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Research article

Some results about semilinear elliptic problems on half-spaces†

Alberto Farina*

LAMFA, CNRS UMR 7352, Université de Picardie Jules Verne, 33 rue Saint-Leu, 80039 Amiens, France

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* **Correspondence:** Email: alberto.farina@u-picardie.fr.

Abstract: We prove some new results about the growth, the monotonicity and the symmetry of (possibly) unbounded non-negative solutions of $-\Delta u = f(u)$ on half-spaces, where f is merely a locally Lipschitz continuous function. Our proofs are based on a comparison principle for solutions of semilinear problems on unbounded slab-type domains and on the moving planes method.

Keywords: qualitative properties of solutions to semilinear elliptic equations; moving planes method; comparison principle

All'amico Sandro con grande affetto e grande stima.

1. Introduction

In this work we study some qualitative properties of the solutions to the elliptic boundary value problem

$$\begin{cases} -\Delta u = f(u) & \text{in} \quad \mathbb{R}^N_+, \\ u \geq 0 & \text{in} \quad \mathbb{R}^N_+, \\ u = 0 & \text{in} \quad \partial\mathbb{R}^N_+, \end{cases} \tag{1.1}$$

where $\mathbb{R}^N_+$ denotes the euclidean half-space $\{x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} : x_N > 0\}$, $N \geq 2$. This type of problem naturally appears in the obtention of *a priori* bounds for positive solutions of nonlinear second order PDE's on smooth bounded domains ([19]), in the study of semilinear problems with small diffusion on smooth bounded domains and in the study of regularity results for some free boundary problems (see e.g., [1,2,5,6]).

In the present work our focus is on the study of the growth of the solutions to (1.1) as well as on their monotonicity and symmetry properties. The situation is quite well understood in the two dimensional case (see [15, 16] and also [4] when u is bounded and positive) while, in the available results for $N \geq 3$ it is always assumed that f is globally Lipschitz continuous (often with $f(0) \geq 0$) and/or that the solution u is positive and bounded (see [2–5,7–13,17,18,21]). For these reasons, in the present work, we concentrate on (possibly) unbounded solutions of (1.1) where f is merely a locally Lipschitz continuous function. The paper is organized as follows. In section 2 we prove a comparison principle for solutions of semilinear problems on unbounded slab-type domains (see Theorem 2.1). By combining this result with the moving planes procedure we prove the monotonicity of the solutions which are bounded (only) on strips. See Theorem 3.1 and Corollary 3.4 in section 3. In section 4 we first establish some results about the growth of an arbitrary solution to (1.1) (see Theorem 4.1 and Theorem 4.4) and then we combine them with those of section 3 to get some new monotonicity and one-dimensional symmetry results (see Theorem 4.5 and Theorem 4.7). In particular, our results cover both the case of some superlinear and subcritical functions f and the case of unbounded solutions with bounded gradient with a general nonlinearity f.

2. A comparison principle

This section is devoted to the proof of a comparison principle for solutions of semilinear problems on unbounded slab-type domains. It is inspired by a result established in [14] and it will be used to obtain the main results of section 3 and 4.

Theorem 2.1 (**Comparison principle in unbounded slabs of small width**).
1) Let $N \geq 2$, $M > 0$ and assume that $f \in C^{0,1}_{loc}([0,+\infty))$. Then there exists $\vartheta = \vartheta(f, M) > 0$ such that, for any $(a,b) \subset \mathbb{R}$ with $0 < b - a < \vartheta$ and any $u, v \in C^2(\mathbb{R}^{N-1} \times [a,b])$ satisfying

$$\begin{cases} -\Delta u - f(u) \leq -\Delta v - f(v) & \text{in} \quad \mathbb{R}^{N-1} \times (a,b), \\ |u|, |v| \leq M & \text{in} \quad \mathbb{R}^{N-1} \times (a,b), \\ u \leq v & \text{on} \quad \partial(\mathbb{R}^{N-1} \times (a,b)), \end{cases} \tag{2.1}$$

we have

$$u \leq v \qquad \text{in} \qquad \mathbb{R}^{N-1} \times (a,b).$$

2) Let $N \geq 2$ and assume that $f \in C^{0,1}([0,+\infty))$. Then there exists $\vartheta = \vartheta(f) > 0$ such that, for any $(a,b) \subset \mathbb{R}$ with $0 < b - a < \vartheta$ and any $u, v \in C^2(\mathbb{R}^{N-1} \times [a,b])$, with at most polynomial growth at infinity and satisfying

$$\begin{cases} -\Delta u - f(u) \leq -\Delta v - f(v) & \text{in} \quad \mathbb{R}^{N-1} \times (a,b), \\ u \leq v & \text{on} \quad \partial(\mathbb{R}^{N-1} \times (a,b)), \end{cases} \tag{2.2}$$

we have

$$u \leq v \qquad \text{in} \qquad \mathbb{R}^{N-1} \times (a,b).$$

3) Let $N \geq 2$ and assume that $f \in C^0([0,+\infty))$ is a non-increasing function. Then, for any $(a,b) \subset \mathbb{R}$ and any $u, v \in C^2(\mathbb{R}^{N-1} \times [a,b])$, with at most polynomial growth at infinity and satisfying

$$\begin{cases} -\Delta u - f(u) \leq -\Delta v - f(v) & \text{in} \quad \mathbb{R}^{N-1} \times (a,b), \\ u \leq v & \text{on} \quad \partial(\mathbb{R}^{N-1} \times (a,b)), \end{cases} \tag{2.3}$$

we have

$$u \leq v \qquad \text{in} \qquad \mathbb{R}^{N-1} \times (a,b).$$

Proof. Set $\Sigma_{a,b} := \mathbb{R}^{N-1} \times (a,b)$. Testing the differential inequality with $w := (u-v)^+\varphi^2$, $\varphi \in C^1_c(\mathbb{R}^{N-1})$, we get

$$\int_{\Sigma_{a,b}} \nabla(u-v)\nabla w \leq \int_{\Sigma_{a,b}} (f(u)-f(v))(u-v)^+\varphi^2$$

and so

$$\begin{aligned} \int_{\Sigma_{a,b}} |\nabla(u-v)^+|^2\varphi^2 &\leq -\int_{\Sigma_{a,b}} 2\varphi(u-v)^+\nabla(u-v)^+\nabla\varphi + \int_{\Sigma_{a,b}} (f(u)-f(v))(u-v)^+\varphi^2 \leq \\ &\leq \int_{\Sigma_{a,b}} 2\Big(\frac{|\nabla(u-v)^+||\varphi|}{\sqrt{2}}\Big)\Big(\sqrt{2}(u-v)^+|\nabla\varphi|\Big) + \int_{\Sigma_{a,b}} (f(u)-f(v))(u-v)^+\varphi^2 \leq \\ &\leq \int_{\Sigma_{a,b}} \frac{|\nabla(u-v)^+|^2\varphi^2}{2} + 2\int_{\Sigma_{a,b}} [(u-v)^+]^2|\nabla\varphi|^2 + \int_{\Sigma_{a,b}} (f(u)-f(v))(u-v)^+\varphi^2. \end{aligned} \tag{2.4}$$

