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Research article

Remarks on the decay/growth rate of solutions to elliptic free boundary problems of obstacle type†

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Abstract: The purpose of this note is to present a "new" approach to the decay rate of the solutions to the no-sign obstacle problem from the free boundary, based on Weiss-monotonicity formula. In presenting the approach we have chosen to treat a problem which is not touched earlier in the existing literature. Although earlier techniques may still work for this problem, we believe this approach gives a shorter proof, and may have wider applications.

Keywords: no-sign obstacle problem; singular elliptic equation; regularity of solutions

1. Introduction

1.1. Problem statement

We consider a singular no-sign obstacle problem of the type

$$\begin{cases} \operatorname{div}(x_1^a \nabla u) = x_1^a f(x)\chi_{\{u\neq 0\}} & \text{in } B_1^+, \\ u = 0 & \text{on } B_1 \cap \{x_1 = 0\}, \end{cases} \tag{1.1}$$

where $a > 1$, χ_D is the characteristic function of D, $B_1 \subset \mathbb{R}^n$ is the unit ball and $B_1^+ = B_1 \cap \{x_1 > 0\}$. The equation is considered in the weak form,

$$\int_{B_1^+} x_1^a \nabla u \nabla \varphi dx = \int_{B_1^+} x_1^a f(x)\varphi\chi_{\{u\neq 0\}} dx,$$

for all $\varphi \in W_0^{1,2}(B_1^+)$. This problem, when the non-negativity assumption $u \geq 0$ is imposed, is already studied in [9]. The above no-sign problem, as a general semilinear PDE with non-monotone r.h.s., introduces certain difficulties and to some extent some challenges. To explain this we shall give a very short review of the existing results and methods for similar type of problems (see also the book [6] and Caffarelli's review of the classical obstacle problem [2]). The general methodology of approaching such problems lies in using the so-called ACF-monotonicity formula (see [8]) or alternatively using John Andersson's dichotomy (see [1] or [3]). Although there are still some chances that both these methods will work for our problem above, we shall introduce a third method here which relies on a softer version of a monotonicity formula (which has a wider applicability) in combination with some elaborated analysis. We refer to this as a Weiss-type monotonicity formula, see (2.1) below.

1.2. Notation

For clarity of exposition we shall introduce some notations and definitions here that are used frequently in the paper. Throughout this paper, $\mathbb{R}^n$ will be equipped with the Euclidean inner product $x \cdot y$ and the induced norm $|x|$, $B_r(x_0)$ will denote the open n-dimensional ball of center x_0, radius r with the boundary $\partial B_r(x_0)$. In addition, $B_r = B_r(0)$ and $\partial B_r = \partial B_r(0)$. $\mathbb{R}^n_+$ stands for half space $\{x \in \mathbb{R}^n : x_1 > 0\}$ as well as $B_r^+ = B_r \cap \mathbb{R}^n_+$. Moreover, in the text we use the n-dimensional Hausdorff measure $\mathcal{H}^n$. For a multi-index $\mu = (\mu_1, \cdots, \mu_n) \in \mathbb{Z}^n_+$, we denote the partial derivative with $\partial^\mu u = \partial_{x_1}^{\mu_1} \cdots \partial_{x_n}^{\mu_n} u$ and $|\mu|_1 = \mu_1 + \cdots + \mu_n$.

For a domain $\Omega \subset \mathbb{R}^n_+$ and $1 \leq p < \infty$, we use the notation $\mathcal{L}^p(\Omega)$ and $W^{m,p}(\Omega)$ as the standard spaces. However, we need some new notation for the weighted spaces

$$\mathcal{L}^p(\Omega; x_1^\theta) := \left\{ u : \int_\Omega x_1^\theta |u(x)|^p \, dx < \infty \right\},$$

where $\theta \in \mathbb{R}$. For $m \in \mathbb{N}$, we define the weighted Sobolev space $W^{m,p}(\Omega; x_1^\theta)$ as the closure of $C^\infty(\overline{\Omega})$ with the following norm,

$$\|u\|_{W^{m,p}(\Omega; x_1^\theta)} := \|u\|_{\mathcal{L}^p(\Omega; x_1^\theta)} + \|x_1 Du\|_{\mathcal{L}^p(\Omega; x_1^\theta)} + \cdots + \|x_1^m D^m u\|_{\mathcal{L}^p(\Omega; x_1^\theta)}.$$

It is noteworthy that for $\theta = 0$, we have $\mathcal{L}^p(\Omega; 1) = \mathcal{L}^p(\Omega)$ but $W^{m,p}(\Omega; 1) \supsetneq W^{m,p}(\Omega)$. Generally, the trace operator has no sense for $\theta > -1$, while functions in $W^{1,p}(\Omega; x_1^\theta)$ have zero traces on $\{x_1 = 0\}$ for $\theta \leq -1$. (Theorem 6 in [7]).

