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Research article

Existence of solutions for a perturbed problem with logarithmic potential in $\mathbb{R}^2$ †

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Abstract: We study a perturbed Schrödinger equation in the plane arising from the coupling of quantum physics with Newtonian gravitation. We obtain some existence results by means of a perturbation technique in Critical Point Theory.

Keywords: variational methods; perturbation methods; finite-dimensional reduction

1. Introduction

The Newton kernel $\Phi_N \colon \mathbb{R}^N \to \mathbb{R}$ is defined by

$$\Phi_N(x) = \begin{cases} \frac{\Gamma\left(\frac{N-2}{2}\right)}{4\pi^{N/2}|x|^{N-2}} & \text{if } N \geq 3 \\ \frac{1}{2\pi} \log \frac{1}{|x|} & \text{if } N = 2. \end{cases}$$

The non-local partial differential equation

$$-\Delta u + au = \left[\Phi_N \star |u|^2\right] u \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where a is a positive function, was proposed in the study of quantum physics of electrons in a ionic crystal (the so-called Pekar polaron model) for $N = 3$. The same equation can also be seen as a coupling of quantum physics with Newtonian gravitation: indeed, the system

$$\begin{cases} \mathrm{i}\psi_t - \Delta\psi + E(x)\psi + \gamma w\psi = 0 \\ \Delta w = |\psi|^2 \end{cases}$$

in the unknown $\psi\colon \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$, $\psi = \psi(x,t)$, reduces to the single equation

$$-\Delta u + au + \gamma \left[\Phi_N \star |u|^2\right] u = 0$$

via the *ansatz* $\psi(x,t) = e^{-i\lambda t} u(x)$ with $\lambda \in \mathbb{R}$ and $a(x) = E(x) + \lambda$. E.H. Lieb proved in [11] that (1.1) possesses, in dimension $N = 3$, a unique ground state solution which is positive and radially symmetric. E. Lenzmann proved in [10] that this solution is also non-degenerate. The analysis of (1.1) in dimension $N = 3$ is heavily based on the algebraic properties of the kernel Φ_3, in particular its *homogeneity*. Lieb's proof of existence carries over to $N = 4$ and $N = 5$, while no solution with finite energy can exist in dimension $N \geq 6$, see [6].

In this note we consider (1.1) in the plane, i.e., when $N = 2$. The appearance of the logarithm in Φ_2 changes drastically the setting of the problem, which has been an open field of study for several years. One of the main obstructions to a straightforward analysis in the planar case is the lack of positivity of the kernel Φ_2.

Some preliminary numerical results contained in [9] encouraged Ph. Choquard, J. Stubbe and M. Vuffray to prove the existence of a unique positive radially symmetric solution by applying a shooting method, see [7]. But only in very recent years have variational methods been used to solve (1.1) for $N = 2$: the formal definition of a Euler functional

$$I(u) = \frac{1}{2} \int_{\mathbb{R}^2} \left(|\nabla u|^2 + a|u|^2\right) + \frac{1}{8\pi} \int_{\mathbb{R}^2 \times \mathbb{R}^2} \log|x-y| |u(x)|^2 |u(y)|^2 \, dx \, dy$$

is not consistent with the metric structure of the Sobolev space $H^1(\mathbb{R}^2)$.

Stubbe proposed in [14] a variational setting for (1.1) in dimension two within the closed subspace

$$X = \left\{ u \in H^1(\mathbb{R}^2) \mid \int_{\mathbb{R}^2} \log(1+|x|) |u(x)|^2 \, dx < +\infty \right\}$$

endowed with the norm

$$\|u\|_X^2 = \int_{\mathbb{R}^2} \left(|\nabla u|^2 + a|u|^2\right) + \int_{\mathbb{R}^2} \log(1+|x|) |u(x)|^2 \, dx.$$

Although this space permits to use variational methods, several difficulties arise from the logarithmic term.

Using this functional approach, S. Cingolani and T. Weth (see [8]) proved some existence results for (1.1) under either a periodicity assumption on the potential a, or the action of a suitable group of transformations. Uniqueness and monotonicity of positive solutions are also proved.

Later on, D. Bonheure, S. Cingolani and J. Van Schaftingen (see [6]) proved that the positive solution u of (1.1) with $a > 0$ is *non-degenerate*, in the sense that the only solutions of the linearized equation associated to (1.1) are the (linear combinations of) the two partial derivatives of u.

