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Research article

Exponential decay of a first order linear Volterra equation†

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Abstract: We consider the linear Volterra equation of the first order in time

$$\dot{u}(t) + \int_0^t g(s)Au(t-s)ds = 0$$

where A is a positive bounded operator on a Hilbert space H. The exponential decay of the related energy is shown to occur, provided that the kernel g is controlled by a negative exponential.

Keywords: Volterra equation; memory kernel; exponential decay; heat equation

1. Introduction

1.1. Setting of the problem

Let H be a real Hilbert space, endowed with scalar product and norm $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, and let $A : H \to H$ be a strictly positive selfadjoint operator. In particular, the square root $A^{1/2}$ of A is well defined, and it is strictly positive selfadjoint as well. If A is not bounded, which can occur only if H is infinite-dimensional, then its domain $\mathfrak{D}(A)$ is strictly contained in H, and we have the dense (not necessarily compact) embeddings

$$\mathfrak{D}(A) \subset \mathfrak{D}(A^{1/2}) \subset H.$$

The work is concerned with the exponential decay of the solutions to the linear Volterra equation of the first order in time

$$\dot{u}(t) + \int_0^t g(s)Au(t-s)ds = 0, \tag{1.1}$$

the *dot* standing for derivative with respect to time. The convolution kernel g, sometimes referred to as *memory kernel*, satisfies the following properties:

- g is a positive, continuous, piecewise smooth, decreasing and summable function on $\mathbb{R}^+ = (0, \infty)$.
- Its derivative g' is negative, increasing and summable on $\mathbb{R}^+$. In particular, g'' is (defined and) positive almost everywhere.

By virtue of the two hypotheses above, g is defined by continuity at $s = 0$ to be

$$g(0) = -\int_0^\infty g'(s)ds < \infty.$$

Without loss of generality, we suppose that g has unit total mass, that is,

$$\int_0^\infty g(s)ds = 1.$$

Within these assumptions, g' may exhibit an integrable singularity at zero, and it may have discontinuities (upward jumps).

- We assume that the discontinuity points, if any, form an increasing (possibly finite) sequence s_n.

Remark 1.1. Equation (1.1) can be seen as a particular instance of its counterpart with *infinite memory*, which reads

$$\dot{u}(t) + \int_0^\infty g(s)Au(t-s)ds = 0. \tag{1.2}$$

The function u is supposed to be an assigned datum for negative times, where is interpreted as the *initial past history* of the problem. Clearly, (1.1) is obtained from (1.2) by merely choosing a null initial past history.

It is well know that, for every initial datum $u_0 \in H$, equation (1.1) has a unique global solution

$$u \in C([0,\infty), H)$$

satisfying the initial condition $u(0) = u_0$. Such a solution is understood in the weak sense if A is an unbounded operator. Actually, this is a byproduct of the existence and uniqueness result, proved in [3], for the more general Eq. (1.2) in the so-called past history framework devised by Dafermos [10]. Besides, the natural energy of the system, given by

$$\mathsf{E}(t) = \frac{1}{2}\Big[\|u(t)\|^2 + g(t)\|A^{1/2}U(t)\|^2 - \int_0^t g'(s)\|A^{1/2}[U(t) - U(t-s)]\|^2 ds\Big], \tag{1.3}$$

where

$$U(t) = \int_0^t u(s)ds,$$

turns out to be a decreasing function, witnessing the dissipative nature of the model.

1.2. Exponential decay of the energy

A relevant question in connection with this model concerns the (uniform) decay properties of the energy. Let us recall the definition.