1.3. Main results

We consider $u \in W^{1,p}(B_1^+, x_1^\theta)$ for some $\theta < -n$ and $n < p$ to be a weak solution of (1.1). This condition provides the continuity of $x_1^{(\theta+n)/p} u$ up to the boundary according to Sobolev embedding Theorem 3.1 in [5]. First, we prove the following a priori regularity result.

Proposition 1.1. *(Appendix A) Let $u \in W^{1,p}(B_1^+, x_1^\theta)$ be a solution of* (1.1) *for some $\theta < -n$, $n < p$ and $f \in \mathcal{L}^\infty(B_1^+)$. Then for each* $\max\{0, 1 + \frac{\theta+n}{p}\} < \beta < 1$ *there exists $C = C(\beta, n, a)$ such that for $r \leq 1/2$,*

$$\sup_{B_r^+(x^0)} |x_1^{\beta-1} u| \leq C r^{2\beta},$$

for all $x_0 \in \{x_1 = 0\}$.

In Appendix A we will prove this proposition. Our main result in this paper concerns the optimal growth rate of solution u of (1.1) at touching free boundary points, which is stated in the following theorem.

Theorem 1.2. *Suppose $u \in W^{1,p}(B_1^+, x_1^\theta)$ is a solution of* (1.1) *for some $\theta < -n$, $n < p$ and $x^0 \in \partial\{u = 0\} \cap \{x_1 = 0\} \cap B_{1/4}^+$. Moreover, if $f \in C^\alpha(\overline{B_1^+})$ for some $\alpha \in (0, 1)$, then*

$$|u(x)| \leq C x_1^2 \left(\left(\frac{|x - x^0| + x_1}{x_1} \right)^{(n+a+4)/2} + 1 \right),$$

for a universal constant $C = C(a, n, [f]_{0,\alpha})$.

2. Monotonicity formula

Our main tool in proving optimal decay for solutions from the free boundary points is Weiss-monotonicity formula, combined with some elaborated techniques. We define the balanced energy functional

$$\Phi_{x^0}(r, u) = r^{-n-2-a} \int_{B_r^+(x^0)} (x_1^a |\nabla u|^2 + 2x_1^a f(x^0) u) dx - 2r^{-n-3-a} \int_{\partial B_r(x^0) \cap \mathbb{R}^n_+} x_1^a u^2 \, d\mathcal{H}^{n-1}. \qquad (2.1)$$

Considering the scaling $u_{r,x^0} = u_r(x) = \frac{u(rx+x^0)}{r^2}$, $\Phi_{x^0}(r, u) = \Phi_0(1, u_r)$. In what follows we prove almost-monotonicity of the energy.

Lemma 2.1 (Almost-Monotonicity Formula)**.** *Let u solve* (1.1) *and be as in Proposition 1.1 and assume that $\nabla u(x^0) = 0$ for some $x^0 \in \{x_1 = 0\}$ and $f \in C^\alpha(\overline{B_1^+})$ for some $\alpha \in (0, 1)$. Then u satisfies, for $r \leq r_0$ such that $B_{r_0}^+(x^0) \subseteq B_1^+$,*

$$\frac{d}{dr} \Phi_{x^0}(r, u) \geq 2r \int_{\partial B_1 \cap \mathbb{R}^n_+} x_1^a \, (\partial_r u_r)^2 \, d\mathcal{H}^{n-1} - C r^{\alpha+\beta-2},$$

where C depends only on $\|f\|_{C^\alpha(\overline{B_1^+})}$ and the constant $C(\beta, n, a)$ in Proposition 1.1.

Proof. Let $u_r(x) := \frac{u(rx+x^0)}{r^2}$, then

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dr} \Phi_{x^0}(r, u) \\
&= \frac{1}{2} \frac{d}{dr} \left[\int_{B_1^+} (x_1^a |\nabla u_r|^2 + 2x_1^a f(x^0) u_r) \, dx - 2 \int_{\partial B_1 \cap \mathbb{R}^n_+} x_1^a u_r^2 \, d\mathcal{H}^{n-1} \right] \\
&= \frac{1}{2} \left[\int_{B_1^+} (2x_1^a \nabla u_r \cdot \nabla \partial_r u_r + 2x_1^a f(x^0) \partial_r u_r) \, dx - 4 \int_{\partial B_1 \cap \mathbb{R}^n_+} x_1^a u_r \partial_r u_r \, d\mathcal{H}^{n-1} \right] \\
&= \int_{B_1^+} \operatorname{div}(x_1^a \partial_r u_r \nabla u_r) - \partial_r u_r \operatorname{div}(x_1^a \nabla u_r) + x_1^a f(x^0) \partial_r u_r \, dx - 2 \int_{\partial B_1 \cap \mathbb{R}^n_+} x_1^a u_r \partial_r u_r \, d\mathcal{H}^{n-1} \\
&= \int_{B_1^+} (f(x^0) - f(rx + x^0) \chi_{\{u_r \neq 0\}}) x_1^a \partial_r u_r \, dx + \int_{\partial B_1^+} x_1^a \partial_r u_r \nabla u_r \cdot \nu \, d\mathcal{H}^{n-1} - 2 \int_{\partial B_1 \cap \mathbb{R}^n_+} x_1^a u_r \partial_r u_r \, d\mathcal{H}^{n-1}
\end{aligned}$$