Motivated by these results, we consider the following perturbed equation, based on (1.1):

$$-\Delta u + au - \frac{1}{2\pi} \left[\log \frac{1}{|\cdot|} \star u^2\right] u = \varepsilon h(x) |u|^{p-1} u \quad \text{in } \mathbb{R}^2. \tag{1.2}$$

The quantity ε plays the rôle of a "small" perturbation, and the function h is a "weight" for the *local* nonlinearity $|u|^{p-1}u$. We refer to the next Sections for the precise assumptions we make.

We will face the problem of constructing solutions to (1.2) by means of a general technique in Critical Point Theory, introduced by A. Ambrosetti and M. Badiale in [1–3]. We refer to [4] for a presentation in book form. For the reader's convenience, we summarize here the main ideas of this method.

Suppose we are given a (real) Hilbert space X and a functional $I_\varepsilon \in C^2(X)$ of the form

$$I_\varepsilon(\#) = I_0(\#) + \varepsilon G(\#).$$

Here $I_0 \in C^2(X)$ is the so-called *unperturbed functional*, while $\varepsilon \in \mathbb{R}$ is a (small) perturbation parameter. We will suppose that there exists a (smooth) manifold Z of dimension $d < \infty$, such that every $z \in Z$ is a critical point of I_0.

Letting $W = (T_z Z)^\perp$ for $z \in Z$, we look for solutions to the equation $I'_\varepsilon(u) = 0$ of the form $u = z + w$, where $z \in Z$ and $w \in W$. We can split the equation $I'_\varepsilon(u) = 0$ into two equations by means of the orthogonal projection $P \colon X \to W$:

$$\begin{cases} PI'_\varepsilon(z+w) = 0 \\ (I-P)I'_\varepsilon(z+w) = 0. \end{cases} \tag{1.3}$$

We will assume that the following conditions hold:

(ND) for all $z \in Z$, we have $T_z Z = \ker I''_0(z)$;

(Fr) for all $z \in Z$, we have that the linear operator $I''_0(z)$ is Fredholm with index zero.

Remark 1.1. The condition **(ND)** can be seen as a *non-degeneracy* assumption, since it is always true that $T_z Z \subset \ker I''_0(z)$, by definition of Z.

It is possible to show that the first equation of system (1.3) can be (uniquely) solved with respect to $w = w(\varepsilon, z)$, with $z \in Z$ and ε sufficiently small. The main result of this perturbation technique can be summarized in the following statement.

Theorem 1.2 ([4])**.** *Suppose that the function $\Phi_\varepsilon \colon Z \to \mathbb{R}$ defined by $\Phi_\varepsilon(z) = I_\varepsilon(z + w(\varepsilon, z))$ possesses, for $|\varepsilon|$ sufficiently small, a critical point $z_\varepsilon \in Z$. Then $u_\varepsilon = z_\varepsilon + w(\varepsilon, z_\varepsilon)$ is a critical point of $I_\varepsilon = I_0 + \varepsilon G$.*

As it should be clear, the perturbation method of Ambrosetti and Badiale leans on the effect of the function h, which *breaks* the invariance of I_0 under translations. As such, the existence of a critical point of the function Φ_ε depends crucially on the behavior of h.

We split our existence results into two categories. The first one assumes that the weight function h is not only bounded, but also sufficiently integrable over $\mathbb{R}^2$; because of this, we can consider this results as a *local* existence result.

Theorem 1.3. *Let $p > 1$ and $h \in L^\infty(\mathbb{R}^2) \cap L^q(\mathbb{R}^2)$ for some $q > 1$. Moreover, suppose that*

(h_1) $\displaystyle\int_{\mathbb{R}^2} h(x)|z_0|^{p+1}\,dx \neq 0.$

Then Eq. (1.2) *has a solution provided $|\varepsilon|$ is small enough.*

It is possibile to drop the integrability condition on h, at the cost of a more delicate analysis of the implicit function $w = w(\varepsilon, z)$ that describes Φ_ε. We have the following *global* result.

Theorem 1.4. *Let $p > 2$ and suppose that h satisfies*

(h_2) $h \in L^\infty(\mathbb{R}^2)$ *and* $\lim_{|x|\to\infty} h(x) = 0$.

