(t,t_0)^{-1} x\right)}\right)(\xi) = \pi^{N/2}\sqrt{\det C(t,t_0)} e^{-\pi^2 \xi^T C(t,t_0)\xi}$$
$$\mathcal{F}\left(\frac{1}{(4\pi)^{N/2}\sqrt{\det C(t,t_0)}} e^{-\left(\frac{1}{4} x^T C(t,t_0)^{-1} x\right)}\right)(\xi) = e^{-4\pi^2 \xi^T C(t,t_0)\xi}.$$

Hence we have computed the antitransform of $G_0(\xi,t,t_0)$, and by (2.6) this also implies

$$\mathcal{F}\left(\frac{1}{(4\pi)^{N/2}\sqrt{\det C(t,t_0)}} e^{-\left(\frac{1}{4}(x-E(t-t_0)x_0)^T C(t,t_0)^{-1}(x-E(t-t_0)x_0)+(t-t_0)\operatorname{Tr} B\right)}\right)(\xi)$$
$$= \exp\left(-4\pi^2\xi^T C(t,t_0)\xi - (t-t_0)\operatorname{Tr} B - 2\pi i \xi^T E(t-t_0)x_0\right).$$

Hence the (so far, "formal") fundamental solution of $\mathcal{L}$ is

$$\Gamma(x,t;x_0,t_0) = \frac{1}{(4\pi)^{N/2}\sqrt{\det C(t,t_0)}} e^{-\left(\frac{1}{4}(x-E(t-t_0)x_0)^T C(t,t_0)^{-1}(x-E(t-t_0)x_0)+(t-t_0)\operatorname{Tr} B\right)},$$

which is the expression given in Theorem 1.4.

3. Comparison between Γ and fundamental solutions of model operators

In this section we will prove Theorem 1.7. The first step is to derive from Proposition 2.2 an analogous control between the quadratic forms associated to the inverse matrices $C_0(t-t_0)^{-1}, C(t,t_0)^{-1}$. The following algebraic fact will help:

Proposition 3.1. *Let C_1, C_2 be two real symmetric positive $N \times N$ matrices. If*

$$\xi^T C_1 \xi \leq \xi^T C_2 \xi \text{ for every } \xi \in \mathbb{R}^N \tag{3.1}$$

then

$$\xi^T C_2^{-1} \xi \leq \xi^T C_1^{-1} \xi \text{ for every } \xi \in \mathbb{R}^N$$

and

$$\det C_1 \leq \det C_2.$$

The first implication is already proved in [18, Remark 2.1.]. For convenience of the reader, we write a proof of both.

Proof. Let us fix some shorthand notation. Whenever (3.1) holds for two symmetric positive matrices, we will write $C_1 \leq C_2$. Note that for every symmetric $N \times N$ matrix G,

$$C_1 \leq C_2 \Longrightarrow GC_1G \leq GC_2G. \tag{3.2}$$

For any symmetric positive matrix C, we can rewrite $C = M^T \Delta M$ with M orthogonal and $\Delta = \operatorname{diag}(\lambda_1, ..., \lambda_n)$. Letting $C^{1/2} = M^T \Delta^{1/2} M$, one can check that $C^{1/2}$ is still symmetric positive, and $C^{1/2} C^{1/2} = I$. Moreover, writing $C^{-1/2} = \left(C^{-1}\right)^{1/2}$ we have

$$C^{-1/2} = M^T \Delta^{-1/2} M, \qquad C^{-1/2} C C^{-1/2} = I.$$

Then, applying (3.2) with $G = C_1^{-1/2}$ we get

$$I = C_1^{-1/2} C_1 C_1^{-1/2} \leq C_1^{-1/2} C_2 C_1^{-1/2}.$$

Next, applying (3.2) to the last inequality with $G = \left(C_1^{-1/2} C_2 C_1^{-1/2}\right)^{-1/2}$ we get

$$\begin{aligned} C_1^{1/2} C_2^{-1} C_1^{1/2} &= \left(C_1^{-1/2} C_2 C_1^{-1/2}\right)^{-1} = \left(C_1^{-1/2} C_2 C_1^{-1/2}\right)^{-1/2} \left(C_1^{-1/2} C_2 C_1^{-1/2}\right)^{-1/2} \\ &\leq \left(C_1^{-1/2} C_2 C_1^{-1/2}\right)^{-1/2} \left(C_1^{-1/2} C_2 C_1^{-1/2}\right) \left(C_1^{-1/2} C_2 C_1^{-1/2}\right)^{-1/2} = I. \end{aligned}$$

Finally, applying (3.2) to the last inequality with $G = C_1^{-1/2}$ we get

$$C_2^{-1} = C_1^{-1/2} \left(C_1^{1/2} C_2^{-1} C_1^{1/2}\right) C_1^{-1/2} \leq C_1^{-1/2} C_1^{-1/2} = C_1^{-1}$$

so the first statement is proved. To show the inequality on determinants, we can write, since $C_1 \leq C_2$,

$$C_2^{-1/2} C_1 C_2^{-1/2} \leq I.$$

Letting M be an orthogonal matrix that diagonalizes $C_2^{-1/2}C_1C_2^{-1/2}$ we get

$$\operatorname{diag}(\lambda_1, ..., \lambda_n) = M^T C_2^{-1/2} C_1 C_2^{-1/2} M \leq I$$

which implies $0 < \lambda_i \leq 1$ for $i = 1, 2, ..., n$ hence also

$$1 \geq \prod_{i=1}^{n} \lambda_i = \det\left(M^T C_2^{-1/2} C_1 C_2^{-1/2} M\right) = \frac{\det C_1}{\det C_2},$$

so we are done. □

Applying Propositions 3.1 and 2.2 we immediately get the following:

Proposition 3.2. *For every $\xi \in \mathbb{R}^N$ and every $t > t_0$ we have*

$$\nu^{-1}\xi^T C_0 (t-t_0)^{-1} \xi \geq \xi^T C(t,t_0)^{-1} \xi \geq \nu \xi^T C_0 (t-t_0)^{-1} \xi \tag{3.3}$$

$$\nu^{-N} \det C_0 (t-t_0) \geq \det C(t,t_0) \geq \nu^N \det C_0 (t-t_0) \tag{3.4}$$

for every $t > t_0$.

We are now in position to give the

Proof of Thm. 1.7. Recall that $C_0(t)$ is defined in (1.18). From the definition of the matrix $C(t,t_0)$ one immediately reads that, letting $C_\nu(t,t_0)$ be the matrix corresponding to the operator $\mathcal{L}_\nu$, one has

$$C_\nu(t,t_0) = \nu C_0(t-t_0) \tag{3.5}$$

hence also

$$\det(C_\nu(t,t_0)) = \nu^N \det C_0(t-t_0). \tag{3.6}$$

From the explicit form of Γ given in (1.12) we read that whenever the matrix $A(t)$ is constant one has

$$\Gamma(x,t;x_0,t_0) = \Gamma(x - E(t-t_0)x_0, t-t_0; 0, 0),$$

in particular this relation holds for Γ_ν. Then (1.12), (3.5), (3.6) imply (1.17). Therefore (3.3) and (3.4) give:

$$\begin{aligned}\Gamma(x,t;x_0,t_0) &= \frac{e^{-\left(\frac{1}{4}(x-E(t-t_0)x_0)^T C(t,t_0)^{-1}(x-E(t-t_0)x_0)+(t-t_0)\operatorname{Tr}B\right)}}{(4\pi)^{N/2}\sqrt{\det C(t,t_0)}} \\ &\leq \frac{e^{-\left(\frac{\nu}{4}(x-E(t-t_0)x_0)^T C_0(t-t_0)^{-1}(x-E(t-t_0)x_0)+(t-t_0)\operatorname{Tr}B\right)}}{(4\pi)^{N/2}\sqrt{\nu^N \det C_0(t-t_0)}} = \frac{1}{\nu^N}\Gamma_{\nu^{-1}}(x,t;x_0,t_0).\end{aligned}$$

Analogously,

$$\Gamma(x,t;x_0,t_0) \geq \frac{\nu^{N/2} e^{-\left(\frac{1}{4\nu}(x-E(t-t_0)x_0)^T C_0(t-t_0)^{-1}(x-E(t-t_0)x_0)+(t-t_0)\operatorname{Tr}B\right)}}{(4\pi)^{N/2}\sqrt{\det C_0(t-t_0)}} = \nu^N \Gamma_\nu(x,t;x_0,t_0)$$

so we have (1.19). □

As anticipated in the introduction, the above comparison result has further useful consequences when combined with some results of [13], where Γ_α is compared with the fundamental solution of the "principal part operator" $\widetilde{\mathcal{L}_\alpha}$ having the same matrix $A = \alpha I_{q,N}$ and a simpler matrix B, actually the matrix obtained from (1.4) annihilating all the $*$ blocks. This operator $\widetilde{\mathcal{L}_\alpha}$ is also 2-homogeneous with respect to dilations and its matrix $C_0(t)$ (which in the next statement is called $C_0^*(t)$) has a simpler form, which gives a useful asymptotic estimate for the matrix of $\mathcal{L}_\alpha$. Namely, the following holds:

Proposition 3.3 (Short-time asymptotics of the matrix $C_0(t)$)**.** *(See [13, (3.14), (3.9), (2.17)]) There exist integers* $1 = \sigma_1 \leq \sigma_2 \leq ... \leq \sigma_N = 2\kappa + 1$ *(with* κ *as in (1.4)), a constant invertible* $N \times N$ *matrix* $C_0^*(1)$ *and a* $N \times N$ *diagonal matrix*

$$D_0(\lambda) = \operatorname{diag}\left(\lambda^{\sigma_1}, \lambda^{\sigma_2}, ..., \lambda^{\sigma_N}\right)$$

such that the following holds. If we let

$$C_0^*(t) = D_0\left(t^{1/2}\right) C_0^*(1)\, D_0\left(t^{1/2}\right),$$

so that

$$\det C_0^*(t) = c_N t^Q$$

where $Q = \sum_{i=1}^N \sigma_i$*, then:*

$$\det C_0(t) = \det C_0^*(t)\,(1 + tO(1)) \;\; as\; t \to 0^+$$
$$x^T C_0(t)^{-1} x = x^T C_0^*(t)^{-1} x\,(1 + tO(1)) \;\; as\; t \to 0^+$$

where in the second equality $O(1)$ *stands for a bounded function on* $\mathbb{R}^N \times (0, 1]$.

The above result allows to prove the following more explicit upper bound on Γ for short times:

Proposition 3.4. *There exist constants* $c, \delta \in (0, 1)$ *such that for* $0 < t_0 - t \leq \delta$ *and every* $x, x_0 \in \mathbb{R}^N$ *we have:*

$$\Gamma(x, t; x_0, t_0) \leq \frac{1}{c\,(t - t_0)^{Q/2}} e^{-c\frac{|x - E(t-t_0)x_0|^2}{t-t_0}}. \tag{3.7}$$

Proof. By (1.19) and the properties of the fundamental solution when the matrix $A(t)$ is constant, we can write:

$$\Gamma(x, t; x_0, t_0) \leq \nu^{-N} \Gamma_{\nu^{-1}}(x - E(t - t_0)x_0, t - t_0; 0, 0). \tag{3.8}$$

On the other hand,

$$\Gamma_\alpha(y, t; 0, 0) = \frac{1}{(4\pi\alpha)^{N/2} \sqrt{\det C_0(t)}} e^{-\left(\frac{1}{4\alpha} y^T C_0(t)^{-1} y + t \operatorname{Tr} B\right)}$$

and by Proposition 3.3 there exist $c, \delta \in (0, 1)$ such that for $0 < t \leq \delta$ and every $y \in \mathbb{R}^N$

$$\det C_0(t) = \det C_0^*(t)\,(1 + tO(1)) \geq c \det C_0^*(t) = c_1 t^Q$$
$$y^T C_0(t)^{-1} y = y^T C_0^*(t)^{-1} y\,(1 + tO(1)) \geq c\, y^T C_0^*(t)^{-1} y$$

$$\geq c\left|D_0\left(t^{-1/2}\right)y\right|^2 = c\sum_{i=1}^{N}\frac{y_i^2}{t^{\sigma_i}} \geq c\frac{|y|^2}{t}.$$

Hence

$$\Gamma(x,t;x_0,t_0) \leq \frac{1}{(4\pi\nu)^{N/2}(t-t_0)^{Q/2}}e^{\frac{\nu}{4}|\mathrm{Tr}\,B|}e^{-\nu c\frac{|x-E(t-t_0)x_0|^2}{t-t_0}} = \frac{1}{c_2(t-t_0)^{Q/2}}e^{-c_2\frac{|x-E(t-t_0)x_0|^2}{t-t_0}}.$$

□

4. Properties of the fundamental solution and Cauchy problem

4.1. Regularity properties of Γ *and asymptotics*

In this section we will prove point (i) of Theorem 1.4.