Then

$$\int_{\Sigma_{a,b}} |\nabla(u-v)^+|^2\varphi^2 \leq 4\int_{\Sigma_{a,b}} [(u-v)^+]^2|\nabla\varphi|^2 + 2\int_{\Sigma_{a,b}} (f(u)-f(v))(u-v)^+\varphi^2. \tag{2.5}$$

On the other hand, by the Poincaré inequality on the interval (a,b) we have

$$\begin{aligned} \int_{\Sigma_{a,b}} |\nabla(u-v)^+|^2\varphi^2 &\geq \int_{\Sigma_{a,b}} |\partial_N(u-v)^+|^2\varphi^2 = \int_{\mathbb{R}^{N-1}} \Big(\int_a^b |\partial_N(u-v)^+|^2 dx_N\Big)\varphi^2(x')dx' \geq \\ &\geq \frac{\pi^2}{(b-a)^2}\int_{\mathbb{R}^{N-1}} \Big(\int_a^b [(u-v)^+]^2 dx_N\Big)\varphi^2(x')dx' = \frac{\pi^2}{(b-a)^2}\int_{\Sigma_{a,b}} [(u-v)^+]^2\varphi^2 \end{aligned} \tag{2.6}$$

and the combination of (2.5) and (2.6) yields

$$\int_{\Sigma_{a,b}} [(u-v)^+]^2\varphi^2 \leq 4\frac{(b-a)^2}{\pi^2}\int_{\Sigma_{a,b}} [(u-v)^+]^2|\nabla\varphi|^2 + 2\frac{(b-a)^2}{\pi^2}\int_{\Sigma_{a,b}} (f(u)-f(v))(u-v)^+\varphi^2. \tag{2.7}$$

Now we distinguish the three cases.

In the case 1), from (2.7) we get

$$\int_{\Sigma_{a,b}} [(u-v)^+]^2\varphi^2 \leq 4\frac{(b-a)^2}{\pi^2}\int_{\Sigma_{a,b}} [(u-v)^+]^2|\nabla\varphi|^2 + 2\frac{(b-a)^2}{\pi^2}L(f,M)\int_{\Sigma_{a,b}} [(u-v)^+]^2\varphi^2 \tag{2.8}$$

where $L(f,M)$ is the Lipschitz constant of f on the interval $[-M,M]$.

Now we set $\vartheta := \frac{\pi}{2\sqrt{1+L(f,M)}} > 0$ and thus, for any $(a,b) \subset \mathbb{R}$ with $b-a < \vartheta$ we have

$$\int_{\Sigma_{a,b}} [(u-v)^+]^2\varphi^2 \leq 8\frac{(b-a)^2}{\pi^2}\int_{\Sigma_{a,b}} [(u-v)^+]^2|\nabla\varphi|^2. \tag{2.9}$$

In the case 2), from (2.7) we get

$$\int_{\Sigma_{a,b}} [(u-v)^+]^2\varphi^2 \leq 4\frac{(b-a)^2}{\pi^2}\int_{\Sigma_{a,b}} [(u-v)^+]^2|\nabla\varphi|^2 + 2\frac{(b-a)^2}{\pi^2}L_f\int_{\Sigma_{a,b}} [(u-v)^+]^2\varphi^2, \tag{2.10}$$

where L_f is the Lipschitz constant of f. So that, for any $(a,b) \subset \mathbb{R}$ with $b-a < \vartheta := \frac{\pi}{2\sqrt{1+L_f}} > 0$, we get (2.9) once again.

In the case 3), from (2.7) we get

$$\int_{\Sigma_{a,b}} [(u-v)^+]^2\varphi^2 \leq 4\frac{(b-a)^2}{\pi^2}\int_{\Sigma_{a,b}} [(u-v)^+]^2|\nabla\varphi|^2 \tag{2.11}$$

since f is non-increasing and so (2.9) is satisfied also in this case. Note that (2.9) holds true for any interval $(a,b) \subset \mathbb{R}$ (i.e., no smallness assumption on the lenght of (a,b) is needed to treat the case 3)).

For $R > 0$ consider $\varphi = \varphi_R \in C^1_c(\mathbb{R}^{N-1})$ such that

$$\begin{cases} 0 \leq \varphi \leq 1 & \text{in } \mathbb{R}^{N-1}, \\ \varphi \equiv 1 & \text{in } B'(0,R) \subset \mathbb{R}^{N-1}, \\ \varphi \equiv 0 & \text{in } \mathbb{R}^{N-1} \setminus B'(0,2R), \\ |\nabla\varphi| \leq \frac{2}{R} & \text{in } \mathbb{R}^{N-1}, \end{cases} \tag{2.12}$$

where $B'(0,R) := \left\{x' \in \mathbb{R}^{N-1} : |x'| < R\right\}$ and define the set $C(R) := \Sigma_{a,b} \cap (B'(0,R) \times \mathbb{R}) = B'(0,R) \times (a,b)$. Using $\varphi = \varphi_R$ in (2.9) we then obtain

$$\begin{aligned} \forall\, R > 0 \qquad & \int_{C(R)} [(u-v)^+]^2 \leq \int_{\Sigma_{a,b}} [(u-v)^+]^2\varphi^2 \leq \\ \leq 8\frac{(b-a)^2}{\pi^2} & \int_{\Sigma_{a,b}} [(u-v)^+]^2|\nabla\varphi|^2 \leq 32\frac{(b-a)^2}{\pi^2R^2}\int_{C(2R)} [(u-v)^+]^2. \end{aligned} \tag{2.13}$$

For $R > 0$ we define the non-decreasing function $h(R) := \int_{C(R)}[(u-v)^+]^2$ and observe that h has at most polynomial growth at infinity thanks to the (growth) assumptions on u and v. Therefore h satisfies

$$\begin{cases} 0 \leq h(R) \leq 32\frac{(b-a)^2}{\pi^2R^2}h(2R) & \forall\, R > 0, \\ h(R) \leq C(1+R^k) & \forall\, R > 0, \end{cases} \tag{2.14}$$

where C and k are positive constants.