Definition 1.2. We say that the energy has an exponential decay if there exist constants $M \geq 1$ and $\omega > 0$, both independent on the initial data, such that

$$\mathsf{E}(t) \leq M\mathsf{E}(0)\mathrm{e}^{-\omega t}.$$

The first result concerning the exponential stability of the energy for the model under consideration, for a kernel g as above (but with g' differentiable) has been obtained under the assumption

$$g''(s) + \delta g'(s) \geq 0, \tag{1.4}$$

for some $\delta > 0$ (see [13]). This is a very popular condition, appearing in several works in connection with equations with memory (see e.g., [9, 11, 17, 21] and references therein). Indeed, within (1.4), one actually proves the exponential stability not only of the energy of (1.1), but of the energy of the linear semigroup generated by the more general model (1.2). It should be observed that, when one has a linear semigroup, the uniform decay of the energy, with respect to the choice of the initial data in any fixed bounded set, is completely equivalent to exponential stability. More recently, in [3] (but see also [12]) a necessary condition for exponential decay (again, of the linear semigroup) has been established:

$$g(s) \leq -Cg'(s), \tag{1.5}$$

for some $C > 0$ and every $s > 0$. The function g in [3], complying with our general assumptions (in particular, $g''(s) \geq 0$ almost everywhere), is allowed to have discontinuities. Condition (1.5) has been shown to be also sufficient in [7], provided that the function g' is not completely flat, namely, it is not a step function. Hence, the problem of the exponential decay of the energy is completely solved for the semigroup generated by (1.2), and in turn the result applies also to (1.1). However, one expects that (1.5) might be too restrictive to obtain the desired conclusion for (1.1) alone. Indeed, the reasonable guess is that the energy has an exponential decay provided that the memory kernel g is controlled by a negative exponential. A result in that direction has been proved in [19], but for the model

$$\dot{u}(t) + \varrho Au(t) + \int_0^t g(s)Au(t-s)ds = 0, \tag{1.6}$$

with $\varrho > 0$. Here, the situation is considerably simpler, due to the presence of the instantaneous dissipation provided by the extra term ϱAu. Instead, when $\varrho = 0$ the dissipation is entirely contributed by the convolution integral, which renders the problem much more challenging.

1.3. Statement of the result

Our goal is to provide a sufficient condition involving a control on the decay of g only, in order for the energy $\mathsf{E}(t)$ of (1.1) to decay exponentially fast. In general, this seems to be quite a hard task. Nonetheless, we can prove a fully satisfactory result if the operator A is bounded (which is always the case if H is finite-dimensional). This, besides solving the problem, for instance, in the case of ordinary differential equations of Volterra type, has interesting and nontrivial applications to the infinite-dimensional case, as we will show in the next section. In the finite-dimensional case, we also recall some results, proving the exponential decay of the solution when the kernel is exponentially decaying, provided that the solution is known to be summable in advance (see [2, 22]) and references therein).

Our main theorem reads as follows.

Theorem 1.3. *Let A be bounded, and assume that $g''(s) > 0$ for almost every $s > 0$. If there exist $C > 0$ and $\delta > 0$ such that*

$$g(s) \leq C\mathrm{e}^{-\delta s}, \tag{1.7}$$

then the energy $\mathsf{E}(t)$ defined in (1.3) *has an exponential decay.*

An example of a kernel g satisfying the assumptions of Theorem 1.3, but not complying with (1.5), can be found in [25]. The idea is to construct the derivative $-g'(s)$, bounded by e^{-s}, in such a way that it remains "almost flat" on arbitrarily large intervals. Actually, the example in [25] needs to be slightly modified, since there g' is constant on such intervals, which turns into having $g''(s) = 0$, whereas we require $g''(s) > 0$.

Before going to proof of the theorem, carried out in Section 3, we discuss two examples.

2. Two relevant examples

In what follows, Ω is a bounded domain of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, and Δ is the Laplace operator with Dirichlet boundary conditions acting on $L^2(\Omega)$, that is, with domain

$$\mathfrak{D}(\Delta) = H^2(\Omega) \cap H_0^1(\Omega).$$

In which case, $-\Delta$ is a strictly positive selfadjoint operator, and

$$\mathfrak{D}((-\Delta)^{1/2}) = H_0^1(\Omega).$$

Here, $L^2(\Omega)$ is the usual Lebesgue space of square-summable functions, whereas $H^2(\Omega)$ and $H_0^1(\Omega)$ are the Sobolev spaces of functions which are square-summable along with their derivatives up to order 2, and up to order 1 and null on the boundary $\partial\Omega$, respectively.