With reference to the explicit form of Γ in (1.12), we start noting that the elements of the matrix

$$E(t-\sigma)A(\sigma)E(t-\sigma)^T$$

are measurable and uniformly essentially bounded for (t,σ,t_0) varying in any region $H \leq t_0 \leq \sigma \leq t \leq K$ for fixed $H,K \in \mathbb{R}$. This implies that the matrix

$$C(t,t_0) = \int_{t_0}^{t} E(t-\sigma)A(\sigma)E(t-\sigma)^T\,d\sigma$$

is Lipschitz continuous with respect to t and with respect to t_0 in any region $H \leq t_0 \leq t \leq K$ for fixed $H,K \in \mathbb{R}$. Moreover, $C(t,t_0)$ and $\det C(t,t_0)$ are jointly continuous in (t,t_0). Recalling that, by Proposition 2.2, the matrix $C(t,t_0)$ is positive definite for any $t > t_0$, we also have that $C(t,t_0)^{-1}$ is Lipschitz continuous with respect to t and with respect to t_0 in any region $H \leq t_0 + \delta \leq t \leq K$ for fixed $H,K \in \mathbb{R}$ and $\delta > 0$, and is jointly continuous in (t,t_0) for $t > t_0$.

From the explicit form of Γ and the previous remarks we conclude that $\Gamma(x,t;x_0,t_0)$ is jointly continuous in $(x,t;x_0,t_0)$ for $t > t_0$, smooth w.r.t. x and x_0 for $t > t_0$ and Lipschitz continuous with respect to t and with respect to t_0 in any region $H \leq t_0 + \delta \leq t \leq K$ for fixed $H,K \in \mathbb{R}$ and $\delta > 0$.

Moreover, every derivative $\frac{\partial^{\alpha+\beta}\Gamma}{\partial x^\alpha \partial^\beta x_0}$ is given by Γ times a polynomial in (x,x_0) with coefficients Lipschitz continuous with respect to t and with respect to t_0 in any region $H \leq t_0 + \varepsilon \leq t \leq K$ for fixed $H,K \in \mathbb{R}$ and $\varepsilon > 0$, and jointly continuous in (t,t_0) for $t > t_0$.

In order to show that Γ and $\frac{\partial^{\alpha+\beta}\Gamma}{\partial x^\alpha \partial x_0^\beta}$ are jointly continuous in the region $\mathbb{R}^{2N+2}_*$ (see (1.13)) we also need to show that these functions tend to zero as $(x,t) \to \left(y,t_0^+\right)$ and $y \neq x_0$. For Γ, this assertion follows by Proposition 3.4: for $y \neq x_0$ and $(x,t) \to \left(y,t_0^+\right)$ we have

$$|x - E(t-t_0)x_0|^2 \to |y - x_0|^2 \neq 0,$$

hence

$$\frac{1}{(t-t_0)^{Q/2}}e^{-c_2\frac{|x-E(t-t_0)x_0|^2}{t-t_0}} \to 0$$

and the same is true for $\Gamma(x,t;x_0,t_0)$.

To prove the analogous assertion for $\frac{\partial^{\alpha+\beta}\Gamma}{\partial x^\alpha \partial x_0^\beta}$ we first need to establish some upper bounds for these derivatives, which will be useful several times in the following.

Proposition 4.1. *For $t > s$, let $C(t,s)^{-1} = \left\{\gamma_{ij}(t,s)\right\}_{i,j=1}^{N}$, let*

$$C'(t,s) = E(t-s)^T C(t,s)^{-1} E(t-s)$$

and let $C'(t,s) = \left\{\gamma'_{ij}(t,s)\right\}_{i,j=1}^{N}$. Then:

(i) For every $x, y \in \mathbb{R}^N$, every $t > s$, $k, h = 1, 2, ..., N$,

$$\partial_{x_k}\Gamma(x,t;y,s) = -\frac{1}{2}\Gamma(x,t;y,s)\cdot\sum_{i=1}^{N}\gamma_{ik}(t,s)(x-E(t-s)y)_i \tag{4.1}$$

$$\partial^2_{x_h x_k}\Gamma(x,t;y,s) = \Gamma(x,t;y,s)\left(\frac{1}{4}\left(\sum_i \gamma_{ik}(t,s)(x-E(t-s)y)_i\right)\cdot\right. \tag{4.2}$$
$$\left.\cdot\left(\sum_j \gamma_{jh}(t,s)(x-E(t-s)y)_j\right) - \frac{1}{2}\gamma_{hk}(t,s)\right)$$

$$\partial_{y_k}\Gamma(x,t;y,s) = -\frac{1}{2}\Gamma(x,t;y,s)\cdot\sum_{i=1}^{N}\gamma'_{ik}(t,s)(y-E(s-t)x)_i \tag{4.3}$$

$$\partial^2_{y_h y_k}\Gamma(x,t;y,s) = \Gamma(x,t;y,s)\cdot\left(\frac{1}{4}\left(\sum_i \gamma'_{ik}(t,s)(y-E(s-t)x)_i\right)\cdot\right. \tag{4.4}$$
$$\left.\cdot\left(\sum_j \gamma'_{jh}(t,s)(y-E(s-t)x)_j\right) - \frac{1}{2}\gamma'_{hk}(t,s)\right).$$

(ii) For every $n, m = 0, 1, 2, ...$ there exists $c > 0$ such that for every $x, y \in \mathbb{R}^N$, every $t > s$

$$\sum_{|\alpha|\le n, |\beta|\le m}\left|\partial_x^\alpha \partial_y^\beta \Gamma(x,t;y,s)\right|$$
$$\le c\Gamma(x,t;y,s)\cdot\left\{1 + \left\|C(t,s)^{-1}\right\| + \left\|C(t,s)^{-1}\right\|^n |x-E(t-s)y|^n\right\}$$
$$\cdot\left\{1 + \|C'(t,s)\| + \|C'(t,s)\|^m |y-E(s-t)x|^m\right\} \tag{4.5}$$

where $\|\cdot\|$ stands for a matrix norm.

Proof. A straightforward computation gives (4.1) and (4.2). Iterating this computation we can also bound

$$\sum_{|\alpha|\le n}\left|\partial_x^\alpha\Gamma(x,t;y,s)\right| \le c\Gamma(x,t;y,s)\cdot$$
$$\cdot\left\{1 + \left\|C(t,s)^{-1}\right\| + \left\|C(t,s)^{-1}\right\|^n |x-E(t-s)y|^n\right\}.$$

To compute y-derivatives of Γ, it is convenient to write

$$(x-E(t-s)y)^T C(t,s)^{-1}(x-E(t-s)y)$$

$$= (y - E(s-t)x)^T C'(t,s)(y - E(s-t)x)$$

with

$$C'(t,s) = E(t-s)^T C(t,s)^{-1} E(t-s).$$

With this notation, we have

$$\Gamma(x,t;y,s) = \frac{1}{(4\pi)^{N/2}\sqrt{\det C(t,s)}} e^{-\left(\frac{1}{4}(y-E(s-t)x)^T C'(t,s)(y-E(s-t)x)+(t-s)\operatorname{Tr} B\right)}$$

and an analogous computation gives (4.3), (4.4) and, by iteration

$$\sum_{|\alpha|\le m} \left|\partial_y^\alpha \Gamma(x,t;y,s)\right| \le c\Gamma(x,t;y,s)\cdot$$
$$\cdot\left\{1 + \|C'(t,s)\| + \|C'(t,s)\|^m |y - E(s-t)x|^m\right\}$$

and finally also (4.5). □

With the previous bounds in hands we can now prove the following:

Theorem 4.2 (Upper bounds on the derivatives of Γ). *(i) For every* $n,m = 0,1,2...$ *and* t,s *ranging in a compact subset of* $\{(t,s) : t \ge s+\varepsilon\}$ *for some* $\varepsilon > 0$ *we have*

$$\sum_{|\alpha|\le n, |\beta|\le m} \left|\partial_x^\alpha \partial_y^\beta \Gamma(x,t;y,s)\right| \tag{4.6}$$
$$\le Ce^{-C'|x-E(t-s)y|^2}\cdot\left\{1 + |x - E(t-s)y|^n + |y - E(s-t)x|^m\right\}$$

for every $x,y\in\mathbb{R}^N$, *for constants* C,C' *depending on* n,m *and the compact set.*

In particular, for fixed $t > s$ *we have*

$$\lim_{|x|\to+\infty} \partial_x^\alpha \partial_y^\beta \Gamma(x,t;y,s) = 0 \text{ for every } y\in\mathbb{R}^N$$
$$\lim_{|y|\to+\infty} \partial_x^\alpha \partial_y^\beta \Gamma(x,t;y,s) = 0 \text{ for every } x\in\mathbb{R}^N$$

for every multiindices α,β.

(ii) For every $n,m = 0,1,2...$ *there exists* $\delta\in(0,1)$, $C,c > 0$ *such that for* $0 < t-s < \delta$ *and every* $x,y\in\mathbb{R}^N$ *we have*

$$\sum_{|\alpha|\le n, |\beta|\le m} \left|\partial_x^\alpha \partial_y^\beta \Gamma(x,t;y,s)\right|$$
$$\le \frac{C}{(t-s)^{Q/2}} e^{-c\frac{|x-E(t-s)y|^2}{t-s}}\cdot\left\{(t-s)^{-\sigma_N} + (t-s)^{-n\sigma_N}|x-E(t-s)y|^n\right\}$$
$$\cdot\left\{(t-s)^{-\sigma_N} + (t-s)^{-m\sigma_N}|y-E(s-t)x|^m\right\}. \tag{4.7}$$

In particular, for every fixed $x_0,y\in\mathbb{R}^N$, $x_0\ne y$, $s\in\mathbb{R}$,

$$\lim_{(x,t)\to(x_0,s^+)} \sum_{|\alpha|+|\beta|\le k} \left|\partial_x^\alpha \partial_y^\beta \Gamma(x,t;y,s)\right| = 0$$

so that Γ *and* $\partial_x^\alpha \partial_y^\beta \Gamma(x,t;y,s)$ *are jointly continuous in the region* $\mathbb{R}^{2N+2}_*$.

Proof. (i) The matrix $C(t,s)$ is jointly continuous in (t,s) and, by Proposition 2.2 is positive definite for any $t > s$. Hence for t, s ranging in a compact subset of $\{(t,s) : t \geq s + \varepsilon\}$ we have

$$\left\|C(t,s)^{-1}\right\|^n + \|C'(t,s)\|^m \leq c$$
$$e^{-\left(\frac{1}{4}(x-E(t-s)y)^T C(t,s)^{-1}(x-E(t-s)y)+(t-s)\operatorname{Tr} B\right)} \leq c_1 e^{-c|x-E(t-s)y|^2}$$

for some $c, c_1 > 0$ only depending on n, m and the compact set. Hence by (4.5) and (1.12) we get (4.6).

Let now t, s be fixed. If y is fixed and $|x| \to \infty$ then (4.6) gives

$$\sum_{|\alpha|\leq n, |\beta|\leq m} \left|\partial_x^\alpha \partial_y^\beta \Gamma(x,t;y,s)\right| \leq C e^{-C'|x|^2}\{1 + |x|^n + |x|^m\} \to 0.$$

If x is fixed and $|y| \to \infty$,

$$\sum_{|\alpha|\leq n, |\beta|\leq m} \left|\partial_x^\alpha \partial_y^\beta \Gamma(x,t;y,s)\right| \leq C e^{-C'|E(t-s)y|^2}\{1 + |E(t-s)y|^n + |E(s-t)x|^m\} \to 0,$$

because when $|y| \to \infty$ also $|E(t-s)y| \to \infty$, since $E(t-s)$ is invertible.