Then for all $|\varepsilon|$ small, Eq. (1.2) *has a solution.*

We highlight that our results differ from those appearing in the literature for several reasons. First of all, the non-degeneracy property appearing in Proposition 2.5 can be used as a basis for further investigation. Moreover, the right-hand side of Eq. (1.2) may (and indeed must) depend on x; no symmetry requirement, like radial symmetry, is needed in our proofs.

The paper is organized as follows. In Section 2 we first give the precise assumptions for our problem, then we recall some known results (classical and not) and we prove some properties for energy functional, such as regularity. In Section 3 we present the proof of the main theorems.

2. Preliminary results

Consider the equation

$$-\Delta u + au - \frac{1}{2\pi}\left[\log\frac{1}{|\cdot|} \star u^2\right]u = \varepsilon h(x)|u|^{p-1}u \quad \text{in } \mathbb{R}^2, \tag{2.1}$$

with $a > 0$, $h \in L^\infty(\mathbb{R}^2)$ and $p > 1$. We introduce the function space

$$X = \left\{u \in H^1(\mathbb{R}^2) \mid \int_{\mathbb{R}^2} |u(x)|^2 \log(1 + |x|)\, dx < \infty\right\},$$

endowed with the norm

$$\|u\|_X^2 = \|u\|_{H^1(\mathbb{R}^2)}^2 + |u|_*^2,$$

where

$$\|u\|_{H^1(\mathbb{R}^2)}^2 = \int_{\mathbb{R}^2} \left(|\nabla u(x)|^2 + a|u(x)|^2\right) dx$$
$$|u|_*^2 = \int_{\mathbb{R}^2} |u(x)|^2 \log(1 + |x|)\, dx.$$

The norm $\|\cdot\|_X$ is associated naturally to an inner product. Let

$$I_\varepsilon(u) = I_0(u) + \varepsilon G(u) \tag{2.2}$$

be the energy functional associated to the equation, where

$$I_0(u) = \frac{1}{2}\int_{\mathbb{R}^2} \left(|\nabla u(x)|^2 + au^2(x)\right) dx - \frac{1}{8\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-y|}|u(x)|^2|u(y)|^2\, dx\, dy$$

and

$$G(u) = -\frac{1}{p+1}\int_{\mathbb{R}^2} h(x)|u(x)|^{p+1}\, dx.$$

We define (see [8, 14]) the symmetric bilinear forms

$$B_1(u,v) = \int_{\mathbb{R}^2\times\mathbb{R}^2} \log(1+|x-y|)u(x)v(y)\,dx\,dy,$$
$$B_2(u,v) = \int_{\mathbb{R}^2\times\mathbb{R}^2} \log\left(1+\frac{1}{|x-y|}\right)u(x)v(y)\,dx\,dy,$$

and

$$B(u,v) = B_1(u,v) - B_2(u,v) = \int_{\mathbb{R}^2\times\mathbb{R}^2} \log(|x-y|)\,u(x)v(y)\,dx\,dy, \tag{2.3}$$

since for all $r > 0$ we have

$$\log r = \log(1+r) - \log\left(1+\frac{1}{r}\right).$$

Remark 2.1. The definitions above are restricted to measurable functions $u, v\colon \mathbb{R}^2 \to \mathbb{R}$ such that the corresponding double integral is well defined in the Lebesgue sense.

In order to find estimates for B_1 and B_2 we recall a classical result of Measure Theory.

Theorem 2.2 (Hardy-Littlewood-Sobolev's inequality [12]). *Let $p > 1$, $q > 1$ and $0 < \lambda < N$ with $\frac{1}{p}+\frac{\lambda}{N}+\frac{1}{q} = 2$. If $f \in L^p(\mathbb{R}^N)$ and $g \in L^q(\mathbb{R}^N)$, then there exists a sharp constant $C(N,\lambda,p)$, independent of f and g, such that*

$$\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|f(x)g(y)|}{|x-y|^\lambda}\,dx\,dy \le C(N,\lambda,p)\|f\|_{L^p(\mathbb{R}^N)}\|g\|_{L^q(\mathbb{R}^N)}. \tag{2.4}$$

The sharp constant satisfies

$$C(N,\lambda,p) \le \frac{N}{(N-\lambda)}\left(\frac{|S^{N-1}|}{N}\right)^{\frac{\lambda}{N}} \frac{1}{pq}\left(\left(\frac{\lambda/N}{1-\frac{1}{p}}\right)^{\frac{\lambda}{N}} + \left(\frac{\lambda/N}{1-\frac{1}{q}}\right)^{\frac{\lambda}{N}}\right).$$