From (2.14) we get $h(R) \leq 32\frac{(b-a)^2}{\pi^2}C(1+2^kR^k)R^{-2}$ for $R > 0$ and thus, by iterating this procedure, we obtain $h(R) \leq \left(32\frac{(b-a)^2}{\pi^2}\right)^m C(1+2^{mk}R^k)R^{-2m}$ for any $R > 0$ and any integer $m \geq 1$. Now we fix $m > k$ and let $R \to +\infty$ to get $\lim_{R\to\infty} h(R) = 0$, which entails $h \equiv 0$. The latter implies $u \leq v$ on $\Sigma_{a,b}$ concluding the proof. □

3. The moving planes method for (possibly) unbounded solutions

Theorem 3.1. *Assume $N \geq 2$, $f \in C^{0,1}_{loc}([0,+\infty))$ with $f(0) \geq 0$ and let $u \in C^2(\overline{\mathbb{R}^N_+})$ be a solution of*

$$\begin{cases} -\Delta u = f(u) & in \quad \mathbb{R}^N_+, \\ u > 0 & in \quad \mathbb{R}^N_+, \\ u = 0 & in \quad \partial\mathbb{R}^N_+. \end{cases} \tag{P}$$

Assume that u is bounded on the slabs $\mathbb{R}^{N-1} \times [0,t]$, for every $t > 0$, i.e., for every $t > 0$ there is a constant $C(t) > 0$ such that $0 \leq u \leq C(t)$ on $\mathbb{R}^{N-1} \times [0,t]$.

Then u is monotone, i.e., $\frac{\partial u}{\partial x_N} > 0$ in $\mathbb{R}^N_+$.

Remark 3.2. When the space dimension $N = 2$, the above monotonicity result holds irrespective of the value of $f(0)$ and without the assumption of boundedness on slabs, see [15, 16].

Proof. The proof is based on the moving planes procedure. For $t > 0$ we set

$$u_t(x', x_N) := u(x', 2t - x_N) \quad \text{and} \quad \Sigma_t := \{(x', x_N) \in \mathbb{R}^N \ : \ 0 < x_N < t\}.$$

We aim at proving that

$$u(x) \leq u_t(x) \quad \forall x \in \Sigma_t, \ \forall t > 0. \tag{3.1}$$

The monotonicity of u will be then a consequence of (3.1) and the strong maximum principle. To prove (3.1) we shall show that

$$\Lambda := \{t > 0 \ : \ u \leq u_\theta \quad \text{in } \Sigma_\theta \quad \forall \theta \leq t\} = (0, +\infty). \tag{3.2}$$

First we prove that Λ is not empty. To this end we observe that, for every $t \in (0,1)$, the functions u and u_t are bounded by $\|u\|_{L^\infty(\mathbb{R}^{N-1}\times[0,2])} := M > 0$. Therefore, we can apply Theorem 2.1 to u and $v := u_t$ on Σ_t to find that $u \leq u_t$ in Σ_t, for all sufficiently small $t > 0$.

Next we plan to prove that $\bar{t} := \sup \Lambda$ is $+\infty$. Assume for contradiction that $\bar{t} < +\infty$ then we can prove the following

Proposition 3.3. *For every $\delta \in (0, \frac{\bar{t}}{2})$ there is $\varepsilon(\delta) > 0$ such that*

$$\forall\, \varepsilon \in (0, \varepsilon(\delta)) \qquad u \leq u_{\bar{t}+\varepsilon} \quad in \quad \mathbb{R}^{N-1} \times [\delta, \bar{t} - \delta] \tag{3.3}$$

Proof of Proposition 3.3. If the claim were not true, there would exist $\delta \in (0, \frac{\bar{t}}{2})$ such that

$$\forall\, k \geq 1 \quad \exists\, \varepsilon_k \in (0, \frac{1}{k}), \exists\, x^k \in \mathbb{R}^{N-1} \times [\delta, \bar{t} - \delta] \quad : \quad u(x^k) > u_{\bar{t}+\epsilon_k}(x^k). \tag{3.4}$$

Observe that the sequence (x^k_N) is bounded and so, up to a subsequence, we may and do suppose that $x^k_N \to \bar{x}_N \in [\delta, \bar{t} - \delta]$, as $k \to \infty$.

For $x \in \mathbb{R}^N_+$ and $k \geq 1$ let us set $u_k(x) := u(x' + (x^k)', x_N)$. By the translation invariance of the equation satisfied by u, the boundedness of u on every strip $\mathbb{R}^{N-1} \times [0,t]$ and standard elliptic estimates we have that the sequence of solutions (u_k) is bounded in $C^{2,\alpha}_{loc}(\overline{\Sigma_t})$, for every $t > 0$ and some $\alpha \in (0,1)$.

Therefore, by the Ascoli-Arzelà theorem (via a diagonal procedure) we can extract a subsequence, still denoted (u_k), which converges in $C^2_{loc}(\overline{\mathbb{R}^N_+})$ to a limit $u^\infty \in C^2(\overline{\mathbb{R}^N_+})$ satisfying

$$\begin{cases} -\Delta u^\infty = f(u^\infty) & \text{in} \quad \mathbb{R}^N_+, \\ u^\infty \geq 0 & \text{in} \quad \mathbb{R}^N_+, \\ u^\infty = 0 & \text{in} \quad \partial\mathbb{R}^N_+. \end{cases} \tag{3.5}$$