(ii) Applying (4.5) together with Proposition 3.4 we get that for some $\delta \in (0,1)$, whenever $0 < t - s < \delta$ we have

$$\begin{aligned}&\sum_{|\alpha|\leq n, |\beta|\leq m} \left|\partial_x^\alpha \partial_y^\beta \Gamma(x,t;y,s)\right| \\ &\leq \frac{1}{c(t-s)^{Q/2}} e^{-c\frac{|x-E(t-s)y|^2}{t-s}} \cdot \left\{1 + \left\|C(t,s)^{-1}\right\| + \left\|C(t,s)^{-1}\right\|^n |x - E(t-s)y|^n\right\} \\ &\cdot \{1 + \|C'(t,s)\| + \|C'(t,s)\|^m |y - E(s-t)x|^m\}.\end{aligned}$$

Next, we recall that by Proposition 3.2 we have

$$\left\|C(t,s)^{-1}\right\| \leq c\left\|C_0(t-s)^{-1}\right\|$$

by Proposition 3.3, for $0 < t - s \leq \delta$

$$\leq c'\left\|C_0^*(t-s)^{-1}\right\| \leq c''(t-s)^{-\sigma_N}$$

and an analogous bound holds for $C'(t,s)$, for small $(t-s)$. Hence we get (4.7).

If now $x_0 \neq y$ are fixed, from (4.7) we deduce

$$\sum_{|\alpha|\leq n, |\beta|\leq m} \left|\partial_x^\alpha \partial_y^\beta \Gamma(x,t;y,s)\right| \leq \frac{C}{(t-s)^{\frac{Q}{2}+(n+m)\sigma_N}} \exp\left(-\frac{c}{t-s}\right) \to 0$$

as $(x,t) \to (x_0, s^+)$. □

With the above theorem, the proof of point (i) in Theorem 1.4 is complete.

Remark 4.3 (Long time behavior of Γ). *We have shown that the fundamental solution $\Gamma(x,t;y,s)$ and its spacial derivatives of every order tend to zero for x or y going to infinity, and tend to zero for $t \to s^+$ and $x \neq y$. It is natural to ask what happens for $t \to +\infty$. However, nothing can be said in general* *about this limit, even when the coefficients a_{ij} are constant, and even in nondegenerate cases. Compare, for $N = 1$, the heat operator*

$$Hu = u_{xx} - u_t,$$

for which

$$\Gamma(x, y; 0, 0) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} \to 0 \textit{ for } t \to +\infty, \textit{ every } x \in \mathbb{R}$$

and the operator

$$Lu = u_{xx} + xu_x - u_t$$

for which (1.12) gives

$$\Gamma(x, t; 0, 0) = \frac{1}{\sqrt{2\pi(1 - e^{-2t})}} e^{-\frac{x^2}{2(1-e^{-2t})}} \to \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \textit{ as } t \to +\infty.$$

4.2. Γ is a solution

In this section we will prove points (ii), (iii), (vi) of Theorem 1.4.

We want to check that our "candidate fundamental solution" with pole at (x_0, t_0), given by (1.12), actually solves the equation outside the pole, with respect to (x, t). Note that, by the results in § 4.1 we already know that Γ is infinitely differentiable w.r.t. x, x_0, and a.e. differentiable w.r.t. t, t_0.

Theorem 4.4. *For every fixed $(x_0, t_0) \in \mathbb{R}^{N+1}$,*

$$\mathcal{L}(\Gamma(\cdot, \cdot; x_0, t_0))(x, t) = 0 \textit{ for a.e. } t > t_0 \textit{ and every } x \in \mathbb{R}^N.$$

Before proving the theorem, let us establish the following easy fact, which will be useful in the subsequent computation and is also interesting in its own:

Proposition 4.5. *For every $t > t_0$ and $x_0 \in \mathbb{R}^N$ we have*

$$\int_{\mathbb{R}^N} \Gamma(x, t; x_0, t_0)\, dx = e^{-(t-t_0)\operatorname{Tr} B} \tag{4.8}$$

$$\int_{\mathbb{R}^N} \Gamma(x_0, t; y, t_0)\, dy = 1.$$

Proof. Let us compute, for $t > t_0$:

$$\int_{\mathbb{R}^N} \Gamma(x, t; x_0, t_0)\, dx$$

$$= \frac{e^{-(t-t_0)\operatorname{Tr} B}}{(4\pi)^{N/2}\sqrt{\det C(t, t_0)}} \int_{\mathbb{R}^N} e^{-\frac{1}{4}(x - E(t-t_0)x_0)^T C^{-1}(t,t_0)(x - E(t-t_0)x_0)} dx$$

letting $x = E(t - t_0)x_0 + 2C(t, t_0)^{1/2} y; dx = 2^N \det C(t, t_0)^{1/2} dy$

$$= \frac{e^{-(t-t_0)\operatorname{Tr} B}}{(4\pi)^{N/2}\sqrt{\det C(t, t_0)}} 2^N \sqrt{\det C(t, t_0)} \int_{\mathbb{R}^N} e^{-|y|^2} dy = e^{-(t-t_0)\operatorname{Tr} B}.$$

Next,

$$\int_{\mathbb{R}^N} \Gamma(x_0, t; y, t_0)\, dy$$

$$= \frac{e^{-(t-t_0)\operatorname{Tr} B}}{(4\pi)^{N/2}\sqrt{\det C(t,t_0)}} \int_{\mathbb{R}^N} e^{-\frac{1}{4}(x_0 - E(t-t_0)y)^T C^{-1}(t,t_0)(x_0 - E(t-t_0)y)} dy$$

letting $y = E(t_0 - t)\left(x_0 - 2C(t,t_0)^{1/2} z\right)$;

$$dy = 2^N \det C(t,t_0)^{1/2} \det E(t_0 - t)\, dz = 2^N \det C(t,t_0)^{1/2} e^{(t-t_0)\operatorname{Tr} B} dz$$

$$= \frac{e^{-(t-t_0)\operatorname{Tr} B}}{(4\pi)^{N/2}\sqrt{\det C(t,t_0)}} 2^N \det C(t,t_0)^{1/2} e^{(t-t_0)\operatorname{Tr} B} \int_{\mathbb{R}^N} e^{-|y|^2} dy = 1.$$

Here in the change of variables we used the relation $\det(\exp B) = e^{\operatorname{Tr} B}$, holding for every square matrix B. □

Proof of Theorem 4.4. Keeping the notation of Proposition 4.1, and exploiting (4.1)–(4.2) we have

$$\begin{aligned} \sum_{k,j=1}^{N} b_{jk} x_k \partial_{x_j} \Gamma(x,t;x_0,t_0) &= \left(\nabla_x \Gamma(x,t;x_0,t_0)\right)^T Bx \\ &= -\frac{1}{2}\Gamma(x,t;x_0,t_0)\left(x - E(t-t_0)x_0\right)^T C(t,t_0)^{-1} Bx. \end{aligned} \tag{4.9}$$

$$\begin{aligned} &\sum_{h,k=1}^{q} a_{hk}(t)\, \partial^2_{x_h x_k} \Gamma(x,t;x_0,t_0) \\ &= \Gamma \left\{ \frac{1}{4} \sum_{i,j=1}^{N} \left(\sum_{h,k=1}^{q} a_{hk}(t)\, \gamma_{ik}(t,t_0)\, \gamma_{jh}(t) \right) \cdot \right. \\ &\left. \cdot \left(x - E(t-t_0)x_0\right)_i \left(x - E(t-t_0)x_0\right)_j - \frac{1}{2} \sum_{h,k=1}^{q} a_{hk}(t)\, \gamma_{hk}(t,t_0) \right\} \\ &= \Gamma(x,t;x_0,t_0) \cdot \left\{ \frac{1}{4} \left(x - E(t-t_0)x_0\right)^T C(t,t_0)^{-1} A(t)\, C^{-1} \left(x - E(t-t_0)x_0\right) \right. \\ &\left. -\frac{1}{2} \operatorname{Tr} A(t)\, C(t,t_0)^{-1} \right\}. \end{aligned} \tag{4.10}$$

$$\begin{aligned} &\partial_t \Gamma(x,t;x_0,t_0) \\ &= -\frac{\partial_t \left(\det C(t,t_0)\right)}{(4\pi)^{N/2}\, 2 \det^{3/2} C(t,t_0)} e^{-\left(\frac{1}{4}(x-E(t-t_0)x_0)^T C(t,t_0)^{-1}(x-E(t-t_0)x_0) + (t-t_0)\operatorname{Tr} B\right)} \\ &- \Gamma(x,t;x_0,t_0) \cdot \\ &\cdot \partial_t \left(\frac{1}{4} \left(x - E(t-t_0)x_0\right)^T C(t,t_0)^{-1} \left(x - E(t-t_0)x_0\right) + (t-t_0)\operatorname{Tr} B \right) \\ &= -\Gamma(x,t;x_0,t_0) \left\{ \frac{\partial_t \left(\det C(t,t_0)\right)}{2 \det C(t,t_0)} \right. \\ &\left. + \frac{1}{4} \partial_t \left(\left(x - E(t-t_0)x_0\right)^T C(t,t_0)^{-1} \left(x - E(t-t_0)x_0\right) \right) + \operatorname{Tr} B \right\}. \end{aligned} \tag{4.11}$$

To shorten notation, from now on, throughout this proof, we will write

$$C \text{ for } C(t,t_0)\text{, and}$$
$$E \text{ for } E(t-t_0)\,.$$

To compute the t-derivative appearing in (4.11) we start writing

$$\begin{aligned}&\partial_t\left((x-Ex_0)^T C^{-1}(x-Ex_0)\right)\\&=2(-\partial_t Ex_0)^T C^{-1}(x-Ex_0)\\&+(x-Ex_0)^T\partial_t\left(C^{-1}\right)(x-Ex_0)\,.\end{aligned}\tag{4.12}$$

First, we note that

$$\partial_t E=-B\exp\left(-(t-t_0)B\right)=-BE.\tag{4.13}$$

Also, note that B commutes with $E(t)$ and B^T commutes with $E(t)^T$. Second, differentiating the identity $C^{\cdot -1}C=I$ we get

$$\partial_t\left(C^{-1}\right)=-C^{-1}\partial_t(C)\,C^{-1}.\tag{4.14}$$

In turn, at least for a.e. t, we have

$$\begin{aligned}\partial_t\left(C(t,t_0)\right)&=E(0)A(t)E(0)^T+\int_{t_0}^{t}\partial_t E(t-\sigma)A(\sigma)E(t-\sigma)^T d\sigma\\&+\int_{t_0}^{t}E(t-\sigma)A(\sigma)\partial_t E(t-\sigma)^T d\sigma\\&=A(t)-BC-CB^T.\end{aligned}$$

By (4.14) this gives

$$\partial_t\left(C^{-1}\right)=-C^{-1}A(t)C^{-1}+C^{-1}B+B^TC^{-1}.\tag{4.15}$$

Inserting (4.13) and (4.15) in (4.12) and then in (4.11) we have

$$\begin{aligned}&\partial_t\left((x-Ex_0)^T C^{-1}(x-Ex_0)\right)\\&=2(BEx_0)^T C^{-1}(x-Ex_0)\\&+(x-Ex_0)^T\left[-C^{-1}A(t)C^{-1}+2B^TC^{-1}\right](x-Ex_0)\,.\end{aligned}$$

$$\begin{aligned}\partial_t\Gamma=-\Gamma\Bigg\{&\frac{\partial_t(\det C)}{2\det C}+\operatorname{Tr}B+\frac{1}{4}\Big[2(BEx_0)^T C^{-1}(x-Ex_0)\\&+(x-Ex_0)^T\left[-C^{-1}A(t)C^{-1}+2B^TC^{-1}\right](x-Ex_0)\Big]\Bigg\}\end{aligned}$$

$$\begin{aligned}&=-\Gamma\Bigg\{\frac{\partial_t(\det C)}{2\det C}+\operatorname{Tr}B-\frac{1}{4}(x-Ex_0)^T C^{-1}A(t)C^{-1}(x-Ex_0)\\&+\frac{1}{2}x^TB^TC^{-1}(x-Ex_0)\Bigg\}.\end{aligned}\tag{4.16}$$