$$= \int_{B_1^+} (f(x^0) - f(rx + x^0)\chi_{\{u_r \neq 0\}}) x_1^a \partial_r u_r \, dx + r \int_{\partial B_1 \cap \mathbb{R}^n_+} x_1^a \, (\partial_r u_r)^2 \, d\mathcal{H}^{n-1}$$
$$= \int_{B_1^+} (f(x^0) - f(rx + x^0))\chi_{\{u_r \neq 0\}} x_1^a \partial_r u_r \, dx + f(x^0) \int_{B_1^+ \cap \{u_r = 0\}} x_1^a \partial_r u_r \, dx + r \int_{\partial B_1 \cap \mathbb{R}^n_+} x_1^a \, (\partial_r u_r)^2 \, d\mathcal{H}^{n-1}.$$

Note that the second integral

$$\int_{B_1^+ \cap \{u_r = 0\}} x_1^a \partial_r u_r \, dx = 0$$

as $|\{u_r = 0 \wedge \nabla u_r \neq 0\}| = 0$ and $\partial_r u_r = 0$ on $\{u_r = 0 \wedge \nabla u_r = 0\}$. Since $|\partial_r u_r| \leq C r^{\beta - 2}$ we infer that

$$\int_{B_1^+} (f(x^0) - f(rx + x^0))\chi_{\{u_r \neq 0\}} x_1^a \partial_r u_r \, dx \geq -C r^{\alpha + \beta - 2}$$

and conclude that

$$\frac{1}{2} \frac{d}{dr} \Phi_{x^0}(r, u) \geq r \int_{\partial B_1 \cap \mathbb{R}^n_+} x_1^a \, (\partial_r u_r)^2 \, d\mathcal{H}^{n-1} - C r^{\alpha + \beta - 2}.$$

□

Definition 2.2. *Let* $\mathbb{HP}_2$ *stand for the class of all two-homogeneous functions* $P \in W^{1,2}(B_1^+; x_1^{a-2})$ *satisfying* $\mathrm{div}(x_1^a \nabla P) = 0$ *in* $\mathbb{R}^n_+$ *with boundary condition* $P = 0$ *on* $x_1 = 0$*. We also define the operator* $\Pi(v, r, x^0)$ *to be the projection of* v_{r,x^0} *onto* $\mathbb{HP}_2$ *with respect to the inner product*

$$\langle v, w \rangle = \int_{\partial B_1 \cap \mathbb{R}^n_+} x_1^a v w \, d\mathcal{H}^{n-1}.$$

We will use the following extension of [10, Lemma 4.1].

Lemma 2.3. *Assume that* $\mathrm{div}(x_1^a \nabla w) = 0$ *in* B_1^+ *with boundary condition* $w = 0$ *on* $x_1 = 0$*, and* $w(0) = |\nabla w(0)| = 0$*. Then*

$$\int_{B_1^+} x_1^a |\nabla w|^2 \, dx - 2 \int_{\partial B_1 \cap \mathbb{R}^n_+} x_1^a w^2 \, d\mathcal{H}^{n-1} \geq 0,$$

and equality implies that $w \in \mathbb{HP}_2$*, i.e., it is homogeneous of degree two.*

Proof. We define an extension of the Almgren frequency,

$$r \mapsto N(w, r) := \frac{r \int_{B_r^+} x_1^a |\nabla w|^2 \, dx}{\int_{\partial B_r^+} x_1^a w^2 \, d\mathcal{H}^{n-1}},$$

$$\frac{1}{2} \frac{N'(w, r)}{N(w, r)} = \frac{\int_{\partial B_r^+} x_1^a (\partial_\nu w)^2 \, d\mathcal{H}^{n-1}}{\int_{\partial B_r^+} x_1^a w \partial_\nu w \, d\mathcal{H}^{n-1}} - \frac{\int_{\partial B_r^+} x_1^a w \partial_\nu w \, d\mathcal{H}^{n-1}}{\int_{\partial B_r^+} x_1^a w^2 \, d\mathcal{H}^{n-1}} \geq 0.$$

Moreover, if $N(w, r) = \kappa$ for $\rho < r < \sigma$, it implies that w is homogeneous of degree κ in $B_\sigma \setminus B_\rho$.