2.1. The Gurtin-Pipkin heat equation

The classical constitutive law ruling the evolution of the relative temperature field u in a rigid isotropic homogeneous heat conductor occupying the domain Ω is the Fourier one, establishing the relation

$$\boldsymbol{q}(t) = -\varrho\nabla u(t), \quad \varrho > 0,$$

between u and the heat flux vector $\boldsymbol{q}$. In [15] (see also [14, 18]), the authors propose the following integral relaxation of the Fourier law, nowadays known as the *Gurtin-Pipkin law*:

$$\boldsymbol{q}(t) = -\int_0^\infty g(s)\nabla u(t-s)ds.$$

Here, the gradient of u is convolved against a fading kernel g, in order to take into account the inertia of the system to a change of state. Recalling that, in absence of external heat sources, the energy balance equation reads

$$\partial_t u(t) + \operatorname{div} \boldsymbol{q}(t) = 0,$$

this leads to the (fully hyperbolic) linear equation with memory

$$\partial_t u(t) - \int_0^\infty g(s)\Delta u(t-s)ds = 0,$$

whose particular case corresponding to having a null initial past history is (see also [20, 23])

$$\partial_t u(t) - \int_0^t g(s)\Delta u(t-s)ds = 0.$$

Such an equation falls within our model, by choosing $H = L^2(\Omega)$ and $A = -\Delta$. Unfortunately, in this case the operator A is unbounded, and so Theorem 1.3 does not apply. A variant of the model is obtained by replacing the Gurtin-Pipkin law with the *Coleman-Gurtin law* [4]

$$\boldsymbol{q}(t) = -\varrho\nabla u(t) - \int_0^\infty g(s)\nabla u(t-s)ds.$$

This, for a null initial past history, yields the equation

$$\partial_t u(t) - \varrho\Delta u(t) - \int_0^t g(s)\Delta u(t-s)ds = 0,$$

which is a concrete realization of (1.6). Now the analogue of Theorem 1.3 holds, due to the results in [19]. Accordingly, extending Theorem 1.3 for unbounded operators would be certainly of great interest.

2.2. *The nonclassical heat equation with memory*

A modified form of the heat equation, studied by several authors in recent years, is the *nonclassical heat equation* (see e.g., [26–28, 31]), given by

$$\partial_t u(t) - \Delta\partial_t u(t) - \Delta u(t) = 0.$$

This is obtained by assuming that the heat (or more generally a diffusive species) behaves as a linearly viscous fluid, which leads to include its velocity gradient in the constitutive law [1]. Namely, setting all the constants equal to one,

$$\boldsymbol{q}(t) = -\nabla u(t) - \nabla\partial_t u(t).$$

The integral relaxation of the nonclassical heat equation becomes

$$\partial_t u(t) - \Delta\partial_t u(t) - \int_0^t g(s)\Delta u(t-s)ds = 0,$$

which provides a more accurate description of the diffusive process in certain materials, such as high-viscosity liquids at low temperatures and polymers [16]. For every initial datum $u_0 \in H^1_0(\Omega)$, the problem admits a unique weak solution (see e.g., [5, 8, 29, 30])

$$u \in C([0, \infty), H^1_0(\Omega)).$$

Applying the operator $(I - \Delta)^{-1}$ to both sides of the equation, we obtain

$$\partial_t u(t) - \int_0^t g(s)(I - \Delta)^{-1} \Delta u(t - s) ds = 0.$$

Since (the bounded extension of)

$$A = (I - \Delta)^{-1} \Delta$$

is a bounded operator on $H^1_0(\Omega)$, this is nothing but a particular realization of the abstract model (1.1) on the Hilbert space $H = H^1_0(\Omega)$, to which now Theorem 1.3 applies.

3. Proof of Theorem 1.3

Since the operator A is strictly positive and bounded, so is its square root $A^{1/2}$. Thus $\|\cdot\|$ and $\|A^{1/2}\cdot\|$ are equivalent norms on H; namely, there exist $c_2 \geq c_1 > 0$ such that

$$c_1 \|u\| \leq \|A^{1/2} u\| \leq c_2 \|u\|, \quad \forall u \in H.$$

In what follows, such an equivalence, as well as the Hölder and the Young inequality, will be used several times without explicit mention.