Exploiting (4.10), (4.9) and (4.16) we can now compute $\mathcal{L}\Gamma$:

$$
\begin{aligned}
&\sum_{h,k=1}^{q} a_{hk}(t)\,\partial^2_{x_h x_k}\Gamma + (\nabla\Gamma)^T Bx - \partial_t\Gamma \\
&= \Gamma\left\{\frac{1}{4}(x - Ex_0)^T C^{-1} A(t) C^{-1}(x - Ex_0) - \frac{1}{2}\operatorname{Tr} A(t) C^{-1}\right. \\
&- \frac{1}{2}\Gamma (x - Ex_0)^T C^{-1} Bx + \frac{\partial_t(\det C)}{2\det C} + \operatorname{Tr} B \\
&\left. - \frac{1}{4}(x - Ex_0)^T C^{-1} A(t) C^{-1}(x - Ex_0) + \frac{1}{2} x^T B^T C^{-1}(x - Ex_0)\right\} \\
&= \Gamma\left\{-\frac{1}{2}\operatorname{Tr} A(t) C^{-1} + \frac{\partial_t(\det C)}{2\det C} + \operatorname{Tr} B\right\}.
\end{aligned}
$$

To conclude our proof we are left to check that, in the last expression, the quantity in braces identically vanishes for $t > t_0$. This, however, is not a straightforward computation, since the term $\partial_t(\det C)$ is not easily explicitly computed. Let us state this fact as a separate ancillary result. □

Proposition 4.6. *For a.e. $t > t_0$ we have*

$$
\frac{\partial_t(\det C(t, t_0))}{2\det C(t, t_0)} = \frac{1}{2}\operatorname{Tr} A(t) C(t, t_0)^{-1} - \operatorname{Tr} B.
$$

To prove this proposition we also need the following

Lemma 4.7. *For every $N \times N$ matrix A, and every $x_0 \in \mathbb{R}^N$ we have:*

$$
\begin{aligned}
&\int_{\mathbb{R}^N} e^{-|x|^2}\left(x^T Ax\right) dx = \frac{\pi^{N/2}}{2}\operatorname{Tr} A \\
&\int_{\mathbb{R}^N} e^{-|x|^2}\left(x_0^T Ax\right) dx = 0.
\end{aligned}
\tag{4.17}
$$

Proof of Lemma 4.7. The second identity is obvious for symmetry reasons. As to the first one, letting $A = \left(a_{ij}\right)_{i,j=1}^N$,

$$
\begin{aligned}
&\int_{\mathbb{R}^N} e^{-|x|^2}\left(x^T Ax\right) dx \\
&= \sum_{i=1}^{N}\left\{\sum_{j=1,\dots,N, j\neq i} a_{ij}\int_{\mathbb{R}^N} e^{-|x|^2} x_i x_j dx + a_{ii}\int_{\mathbb{R}^N} e^{-|x|^2} x_i^2 dx\right\} \\
&= \sum_{i=1}^{N}\left\{0 + a_{ii}\left(\int_{\mathbb{R}^{N-1}} e^{-|w|^2} dw\right)\left(\int_{\mathbb{R}} e^{-x_i^2} x_i^2 dx_i\right)\right\} \\
&= \sum_{i=1}^{N} a_{ii}\pi^{\frac{N-1}{2}}\left(\int_{\mathbb{R}} e^{-t^2} t^2 dt\right) = \pi^{\frac{N-1}{2}}\cdot\frac{\sqrt{\pi}}{2}\sum_{i=1}^{N} a_{ii} = \frac{\pi^{N/2}}{2}\operatorname{Tr} A
\end{aligned}
$$

where the integrals corresponding to the terms with $i \neq j$ vanish for symmetry reasons. □

Proof of Proposition 4.6. Taking $\frac{\partial}{\partial t}$ in the identity (4.8) we have, by (4.16), for almost every $t > t_0$,

$$
\begin{aligned}
&-e^{-(t-t_0)\operatorname{Tr}B}\operatorname{Tr}B = \int_{\mathbb{R}^N} \frac{\partial \Gamma}{\partial t}(x,t;x_0,t_0)\,dx \\
&= -\int_{\mathbb{R}^N} \Gamma(x,t;x_0,t_0)\left\{\frac{\partial_t(\det C)}{2\det C} + \operatorname{Tr}B + \frac{1}{2}x^T B^T C^{-1}(x-Ex_0)\right. \\
&\left. -\frac{1}{4}(x-Ex_0)^T C^{-1}A(t)C^{-1}(x-Ex_0)\right\}dx
\end{aligned}
$$

$$
\begin{aligned}
&= -\left\{\frac{\partial_t(\det C)}{2\det C} + \operatorname{Tr}B\right\}e^{-(t-t_0)\operatorname{Tr}B} - \int_{\mathbb{R}^N}\Gamma(x,t;x_0,t_0)\left\{\frac{1}{2}x^T B^T C^{-1}(x-Ex_0)\right. \\
&\left. -\frac{1}{4}(x-Ex_0)^T C^{-1}A(t)C^{-1}(x-Ex_0)\right\}dx
\end{aligned}
$$

hence

$$
\begin{aligned}
&\frac{\partial_t(\det C)}{2\det C}\cdot e^{-(t-t_0)\operatorname{Tr}B} = -\frac{e^{-(t-t_0)\operatorname{Tr}B}}{(4\pi)^{N/2}\sqrt{\det C}}\cdot \\
&\cdot\int_{\mathbb{R}^N} e^{-\frac{1}{4}(x-Ex_0)^T C^{-1}(x-Ex_0)}\left\{\frac{1}{2}x^T B^T C^{-1}(x-Ex_0)\right. \\
&\left. -\frac{1}{4}(x-Ex_0)^T C^{-1}A(t)C^{-1}(x-Ex_0)\right\}dx
\end{aligned}
$$

and letting again $x = Ex_0 + 2C^{1/2}y$ inside the integral

$$
\begin{aligned}
&\frac{\partial_t(\det C)}{2\det C} = -\frac{1}{\pi^{N/2}}\int_{\mathbb{R}^N} e^{-|y|^2}\cdot \\
&\cdot\left\{\left(x_0^T E^T + 2y^T C^{1/2}\right)B^T C^{-1/2}y - y^T C^{-1/2}A(t)C^{-1/2}y\right\}dy \\
&= -\frac{1}{\pi^{N/2}}\frac{\pi^{N/2}}{2}\left(0 + 2\operatorname{Tr}C^{1/2}B^T C^{-1/2} + \operatorname{Tr}C^{-1/2}A(t)C^{-1/2}\right) \\
&= -\operatorname{Tr}C^{1/2}B^T C^{-1/2} + \frac{1}{2}\operatorname{Tr}C^{-1/2}A(t)C^{-1/2}.
\end{aligned}
$$

where we used Lemma 4.7. Finally, since similar matrices have the same trace,

$$
\begin{aligned}
&-\operatorname{Tr}C^{1/2}BC^{-1/2} + \frac{1}{2}\operatorname{Tr}C^{-1/2}A(t)C^{-1/2} \\
&= -\operatorname{Tr}B + \frac{1}{2}\operatorname{Tr}A(t)C^{-1},
\end{aligned}
$$

so we are done. □

The proof of Proposition 4.6 also completes the proof of Theorem 4.4.

Remark 4.8. *Since, by Theorem 4.4, we can write*

$$
\partial_t\Gamma(x,t,x_0,t_0) = \sum_{i,j=1}^{q} a_{ij}(t)\,\partial^2_{x_i x_j}\Gamma(x,t,x_0,t_0) + \sum_{k,j=1}^{N} b_{jk}x_k\partial_{x_j}\Gamma(x,t,x_0,t_0),
$$

the function $\partial_t\Gamma$ satisfies upper bounds analogous to those proved in Theorem 4.2 for $\partial^2_{x_i x_j}\Gamma$.

Let us now show that Γ satisfies, with respect to the other variables, the transposed equation, that is:

Theorem 4.9. *Letting*

$$\mathcal{L}^{*}u=\sum_{i,j=1}^{q}a_{ij}(s)\,\partial_{y_iy_j}^{2}u-\sum_{k,j=1}^{N}b_{jk}y_k\partial_{j_j}u-u\operatorname{Tr}B+\partial_s u$$

we have, for every fixed (x,t)

$$\mathcal{L}^{*}\left(\Gamma\left(x,t;\cdot,\cdot\right)\right)(y,s)=0$$

for a.e. $s<t$ *and every* y.

Proof. We keep the notation used in the proof of Proposition 4.1:

$$C'(t,s)=E(t-s)^{T}\,C(t,s)^{-1}\,E(t-s)$$
$$\Gamma(x,t;y,s)=\frac{1}{(4\pi)^{N/2}\sqrt{\det C(t,s)}}e^{-\left(\frac{1}{4}(y-E(s-t)x)^{T}C'(t,s)(y-E(s-t)x)+(t-s)\operatorname{Tr}B\right)}.$$

Exploiting (4.3) and (4.4) we have, by a tedious computation which is analogous to that in the proof of Theorem 4.4,

$$\begin{aligned}\mathcal{L}^{*}\Gamma(x,t;y,s)&=\frac{1}{2}\Gamma(x,t;y,s)\Bigg\{-\operatorname{Tr}A(s)\,C'(t,s)-\frac{\partial_s(\det C(t,s))}{\det C(t,s)}\\&+y^{T}B^{T}C'(t,s)\,y-y^{T}B^{T}E(t-s)^{T}\,C(t,s)^{-1}\,x\\&+(BE(t-s)\,y)^{T}\,C(t,s)^{-1}\,(x-E(t-s)\,y)\Big\}\\&=\frac{1}{2}\Gamma(x,t;y,s)\left\{-\operatorname{Tr}A(s)\,C'(t,s)-\frac{\partial_s(\det C(t,s))}{\det C(t,s)}\right\}.\end{aligned}$$

So we are done provided that: □

Proposition 4.10. *For a.e.* $s<t$ *we have*

$$\frac{\partial_s(\det C(t,s))}{2\det C(t,s)}=-\operatorname{Tr}A(s)\,C'(t,s).$$

Proof. Taking $\frac{\partial}{\partial s}$ in the identity (4.8) we have, by (4.16), for almost every $s<t$,

$$\begin{aligned}&e^{-(t-s)\operatorname{Tr}B}\operatorname{Tr}B=\int_{\mathbb{R}^N}\frac{\partial\Gamma}{\partial s}(x,t;x_0,s)\,dx\\&=-\int_{\mathbb{R}^N}\Gamma(x,t;x_0,s)\cdot\\&\cdot\Bigg\{\frac{\partial_s(\det C)}{2\det C}-\operatorname{Tr}B-\frac{1}{2}(BE(t-s)\,x_0)^{T}\,C(t,s)^{-1}\,(x-E(t-s)\,x_0)\\&+\frac{1}{4}(E(s-t)\,x-x_0)^{T}\,C'(t,s)\,A(s)\,C'(t,s)\,(E(s-t)\,x-x_0)\Bigg\}\,dx\end{aligned}$$

$$= -\left\{\frac{\partial_s(\det C)}{2\det C} - \operatorname{Tr} B\right\} e^{-(t-s)\operatorname{Tr} B}$$

$$- \int_{\mathbb{R}^N} \Gamma(x,t;x_0,s)\left\{-\frac{1}{2}(BE(t-s)x_0)^T C(t,s)^{-1}(x-E(t-s)x_0)\right.$$

$$\left.+\frac{1}{4}(E(s-t)x-x_0)^T C'(t,s)A(s)C'(t,s)(E(s-t)x-x_0)\right\} dx$$

hence

$$\frac{\partial_s(\det C)}{2\det C} = -\frac{1}{(4\pi)^{N/2}\sqrt{\det C}}\int_{\mathbb{R}^N} e^{-\frac{1}{4}(x-E(t-s)x_0)^T C(t,s)^{-1}(x-E(t-s)x_0)}\cdot$$

$$\cdot\left\{-\frac{1}{2}(BE(t-s)x_0)^T C(t,s)^{-1}(x-E(t-s)x_0)\right.$$

$$\left.+\frac{1}{4}(E(s-t)x-x_0)^T C'(t,s)A(s)C'(t,s)(E(s-t)x-x_0)\right\} dx$$

and letting again $x = E(t-s)x_0 + 2C^{1/2}(t,s)y$ inside the integral, applying Lemma 4.7 and (4.17), with some computation we get

$$\frac{\partial_s(\det C)}{\det C} = -\operatorname{Tr} C^{-1/2}(t,s)E(t-s)A(s)E(t-s)^T C(t,s)^{-1/2}.$$

Since $C^{-1/2}(t,s)E(t-s)A(s)E(t-s)^T C(t,s)^{-1/2}$ and $A(s)C'(t,s)$ are similar, they have the same trace, so the proof is concluded. □

4.3. The Cauchy problem

In this section we will prove points (iv), (v), (vii) of Theorem 1.4.