If $p = q = \frac{2N}{2N-\lambda}$, then

$$C(N,\lambda,p) = C(N,\lambda) = \pi^{\frac{\lambda}{2}} \frac{\Gamma(N/2-\lambda/2)}{\Gamma(N-\lambda/2)}\left(\frac{\Gamma(N/2)}{\Gamma(N)}\right)^{-1+\lambda/N}.$$

In this case there is equality in (2.4) *if and only if $g \equiv cf$ with c constant and*

$$f(x) = A\left(\gamma^2+|x-\alpha|^2\right)^{-(2N-\lambda)/2}$$

for some $A \in \mathbb{R}$, $\gamma \in \mathbb{R}\setminus\{0\}$ and $\alpha \in \mathbb{R}^N$.

We note that, since

$$\log(1+|x-y|) \le \log(1+|x|+|y|) \le \log(1+|x|) + \log(1+|y|),$$

we have by Schwarz's inequality

$$|B_1(uv,wz)| \le \int_{\mathbb{R}^2\times\mathbb{R}^2} \left[\log(1+|x|)+\log(1+|y|)\right]|u(x)v(x)||w(y)z(y)|\,dx\,dy$$

$$\leq |u|_*|v|_*\|w\|_{L^2(\mathbb{R}^2)}\|z\|_{L^2(\mathbb{R}^2)} + \|u\|_{L^2(\mathbb{R}^2)}\|v\|_{L^2(\mathbb{R}^2)}|w|_*|z|_* \tag{2.5}$$

for u, v, w, $z \in X$. Next, since $0 \leq \log(1+r) \leq r$ for all $r > 0$, we have by Hardy-Littlewood-Sobolev's inequality

$$|B_2(u,v)| \leq \int_{\mathbb{R}^2\times\mathbb{R}^2} \frac{1}{|x-y|} u(x)v(y)\,dx\,dy \leq C\|u\|_{L^{\frac{4}{3}}(\mathbb{R}^2)}\|v\|_{L^{\frac{4}{3}}(\mathbb{R}^2)}, \tag{2.6}$$

for $u, v \in L^{\frac{4}{3}}(\mathbb{R}^2)$, for some constant $C > 0$. In particular, from (2.5) we have

$$B_1(u^2,u^2) \leq 2|u|_*^2\|u\|_{L^2(\mathbb{R}^2)}^2 \tag{2.7}$$

for all $u \in X$ and from (2.6) we have

$$B_2(u^2,u^2) \leq C\|u\|_{L^{\frac{8}{3}}(\mathbb{R}^2)}^4 \tag{2.8}$$

for all $u \in L^{\frac{8}{3}}(\mathbb{R}^2)$.

Proposition 2.3. *The functional I_ε is of class $C^2(X)$.*

Proof. The proof is similar to [8, Lemma 2.2], so we just sketch the main ideas. Recalling (2.7), (2.8), the assumption $h \in L^\infty(\mathbb{R}^2)$ and the fact that X is compactly embedded into $L^s(\mathbb{R}^2)$, $s \in [2,+\infty)$ (see [5, 6, 8] for a proof) we have

$$\begin{aligned}
|I_\varepsilon(u)| &\leq \frac{1}{2}\|u\|_{H^1(\mathbb{R}^2)} + \frac{1}{8\pi}B(u^2,u^2) + \frac{1}{p+1}\varepsilon\|h\|_\infty\|u\|_{L^{p+1}(\mathbb{R}^2)}^{p+1} \\
&\leq \frac{1}{2}\|u\|_{H^1(\mathbb{R}^2)} + \frac{1}{4\pi}|u|_*^2\|u\|_{L^2(\mathbb{R}^2)}^2 + \frac{1}{8\pi}C\|u\|_{L^{\frac{4}{3}}(\mathbb{R}^2)}^4 \\
&\quad + \|h\|_\infty\|u\|_X^{p+1} < +\infty.
\end{aligned}$$

The first Gâteaux derivative of I_ε along v is

$$\begin{aligned}
I'_\varepsilon(u)v = &\int_{\mathbb{R}^2} (\nabla u\cdot\nabla v + auv)\;dx - \frac{1}{2\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-y|}u^2(x)u(y)v(y)\,dx\,dy \\
&- \varepsilon\int_{\mathbb{R}^2} h(x)|u|^{p-1}uv\,dx.
\end{aligned} \tag{2.9}$$