3.1. The equation revisited

It is more convenient to rewrite (1.1) in a different form. To this end, we introduce the auxiliary variable $\eta^t(s)$, for $t \geq 0$ and $s > 0$, formally defined as

$$\eta^t(s) = \int_{t-s}^t u(y) dy,$$

where we put $u(s) = 0$ for $s < 0$. With this position, calling

$$\mu(s) = -g'(s),$$

Equation (1.1) turns into the system

$$\begin{cases} \dot{u}(t) + \displaystyle\int_0^\infty \mu(s) A \eta^t(s) ds = 0, \\ \dot{\eta}^t = -\partial_s \eta^t + u(t), \end{cases}$$

where the latter equation is complemented with the further condition $\eta^0(s) = 0$, following from the formal definition of η itself. In order to frame the system above in the correct functional setting, we introduce the so-called memory space $\mathcal{M}$, that is, the Hilbert space of H-valued square-summable functions with respect to the measure $\mu(s)ds$, defined as

$$\mathcal{M} = L^2_\mu(\mathbb{R}^+; H) \qquad \text{normed by} \qquad \|\eta\|^2_{\mathcal{M}} = \int_0^\infty \mu(s)\|A^{1/2}\eta(s)\|^2 ds.$$

Then, defining the Hilbert space

$$\mathcal{H} = H \times \mathcal{M} \qquad \text{normed by} \qquad \|(u,\eta)\|^2_{\mathcal{H}} = \|u\|^2 + \|\eta\|^2_{\mathcal{M}},$$

we consider the abstract evolution equation on $\mathcal{H}$

$$\begin{cases} \dot{u}(t) + \displaystyle\int_0^\infty \mu(s)A\eta^t(s)ds = 0, \\ \dot{\eta}^t = T\eta^t + u(t), \end{cases} \tag{3.1}$$

where T is the infinitesimal generator of the right-translation semigroup on $\mathcal{M}$, that is,

$$T\eta = -\partial_s\eta \qquad \text{with domain} \qquad \mathfrak{D}(T) = \{\eta \in \mathcal{M} : \partial_s\eta \in \mathcal{M},\ \eta(0) = 0\}.$$

It is well known that (3.1) generates a linear contraction semigroup on $\mathcal{H}$ (see [3]). In other words, for *every* pair of initial data $(u_0, \eta_0) \in \mathcal{H}$, there is a unique global solution

$$(u(t), \eta^t) \in C([0,\infty), \mathcal{H})$$

satisfying the initial condition $(u(0), \eta^0) = (u_0, \eta_0)$, and whose energy

$$\mathsf{E}(t) = \frac{1}{2}\Big[\|u(t)\|^2 + \int_0^\infty \mu(s)\|A^{1/2}\eta^t(s)\|^2 ds\Big],$$

is decreasing. Besides, denoting as before

$$U(t) = \int_0^t u(s)ds,$$

the η-component of the solutions fulfills the explicit representation

$$\eta^t(s) = \begin{cases} U(t) - U(t-s) & \text{if } s \le t, \\ \eta_0(s-t) + U(t) & \text{if } s > t. \end{cases}$$

According to [3], the following holds:

Theorem 3.1. *A function $u(t)$ is the solution to* (1.1) *with initial datum $u_0 \in H$ if and only if the function $(u(t), \eta^t)$ is the solution to* (3.1) *with initial datum* $(u_0, 0)$. *In which case, the representation formula for η becomes*

$$\eta^t(s) = \begin{cases} U(t) - U(t-s) & \textit{if } s \le t, \\ U(t) & \textit{if } s > t. \end{cases} \tag{3.2}$$

Besides, the energy $\mathsf{E}(t)$ written above coincides with the energy defined in (1.3).