We are going to show that the Cauchy problem can be solved, by means of our fundamental solution Γ. Just to simplify notation, let us now take $t_0 = 0$ and let $C(t) = C(t,0)$. We have the following:

Theorem 4.11. *Let*

$$u(x,t) = \int_{\mathbb{R}^N} \Gamma(x,t;y,0)f(y)\,dy$$

$$= \frac{e^{-t\operatorname{Tr} B}}{(4\pi)^{N/2}\sqrt{\det C(t)}}\int_{\mathbb{R}^N} e^{-\frac{1}{4}(x-E(t)y)^T C(t)^{-1}(x-E(t)y)} f(y)\,dy. \tag{4.18}$$

Then:

(a) if $f \in L^p(\mathbb{R}^N)$ for some $p \in [1,\infty]$ or $f \in C_b^0(\mathbb{R}^N)$ (bounded continuous) then u solves the equation $\mathcal{L}u = 0$ in $\mathbb{R}^N \times (0,\infty)$ and $u(\cdot,t) \in C^\infty(\mathbb{R}^N)$ for every fixed $t > 0$.

(b) if $f \in C^0(\mathbb{R}^N)$ and there exists $C > 0$ such that (1.15) holds, then there exists $T > 0$ such that u solves the equation $\mathcal{L}u = 0$ in $\mathbb{R}^N \times (0,T)$ and $u(\cdot,t) \in C^\infty(\mathbb{R}^N)$ for every fixed $t \in (0,T)$.

The initial condition f is attained in the following senses:

(i) For every $p \in [1,+\infty)$, if $f \in L^p(\mathbb{R}^N)$ we have $u(\cdot,t) \in L^p(\mathbb{R}^N)$ for every $t > 0$, and

$$\|u(\cdot,t) - f\|_{L^p(\mathbb{R}^N)} \to 0 \text{ as } t \to 0^+.$$

(ii) If $f \in L^{\infty}\left(\mathbb{R}^N\right)$ *and* f *is continuous at some point* $x_0 \in \mathbb{R}^N$ *then*

$$u(x,t) \to f(x_0) \text{ as } (x,t) \to (x_0, 0).$$

(iii) If $f \in C_*^0\left(\mathbb{R}^N\right)$ *(i.e., vanishing at infinity) then*

$$\sup_{x\in\mathbb{R}^N} |u(x,t) - f(x)| \to 0 \text{ as } t \to 0^+.$$

(iv) If $f \in C^0\left(\mathbb{R}^N\right)$ *and satisfies (1.15), then*

$$u(x,t) \to f(x_0) \text{ as } (x,t) \to (x_0, 0).$$

Proof. From Theorem 4.2, (i), we read that for (x,t) ranging in a compact subset of $\mathbb{R}^N \times (0,+\infty)$, and every $y \in \mathbb{R}^N$,

$$\sum_{|\alpha|\le n} \left|\partial_x^{\alpha}\Gamma(x,t;y,0)\right| \le ce^{-c_1|y|^2} \cdot \{1 + |y|^n\}$$

for suitable constants $c, c_1 > 0$. Moreover, by Remark 4.8, $|\partial_t\Gamma|$ also satisfies this bound (with $n = 2$). This implies that for every $f \in L^p\left(\mathbb{R}^N\right)$ for some $p \in [1,\infty]$, (in particular for $f \in C_b^0\left(\mathbb{R}^N\right)$) the integral defining u converges and $\mathcal{L}u$ can be computed taking the derivatives inside the integral. Moreover, all the derivatives $u_{x_i}, u_{x_i x_j}$ are continuous, while u_t is defined only almost everywhere, and locally essentially bounded. Then by Theorem 4.4 we have $\mathcal{L}u(x,t) = 0$ for a.e. $t > 0$ and every $x \in \mathbb{R}^N$. Also, the x-derivatives of every order can be actually taken under the integral sign, so that $u(\cdot,t) \in C^{\infty}\left(\mathbb{R}^N\right)$. This proves (a). Postponing for a moment the proof of (b), to show that u attains the initial condition (points (i)–(iii)) let us perform, inside the integral in (4.18), the change of variables

$$\begin{aligned} C(t)^{-1/2}(x - E(t)y) &= 2z \\ y &= E(-t)\left(x - 2C(t)^{1/2}z\right) \\ dy &= 2^N e^{t\,\mathrm{Tr}\,B} \det C(t)^{1/2}\, dz \end{aligned}$$

so that

$$u(x,t) = \frac{1}{\pi^{N/2}} \int_{\mathbb{R}^N} e^{-|z|^2} f\left(E(-t)\left(x - 2C(t)^{1/2}z\right)\right) dz$$

and, since $\int_{\mathbb{R}^N} \frac{e^{-|z|^2}}{\pi^{N/2}} dz = 1$,

$$|u(x,t) - f(x)| \le \int_{\mathbb{R}^N} \frac{e^{-|z|^2}}{\pi^{N/2}} \left| f\left(E(-t)\left(x - 2C(t)^{1/2}z\right)\right) - f(x)\right| dz.$$

Let us now proceed separately in the three cases.

(i) By Minkowsky's inequality for integrals we have

$$\|u(\cdot,t) - f\|_{L^p(\mathbb{R}^N)} \le \int_{\mathbb{R}^N} \frac{e^{-|z|^2}}{\pi^{N/2}} \left\| f\left(E(-t)\left(\cdot - 2C(t)^{1/2}z\right)\right) - f(\cdot)\right\|_{L^p(\mathbb{R}^N)} dz.$$

Next,

$$\left\| f\left(E(-t)\left(\cdot - 2C(t)^{1/2}z\right)\right) - f(\cdot)\right\|_{L^p(\mathbb{R}^N)}$$

$$\le \left\| f\left(E(-t)\left(\cdot - 2C(t)^{1/2} z\right)\right)\right\|_{L^p(\mathbb{R}^N)} + \|f\|_{L^p(\mathbb{R}^N)}$$
$$= \|f(E(-t)(\cdot))\|_{L^p(\mathbb{R}^N)} + \|f\|_{L^p(\mathbb{R}^N)} \le c\,\|f\|_{L^p(\mathbb{R}^N)}$$

for $0 < t < 1$, since

$$\|f(E(-t)(\cdot))\|^p_{L^p(\mathbb{R}^N)} = \int_{\mathbb{R}^N} |f(E(-t)(x))|^p\, dx$$

letting $E(-t)\,x = y; x = E(t)\,y; dx = e^{-t\operatorname{Tr} B} dy$,

$$= e^{-t\operatorname{Tr} B}\,\|f\|_{L^p(\mathbb{R}^N)} \le e^{|\operatorname{Tr} B|}\,\|f\|_{L^p(\mathbb{R}^N)} \text{ for } 0 < t < 1.$$

This means that for every $t \in (0, 1)$ we have

$$\frac{e^{-|z|^2}}{\pi^{N/2}} \left\| f\left(E(-t)\left(\cdot - 2C(t)^{1/2} z\right)\right) - f(\cdot)\right\|_{L^p(\mathbb{R}^N)} \le c\,\|f\|_{L^p(\mathbb{R}^N)} \frac{e^{-|z|^2}}{\pi^{N/2}} \in L^1\left(\mathbb{R}^N\right).$$

Let us show that for a.e. fixed $z \in \mathbb{R}^N$ we also have

$$\frac{e^{-|z|^2}}{\pi^{N/2}} \left\| f\left(E(-t)\left(\cdot - 2C(t)^{1/2} z\right)\right) - f(\cdot)\right\|_{L^p(\mathbb{R}^N)} \to 0 \text{ as } t \to 0^+,$$

this will imply the desired result by Lebesgue's theorem.

$$\left\| f\left(E(-t)\left(\cdot - 2C(t)^{1/2} z\right)\right) - f(\cdot)\right\|_{L^p(\mathbb{R}^N)}$$
$$\le \left\| f\left(E(-t)\left(\cdot - 2C(t)^{1/2} z\right)\right) - f(E(-t)\,\cdot)\right\|_{L^p(\mathbb{R}^N)} + \|f(E(-t)(\cdot)) - f\|_{L^p(\mathbb{R}^N)}.$$

Now:

$$\left\| f\left(E(-t)\left(\cdot - 2C(t)^{1/2} z\right)\right) - f(E(-t)\,\cdot)\right\|^p_{L^p(\mathbb{R}^N)}$$
$$= \int_{\mathbb{R}^N} \left| f\left(E(-t)\left(x - 2C(t)^{1/2} z\right)\right) - f(E(-t)\,x)\right|^p dx$$
$$= e^{t\operatorname{Tr} B} \int_{\mathbb{R}^N} \left| f\left(y - 2E(-t)\,C(t)^{1/2} z\right) - f(y)\right|^p dy \to 0$$

for z fixed and $t \to 0^+$, because $2E(-t)\,C(t)^{1/2} z \to 0$ and the translation operator is continuous on $L^p\left(\mathbb{R}^N\right)$.

It remains to show that

$$\|f(E(-t)(\cdot)) - f\|_{L^p(\mathbb{R}^N)} \to 0 \text{ as } t \to 0^+,$$

which is not straightforward. For every fixed $\varepsilon > 0$, let ϕ be a compactly supported continous function such that $\|f - \phi\|_p < \varepsilon$, then

$$\|f(E(-t)(\cdot)) - f\|_p \le \|f(E(-t)(\cdot)) - \phi(E(-t)(\cdot))\|_p$$
$$+ \|\phi(E(-t)(\cdot)) - \phi\|_p + \|f - \phi\|_p$$

and

$$\|f(E(-t)(\cdot)) - \phi(E(-t)(\cdot))\|_p = \left(e^{t\operatorname{Tr}B}\right)^{1/p}\|f-\phi\|_p \le \left(e^{|\operatorname{Tr}B|}\right)^{1/p}\varepsilon$$

for $t \in (0,1)$. Let $\operatorname{sprt}\phi \subset B_R(0)$, then for every $t \in (0,1)$ we have $|E(-t)(x)| \le c|x|$ so that

$$\|\phi(E(-t)(\cdot)) - \phi\|^p_{L^p(\mathbb{R}^N)} = \int_{|x|<CR} |\phi(E(-t)(x)) - \phi(x)|^p\,dx.$$

Since for every $x \in \mathbb{R}^N$, $\phi(E(-t)(x)) \to \phi(x)$ as $t \to 0^+$ and

$$|\phi(E(-t)(x)) - \phi(x)|^p \le 2\max|\phi|^p$$

which is integrable on $B_{CR}(0)$, by uniform continuity of ϕ,

$$\|\phi(E(-t)(\cdot)) - \phi\|_{L^p(\mathbb{R}^N)} \to 0 \text{ as } t \to 0^+,$$

hence for t small enough

$$\|f(E(-t)(\cdot)) - f\|_p \le c\varepsilon,$$

and we are done.

(ii) Let $f \in L^\infty\left(\mathbb{R}^N\right)$, and let f be continuous at some point $x_0 \in \mathbb{R}^N$ then

$$|u(x,t) - f(x_0)| \le \int_{\mathbb{R}^N} \frac{e^{-|z|^2}}{\pi^{N/2}}\left|f\left(E(-t)\left(x - 2C(t)^{1/2}z\right)\right) - f(x_0)\right|dz.$$

Now, for fixed $z \in \mathbb{R}^N$ and $(x,t) \to (x_0,0)$ we have

$$E(-t)\left(x - 2C(t)^{1/2}z\right) \to x_0$$
$$f\left(E(-t)\left(x - 2C(t)^{1/2}z\right)\right) \to f(x_0)$$

while

$$\frac{e^{-|z|^2}}{\pi^{N/2}}\left|f\left(E(-t)\left(x - 2C(t)^{1/2}z\right)\right) - f(x_0)\right| \le 2\|f\|_{L^\infty(\mathbb{R}^N)}\frac{e^{-|z|^2}}{\pi^{N/2}} \in L^1\left(\mathbb{R}^N\right)$$

hence by Lebesgue's theorem

$$|u(x,t) - f(x_0)| \to 0.$$

(iii) As in point (i) we have

$$\|u(\cdot,t) - f\|_{L^\infty(\mathbb{R}^N)} \le \int_{\mathbb{R}^N} \frac{e^{-|z|^2}}{\pi^{N/2}}\left\|f\left(E(-t)\left(\cdot - 2C(t)^{1/2}z\right)\right) - f(\cdot)\right\|_{L^\infty(\mathbb{R}^N)}dz$$

and as in point (ii)

$$\frac{e^{-|z|^2}}{\pi^{N/2}}\left\|f\left(E(-t)\left(\cdot - 2C(t)^{1/2}z\right)\right) - f(\cdot)\right\|_{L^\infty(\mathbb{R}^N)} \le 2\|f\|_{L^\infty(\mathbb{R}^N)}\frac{e^{-|z|^2}}{\pi^{N/2}} \in L^1\left(\mathbb{R}^N\right).$$

Let us show that for every fixed z we have

$$\left\|f\left(E(-t)\left(\cdot - 2C(t)^{1/2}z\right)\right) - f(\cdot)\right\|_{L^\infty(\mathbb{R}^N)} \to 0 \text{ as } t \to 0^+,$$

hence by Lebesgue's theorem we will conclude the desired assertion.