Furthermore, by the definition of Λ, (3.4) and the uniform convergence, we have that $u^\infty \leq u^\infty_{\bar{t}}$ on $\Sigma_{\bar{t}}$ and $u^\infty(0', \bar{x}_N) \geq u^\infty_{\bar{t}}(0', \bar{x}_N)$ and so

$$u^\infty(0', \bar{x}_N) = u^\infty_{\bar{t}}(0', \bar{x}_N). \tag{3.6}$$

Then,

$$\begin{cases} \Delta(u^\infty_{\bar{t}} - u^\infty) = f(u^\infty_{\bar{t}}) - f(u^\infty) \leq C(u^\infty_{\bar{t}} - u^\infty) & \text{in} \quad \Sigma_{\bar{t}}, \\ u^\infty_{\bar{t}} - u^\infty \geq 0 & \text{in} \quad \Sigma_{\bar{t}}, \end{cases} \tag{3.7}$$

where C is the Lipschitz constant of f on the interval $[0, \|u\|_{L^\infty(\mathbb{R}^{N-1}\times[0,2\bar{t}])}]$ and so $u^\infty_{\bar{t}} \equiv u^\infty$ on $\Sigma_{\bar{t}}$ by (3.6) and the strong maximum principle. In particular $u^\infty_{\bar{t}} \equiv 0$ on the set $\{ x_N = \bar{x}_N \}$ and so $u^\infty_{\bar{t}} \equiv 0$ on $\mathbb{R}^N_+$ thanks to (3.5) and the strong maximum principle (recall that $f(0) \geq 0$ is in force). We observe that $0 = -\Delta u^\infty = f(u^\infty) = f(0)$ and we set

$$v_k(x) := \frac{u_k(x)}{u_k(0', x^k_N)} = \frac{u(x' + (x^k)', x_N)}{u_k(0', x^k_N)} \tag{3.8}$$

so that $v_k(0', x^k_N) = 1$ for every $k \geq 1$. Then,

$$-\Delta v_k = \frac{f(u_k)}{u_k(0', x^k_N)} = \frac{f(u_k)}{u_k} \frac{u_k}{u_k(0', x^k_N)} = \frac{f(u_k)}{u_k} v_k = \frac{f(u_k) - f(0)}{u_k} v_k = c_k(x) v_k \tag{3.9}$$

with $(c_k)_{k\geq 1}$ uniformly bounded on every slab $\mathbb{R}^{N-1} \times [0, t]$, $t > 0$. We can therefore apply the Harnack inequality to v_k to get, for every compact set $K_n := \overline{B}(0', n) \times [0, n]$,

$$\sup_{K_n \cap \{x_N \geq \delta\}} v_k \leq C_H(n) \inf_{K_n \cap \{x_N \geq \delta\}} v_k \leq C_H(n) \qquad \forall\, n \geq \bar{t},\ \forall k \geq 1, \tag{3.10}$$

where in the latter we have used the fact that $(0', x^k_N) \in K_n$ for $k \geq 1$ and $n > \bar{t}$.

Moreover, by the definition of Λ, we know that $\frac{\partial u}{\partial x_N} > 0$ in $\Sigma_{\bar{t}}$ and so

$$\sup_{K_n} v_k \leq C_H(n) \sup_{K_n \cap \{x_N \geq \delta\}} v_k \leq C_H(n) \qquad \forall\, n \geq \bar{t},\ \forall k \geq 1. \tag{3.11}$$

Now we set $\alpha_k := u_k(0', x^k_N)$, $f_k(t) := \frac{f(\alpha_k t)}{\alpha_k}$, we rewrite (3.9) as

$$-\Delta v_k = \frac{f(\alpha_k v_k)}{\alpha_k} = f_k(v_k) \tag{3.12}$$

and we observe that the family $(f_k)_{k\geq 1}$ is relatively compact in $C^0_{loc}([0, +\infty))$ since $f_k(0) = 0$ and

$$\forall\, \eta > 0 \quad \exists\, C(\eta) > 0 \quad : \quad \forall\, k \geq 1, \quad \forall\, t, t' \in [0, \eta] \qquad |f_k(t) - f_k(t')| \leq C(\eta)|t - t'|$$

(the latter is satisfied with $C(\eta)$ being the Lipschitz constant of f on the segment $[0, \eta\|u\|_{L^\infty(\mathbb{R}^{N-1}\times[0,\bar{t}])}]$). Thus, up to a subsequence, $f_k \to f^\infty$ in $C^0_{loc}([0,+\infty)$ with $f^\infty \in C^{0,1}_{loc}([0,+\infty)$ and$f^\infty(0) = 0$.

In view of (3.11) and (3.12) we can use, once again, elliptic estimates and the Ascoli-Arzelà Theorem to find a subsequence (still denoted by (v_k)) which converges in $C^2_{loc}(\overline{\mathbb{R}^N_+})$ to a limit $v^\infty \in C^2(\overline{\mathbb{R}^N_+})$. By gathering together all those informations we finally get that

$$\begin{cases} -\Delta v^\infty = f^\infty(v^\infty) & \text{in} \quad \mathbb{R}^N_+, \\ v^\infty \geq 0 & \text{in} \quad \mathbb{R}^N_+, \\ v^\infty = 0 & \text{in} \quad \partial\mathbb{R}^N_+, \\ v^\infty(0', \bar{x}_N) = 1 \end{cases} \tag{3.13}$$

and

$$\begin{cases} \Delta(v^\infty_{\bar{t}} - v^\infty) = f^\infty(v^\infty_{\bar{t}}) - f^\infty(v^\infty) = c^\infty(x)(v^\infty_{\bar{t}} - v^\infty) & \text{in} \quad \Sigma_{\bar{t}}, \\ v^\infty_{\bar{t}} - v^\infty \geq 0 & \text{in} \quad \Sigma_{\bar{t}}, \\ v^\infty(0', \bar{x}_N) = v^\infty_{\bar{t}}(0', \bar{x}_N), \end{cases} \tag{3.14}$$

with c^∞ locally bounded on $\mathbb{R}^N_+$.

The strong maximum principle and (3.13) imply that $v^\infty > 0$ in $\mathbb{R}^N_+$ while another application of the strong maximum principle to (3.14) yields $v^\infty_{\bar{t}} \equiv v^\infty$ in $\overline{\Sigma_{\bar{t}}}$ and so v^∞ must vanish somewhere in $\mathbb{R}^N_+$. The latter contradicts $v^\infty > 0$ in $\mathbb{R}^N_+$ and concludes the proof of proposition 3.3.