We add and subtract $\int_{\mathbb{R}^2} u(x)v(x)\log(1+|x|)\,dx$ to recover the scalar product of X, so we obtain

$$I'_\varepsilon(u)v = (u|v)_X - \frac{1}{2\pi}B(u^2,uv) - \int_{\mathbb{R}^2} u(x)v(x)\log(1+|x|)\,dx - \varepsilon\int_{\mathbb{R}^2} h(x)|u|^{p-1}uv\,dx.$$

Now,

$$\begin{aligned}
|I'_\varepsilon(u)v| &\leq (u|v)_X + \frac{1}{2\pi}B(u^2,uv) + \int_{\mathbb{R}^2} u(x)v(x)\log(1+|x|)\,dx + \varepsilon\int_{\mathbb{R}^2} |h(x)||u|^p|v|\,dx \\
&\leq (u|v)_X + \frac{1}{2\pi}|u|_*^2\|uv\|_{L^2(\mathbb{R}^2)} + \frac{1}{2\pi}\|u\|_{L^2(\mathbb{R}^2)}^2|uv|_*
\end{aligned}$$

$$+\frac{1}{2\pi}C\|u\|^2_{L^{\frac{8}{3}}(\mathbb{R}^2)}\|uv\|_{L^{\frac{4}{3}}(\mathbb{R}^2)}+\varepsilon\|h\|_\infty\|u\|^p_{pp'}\|v\|_p$$

The second Gâteaux derivative of $I_\varepsilon(u)$ along (v,w) is

$$\begin{aligned}I''_\varepsilon(u)(v,w)=&\int_{\mathbb{R}^2}(\nabla v\cdot\nabla w+avw)\,dx-\frac{1}{2\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2}\log\frac{1}{|x-y|}u^2(x)v(y)w(y)\,dx\,dy\\&-\frac{1}{\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2}\log\frac{1}{|x-y|}u(x)v(x)u(y)w(y)\,dx\,dy\\&-(p-1)\varepsilon\int_{\mathbb{R}^2}h(x)|u|^{p-1}w\,dx-\varepsilon\int_{\mathbb{R}^2}h(x)|u|^{p-1}vw\,dx.\end{aligned}$$

In this case, we add and subtract $\int_{\mathbb{R}^2}v(x)w(x)\log(1+|x|)\,dx$, hence

$$\begin{aligned}I''_\varepsilon(u)(v,w)=(v|w)_X-\int_{\mathbb{R}^2}v(x)w(x)\log(1+|x|)\,dx-\frac{1}{2\pi}B(u^2,vw)-\frac{1}{\pi}B(uv,uw)\\-(p-1)\varepsilon\int_{\mathbb{R}^2}h(x)|u|^{p-1}w\,dx-\varepsilon\int_{\mathbb{R}^2}h(x)|u|^{p-1}vw\,dx.\end{aligned}$$

Finally,

$$\begin{aligned}|I''_\varepsilon(u)(v,w)|\le&(v|w)_X+\int_{\mathbb{R}^2}v(x)w(x)\log(1+|x|)\,dx+\frac{1}{2\pi}B(u^2,vw)+\frac{1}{\pi}B(uv,uw)\\&+(p-1)\varepsilon\int_{\mathbb{R}^2}|h(x)||u|^{p-1}|w|\,dx+\varepsilon\int_{\mathbb{R}^2}|h(x)||u|^{p-1}|v||w|\,dx\\\le&(v|w)_X+\int_{\mathbb{R}^2}v(x)w(x)\log(1+|x|)\,dx+\frac{1}{2\pi}|u|^2_*\|vw\|_{L^2(\mathbb{R}^2)}\\&+\frac{1}{2\pi}\|u\|^2_{L^2(\mathbb{R}^2)}|v|_*|w|_*+\frac{1}{2\pi}C\|u\|^2_{L^{\frac{8}{3}}(\mathbb{R}^2)}\|vw\|_{L^{\frac{4}{3}}(\mathbb{R}^2)}\\&+\frac{1}{\pi}|u|_*|v|_*\|u\|_{L^2(\mathbb{R}^2)}\|w\|_{L^2(\mathbb{R}^2)}+\frac{1}{\pi}\|u\|_{L^2(\mathbb{R}^2)}\|v\|_{L^2(\mathbb{R}^2)}|u|_*|w|_*\\&+\frac{1}{\pi}C\|u\|_{L^2(\mathbb{R}^2)}\|v\|_{L^2(\mathbb{R}^2)}\|u\|_{L^2(\mathbb{R}^2)}\|w\|_{L^2(\mathbb{R}^2)}\\&+(p-1)\varepsilon\|h\|_\infty\|u\|^{p-1}_{(p-1)p'}\|w\|_p+\varepsilon\|h\|_\infty\|u\|^{p-1}_{(p-1)p'}\|vw\|_p.\end{aligned}$$