Now supposing towards a contradiction that $N(w, s) < 2$ for some $s \in (0, 1]$, and defining $w_r(x) := \frac{w(rx)}{\|w(rx)\|_{\mathcal{L}^2(\partial B_1^+, x_1^b)}}$, we infer from $N(w, s) < 2$ that ∇w_r is bounded in $\mathcal{L}^2(B_1^+; x_1^a)$ and so $\nabla w_{r_m} \rightharpoonup \nabla w_0$ weakly in $\mathcal{L}^2(B_1^+; x_1^a)$ and $w_{r_m} \to w_0$ strongly in $\mathcal{L}^2(\partial B_1^+; x_1^a)$ as a sequence $r_m \to 0$. Consequently, w_0 satisfies $\operatorname{div}(x_1^a \nabla w_0) = 0$ in B_1^+, $w_0(0) = |\nabla w_0(0)| = 0$ and $w_0 = 0$ on $x_1 = 0$ as well as $\|w_0\|_{\mathcal{L}^2(\partial B_1^+; x_1^a)} = 1$. Furthermore, for all $r \in (0,1)$ we have

$$N(w_0, r) = \lim_{r_m \to 0} N(w_{r_m}, r) = \lim_{r_m \to 0} N(w, r r_m) = N(w, 0+),$$

and so w_0 must be a homogeneous function of degree $\kappa := N(w, 0+) < 2$. Note that for every multi-index $\mu \in \{0\} \times \mathbb{Z}_+^{n-1}$, the higher order partial derivative $\zeta = \partial^\mu w_0$ satisfies the equation $\operatorname{div}(x_1^a \nabla \zeta) = 0$ in $\mathbb{R}_+^n$. From the integrability and homogeneity we infer that $\partial^\mu w_0 \equiv 0$ for $\kappa - |\mu|_1 < -\frac{n}{2}$, otherwise

$$\int_{B_1^+} |\partial^\mu w_0|^2 dx = \left(\int_0^1 r^{2(\kappa - |\mu|_1) + n - 1} dr \right) \int_{\partial B_1 \cap \mathbb{R}_+^n} |\partial^\mu w_0|^2 d\mathcal{H}^{n-1}$$

can not be bounded. Thus $x' \mapsto w_0(x_1, x')$ is a polynomial, and we can write $w_0(x_1, x') = x_1^\kappa p(\frac{x'}{x_1})$. Consider the multi-index μ such that $|\mu|_1 = \deg p$, so $\partial^\mu w_0 = x_1^{\kappa - |\mu|_1} \partial^\mu p$ is a solution of $\operatorname{div}(x_1^a \nabla \zeta) = 0$ in $\mathbb{R}_+^n$. Therefore, $\partial^\mu w_0 \in W^{1,2}(B_1^+; x_1^\theta)$ for $-1 < \theta$ according to Proposition A.1, which implies that $2(\kappa - |\mu|_1) + \theta > -1$. So, $\deg p < \kappa + \frac{\theta+1}{2}$.

Substituting $w_0(x) = x_1^{\kappa-1}(\alpha x_1 + \ell \cdot x')$ for $\kappa > 1$ in the equation and comparing with $w_0(0) = |\nabla w_0(0)| = 0$ we arrive at the only nonzero possible case being $\kappa + a = 2$, which contradicts $a > 1$. The case $\kappa < 1$ leads to $\deg p = 0$ and $w_0(x) = \alpha x_1^\kappa$, which implies $\kappa + a = 1$ and a contradiction to $a > 1$. □

3. Decay rate of solutions close to degenerate points

Proposition 3.1. *Let $f \in C^\alpha(\overline{B_1^+})$ and u be solution of* (1.1) *satisfying the condition in Proposition 1.1. Then the function*

$$r \mapsto r^{-n-3-a} \int_{\partial B_r^+(x^0)} x_1^a u^2(x) \, d\mathcal{H}^{n-1}$$

is bounded on $(0, 1/8)$*, uniformly in* $x^0 \in \partial\{u = 0\} \cap \{x_1 = 0\} \cap B_{1/8}$.

Proof. Let us divide the proof into steps.

Step 1 We claim that there exists a constant $C_1 < \infty$ such that for all $x^0 \in \partial\{u = 0\} \cap \{x_1 = 0\} \cap B_{1/8}$ and $r \leq 1/8$,

$$f(x^0) \int_{B_1^+} x_1^a u_{x^0,r}(x) \, dx \geq -C_1,$$

where $u_{x^0,r} := \frac{u(rx+x^0)}{r^2}$. To prove this we observe that $w := u_{x^0,r}$ satisfies

$$\operatorname{div}(x_1^a \nabla w) = x_1^a f_r(x) := x_1^a f(x^0 + rx) \chi_{\{u_{x^0,r} \neq 0\}}, \qquad \text{in } B_1^+.$$

Moreover, for $\phi(\rho) := \rho^{-n-a+1} \int_{\partial B_\rho^+} x_1^a w(x) \, d\mathcal{H}^{n-1}$ we have

$$\phi'(\rho) = \int_{\partial B_1^+} x_1^a \nabla w(\rho x) \cdot x \, d\mathcal{H}^{n-1} = \rho^{-n-a+1} \int_{B_\rho^+} \operatorname{div}(x_1^a \nabla w(x)) \, dx$$

$$= \rho^{-n-a+1} \int_{B_\rho^+} x_1^a f_r(x)\, dx.$$

If $f(x^0) \geq \frac{1}{8^\alpha}[f]_{0,\alpha}$ then $f_r \geq 0$ for $r \leq 1/8$. Therefore ϕ is increasing and $\phi(\rho) \geq \phi(0) = 0$ (recall that $w(0) = 0$). Similarly, if $f(x^0) \leq -\frac{1}{8^\alpha}[f]_{0,\alpha}$, we obtain that $\phi(\rho) \leq 0$. Therefore the claim is true for $C_1 = 0$ in these cases.