Therefore, rather than the solutions to (1.1), from now on we will consider instead the trajectories of the semigroup generated by (3.1), but *only* those arising from initial data of the form $(u_0, 0)$. Besides, as shown in [3], for every initial datum $(u_0, 0)$ the energy $\mathsf{E}(t)$ fulfills the differential identity*

$$\frac{d}{dt}\mathsf{E}(t) + \frac{1}{2}\Big[\int_0^\infty -\mu'(s)\|A^{1/2}\eta^t(s)\|^2 ds + \mathbb{J}[\eta^t]\Big] = 0, \tag{3.3}$$

having set

$$\mathbb{J}[\eta^t] = \sum_n [\mu(s_n^-) - \mu(s_n^+)]\|A^{1/2}\eta^t(s_n)\|^2 \geq 0.$$

3.2. Auxiliary energy functionals

Throughout the rest of the paper, $K \geq 0$ will stand for a *generic* constant, depending on the structural parameters but *independent* of the initial energy $\mathsf{E}(0)$.

In order to obtain a satisfactory energy inequality, we need to introduce suitable energy-like functionals. The first of them, devised in [24] and subsequently used in [6] in the context of viscoelasticity, reads as follows:

$$\Phi(t) = -\frac{1}{\kappa}\int_0^\infty \psi(s)\langle u(t), \eta^t(s)\rangle ds.$$

Here,

$$\kappa = g(0) = \int_0^\infty \mu(s)ds > 0,$$

while $\psi(s)$ is the *truncated kernel* given by

$$\psi(s) = \mu(s_*)\chi_{(0,s_*]}(s) + \mu(s)\chi_{(s_*,\infty)}(s),$$

for some fixed $s_* > 0$, smaller than the first jump point whenever exists, and small enough that

$$\int_0^{s_*} \mu(s)ds \leq \frac{\kappa}{4}. \tag{3.4}$$

The introduction of ψ is needed to handle the possible singularity of μ at the origin. It is readily seen that

$$|\Phi(t)| \leq K\mathsf{E}(t). \tag{3.5}$$

Defining the *μ-measure* of a (Lebesgue measurable) set $\mathcal{S} \subset \mathbb{R}^+$ by

$$\mathfrak{m}(\mathcal{S}) = \frac{1}{\kappa}\int_{\mathcal{S}} \mu(s)ds,$$

the following holds.

Lemma 3.2. *There exists a structural constant $K_1 > 0$ such that for every $\mathcal{S} \subset \mathbb{R}^+$*

$$\frac{d}{dt}\Phi + \frac{1}{2}\|u\|^2 \leq 2\mathfrak{m}(\mathcal{S})\int_{\mathcal{S}} \mu(s)\|A^{1/2}\eta(s)\|^2 ds + 2\int_{\mathbb{R}^+\setminus\mathcal{S}} \mu(s)\|A^{1/2}\eta(s)\|^2 ds$$
$$+K_1\Big[\int_0^\infty -\mu'(s)\|A^{1/2}\eta(s)\|^2 ds + \mathbb{J}[\eta]\Big].$$

*Indeed, initial data of this form actually belong to the domain of the infinitesimal generator of the semigroup.

Proof. We compute the time derivative of Φ as

$$\begin{aligned}\frac{d}{dt}\Phi &= -\frac{1}{\kappa}\int_0^\infty \psi(s)\langle \dot{u}, \eta(s)\rangle ds - \frac{1}{\kappa}\int_0^\infty \psi(s)\langle u, \dot{\eta}(s)\rangle ds\\ &= \frac{1}{\kappa}\int_0^\infty \psi(s)\Big(\int_0^\infty \mu(\sigma)\langle A^{1/2}\eta(\sigma), A^{1/2}\eta(s)\rangle d\sigma\Big) ds\\ &\quad - \frac{1}{\kappa}\int_0^\infty \psi(s)\langle u, T\eta(s)\rangle ds - \frac{1}{\kappa}\|u\|^2\int_0^\infty \psi(s) ds.\end{aligned}$$