For every $\varepsilon > 0$ we can pick $\phi \in C_c^0\left(\mathbb{R}^N\right)$ such that $\|f - \phi\|_\infty < \varepsilon$, then

$$\begin{aligned}\|f(E(-t)(\cdot)) - f\|_\infty &\le \|f(E(-t)(\cdot)) - \phi(E(-t)(\cdot))\|_\infty + \|\phi(E(-t)(\cdot)) - \phi\|_\infty + \|f - \phi\|_\infty \\ &< 2\varepsilon + \|\phi(E(-t)(\cdot)) - \phi\|_\infty .\end{aligned}$$

Since ϕ is compactly supported, there exists $R > 0$ such that for every $t \in (0, 1)$ we have $\phi(E(-t)(x)) - \phi(x) \neq 0$ only if $|x| < R$.

$$|E(-t)(x) - x| \le |E(-t) - I|\, R.$$

Since ϕ is uniformly continuous, for every $\varepsilon > 0$ there exists $\delta > 0$ such that for $0 < t < \delta$ we have

$$|\phi(E(-t)(x)) - \phi(x)| < \varepsilon$$

whenever $|x| < R$. So we are done.

Let us now prove (b). To show that u is well defined, smooth in x, and satisfies the equation, for $|x| \le R$ let us write

$$\begin{aligned}u(x,t) &= \int_{|y|<2R} \Gamma(x,t;y,0)\, f(y)\, dy + \int_{|y|>2R} \Gamma(x,t;y,0)\, f(y)\, dy \\ &\equiv I(x,t) + II(x,t).\end{aligned}$$

Since f is bounded for $|y| < 2R$, reasoning like in the proof of point (a) we see that $\mathcal{L}I(x,t)$ can be computed taking the derivatives under the integral sign, so that $\mathcal{L}I(x,t) = 0$. Moreover, the function $x \mapsto I(x,t)$ is $C^\infty\left(\mathbb{R}^N\right)$.

To prove the analogous properties for $II(x,t)$ we have to apply Theorem 4.2, (ii): there exists $\delta \in (0,1), C, c > 0$ such that for $0 < t < \delta$ and every $x, y \in \mathbb{R}^N$ we have, for $n = 0, 1, 2, ...$

$$\sum_{|\alpha|\le n} \left|\partial_x^\alpha \Gamma(x,t;y,0)\right| \le \frac{C}{t^{Q/2}} e^{-c\frac{|x-E(t)y|^2}{t}} \cdot \left\{t^{-\sigma_N} + t^{-n\sigma_N} |x - E(t)\, y|^n\right\}.$$

Recall that $|x| < R$ and $|y| > 2R$. For δ small enough and $t \in \left(\frac{\delta}{2}, \delta\right)$ we have

$$\sum_{|\alpha|\le n} \left|\partial_x^\alpha \Gamma(x,t;y,0)\right| \le C e^{-c\frac{|y|^2}{t}} \cdot \{1 + |y|^n\}$$

with constants depending on δ, n. Therefore, if α is the constant appearing in the assumption (1.15),

$$\begin{aligned}&\int_{|y|>2R} \sum_{|\alpha|\le n} \left|\partial_x^\alpha \Gamma(x,t;y,0)\right| |f(y)|\, dy \\ &\le C \int_{|y|>2R} e^{-c\frac{|y|^2}{\delta}} \cdot \{1 + |y|^n\}\, e^{\alpha|y|^2} |f(y)|\, e^{-\alpha|y|^2} dy \\ &\le C \sup_{y\in\mathbb{R}^N} \left(e^{\left(-\frac{c}{\delta}+\alpha\right)|y|^2} \{1 + |y|^n\}\right) \cdot \int_{\mathbb{R}^N} |f(y)|\, e^{-\alpha|y|^2} dy\end{aligned}$$

which shows that for δ small enough $\mathcal{L}II(x,t)$ can be computed taking the derivatives under the integral sign, so that $\mathcal{L}II(x,t)=0$. Moreover, the function $x\mapsto II(x,t)$ is $C^{\infty}\left(\mathbb{R}^N\right)$. This proves (b).

(iv). For $|x_0|\leq R$ let us write

$$u(x,t)=\int_{|y|<2R}\Gamma(x,t;y,0)\,f(y)\,dy+\int_{|y|>2R}\Gamma(x,t;y,0)\,f(y)\,dy\equiv I+II.$$

Applying point (ii) to $f(y)\,\chi_{B_{2r}(0)}$ we have

$$I=\int_{|y|<2R}\Gamma(x,t;y,0)\,f(y)\,dy\to f(x_0)$$

as $(x,t)\to(x_0,0)$. Let us show that $II\to 0$. By (3.7) we have

$$|II|\leq\int_{|y|>2R}\frac{1}{ct^{Q/2}}e^{-c\frac{|x-E(t)y|^2}{t}}\,|f(y)|\,dy.$$

For y fixed with $|y|>2R$, hence $|x_0-y|\neq 0$, we have

$$\lim_{(x,t)\to(x_0,0)}\frac{1}{t^{Q/2}}e^{-c\frac{|x-E(t)y|^2}{t}}=\lim_{(x,t)\to(x_0,0)}\frac{1}{t^{Q/2}}e^{-c\frac{|x_0-y|^2}{t}}=0.$$

Since $|y|>2R$, $|x_0|<R$, for $x\to x_0$ we can assume $|x|<\frac{3}{2}R$ and for t small enough we have $|x-E(t)\,y|\geq c\,|y|$ for some $c>0$, hence

$$\begin{aligned}\frac{1}{ct^{Q/2}}e^{-c\frac{|x-E(t)y|^2}{t}}\,|f(y)|\,\chi_{\{|y|>2R\}}&\leq\frac{1}{ct^{Q/2}}e^{-c_1\frac{|y|^2}{t}}e^{\alpha|y|^2}\chi_{\{|y|>2R\}}\,|f(y)|\,e^{-\alpha|y|^2}\\&\leq\frac{1}{ct^{Q/2}}e^{\left(\alpha-\frac{c_1}{t}\right)|y|^2}\chi_{\{|y|>2R\}}\left\{|f(y)|\,e^{-\alpha|y|^2}\right\}\end{aligned}$$

for t small enough

$$\begin{aligned}&\leq\frac{1}{ct^{Q/2}}e^{-\frac{c_1}{2t}|y|^2}\chi_{\{|y|>2R\}}\left\{|f(y)|\,e^{-\alpha|y|^2}\right\}\\&\leq\frac{1}{ct^{Q/2}}e^{-\frac{2c_1}{t}R^2}\left\{|f(y)|\,e^{-\alpha|y|^2}\right\}\leq c\,|f(y)|\,e^{-\alpha|y|^2}\in L^1\left(\mathbb{R}^N\right).\end{aligned}$$

Hence by Lebesgue's theorem $II\to 0$ as $(x,t)\to(x_0,0)$, and we are done. □

Remark 4.12. *If f is an unbounded continuous function satisfying (1.15), the solution of the Cauchy problem can blow up in finite time, already for the heat operator: the solution of*

$$\begin{cases}u_t-u_{xx}=0\text{ in }\mathbb{R}\times(0,+\infty)\\u(x,0)=e^{x^2}\end{cases}$$

is given by

$$u(x,t)=\frac{1}{\sqrt{4\pi t}}\int_{\mathbb{R}}e^{-\frac{(x-y)^2}{4t}}e^{y^2}dy=\frac{e^{\frac{x^2}{1-4t}}}{\sqrt{1-4t}}\text{ for }0<t<\frac{1}{4},$$

with $u(x,t)\to+\infty$ for $t\to\left(\frac{1}{4}\right)^-$.

We next prove a uniqueness results for the Cauchy problem (1.9). In the following we consider solutions defined in some possibly bounded time interval $[0, T)$.

Theorem 4.13 (Uniqueness)**.** *Let $\mathcal{L}$ be an operator of the form (1.1) satisfying the assumptions (H1)–(H2), let $T \in (0, +\infty]$, and let either $f \in C(\mathbb{R}^N)$, or $f \in L^p(\mathbb{R}^N)$ with $1 \le p < +\infty$.*

If u_1 and u_2 are two solutions to the same Cauchy problem

$$\begin{cases} \mathcal{L}u = 0 \text{ in } \mathbb{R}^N \times (0, T), \\ u(\cdot, 0) = f, \end{cases} \tag{4.19}$$

satisfying (1.16) for some $C > 0$, then $u_1 \equiv u_2$ in $\mathbb{R}^N \times (0, T)$.

Proof. Because of the linearity of $\mathcal{L}$, it is enough to prove that if the function $u := u_1 - u_2$ satisfies (4.19) with $f = 0$ and (1.16), then $u(x, t) = 0$ for every $(x, t) \in \mathbb{R} \times (0, +\infty)$. We will prove that $u = 0$ in a suitably thin strip $\mathbb{R} \times (0, t_1)$, where t_1 only depends on $\mathcal{L}$ and C, the assertion then will follow by iterating this argument.

Let $t_1 \in (0, T]$ be a fixed mumber that will be specified later. For every positive R we consider a function $h_R \in C^\infty(\mathbb{R}^N)$, such that $h_R(\xi) = 1$ whenever $|\xi| \le R$, $h_R(\xi) = 0$ for every $|\xi| \ge R + 1/2$ and that $0 \le h_R(\xi) \le 1$. We also assume that all the first and second order derivatives of h_R are bounded by a constant that doesn't depend on R. We fix a point $(y, s) \in \mathbb{R}^N \times (0, t_1)$, and we let v denote the function

$$v(\xi, \tau) := h_R(\xi) \Gamma(y, s; \xi, \tau).$$

For $\varepsilon \in (0, t_1/2)$ we define the domain

$$Q_{R,\varepsilon} := \left\{(\xi, \tau) \in \mathbb{R}^N \times (0, t_1) : |\xi| < R + 1, \varepsilon < \tau < s - \varepsilon\right\}$$

and we also let $Q_R = Q_{R,0}$. Note that in $Q_{R,\varepsilon}$ the function $v(\xi, \tau)$ is smooth in ξ and Lipschitz continuous in τ.