In the case $|f(x^0)| \leq \frac{1}{8^\alpha}[f]_{0,\alpha}$, then $|f_r(x)| \leq \frac{2}{8^\alpha}[f]_{0,\alpha}$ and then

$$|\phi'(\rho)| \leq 2^{1-3\alpha}[f]_{0,\alpha}\rho^{-n-a+1} \int_{B_\rho^+} x_1^a\, dx \leq 2^{1-3\alpha}[f]_{0,\alpha}\rho.$$

So, $|\phi(\rho)| \leq 2^{-3\alpha}[f]_{0,\alpha}\rho^2$ and

$$\begin{aligned}\left| f(x^0) \int_{B_1^+} x_1^a u_{x^0,r}(x)\, dx \right| &= \left| f(x^0) \int_0^1 \rho^{n+a-1}\phi(\rho) d\rho \right| \\ &\leq \frac{2^{-6\alpha}[f]_{0,\alpha}^2}{n+a+2} =: C_1.\end{aligned}$$

Step 2 We claim that there exists a constant $C_2 < \infty$ such that

$$\mathrm{dist}_{\mathcal{L}^2(\partial B_1 \cap \mathbb{R}^n_+; x_1^a)}(u_{x^0,r}, \mathbb{HP}_2) \leq C_2,$$

for every $x^0 \in \partial\{u = 0\} \cap \{x_1 = 0\} \cap B_{1/8}$, $r \leq 1/8$. Suppose towards a contradiction that this is not true, then there exists a sequence u^m, $x^m \to \bar{x}$ and $r_m \to 0$ such that

$$M_m = \|u^m_{x^m,r_m} - \Pi(u^m, r_m, x^m)\|_{\mathcal{L}^2(\partial B_1 \cap \mathbb{R}^n_+; x_1^a)} \to \infty, \qquad m \to \infty.$$

Let $u_m := u^m_{x^m,r_m}$ and $p_m = \Pi(u^m, r_m, x^m)$ and $w_m = \frac{u_m - p_m}{M_m}$. Then, since $u_m(0) = |\nabla u_m(0)| = 0$ and by the monotonicity formula and the result of previous step, we find that

$$\begin{aligned}&\int_{B_1^+} x_1^a |\nabla w_m|^2\, dx - 2\int_{\partial B_1^+} x_1^a w_m^2\, d\mathcal{H}^{n-1} \\ &= \frac{1}{M_m^2}\left[\Phi_{x^m}(r_m) - 2\int_{B_1^+} f(x^m) x_1^a u_m\, dx\right] \\ &\quad + \frac{1}{M_m^2}\int_{\partial B_1^+} x_1^a \left(p_m \nabla p_m \cdot \nu - 2u_m \nabla p_m \cdot \nu - 2p_m^2 + 4u_m p_m\right) d\mathcal{H}^{n-1} \\ &\leq \frac{1}{M_m^2}(\Phi_{x^m}(r_m) + 2C_1) \leq \frac{1}{M_m^2}\left(\Phi_{x^m}\left(\frac{1}{2}\right) + 2C_1\right) \to 0, \qquad m \to \infty.\end{aligned} \tag{3.1}$$

Passing to a subsequence such that $\nabla w_m \rightharpoonup \nabla w$ in $\mathcal{L}^2(B_1^+; x_1^a)$ as $m \to \infty$, the compact embedding on the boundary implies that $\|w\|_{\mathcal{L}^2(\partial B_1 \cap \mathbb{R}^n_+; x_1^a)} = 1$, and

$$\int_{B_1^+} x_1^a |\nabla w|^2 dx \leq 2\int_{\partial B_1} x_1^a w^2\, d\mathcal{H}^{n-1} \tag{3.2}$$

and that

$$\int_{\partial B_1} wp\, d\mathcal{H}^{n-1} = 0, \qquad \forall p \in \mathbb{HP}_2. \tag{3.3}$$

Since $\operatorname{div}(x_1^a \nabla w_m) = \frac{x_1}{M_m} f(x^m + r\cdot)\chi_{\{u_m \neq 0\}}$, it follows that $\operatorname{div}(x_1^a \nabla w) = 0$ in B_1^+. Moreover, we obtain from L^p-theory that $w_m \to w$ in $C^{1,\alpha}_{\mathrm{loc}}(B_1^+)$ for each $\alpha \in (0,1)$ as $m \to \infty$. Consequently $w(0) = |\nabla w(0)| = 0$. Thus we can apply Lemma 2.3 and obtain from (3.2) that w is homogeneous of degree 2, contradicting (3.3) and $\|w\|_{\mathcal{L}^2(\partial B_1)} = 1$. This proves the claim.