Now we are ready to prove that $\bar{t} = +\infty$. By proposition 3.3 we know that for every $\delta \in (0, \frac{\bar{t}}{2})$ there is $\varepsilon(\delta) \in (0, \delta)$ such that

$$\forall\, \varepsilon \in (0, \varepsilon(\delta)) \qquad u \leq u_{\bar{t}+\varepsilon} \quad \text{in} \quad \mathbb{R}^{N-1} \times [\delta, \bar{t} - \delta]. \tag{3.15}$$

Now we set $M := \|u\|_{L^\infty(\mathbb{R}^{N-1}\times[0,2\bar{t}])} > 0$ and choose $2\delta < \min\{\frac{\bar{t}}{2}, \vartheta(M, f)\}$ so that we can apply Theorem 2.1 to u and $u_{\bar{t}+\epsilon}$ on the sets $\mathbb{R}^{N-1} \times (0, \delta)$ and $\mathbb{R}^{N-1} \times (\bar{t} - \delta, \bar{t} + \epsilon)$. This implies

$$\forall\, \varepsilon \in (0, \varepsilon(\delta)) \qquad u \leq u_{\bar{t}+\varepsilon} \quad \text{in} \quad \Sigma_{\bar{t}+\epsilon} \tag{3.16}$$

which clearly contradicts the definition $\bar{t}$. Therefore $\bar{t} = +\infty$ so that, for every $t > 0$,

$$\begin{cases} \Delta(u_t - u) = f(u_t) - f(u) = c^\infty_t(x)(u_t - u) & \text{in} \quad \Sigma_t, \\ u_t - u \geq 0 & \text{in} \quad \Sigma_t, \end{cases} \tag{3.17}$$

with c^∞_t locally bounded on Σ_t. Again, as before, the maximum principle and the assumption $u > 0$ in $\mathbb{R}^N_+$ imply that

$$\forall\, t > 0 \qquad u_t - u > 0 \qquad \text{in} \quad \Sigma_t$$

and the Hopf's lemma tell us that

$$\forall\, t > 0, \quad \forall\, x' \in \mathbb{R}^{N-1} \qquad -2\frac{\partial u}{\partial x_N}(x', t) = \frac{\partial(u_t - u)}{\partial x_N}(x', t) < 0.$$

The latter proves the desired conclusion. □

An inspection of the first part of the proof of Theorem 3.1 immediately reveals that the moving planes procedure can always be started irrespectively of the value of $f(0)$ provided u is bounded on a single slab $\mathbb{R}^{N-1} \times [0, t_0]$. More precisely we have the following

Corollary 3.4 (**Starting the moving planes method**). *Assume $N \geq 2$, $f \in C^{0,1}_{loc}([0, +\infty))$ and let $u \in C^2(\overline{\mathbb{R}^N_+})$ be a solution of*

$$\begin{cases} -\Delta u = f(u) & in \quad \mathbb{R}^N_+, \\ u \geq 0 & in \quad \mathbb{R}^N_+, \\ u = 0 & in \quad \partial\mathbb{R}^N_+. \end{cases} \tag{3.18}$$

Assume that there exists $t_0 > 0$ such that u is bounded on the slab $\mathbb{R}^{N-1} \times [0, t_0]$. Then there exists $t_1 \in (0, t_0)$ such that

$$\forall\, t \in (0, t_1) \qquad u \leq u_t \quad in \quad \Sigma_t, \tag{3.19}$$

$$\frac{\partial u}{\partial x_N} \geq 0 \quad in \quad \Sigma_{t_1}. \tag{3.20}$$

Furthermore, if $u \not\equiv 0$, there exists $t_2 \in (0, t_1)$ such that

$$\forall\, t \in (0, t_2) \qquad 0 < u < u_t \quad in \quad \Sigma_t, \tag{3.21}$$

$$\frac{\partial u}{\partial x_N} > 0 \quad in \quad \Sigma_{t_2}. \tag{3.22}$$

Remark 3.5. When the space dimension $N = 2$, the above monotonicity result holds even without the assumption of boundedness on the slab $\mathbb{R}^{N-1} \times [0, t_0]$, see [15, 16].

Proof. Just note that at the beginning of the proof of Theorem 3.1 we have never used anything about the value of $f(0)$ to prove that $\Lambda := \{t > 0 \ : \ u \leq u_\theta \quad \text{in } \Sigma_\theta \quad \forall \theta \leq t\}$ is not empty. This immediately yields (3.19) and (3.20). Let now suppose that $u \not\equiv 0$. Then, $u > 0$ in $\mathbb{R}^N_+$ if $f(0) \geq 0$ (by the strong maximum principle) and $u > 0$ in Σ_{t_2}, for some small $t_2 > 0$, if $f(0) < 0$ thanks to Theorem 6.1. of [15]. As before, this information and the strong maximum principle imply (3.21) and (3.22). □

4. Boundedness, monotonicity and symmetry

Next we prove a result which provides natural assumptions ensuring that all solutions u of problem (P) are automatically bounded on the slabs $\mathbb{R}^{N-1} \times [0, t]$, for every $t > 0$.

Theorem 4.1. *Assume $N \geq 2$, $f \in C^0([0, +\infty))$ and let $u \in C^2(\overline{\mathbb{R}^N_+})$ be a solution of (3.18). Then u is bounded on the slabs $\mathbb{R}^{N-1} \times [0, t]$, for every $t > 0$, if one of the following assumptions holds true :*

- *(H_1) (Superlinear nonlinearities) f satisfies $\lim_{t\to\infty} \frac{f(t)}{t} = \infty$ and $f(t) \leq a_0(1 + t^r)$ for $t \geq 0$, where $r \in (1, \frac{N+1}{N-1})$ and $a_0 > 0$;*
- *(H_2) $\nabla u \in L^\infty(\mathbb{R}^N_+)$;*
- *(H_3) u has at most linear growth at infinity and $f(u(x)) \leq 0$ for every $x \in \overline{\mathbb{R}^N_+}$.*

When either (H_1) or (H_3) is in force, the bound on u on the slab $\mathbb{R}^{N-1} \times [0, t]$ is independent of the solution u (it actually depends on f, N and t only).

Remark 4.2. i) It will be clear from the proof that the conclusion of the theorem above holds true if (H_2) is replaced by : $|\nabla u|$ *is bounded on the strips* Σ_t*,* $t > 0$.

ii) Some control on the solution is however needed, even when $f(u(x)) \leq 0$ for every $x \in \mathbb{R}^N_+$. Indeed, the positive function $u(x) = x_N e^{x_1}$ solves $-\Delta u = -u \leq 0$ on $\mathbb{R}^N_+$, $u = 0$ on $\partial\mathbb{R}^N_+$, but it is unbounded on any slab Σ_t, $t > 0$.