By (1.1) and (1.10) we can compute the following Green identity, with u and v as above.

$$\begin{aligned} & v\mathcal{L}u - u\mathcal{L}^*v \\ & = \sum_{i,j=1}^{q} a_{ij}(t) \left(v\partial^2_{x_i x_j} u - u\partial^2_{x_i x_j} v\right) + \sum_{k,j=1}^{N} b_{jk} x_k \left(v\partial_{x_j} u + u\partial_{x_j} v\right) \\ & - (v\partial_t u + u\partial_t v) + uv \operatorname{Tr} B \\ & = \sum_{i,j=1}^{q} \partial_{x_i} \left(a_{ij}(t) \left(v\partial_{x_j} u - u\partial_{x_j} v\right)\right) + \sum_{k,j=1}^{N} \partial_{x_j} \left(b_{jk} x_k uv\right) - \partial_t (uv). \end{aligned}$$

We now integrate the above identity on $Q_{R,\varepsilon}$ and apply the divergence theorem, noting that $v, \partial_{x_1} v, \ldots, \partial_{x_N} v$ vanish on the lateral part of the boundary of $Q_{R,\varepsilon}$, by the properties of h_R. Hence:

$$\begin{aligned} & \int_{Q_{R,\varepsilon}} v(\xi, \tau)\mathcal{L}u(\xi, \tau) - u(\xi, \tau)\mathcal{L}^* v(\xi, \tau) d\xi\, d\tau \\ & = \int_{\mathbb{R}^N} u(\xi, \varepsilon) v(\xi, \varepsilon) d\xi - \int_{\mathbb{R}^N} u(\xi, s - \varepsilon) v(\xi, s - \varepsilon) d\xi. \end{aligned} \tag{4.20}$$

Concerning the last integral, since the function $y \mapsto h_R(y)u(y,s)$ is continuous and compactly supported, by Theorem 4.11, (iii) we have that

$$\int_{\mathbb{R}^N} u(\xi, s-\varepsilon)v(\xi, s-\varepsilon)d\xi = \int_{\mathbb{R}^N} u(\xi, s-\varepsilon)h_R(\xi)\Gamma(y,s;\xi,s-\varepsilon)d\xi \to h_R(y)u(y,s)$$

as $\varepsilon \to 0^+$. Moreover

$$\int_{\mathbb{R}^N} u(\xi,\varepsilon)v(\xi,\varepsilon)d\xi = \int_{\mathbb{R}^N} u(\xi,\varepsilon)h_R(\xi)\Gamma(y,s;\xi,\varepsilon)d\xi \to 0,$$

as $\varepsilon \to 0^+$, since Γ is a bounded function whenever $(\xi,\varepsilon) \in \mathbb{R}^N \times (0, s/2)$, and $u(\cdot,\varepsilon)h_R \to 0$ either uniformly, if the inital datum is assumed by continuity, or in the L^p norm. Using the fact that $\mathcal{L}u = 0$ and $u(\cdot, 0) = 0$, we conclude that, as $|y| < R$, (4.20) gives

$$u(y,s) = \int_{Q_R} u(\xi,\tau)\mathcal{L}^* v(\xi,\tau)d\xi\, d\tau. \tag{4.21}$$

Since $\mathcal{L}^*\Gamma(y,s;\xi,\tau) = 0$ whenever $\tau < s$, we have

$$\begin{aligned}\mathcal{L}^*(h_R\Gamma) &= \sum_{i,j=1}^{q} a_{ij}(\tau)\,\partial^2_{\xi_i\xi_j}(h_R\Gamma) - \sum_{k,j=1}^{N} b_{jk}\xi_k\partial_{\xi_j}(h_R\Gamma) - h_R(\Gamma\,\mathrm{Tr}\,B + \partial_\tau\Gamma)\\ &= \Gamma\sum_{i,j=1}^{q} a_{ij}(\tau)\,\partial^2_{\xi_i\xi_j}h_R + 2\sum_{i,j=1}^{q} a_{ij}(\tau)\left(\partial_{\xi_i}h_R\right)\left(\partial_{\xi_j}\Gamma\right) - \Gamma\sum_{k,j=1}^{N} b_{jk}\xi_k\partial_{\xi_j}h_R\end{aligned}$$

therefore the identity (4.21) yields, since $\partial_{\xi_i}h_R = 0$ for $|\xi| \le R$,

$$\begin{aligned}u(y,s) = \int_{Q_R\setminus Q_{R-1}} u(\xi,\tau)&\Bigg\{\sum_{i,j=1}^{q} a_{i,j}(\tau)\cdot\\ &\cdot\left[2\partial_{\xi_i}h_R(\xi)\partial_{\xi_j}\Gamma(y,s;\xi,\tau) + \Gamma(y,s;\xi,\tau)\partial_{\xi_i\xi_j}h_R(\xi)\right]\\ &-\sum_{k,j=1}^{N} b_{jk}\xi_k\partial_{\xi_j}h_R(\xi)\Gamma(y,s;\xi,\tau)\Bigg\}d\xi\,d\tau.\end{aligned} \tag{4.22}$$

We claim that (4.22) implies

$$|u(y,s)| \le \int_{Q_R\setminus Q_{R-1}} C_1|u(\xi,\tau)|e^{-C|\xi|^2}d\xi\,d\tau, \tag{4.23}$$

for some positive constant C_1 only depending on the operator $\mathcal{L}$ and on the uniform bound f the derivatives of h_R, provided that t_1 is sufficiently small. Our assertion then follows by letting $R \to +\infty$.

So we are left to prove (4.23). By Proposition 3.4 we know that, for suitable constants $\delta \in (0,1)$, $c_1, c_2 > 0$, for $0 < s-\tau \le \delta$ and every $y, \xi \in \mathbb{R}^N$ we have:

$$\Gamma(y,s,\xi,\tau) \le \frac{c_1}{(s-\tau)^{Q/2}}e^{-c_2\frac{|y-E(s-\tau)\xi|^2}{s-\tau}}. \tag{4.24}$$

Moreover, from the computation in the proof of Theorem 4.9 we read that

$$\nabla_{\xi}\Gamma(y,s;\xi,\tau) = -\frac{1}{2}\Gamma(y,s;\xi,\tau)\,C'(s,\tau)(\xi - E(\tau - s)\,y)$$

where

$$C'(s,\tau) = E(s-\tau)^T\,C(s,\tau)^{-1}\,E(s-\tau).$$

Hence

$$\nabla_{\xi}\Gamma(y,s;\xi,\tau) = \frac{1}{2}\Gamma(y,s;\xi,\tau)\,E(s-\tau)^T\,C(s,\tau)^{-1}(y - E(s-\tau)\,\xi).$$

By (3.3) we have inequality for matrix norms

$$\left\|C(s,\tau)^{-1}\right\| \leq c\left\|C_0(s-\tau)^{-1}\right\|$$

and, for $0 < s-\tau \leq \delta$

$$\leq c\left\|C_0^{*(s-\tau)-1}\right\| \leq c\,\|D_0(s-\tau)\|^{-1}$$

hence

$$\begin{aligned}\left|\nabla_{\xi}\Gamma(y,s;\xi,\tau)\right| &\leq c\Gamma(y,s;\xi,\tau)\,\|D_0(s-\tau)\|^{-1}\,|y - E(s-\tau)\,\xi| \\ &\leq \frac{c_1}{(s-\tau)^{\frac{Q}{2}+\sigma_N}}e^{-c_2\frac{|y-E(s-\tau)\xi|^2}{s-\tau}}\,|y - E(s-\tau)\,\xi|. \qquad (4.25)\end{aligned}$$

Now, in the integral in (4.22) we have $R < |\xi| < R+1$. Then for $|y| < R/2$ and $0 < s-\tau \leq \delta < 1$ we have

$$\begin{aligned}\frac{|\xi|}{2} &\leq |\xi| - |y| \leq |y-\xi| \leq |y - E(s-\tau)\,\xi| + |E(s-\tau)\,\xi - \xi| \\ &\leq |y - E(s-\tau)\,\xi| + \|E(s-\tau) - I\|\,|\xi| \leq |y - E(s-\tau)\,\xi| + \frac{|\xi|}{4}.\end{aligned}$$

Hence

$$|y - E(s-\tau)\,\xi| \geq \frac{|\xi|}{4}.$$

Moreover

$$|y - E(s-\tau)\,\xi| \leq |y| + c\,|\xi| \leq c_1\,|\xi|.$$

Hence (4.24)–(4.25) give

$$\begin{aligned}\Gamma(y,s,\xi,\tau) &\leq \frac{c_1}{(s-\tau)^{Q/2}}e^{-c_3\frac{|\xi|^2}{s-\tau}} \\ \left|\partial_{\xi_j}\Gamma(y,s;\xi,\tau)\right| &\leq \frac{c_1}{(s-\tau)^{\frac{Q}{2}+\sigma_N}}\,|\xi|\,e^{-c_3\frac{|\xi|^2}{s-\tau}}.\end{aligned}$$

Therefore (4.22) gives

$$|u(y,s)| \leq \int_{Q_R\setminus Q_{R-1}} |u(\xi,\tau)| \left\{\frac{c_1}{(s-\tau)^{\frac{Q}{2}+\sigma_N}}\,|\xi|\,e^{-c_3\frac{|\xi|^2}{s-\tau}}\right\} d\xi\,d\tau.$$

We can assume $R > 1$, writing, for $0 < s - \tau < 1$ and every $\xi \in \mathbb{R}^N$ with $|\xi| > 1$,

$$\frac{c_1}{(s-\tau)^{\frac{Q}{2}+\sigma_N}} |\xi| \, e^{-c_3 \frac{|\xi|^2}{s-\tau}} = \frac{c_1}{(s-\tau)^{\frac{Q}{2}+\sigma_N}} |\xi| \, e^{-c_3 \frac{1}{s-\tau}} e^{-c_3 \frac{|\xi|^2-1}{s-\tau}}$$
$$\leq c\,|\xi| \, e^{-c_3 \frac{|\xi|^2-1}{s-\tau}} \leq c\,|\xi| \, e^{-c_3\left(|\xi|^2-1\right)} = c_4 \, |\xi| \, e^{-c_3|\xi|^2} \leq c_5 e^{-c_6|\xi|^2}.$$

This implies the Claim, so we are done. □

The link between the existence result of Theorem 4.11 and the uniqueness result of Theorem 4.13 is completed by the following

Proposition 4.14. *(a) Let f be a bounded continuous function on $\mathbb{R}^N$, or a function belonging to $L^p\left(\mathbb{R}^N\right)$ for some $p \in [1, \infty)$. Then the function*

$$u(x,t) = \int_{\mathbb{R}^N} \Gamma(x,t;y,0) \, f(y) \, dy$$

satisfies the condition (1.16) for every fixed constants $T, C > 0$.

(b) If $f \in C^0\left(\mathbb{R}^N\right)$ satisfies the condition (1.15) for some constant $\alpha > 0$ then the function u satisfies (1.16) for some $T, C > 0$.

This means that in the class of functions satisfying (1.16) there exists one and only one solution to the Cauchy problem, under any of the above assumptions on the initial datum f.

Proof. (a) If f is bounded continuous we simply have

$$|u(x,t)| \leq \|f\|_{C_b^0(\mathbb{R}^N)} \int_{\mathbb{R}^N} \Gamma(x,t;y,0) \, dy = \|f\|_{C_b^0(\mathbb{R}^N)}$$

by Proposition 4.5. Hence (1.16) holds for every fixed $C, T > 0$.

Let now $f \in L^p\left(\mathbb{R}^N\right)$ for some $p \in [1, \infty)$. Let us write

$$\begin{aligned} u(x,t) &= \frac{e^{-t \operatorname{Tr} B}}{(4\pi)^{N/2} \sqrt{\det C(t)}} \int_{\mathbb{R}^N} e^{-\frac{1}{4}(x-E(t)y)^T C(t)^{-1}(x-E(t)y)} f(y) \, dy \\ &= \frac{e^{-t \operatorname{Tr} B}}{(4\pi)^{N/2} \sqrt{\det C(t)}} \int_{\mathbb{R}^N} e^{-\frac{1}{4}(E(-t)x-y)^T C'(t)(E(-t)x-y)} f(y) \, dy \\ &= \frac{e^{-t \operatorname{Tr} B}}{(4\pi)^{N/2} \sqrt{\det C(t)}} (k_t * f)(E(-t)x) \end{aligned}$$

having set

$$k_t(x) = e^{-\frac{1}{4}x^T C'(t) x}.$$

Then

$$\begin{aligned} &\int_0^T \left(\int_{\mathbb{R}^N} |u(x,t)| \, e^{-C|x|^2} dx \right) dt \\ &\leq \int_0^T \frac{e^{-t \operatorname{Tr} B}}{(4\pi)^{N/2} \sqrt{\det C(t)}} \left(\int_{\mathbb{R}^N} |(k_t * f)(E(-t)x)| \, e^{-C|x|^2} dx \right) dt. \end{aligned} \tag{4.26}$$