Step 3 We will show that there exists constant C_2 such that for all $x^0 \in \partial\{u = 0\} \cap \{x_1 = 0\}$ satisfying

$$\liminf_{r \to 0^+} \frac{|B_r^+(x^0) \cap \{u = 0\}|}{|B_r^+|} = 0, \tag{3.4}$$

we have

$$\Phi_{x^0}(0+) - \int_{B_1^+} x_1^a f(x^0) u_{x^0,r}\, dx \geq -C_2 r^\alpha |f(x^0)|.$$

In order to see this, we can observe that

$$\begin{aligned}
\int_{B_1^+} x_1^a f(x^0) u_{x^0,r}(x) dx &= f(x^0) \int_0^1 \int_0^1 \partial_s \Big[\int_{\partial B_\rho^+} (sx_1)^a u_{x^0,r}(sx)\, d\mathcal{H}^{n-1}(x) \Big] ds\, d\rho \\
&= f(x^0) \int_0^1 \rho \int_0^1 \int_{\partial B_\rho^+} (sx_1)^a \nabla u_{x^0,r}(sx) \cdot \nu\, d\mathcal{H}^{n-1}\, ds\, d\rho \\
&= f(x^0) \int_0^1 \rho \int_0^1 s \int_{B_\rho^+} \operatorname{div}((sx_1)^a \nabla u_{x^0,r}(sx))\, dx\, ds\, d\rho \\
&= f(x^0) \int_0^1 \rho \int_0^1 s \int_{B_\rho^+} (sx_1)^a f(rsx) \chi_{\Omega_{x^0,r}}(sx)\, dx\, ds\, d\rho \\
&= f(x^0)^2 \int_0^1 \rho^{1+a+n} \int_0^1 s^{1+a} \int_{B_1^+} x_1^a\, dx\, ds\, d\rho \\
&\quad + f(x^0) \int_0^1 \rho^{1+a+n} \int_0^1 s^{1+a} \int_{B_1^+} x_1^a \Big(f(rs\rho x) - f(x^0) \Big) dx\, ds\, d\rho \\
&\leq \frac{f(x^0)^2}{(n+a+2)(a+2)} \int_{B_1^+} x_1^b\, dx + C_2 r^\alpha |f(x^0)|.
\end{aligned}$$

Now by condition (3.4), consider a sequence $r_m \to 0$ such that $\frac{|B_{r_m}^+(x^0) \cap \{v=0\}|}{|B_{r_m}^+|} \to 0$ and assume that $\nabla(u_{x^0,r_m} - p_{x^0,r_m}) \rightharpoonup \nabla w$ in $\mathcal{L}^2(B_1^+; x_1^a)$ as $m \to \infty$. Observe now $\operatorname{div}(x_1^a w) = f(x^0)$ in B_1^+ and by similar calculation as above we will have

$$\int_{B_1^+} x_1^a f(x^0) w\, dx = \frac{f(x^0)^2}{(n+a+2)(a+2)} \int_{B_1^+} x_1^a\, dx.$$

On the other hand,

$$\Phi_{x^0}(0+) = \lim_{m \to \infty} \Phi_{x^0}(r_m)$$

$$\geq \int_{B_1^+} (x_1^a|\nabla w|^2 + 2f(x^0)x_1^a w)dx - 2\int_{\partial B_1^+} x_1^a w^2 \, d\mathcal{H}^{n-1}$$
$$= \int_{B_1^+} (-w \operatorname{div}(x_1^a \nabla w) + 2f(x^0)x_1^a w)dx$$
$$= \int_{B_1^+} x_1^a f(x^0) w \, dx = \frac{f(x^0)^2}{(n+a+2)(a+2)} \int_{B_1^+} x_1^a \, dx.$$

Step 4 In this step, we prove the proposition for the points satisfying condition (3.4). For these points, we have

$$\frac{1}{2}\partial_r\Big[\int_{\partial B_1^+} x_1^a u_{x^0,r}^2 \, d\mathcal{H}^{n-1}\Big] = \int_{\partial B_1^+} x_1^a u_{x^0,r} \partial_r u_{x^0,r} \, d\mathcal{H}^{n-1} = \frac{1}{r}\int_{\partial B_1^+} x_1^a u_{x^0,r}(\nabla u_{x^0,r}\cdot x - 2u_{x^0,r}) \, d\mathcal{H}^{n-1}$$
$$= \frac{1}{r}\int_{B_1^+} (x_1^a |\nabla u_{x^0,r}|^2 + u_{x^0,r}\operatorname{div}(x_1^a \nabla u_{x^0,r}))dx - \frac{2}{r}\int_{\partial B_1^+} x_1^a u_{x^0,r}^2 \, d\mathcal{H}^{n-1}$$
$$= \frac{1}{r}\left(\Phi_{x^0}(r) - 2\int_{B_1^+} f(x^0) x_1^a u_{x^0,r}(x) \, dx + \int_{B_1^+} x_1^a f(rx) u_{x^0,r}(x) \, dx\right)$$
$$= \frac{1}{r}\left(\Phi_{x^0}(r) - \int_{B_1^+} f(x^0)x_1^a u_{x^0,r} dx\right) + \frac{1}{r}\left(\int_{B_1^+} (f(rx) - f(x^0))x_1^a u_{x^0,r}(x) \, dx\right)$$
$$\geq \frac{1}{r}\left(\Phi_{x^0}(0^+) - \int_{B_1^+} f(x^0)x_1^a u_{x^0,r} dx\right) - Cr^{\alpha+\beta-2} - C_3 r^{\alpha+\beta-2} \geq -C_2 r^{\alpha-1} - Cr^{\alpha+\beta-2} - C_3 r^{\alpha+\beta-2}$$