Using (3.4) and the equality $\psi(s) = \mu(s)$ for $s \geq s_*$,

$$-\frac{1}{\kappa}\|u\|^2\int_0^\infty \psi(s) ds \leq -\frac{1}{\kappa}\|u\|^2\int_{s_*}^\infty \mu(s) ds \leq -\frac{3}{4}\|u\|^2.$$

Moreover, integrating by parts in s, we have (see e.g. [24])

$$\begin{aligned}-\frac{1}{\kappa}\int_0^\infty \psi(s)\langle u, T\eta(s)\rangle ds &= -\frac{1}{\kappa}\int_{s_*}^\infty \mu'(s)\langle u, \eta(s)\rangle ds + \frac{1}{\kappa}\sum_n [\mu(s_n^-) - \mu(s_n^+)]\langle u, \eta(s_n)\rangle\\ &\leq \frac{1}{\kappa}\|u\|\Big[\int_{s_*}^\infty -\mu'(s)\|\eta(s)\| ds + \sum_n [\mu(s_n^-) - \mu(s_n^+)]\|\eta(s_n)\|\Big]\\ &\leq \frac{1}{4}\|u\|^2 + K_1\Big[\int_0^\infty -\mu'(s)\|A^{1/2}\eta(s)\|^2 ds + \mathbb{J}[\eta]\Big],\end{aligned}$$

for some structural constant $K_1 > 0$. Finally, for every measurable set $\mathcal{S} \subset \mathbb{R}^+$,

$$\begin{aligned}&\frac{1}{\kappa}\int_0^\infty \psi(s)\Big(\int_0^\infty \mu(\sigma)\langle A^{1/2}\eta(\sigma), A^{1/2}\eta(s)\rangle d\sigma\Big) ds\\ &\leq \frac{1}{\kappa}\Big[\int_{\mathcal{S}} \mu(s)\|A^{1/2}\eta(s)\| ds + \int_{\mathbb{R}^+\setminus\mathcal{S}} \mu(s)\|A^{1/2}\eta(s)\| ds\Big]^2\\ &\leq \frac{2}{\kappa}\Big[\int_{\mathcal{S}} \mu(s)\|A^{1/2}\eta(s)\| ds\Big]^2 + \frac{2}{\kappa}\Big[\int_{\mathbb{R}^+\setminus\mathcal{S}} \mu(s)\|A^{1/2}\eta(s)\| ds\Big]^2\\ &\leq 2\mathfrak{m}(\mathcal{S})\int_{\mathcal{S}} \mu(s)\|A^{1/2}\eta(s)\|^2 ds + 2\int_{\mathbb{R}^+\setminus\mathcal{S}} \mu(s)\|A^{1/2}\eta(s)\|^2 ds.\end{aligned}$$

Collecting all the estimates above, the conclusion follows. □

Next, we define the further functionals

$$\Psi(t) = \int_0^\infty g(s)\|A^{1/2}\eta^t(s)\|^2 ds,$$
$$\Lambda(t) = \int_0^\infty \mathrm{e}^{-\delta s}\|A^{1/2}\eta^t(s)\|^2 ds,$$

where $\delta > 0$ is the constant appearing in (1.7). Note that Λ is well-defined. Indeed, since the energy is decreasing, for every $t \geq 0$ we have

$$\|u(t)\|^2 \leq 2\mathsf{E}(0),$$

and by the representation formula (3.2) we conclude that

$$\|A^{1/2}\eta^t(s)\|^2 \le K\mathsf{E}(0)s^2 \quad \Rightarrow \quad \Lambda(t) \le K\mathsf{E}(0).$$

Besides, it is apparent from (1.7) that

$$\Psi(t) \le C\Lambda(t). \tag{3.6}$$

Lemma 3.3. *There exists a structural constant $K_2 > 0$ such that*

$$\frac{d}{dt}[\Psi + \Lambda] + \|\eta\|^2_{\mathcal{M}} + \frac{\delta}{2}\Lambda \le K_2\|u\|^2.$$