Proof. When (H_1) is in force we use Theorem 2 of the recent work [23]. To this end we first observe that the assumptions on f imply that $f(s) \geq -A$ for every $s \geq 0$ and some $A > 0$. Then, for $R > 1$ we set $\Omega := B(0', 1) \times (0, 2R)$ and observe that, for any $z' \in \mathbb{R}^{N-1}$, the function $v(x) := u(x' + z', x_N)$ solves

$$\begin{cases} -\Delta v = f(v) & \text{in} \quad \Omega, \\ v \geq 0 & \text{in} \quad \Omega, \\ v = 0 & \text{on} \quad T := B(0', 1) \times 0. \end{cases} \tag{4.1}$$

Now, we fix $q > N$ and we apply Theorem 2 of [23] to v with $A^{(1)} = A^{(2)} = Id$ (hence $\lambda = 1$ and $\Lambda = \Lambda(q, N, \Omega)$), $b \equiv 0$, $h = A$, $f(x, s) = f(s) + A \geq 0$, $g(x, s) = f^+(s)$, $\xi(s) = s$, $\beta = 1$, $\Omega' = \Omega$ and $\omega = B(z_0, \frac{1}{4})$, where $z_0 = (0', 1)$. This leads to

$$v(x) \leq C \qquad \forall\, x \in \Omega,$$

where C is a positive constant depending only on N, q, r, Ω, T, f. Since z' is an arbitrary point of $\mathbb{R}^{N-1}$ we then have

$$v(x) \leq C \qquad \forall\, x \in \mathbb{R}^{N-1} \times [0, 2R],$$

where $C > 0$ depends only on R, N, q, r, T, f. The latter gives the desired conclusion since $R > 1$ is arbitrary.

When (H_2) holds true, the conclusion is clear thanks to the boundary condition satisfied by u and the mean value theorem.

When (H_3) is satisfied we use the following consequence of the maximum principle. Hereafter, for $z \in \partial\mathbb{R}^N_+$ and $R > 0$, we set $B^+(z, R) := B(z, R) \cap \mathbb{R}^N_+$.

Lemma 4.3. *Assume* $N \geq 2$ *and let* $v \in C^2(\overline{B^+(0, R)})$ *be any solution of*

$$\begin{cases} -\Delta v \leq 0 & \text{in} \quad B^+(0, R), \\ v \geq 0 & \text{in} \quad B^+(0, R), \\ v = 0 & \text{in} \quad \overline{B^+(0, R)} \cap \partial\mathbb{R}^N_+ \end{cases} \tag{4.2}$$

Then

$$0 \leq v(x) \leq 4N\Big(\sup_{B^+(0,R)} v\Big)\frac{x_N}{R} \qquad \forall\, x \in B^+\Big(0, \frac{3R}{4}\Big). \tag{4.3}$$

Proof of Lemma 4.3. If $x \in B^+(0, \frac{3R}{4})$ and $x_N \geq \frac{3R}{4}$, then (4.3) is clearly true. If $x = (x', x_N) \in B^+(0, \frac{3R}{4})$ and $x_N < \frac{3R}{4}$, we set $z = (x', 0) \in \partial\mathbb{R}^N_+$, $S := \sup_{B^+(0,R)} v$, $r = R - |z|$ and observe that $0 < x_N < \frac{R}{4} < r < R$. Then, for $y \in B^+(z, r)$, we consider the harmonic function $H(y) := S\Big(\frac{|y-z|^2}{r^2} + N(\frac{y_N}{r} - \frac{y_N^2}{r^2})\Big)$, which also satisfies $H \geq v$ on $\partial B^+(z, r)$. Therefore, $0 \leq v \leq H$ on $B^+(z, r)$, by the maximum principle. In particular, for $y = x$, we get $0 \leq v(x) \leq S\Big(\frac{x_N^2}{r^2} + N(\frac{x_N}{r} - \frac{x_N^2}{r^2})\Big) \leq S\,N(\frac{x_N}{r}) \leq 4N\Big(\sup_{B^+(0,R)} v\Big)\frac{x_N}{R}$. Which concludes the proof of the Lemma.

By (H_3) there is $a_0 > 0$ such that $u(x) \leq a_0(1 + |x|)$ for every $x \in \mathbb{R}^N_+$. Let $x \in \mathbb{R}^N_+$ and pick $R = 2|x| + 1$ and observe that $x \in B^+(0, \frac{3R}{4})$. Thus, an application of the above Lemma 4.3 yields $0 \leq u(x) \leq 4N(a_0(1 + R)\frac{x_N}{R} \leq (8a_0N)x_N$. This concludes the proof of the Theorem. □

By gathering together the previous results we can deduce various consequences. We start with

Theorem 4.4. *Assume $N \geq 2$, $f \in C^0([0, +\infty))$ and let $u \in C^2(\overline{\mathbb{R}^N_+})$ be a solution of* (3.18).

i) If f satisfies $\lim_{t\to\infty} \frac{f(t)}{t^r} = \ell \in (0, +\infty)$ for some $r \in (1, \frac{N+1}{N-1})$, then u is bounded on $\mathbb{R}^N_+$.
ii) If f satisfies $t^r - t \leq f(t) \leq \Lambda(t^r + 1)$ for $t \geq 0$, where $r \in (1, \frac{N+1}{N-1})$ and $\Lambda > 1$, then u is bounded on $\mathbb{R}^N_+$.

In both cases the bound on u is universal, i.e, it depends on f and N only.

iii) If $\nabla u \in L^\infty(\mathbb{R}^N_+)$ and f satisfies $\lim_{t\to\infty} \frac{f(t)}{t^p} = \ell \in (0, +\infty)$ for some $p \in (1, \overline{p}(N))$, then u is bounded on $\mathbb{R}^N_+$.
Here $\overline{p}(N)$ is the Sobolev exponent, i.e., $\overline{p}(N) = \frac{N+2}{N-2}$ if $N \geq 3$ and $p_S(2) = +\infty$.
iv) If $\nabla u \in L^\infty(\mathbb{R}^N_+)$ and f satisfies $t^p - t \leq f(t) \leq \Lambda(t^p + 1)$ for $t \geq 0$, where $p \in (1, \underline{p}(N))$ and $\Lambda > 1$, then u is bounded on $\mathbb{R}^N_+$.
Here $\underline{p}(N)$ is the Serrin exponent, i.e., $\underline{p}(N) = \frac{N}{N-2}$ if $N \geq 3$ and $p_S(2) = +\infty$.