Thus $r \mapsto \partial_r\left[\int_{\partial B_1^+} x_1^a u_{x^0,r}^2 \, d\mathcal{H}^{n-1}\right]$ is integrable and we obtain uniform boundedness of $\int_{\partial B_1^+} x_1^a u_{x^0,r}^2 d\mathcal{H}^{n-1} = r^{-n-3-b}\int_{\partial B_r(x^0)^+} x_1^a u^2 \, d\mathcal{H}^{n-1}$ for all points with property (3.4). It follows that the boundedness holds uniformly on the closure of those points x^0.

Step 5 We now consider the case

$$\liminf_{r\to 0^+} \frac{|B_r^+(x^0) \cap \{u=0\}|}{|B_r^+|} > 0.$$

Let us assume towards a contradiction that there are sequences u^m, r_m and x^m such that and $M_m = \|u_{x^m,r_m}\|_{\mathcal{L}^2(\partial B_1^+)} \to +\infty$ as $m\to\infty$. Setting $w_m = \frac{u^m_{x^m,r_m}}{M_m}$ we obtain, as in Step 2, that a subsequence of w_m converges weakly in $W^{1,2}(B_1^+; x_1^{a-2})$ to a function w, with $\|w\|_{\mathcal{L}^2(\partial B_1^+; x_1^a)} = 1$, $w(0) = |\nabla w(0)| = 0$, $\operatorname{div}(x_1^a \nabla w) = 0$ and

$$\int_{B_1^+} x_1^a |\nabla w|^2 dx \leq 2\int_{\partial B_1^+} x_1^a w^2 \, d\mathcal{H}^{n-1}.$$

According to Lemma 2.3, $w \in \mathbb{HP}_2$. In addition we now know that

$$\int_{B_1^+} \chi_{\{w=0\}} \geq \limsup_{r_m\to 0^+} \int_{B_1^+} \chi_{\{u_{x^0,r_m}=0\}} > 0.$$

This however contradicts the analyticity of w inside B_1^+, knowing that $\|w\|_{\mathcal{L}^2(\partial B_1; x_1^a)} = 1$. □

Now we are ready to prove the main result of the article.

Proof of Theorem 1.2. From Theorem 8.17 in [4], we know that if $\mathrm{div}(b(x)\nabla w) = g$ such that $1 \leq b(x) \leq 5^a$, then there exists a universal constant $C = C(a, n)$ such that

$$\|w\|_{\mathcal{L}^\infty(B_{R/2})} \leq C(a,n)\left(R^{-n/2}\|w\|_{\mathcal{L}^2(B_R)} + R^2\|g\|_{\mathcal{L}^\infty(B_R)}\right).$$

Now for $x^0 \in \partial\{u = 0\}\cap\{x_1 = 0\}\cap B^+_{1/8}$ and an arbitrary point $y \in \partial B^+_r(x^0)$, we apply the above estimate for $R = 2\delta/3$, $w = (\delta/3)^a u$ and equation $\mathrm{div}\,(b(x)\nabla w) = x_1^a f\chi_{\{u\neq 0\}}$, where $\delta = y_1$ and $b(x) = \frac{x_1^a}{(\delta/3)^a}$. Note that $1 \leq b(x) \leq 5^a$ in $B_{2\delta/3}(y)$ and

$$|u(y)| \leq C(a,n)\left((2\delta/3)^{-n/2}\|u\|_{\mathcal{L}^2(B_{2\delta/3}(y))} + (2\delta/3)^2 5^a\|f\|_{\mathcal{L}^\infty(B_{2\delta/3}(y))}\right).$$

According to Proposition 3.1,

$$\begin{aligned}\|u\|^2_{\mathcal{L}^2(B_{2\delta/3}(y))} &\leq \left(\frac{3}{\delta}\right)^a \int_{B_{2\delta/3}(y)} x_1^a|u|^2\,dx\\ &\leq \left(\frac{3}{\delta}\right)^a \int_{B_{r+2\delta/3}(x^0)} x_1^a|u|^2\,dx\\ &\leq C\left(\frac{3}{\delta}\right)^a (r+2\delta/3)^{n+a+4}.\end{aligned}$$

Hence,

$$|u(y)| \leq C\left(\delta^{-(n+a)/2}(r+\delta)^{(n+a+4)/2} + \delta^2\right) \leq Cy_1^2\left(\left(\frac{r+y_1}{y_1}\right)^{(n+a+4)/2} + 1\right). \qquad \square$$

From this theorem it follows that solutions have quadratic growth inside cones.