Proof. A direct calculation provides the identity

$$\frac{d}{dt}[\Psi + \Lambda] + \|\eta\|^2_{\mathcal{M}} + \delta\Lambda = 2\int_0^\infty [g(s) + \mathrm{e}^{-\delta s}]\langle A^{1/2}u, A^{1/2}\eta(s)\rangle ds.$$

Owing to (1.7), the right-hand side above is less than or equal to

$$2(C+1)\|A^{1/2}u\|\int_0^\infty \mathrm{e}^{-\delta s}\|A^{1/2}\eta(s)\|ds \le \frac{\delta}{2}\Lambda + K\|u\|^2,$$

completing the proof. □

3.3. Conclusion of the proof

For $a > 0$ to be suitably fixed later and $b = \frac{1}{4K_2}$, we introduce the functional

$$\mathsf{L}(t) = a\mathsf{E}(t) + \Phi(t) + b[\Psi(t) + \Lambda(t)].$$

Due to (3.5)–(3.6), up to choosing a sufficiently large, we have the controls

$$\frac{1}{a}[\mathsf{E}(t) + \Lambda(t)] \le \mathsf{L}(t) \le K[\mathsf{E}(t) + \Lambda(t)]. \tag{3.7}$$

Next, for $n \in \mathbb{N}$, we define the set

$$\mathcal{S}_n = \{s \in \mathbb{R}^+ : \ n\mu'(s) + \mu(s) > 0\}.$$

Choosing $\mathcal{S} = \mathcal{S}_n$ in Lemma 3.2, and noting that

$$2\int_{\mathbb{R}^+\setminus\mathcal{S}_n} \mu(s)\|A^{1/2}\eta(s)\|^2 ds \le -2n\int_0^\infty \mu'(s)\|A^{1/2}\eta(s)\|^2 ds,$$

we get

$$\frac{d}{dt}\Phi + \frac{1}{2}\|u\|^2 \le 2\mathfrak{m}(\mathcal{S}_n)\|\eta\|^2_{\mathcal{M}} + (2n + K_1)\Big[\int_0^\infty -\mu'(s)\|A^{1/2}\eta(s)\|^2 ds + \mathbb{J}[\eta]\Big].$$

Therefore, in light of the energy identity (3.3) and Lemma 3.3, the functional L fulfills

$$\frac{d}{dt}\mathsf{L} + \frac{1}{4}\|u\|^2 + (b - 2\mathfrak{m}(\mathcal{S}_n))\|\eta\|^2_{\mathcal{M}} + \frac{b\delta}{2}\Lambda$$

$$\leq \Big(2n + K_1 - \frac{a}{2}\Big)\Big[\int_0^\infty -\mu'(s)\|A^{1/2}\eta(s)\|^2 ds + \mathbb{J}[\eta]\Big].$$

At this point, we observe that

$$\lim_{n\to\infty} \mathfrak{m}(\mathcal{S}_n) = 0.$$

Indeed, $\mu'(s) < 0$ almost everywhere by assumption, which implies that the sets $\mathcal{S}_n$ are (decreasingly) nested, and their intersection has null measure. Hence, choosing first n sufficiently large that

$$b - 2\mathfrak{m}(\mathcal{S}_n) > 0,$$

and then choosing

$$a = 4n + 2K_1,$$

up to possibly increasing n such that (3.7) holds, we obtain the differential inequality

$$\frac{d}{dt}\mathsf{L} + \varepsilon[\mathsf{E} + \Lambda] \leq 0,$$

for some $\varepsilon > 0$. Invoking (3.7), up to reducing $\varepsilon > 0$ accordingly, we end up with

$$\frac{d}{dt}\mathsf{L} + \varepsilon\mathsf{L} \leq 0.$$

Being the initial value $\eta^0(s) \equiv 0$, we have that $\mathsf{L}(0) = a\mathsf{E}(0)$. Therefore, the Gronwall lemma and a further exploitation of (3.7) entail

$$\mathsf{E}(t) \leq a\mathsf{L}(t) \leq a^2\mathsf{E}(0)\mathrm{e}^{-\varepsilon t}.$$

This finishes the proof of Theorem 1.3. □

Conflict of interest

The authors declare no conflict of interest.

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