Applying Hölder inequality with $q^{-1}+p^{-1}=1$ and Young's inequality we get:

$$\int_{\mathbb{R}^N} |(k_t * f)(E(-t)x)| e^{-C|x|^2} dx$$

$$E(-t)x = y; x = E(t)y; dx = e^{-t \operatorname{Tr} B} dy$$

$$= e^{-t \operatorname{Tr} B} \int_{\mathbb{R}^N} |(k_t * f)(y)| e^{-C|E(t)y|^2} dy$$

$$\leq e^{-t \operatorname{Tr} B} \|k_t * f\|_{L^p(\mathbb{R}^N)} \left\| e^{-C|E(t)y|^2} \right\|_{L^q(\mathbb{R}^N)}$$

$$\leq c(q,T) e^{-t \operatorname{Tr} B} \|f\|_{L^p(\mathbb{R}^N)} \|k_t\|_{L^1(\mathbb{R}^N)} \tag{4.27}$$

and inserting (4.27) into (4.26) we have

$$\int_0^T \left(\int_{\mathbb{R}^N} |u(x,t)| e^{-C|x|^2} dx \right) dt$$

$$\leq \int_0^T \frac{e^{-t \operatorname{Tr} B}}{(4\pi)^{N/2} \sqrt{\det C(t)}} c(q,T) e^{-t \operatorname{Tr} B} \|f\|_{L^p(\mathbb{R}^N)} \int_{\mathbb{R}^N} e^{-\frac{1}{4} x^T C'(t) x} dx dt$$

$$= c(q,T) \|f\|_{L^p(\mathbb{R}^N)} \int_0^T \int_{\mathbb{R}^N} \frac{e^{-t \operatorname{Tr} B}}{(4\pi)^{N/2} \sqrt{\det C(t)}} e^{-t \operatorname{Tr} B} e^{-\frac{1}{4} x^T C'(t) x} dx dt$$

$$x = E(-t) w; dx = e^{t \operatorname{Tr} B} dw$$

$$= c(q,T) \|f\|_{L^p(\mathbb{R}^N)} \int_0^T \int_{\mathbb{R}^N} \frac{e^{-t \operatorname{Tr} B}}{(4\pi)^{N/2} \sqrt{\det C(t)}} e^{-\frac{1}{4} (E(-t)w)^T C'(t) E(-t) w} dw dt$$

$$= c(q,T) \|f\|_{L^p(\mathbb{R}^N)} \int_0^T \int_{\mathbb{R}^N} \Gamma(w,t;0,0) dw dt$$

$$= c(q,T) \|f\|_{L^p(\mathbb{R}^N)} \int_0^T e^{-t \operatorname{Tr} B} dt \leq c(q,T) \|f\|_{L^p(\mathbb{R}^N)}$$

by (4.8). Hence (1.16) still holds for every fixed $C, T > 0$.

(b) Assume that

$$\int_{\mathbb{R}^N} |f(y)| e^{-\alpha|y|^2} dy < \infty$$

for some $\alpha > 0$ and, for $T \in (0,1)$, $\beta > 0$ to be chosen later, let us bound:

$$\int_0^T \left(\int_{\mathbb{R}^N} |u(x,t)| e^{-\beta|x|^2} dx \right) dt$$

$$\leq \int_0^T \left(\int_{\mathbb{R}^N} \left(\frac{e^{-t \operatorname{Tr} B}}{(4\pi)^{N/2} \sqrt{\det C(t)}} \int_{\mathbb{R}^N} e^{-\frac{1}{4}(x-E(t)y)^T C(t)^{-1}(x-E(t)y)} |f(y)| dy \right) e^{-\beta|x|^2} dx \right) dt$$

$$y = E(-t)\left(x - 2C(t)^{1/2} z\right); dy = e^{t \operatorname{Tr} B} 2^N \det C(t)^{1/2} dz$$

$$= \int_0^T \int_{\mathbb{R}^N} \frac{e^{-|z|^2}}{\pi^{N/2}} \left(\int_{\mathbb{R}^N} \left| f\left(E(-t)\left(x - 2C(t)^{1/2} z\right)\right) \right| e^{-\beta|x|^2} dx \right) dz dt$$

$$E(-t)\left(x - 2C(t)^{1/2} z\right) = w; e^{t \operatorname{Tr} B} dx = dw$$

$$
\begin{aligned}
&= \int_0^T \int_{\mathbb{R}^N} \frac{e^{-|z|^2}}{\pi^{N/2}} \left(\int_{\mathbb{R}^N} e^{-t \operatorname{Tr} B} \left| f(w) \right| e^{-\beta \left| E(t) w + 2C(t)^{1/2} z \right|^2} dw \right) dz dt \\
&= \int_0^T e^{-t \operatorname{Tr} B} \int_{\mathbb{R}^N} \frac{e^{-|z|^2}}{\pi^{N/2}} \cdot \\
&\cdot \left(\int_{\mathbb{R}^N} \left| f(w) \right| e^{-\beta \left(|E(t)w|^2 + 4 \left| C(t)^{1/2} z \right|^2 + 2 (E(t)w)^T C(t)^{1/2} z \right)} dw \right) dz dt \\
&= \int_0^T \frac{e^{-t \operatorname{Tr} B}}{\pi^{N/2}} \left(\int_{\mathbb{R}^N} \left| f(w) \right| e^{-\beta |E(t)w|^2} \cdot \right. \\
&\cdot \left. \left(\int_{\mathbb{R}^N} e^{-|z|^2} e^{-4\beta \left| C(t)^{1/2} z \right|^2} e^{-2\beta (E(t)w)^T C(t)^{1/2} z} dz \right) dw \right) dt.
\end{aligned}
$$

Next, for $0 < t < 1$ we have, since $\|C(t)\| \leq ct$,

$$
\left| -2\beta \left(E(t) w \right)^T C(t)^{1/2} z \right| \leq c_1 \beta |w| \sqrt{t} |z|
$$

so that

$$
\begin{aligned}
&\int_0^T \left(\int_{\mathbb{R}^N} |u(x,t)| e^{-\beta |x|^2} dx \right) dt \\
&\leq \frac{e^{|\operatorname{Tr} B|}}{\pi^{N/2}} \int_0^T \left(\int_{\mathbb{R}^N} |f(w)| e^{-\beta |E(t)w|^2} \left(\int_{\mathbb{R}^N} e^{-|z|^2} e^{c_1 \beta |w| \sqrt{t} |z|} dz \right) dw \right) dt.
\end{aligned}
$$

Next,

$$
\begin{aligned}
\int_{\mathbb{R}^N} e^{-|z|^2} e^{c_1 \beta |w| \sqrt{t} |z|} dz &= c_n \int_0^{+\infty} e^{-\rho^2 + c_1 \beta |w| \sqrt{t} \rho} \rho^{n-1} d\rho \\
&\leq c \int_0^{+\infty} e^{-\frac{\rho^2}{2} + c_1 \beta \rho \sqrt{t}} d\rho = c e^{c_2 \beta^2 t |w|^2}
\end{aligned}
$$

and

$$
\int_0^T \left(\int_{\mathbb{R}^N} |u(x,t)| e^{-\beta |x|^2} dx \right) dt \leq c \int_0^T \left(\int_{\mathbb{R}^N} |f(w)| e^{-\beta |E(t)w|^2} e^{c_2 \beta^2 t |w|^2} dw \right) dt.
$$

Since $E(t)$ is invertible and $E(0) = 1$, for T small enough and $t \in (0,T)$ we have $|E(t)w| \geq \frac{1}{2} |w|$ so that

$$
e^{-\beta |E(t)w|^2} e^{c_2 \beta^2 t |w|^2} \leq e^{-|w|^2 \beta \left(\frac{1}{2} - c_2 t \beta \right)}.
$$

We now fix $\beta = 4\alpha$ and then fix T small enough such that $\frac{1}{2} - c_2 T \beta \geq \frac{1}{4}$, so that for $t \in (0,T)$ we have

$$
e^{-|w|^2 \beta \left(\frac{1}{2} - c_2 t \beta \right)} \leq e^{-|w|^2 \beta \left(\frac{1}{2} - c_2 T \beta \right)} \leq e^{-\alpha |w|^2}
$$

and

$$
\int_0^T \left(\int_{\mathbb{R}^N} |u(x,t)| e^{-\beta |x|^2} dx \right) dt \leq c \int_0^T \left(\int_{\mathbb{R}^N} |f(w)| e^{-\alpha |w|^2} dw \right) dt < \infty.
$$

So we are done. □

The previous uniqueness property for the Cauchy problem also implies the following replication property for the heat kernel:

Corollary 4.15. *For every $x, y \in \mathbb{R}^N$ and $s < \tau < t$ we have*

$$\Gamma(x, t; y, s) = \int_{\mathbb{R}^N} \Gamma(x, t; z, \tau) \Gamma(z, \tau; y, s)\, dz.$$

Proof. Let

$$u(x, t) = \int_{\mathbb{R}^N} \Gamma(x, t; z, \tau) \Gamma(z, \tau; y, s)\, dz$$
$$f(z) = \Gamma(z, \tau; y, s)$$

for $y \in \mathbb{R}^N$ fixed, $\tau > s$ fixed. By Theorem 1.4, (i), $f \in C_*^0\left(\mathbb{R}^N\right)$. Hence by Theorem 4.11, point (iii), u solves the Cauchy problem

$$\begin{cases} \mathcal{L}u(x, t) = 0 \text{ for } t > \tau \\ u(x, \tau) = \Gamma(x, \tau; y, s) \end{cases}$$

where the initial datum is assumed continuously, uniformly as $t \to \tau$. Since $v(x, t) = \Gamma(x, t; y, s)$ solves the same Cauchy problem, by Theorem 4.13 the assertion follows. □

Acknowledgments

The authors thank the anonymous referee for her/his suggestions that improved this manuscript. This research was partially supported by the grant of Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM). The second author acknowledges financial support from the FAR2019 project: "Risk assessment in the EU: new indices based on machine learning methods" funded by UNIMORE.

Conflict of interest

The authors declare no conflict of interest.

References

1. Anceschi F, Polidoro S (2020) A survey on the classical theory for Kolmogorov equation. *Le Matematiche* 75: 221–258.
2. Bramanti M (2014) *An Invitation to Hypoelliptic Operators and Hörmander's Vector Fields*, Cham: Springer.
3. Da Prato G (1998) *Introduction to Stochastic Differential Equations*, 2Eds., Scuola Normale Superiore, Pisa.
4. Delarue F, Menozzi S (2010) Density estimates for a random noise propagating through a chain of differential equations. *J Funct Anal* 259: 1577–1630.

5. Di Francesco M, Pascucci S (2005) On a class of degenerate parabolic equations of Kolmogorov type. *Appl Math Res Express* 2005: 77–116.
6. Di Francesco M, Polidoro S (2006) Schauder estimates, Harnack inequality and Gaussian lower bound for Kolmogorov-type operators in non-divergence form. *Adv Differential Equ* 11: 1261–1320.
7. Farkas B, Lorenzi L (2009) On a class of hypoelliptic operators with unbounded coefficients in $\mathbb{R}^N$. *Commun Pure Appl Anal* 8: 1159–1201.
8. Hörmander L (1967) Hypoelliptic second order differential equations. *Acta Math* 119: 147–171.
9. Il'in AM (1964) On a class of ultraparabolic equations. *Dokl Akad Nauk SSSR* 159: 1214–1217.
10. Kolmogorov AN (1934) Zur Theorie der Brownschen Bewegung. *Ann Math* 35: 116–117.
11. Kupcov LP (1972) The fundamental solutions of a certain class of elliptic-parabolic second order equations. *Differencial'nye Uravnenija* 8: 1649–1660.
12. Kuptsov LP (1982) Fundamental solutions of some second-order degenerate parabolic equations. *Mat Zametki* 31: 559–570.
13. Lanconelli E, Polidoro S (1994) On a class of hypoelliptic evolution operators. *Rend Sem Mat* 52: 29–63.
14. Lunardi A (1997) Schauder estimates for a class of degenerate elliptic and parabolic operators with unbounded coefficients in $\mathbb{R}^n$. *Ann Scuola Norm Sup Pisa* 24: 133–164.
15. Manfredini M (1997) The Dirichlet problem for a class of ultraparabolic equations. *Adv Differential Equ* 2: 831–866.
16. Pascucci A, Pesce A (2019) On stochastic Langevin and Fokker-Planck equations: the two-dimensional case. *arXiv:1910.05301*.
17. Pascucci A, Pesce A (2020) The parametrix method for parabolic SPDEs. *Stochastic Process Appl*, To appear.
18. Polidoro S (1994) On a class of ultraparabolic operators of Kolmogorov-Fokker-Planck type. *Matematiche* 49: 53–105.
19. Sonin IM (1967) A class of degenerate diffusion processes. *Trans Theory Probab Appl* 12: 490–496.
20. Weber M (1951) The fundamental solution of a degenerate partial differential equation of parabolic type. *T Am Math Soc* 71: 24–37.