Proof. If f satisfies the assumption of item i), then f also satisfies the assumption (H_1) of Theorem 4.1. Thus u is bounded on the slab $\mathbb{R}^{N-1} \times [0, 1]$ by a constant depending only on N and f. On the other hand, by Theorem 2.1 of [20], applied with $\Omega = \mathbb{R}^N_+$, we have that $u(x) \leq C(N, f)(1 + dist^{-\frac{2}{r-1}}(x, \partial\Omega))$ for every $x \in \Omega = \mathbb{R}^N_+$. Hence u is bounded on the set $\mathbb{R}^{N-1} \times [1, +\infty)$ by the universal constant $2C(N, f)$. This gives the conclusion.

If f satisfies the assumption of item ii), then f also satisfies the assumption (H_1) of Theorem 4.1 and so, as before, u is bounded on the slab $\mathbb{R}^{N-1} \times [0, 4]$ by a constant depending only on N and f. On the other hand the following standard integral estimate holds true for u

$$\int_{B(x_0,1)} u^r \leq C(N, r) \tag{4.4}$$

for all x_0 such that $\overline{B(x_0, 2)} \subset \mathbb{R}^N_+$. Here $C(N, r)$ is a positive constant independent on x_0 and u (it actually depends on N and r only). To this end, we first observe that the functions $u_{x_0}(x) := u(x + x_0)$ satisfy $-\Delta u_{x_0} \geq u^r_{x_0} - u_{x_0}$ on $B(0, 2)$ and then we multiply the previous differential inequality by ϕ_1 (a positive first eigenfunction of $-\Delta$ with homogeneous Dirichlet boundary conditions in $B(0, 2)$) and integrate by parts to get

$$\int_{B(0,2)} u^r_{x_0}\phi_1 - \int_{B(0,2)} u_{x_0}\phi_1 \leq -\int_{B(0,2)} \Delta u_{x_0}\phi_1 \leq -\int_{B(0,2)} u_{x_0}\Delta\phi_1 = \lambda_1 \int_{B(0,2)} u_{x_0}\phi_1$$

where $\lambda_1 > 0$ is the first eigenvalue of $-\Delta$ with homogeneous Dirichlet boundary conditions in $B(0, 2)$. From the latter, after an application of Holder inequality, we obtain

$$\Big(\inf_{B(0,1)} \phi_1\Big) \int_{B(0,1)} u^r_{x_0} \leq \int_{B(0,2)} u^r_{x_0}\phi_1 \leq (1 + \lambda_1)^{\frac{r}{r-1}} \int_{B(0,2)} \phi_1^{\frac{r}{r-1}}$$

and so

$$\int_{B(x_0,1)} u^r = \int_{B(0,1)} u^r_{x_0} \leq \Big(\inf_{B(0,1)} \phi_1\Big)^{-1} (1+\lambda_1)^{\frac{r}{r-1}} \int_{B(0,2)} \phi_1^{\frac{r}{r-1}} := C(N,r)$$

as claimed. From (4.4) we then get

$$\Big(\inf_{B(x_0,1)} u\Big)^r \leq \frac{1}{|B(x_0,1)|} \int_{B(x_0,1)} u^r \leq \frac{C(N,r)}{|B(0,1)|}$$

hence, for all x_0 such that $\overline{B(x_0,2)} \subset \mathbb{R}^N_+$,

$$\inf_{B(x_0,1)} u \leq C'(N,r) \tag{4.5}$$

where $C'(N,r)$ is a positive constant independent on x_0 and u.

Combining (4.5) with the Harnack inequality (see e.g. item (b) of Theorem 4.1 and item (b) of Theorem 4.3 of [22]), applied to every ball $B(x_0,1)$ where $x_0 \in \mathbb{R}^{N-1} \times [3,+\infty)$, we obtain

$$u(x_0) \leq \sup_{B(x_0,1)} u \leq C(r,\Lambda,R=1) \inf_{B(x_0,1)} u \leq C(r,\Lambda,R=1)C'(N,r) := C''(N,f)$$

where $C''(N,r)$ is a positive constant independent on x_0 and u. The desired conclusion then follows.

The cases iii) and iv) are treated as the cases i) and ii) with the only difference that we use that (H_2) of Theorem 4.1 is now in force. □

Theorem 4.5. *Assume $N \geq 2$, $f \in C^{0,1}_{loc}([0,+\infty))$ with $f(0) \geq 0$ and let $u \in C^2(\overline{\mathbb{R}^N_+})$ be a solution of (P). If either the condition (H_1) or (H_2) or (H_3) of Theorem 4.1 is satisfied, then u is monotone, i.e., $\frac{\partial u}{\partial x_N} > 0$ in $\mathbb{R}^N_+$.*

Remark 4.6. In the case $N = 2$ the conclusion of the theorem above was already known to hold under the sole assumption that f is locally lipschitz continuous, see [15, 16].

Proof. Theorem 4.1 implies that u is bounded on every slab. The conclusion then follows by applying Theorem 3.1. □

Theorem 4.7. *Assume $f \in C^{0,1}_{loc}([0,+\infty))$ and let $u \in C^2(\overline{\mathbb{R}^N_+})$ be a solution of* (3.18).

a) Assume $N = 2, 3$ and let us suppose that one of the following assumptions holds true :

- *i) $f(0) \geq 0$ and $\lim_{t\to\infty} \frac{f(t)}{t^r} = \ell \in (0,+\infty)$ for some $r \in (1, \frac{N+1}{N-1})$.*
- *ii) $t^r - t \leq f(t) \leq \Lambda(t^r + 1)$ for $t \geq 0$, where $r \in (1, \frac{N+1}{N-1})$ and $\Lambda > 1$.*
- *iii) $\nabla u \in L^\infty(\mathbb{R}^N_+)$ and $f(0) \geq 0$, $\lim_{t\to\infty} \frac{f(t)}{t^p} = \ell \in (0,+\infty)$ for some $p \in (1, \overline{p}(N))$.*
- *iv) $\nabla u \in L^\infty(\mathbb{R}^N_+)$ and $t^p - t \leq f(t) \leq \Lambda(t^p + 1)$ for $t \geq 0$, where $p \in (1, \underline{p}(N))$ and $\Lambda > 1$.*

Here $\underline{p}$ and $\overline{p}$ are as in Theorem 4.4.