Corollary 3.2. *Suppose u is a solution of* (1.1) *satisfying the condition in Proposition 1.1 and $x^0 \in \partial\{u = 0\} \cap \{x_1 = 0\} \cap B^+_{1/8}$. Then, for every constant $\tau > 0$,*

$$\sup_{B_r(x^0)\cap C} |u| \leq C\left(\left(\frac{1}{\tau}+1\right)^{\frac{n+a+4}{2}} + 1\right)r^2,$$

where $C := \{x : x_1 \geq \tau|x - x^0|\}$.

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Conflict of interest

The authors declare no conflict of interest.

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A. A priori regularity of the problem

Let u be a solution of (1.1) for $f \in \mathcal{L}^\infty(B_1^+)$. We are going to show a priori regularity for solutions to (1.1). Consider the operator $\mathcal{L}_{a,c}u := x_1^2\Delta u + ax_1\partial_1 u - cu$. The following proposition is the regularity result related to this operator which has been proven by Krylov [5, Theorem 2.7, Theorem 2.8].

Proposition A.1. *i) For any $a \in \mathbb{R}$, $p > 1$ and $\theta \in \mathbb{R}$ there exists a constant $c_0 > 0$ such that for any $c \geq c_0$ the operator $\mathcal{L}_{a,c}$ is a bounded one-to-one operator from $W^{2,p}(\mathbb{R}^n_+; x_1^\theta)$ onto $\mathcal{L}^p(\mathbb{R}^n_+; x_1^\theta)$ and its inverse is also bounded, in particular for any $u \in W^{2,p}(\mathbb{R}^n_+; x_1^\theta)$*

$$\|u\|_{W^{2,p}(\mathbb{R}^n_+;x_1^\theta)} \leq C\|\mathcal{L}_{a,c}u\|_{\mathcal{L}^p(\mathbb{R}^n_+;x_1^\theta)},$$

where C is independent of u and c.
ii) The statement in i) is valid for the operator $\mathcal{L}_{a,0}$ when $-1 < \theta < a-2$ and $a > 1$ or either $a-2 < \theta < -1$ and $a < 1$.

Now we can deduce a priori regularity result for u as follows.

Proof of Proposition 1.1. Notice that $x_1^{\beta-1}u \in C(\overline{B_1^+})$ due to Sobolev embedding Theorem 3.1 in [5]. Then if the statement of proposition fails, there exists a sequence u_j of solutions (1.1), $x^j \in \{x_1 = 0\}$

and $r_j \to 0$ such that

$$\sup_{B_r^+(x^j)} |x_1^{\beta-1} u_j| \leq j r^{1+\beta/2}, \quad \forall r \geq r_j, \qquad \sup_{B_{r_j}^+(x^j)} |x_1^{\beta-1} u_j| = j r_j^{1+\beta/2}.$$

In particular, the function $\tilde{u}_j(x) = \frac{u_j(x^j + r_j x)}{j r_j^{1+\beta/2}}$, satisfies

$$\sup_{B_R^+} |x_1^{\beta-1} \tilde{u}_j| \leq R^{1+\beta/2}, \qquad \text{for } 1 \leq R \leq \frac{1}{r_j}, \tag{A.1}$$

and with equality for $R = 1$, along with

$$\mathcal{L}_{a,c_0} \tilde{u}_j = \frac{r_j^{1-\beta/2}}{j} f(x^j + r_j x) - c_0 \tilde{u}_j, \tag{A.2}$$

where c_0 is defined in Proposition A.1. According to (A.1), the right hand side of (A.2) is uniformly bounded in $\mathcal{L}^p(B_R^+; x_1^\theta)$ for $p(\beta - 1) - 1 < \theta \leq -1$. From here and Proposition A.1 we conclude that $\{\tilde{u}_j\}$ is bounded in $W^{2,p}(B_R^+; x_1^\theta)$ for some $\theta \leq -1$ and there is a convergent subsequence, tending to a function u_0 with properties

$$\sup_{B_R^+} |x_1^{\beta-1} u_0| \leq R^{1+\beta/2}, \quad \forall R \geq 1, \quad \sup_{B_1^+} |x_1^{\beta-1} u_0| = 1, \quad \operatorname{div}(x_1^a \nabla u_0) = 0, \tag{A.3}$$

as well as the condition $\theta \leq -1$ insures that the trace operator is well defined and u_0 is zero on $\{x_1 = 0\}$. The Liouville type theorem in [9, Lemma 20]) implies that

$$|D^2 u_0(x^0)| \leq \frac{C}{R^2} \sup_{B_R^+(x^0)} |u_0| \leq \frac{C(R + |x_0|)^{2-\beta/2}}{R^2} \to 0.$$

Therefore, u_0 is a linear function, which contradicts (A.3). □