Then, either $u \equiv 0$ and $f(0) = 0$, or u is positive, bounded, monotone and one-dimensional on $\mathbb{R}^N_+$.

b) Assume $N \geq 2$.

- *i) If $t^r \leq f(t) \leq \Lambda t^r$ for $t \geq 0$, where $r \in (1, \frac{N+1}{N-1})$ and $\Lambda > 1$, then $u \equiv 0$ in $\mathbb{R}^N_+$.*
- *ii) if $\nabla u \in L^\infty(\mathbb{R}^N_+)$ and $t^p \leq f(t) \leq \Lambda t^p$ for $t \geq 0$, where $p \in (1, \underline{p}(N))$ and $\Lambda > 1$, then $u \equiv 0$ in $\mathbb{R}^N_+$.*

Remark 4.8. For $N = 2$: item a) i) holds true for any $r > 1$ (see [16]), item a) iii) holds true for any locally Lipschitz function f satisfying $f(0) \geq 0$ (see [15]) while item a) iv) and item b) ii) hold true for any locally Lipschitz function f (see [15]).

Proof. Note that $f(0) \geq 0$ in any case. Then, by the strong maximum principle, either $u \equiv 0$ and so $f(0) = 0$, or $u > 0$ on $\mathbb{R}^N_+$. Then, to conclude the proof of item a) we just need to treat the case $u > 0$. By Theorem 4.4 and Theorem 4.5 u is bounded and monotone. Since a solution for $N = 2$ can be seen as a solution for $N = 3$, the one-dimensional symmetry of u then follows from Theorem 1.5 of [18] (or from Theorem 1.5 of [4] if $f \in C^1$). Let us now turn to item b) and suppose for contradiction that $u > 0$. If $N = 2, 3$ then, thanks to item a), u would be a 1D, bounded and monotone increasing solution to $-u'' = f(u)$ on $\mathbb{R}^+$, which is clearly impossible. If $N \geq 4$, u would be bounded and monotone increasing by Theorem 4.5. But this is in contradiction with the last sentence of item (a) of Theorem IV of [22] (which implies that $u \to 0$ as $x_N \to +\infty$). Thus $u \equiv 0$ on $\mathbb{R}^N_+$, as claimed. □

Conflict of interest

The authors declare no conflict of interest.

References

1. Berestycki H, Caffarelli LA, Nirenberg L (1990) Uniform estimates for regularization of free boundary problems, In: *Analysis and Partial Differential Equations*, New York: Dekker, 567–617.
2. Berestycki H, Caffarelli LA, Nirenberg L (1993) Symmetry for elliptic equations in the halfspace, In: *Boundary Value Problems for PDEs and Applications*, Paris: Masson, 27–42.
3. Berestycki H, Caffarelli LA, Nirenberg L (1996) Inequalities for second order elliptic equations with applications to unbouded domains. *Duke Math J* 81: 467–494.
4. Berestycki H, Caffarelli LA, Nirenberg L (1997) Further qualitative properties for elliptic equations in unbouded domains. *Ann Scuola Norm Sup Pisa Cl Sci* 25: 69–94.
5. Berestycki H, Caffarelli LA, Nirenberg L (1997) Monotonicity for elliptic equations in an unbounded Lipschitz domain. *Commun Pure Appl Math* 50: 1089-1111.
6. Caffarelli LA, Salsa S (2005) *A Geometric Approach To Free Boundary Problems*, AMS.
7. Chen Z, Lin CS, Zou W (2014) Monotonicity and nonexistence results to cooperative systems in the half space. *J Funct Anal* 266: 1088–1105.
8. Cortázar C, Elgueta M, García-Melián J (2016) Nonnegative solutions of semilinear elliptic equations in half-spaces. *J Math Pure Appl* 106: 866-–876.
9. Dancer EN (1992) Some notes on the method of moving planes. *B Aust Math Soc* 46: 425–434.
10. Dancer EN (2009) Some remarks on half space problems. *Disc Cont Dyn Sist* 25: 83–88.
11. Farina A (2003) Rigidity and one-dimensional symmetry for semilinear elliptic equations in the whole of $\mathbb{R}^N$ and in half spaces. *Adv Math Sci Appl* 13: 65–82.
12. Farina A (2007) On the classification of solutions of the Lane-Emden equation on unbounded domains of $\mathbb{R}^N$. *J Math Pure Appl* 87: 537–561.

13. Farina A (2015) Some symmetry results and Liouville-type theorems for solutions to semilinear equations. *Nonlinear Anal Theor* 121: 223–229.

14. Farina A, Montoro L, Sciunzi B (2012) Monotonicity and one-dimensional symmetry for solutions of $-\Delta_p u = f(u)$ in half-spaces. *Calc Var Partial Dif* 43: 123–145.

15. Farina A, Sciunzi B (2016) Qualitative properties and classification of nonnegative solutions to $-\Delta u = f(u)$ in unbounded domains when $f(0) < 0$. *Rev Mat Iberoam* 32: 1311–1330.

16. Farina A, Sciunzi B (2017) Monotonicity and symmetry of nonnegative solutions to $-\Delta u = f(u)$ in half-planes and strips. *Adv Nonlinear Stud* 17: 297–310.

17. Farina A, Soave N (2013) Symmetry and uniqueness of nonnegative solutions of some problems in the halfspace. *J Math Anal Appl* 403: 215–233.

18. Farina A, Valdinoci E (2010) Flattening results for elliptic PDEs in unbounded domains with applications to overdetermined problems. *Arch Ration Mech Anal* 195: 1025–1058.

19. Gidas B, Spruck J (1981) A priori bounds for positive solutions of nonlinear elliptic equations. *Commun Part Diff Eq* 6: 883–901.

20. Polácik PP, Quittner P, Souplet P (2007) Singularity and decay estimates in superlinear problems via Liouville-type theorems. I. Elliptic equations and systems. *Duke Math J* 139: 555–579.

21. Quaas A, Sirakov B (2006) Existence results for nonproper elliptic equations involving the Pucci operator. *Commun Part Diff Eq* 31: 987–1003.

22. Serrin J, Zou H (2002) Cauchy–Liouville and universal boundedness theorems for quasilinear elliptic equations and inequalities. *Acta Math* 189: 79–142.

23. Sirakov B (2019) A new method of proving a priori bounds for superlinear elliptic PDE. *arXiv:1904.03245*.

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