It is now standard to conclude that the first and the second Gâteaux derivatives are continuous (with respect to $u\in X$), so that $I_\varepsilon\in C^2(X)$. □

Critical points of the unperturbed functional $I'_0(u)=0$ are solutions of the equation

$$-\Delta u+au-\frac{1}{2\pi}\left[\log\frac{1}{|\cdot|}\star u^2\right]u=0\quad\text{in }\mathbb{R}^2,\tag{2.10}$$

which admits for every $a>0$ a unique — up to translations — radially symmetric solution $u\in X$ (see [8], Theorem 1.3).

Since (2.10) is invariant under translations, we can consider $z_\xi(x)=u(x-\xi)$, with $\xi\in\mathbb{R}^2$. The manifold

$$Z=\left\{z_\xi\mid\xi\in\mathbb{R}^2\right\}$$

is therefore a critical manifold for I_0. We want to show that the manifold Z satisfies the properties **(ND)** and **(Fr)**.

The **(ND)** property follows from the following theorem, proved in [6].

Theorem 2.4. *If $a > 0$ and $u \in X$ is a radial solution of* (2.10) *then there exists $\mu \in (0, \infty)$ such that*

$$u(x) = \frac{(\mu + o(1))}{\sqrt{|x|}\,(\log|x|)^{1/4}} \exp\left(-\sqrt{M} e^{-\frac{a}{M}} \int_1^{e^{\frac{a}{M}}|x|} \sqrt{\log s}\, ds \right)$$

with

$$M = \frac{1}{2\pi} \int_{\mathbb{R}^2} |u|^2.$$

To prove that **(Fr)** holds, we actually show that $I_0''(z_\xi)$ is a compact perturbation of the identity operator.

Proposition 2.5. *$I_0''(z_\xi) = L - K$, where L is a continuous invertible operator and K is a continuous linear compact operator in X.*

Proof. We recall that

$$\begin{aligned} I_0''(z_\xi)(v, w) = &\int_{\mathbb{R}^2} (\nabla v \cdot \nabla w + avw)\, dx - \frac{1}{2\pi} \int_{\mathbb{R}^2 \times \mathbb{R}^2} \log \frac{1}{|x-y|} z_\xi^2(x) v(y) w(y)\, dx\, dy \\ &- \frac{1}{\pi} \int_{\mathbb{R}^2 \times \mathbb{R}^2} \log \frac{1}{|x-y|} z_\xi(x) v(x) z_\xi(y) w(y)\, dx\, dy \end{aligned}$$

We add and subtract $\int_{\mathbb{R}^2} \log(1+|x|) v(x) w(x)\, dx$, hence

$$\begin{aligned} I_0''(z_\xi)(v, w) = &\int_{\mathbb{R}^2} (\nabla v \cdot \nabla w + avw)\, dx + \int_{\mathbb{R}^2} \log(1+|x|) v(x) w(x)\, dx \\ &- \int_{\mathbb{R}^2} \log(1+|x|) v(x) w(x)\, dx \\ &- \frac{1}{2\pi} \int_{\mathbb{R}^2 \times \mathbb{R}^2} \log \frac{1}{|x-y|} z_\xi^2(x) v(y) w(y)\, dx\, dy \\ &- \frac{1}{\pi} \int_{\mathbb{R}^2 \times \mathbb{R}^2} \log \frac{1}{|x-y|} z_\xi(x) v(x) z_\xi(y) w(y)\, dx\, dy, \end{aligned}$$

and so

$$\begin{aligned} I_0''(z_\xi)(v, w) = (v|w)_X &- \int_{\mathbb{R}^2} \log(1+|x|) v(x) w(x)\, dx - \frac{1}{2\pi} \int_{\mathbb{R}^2 \times \mathbb{R}^2} \log \frac{1}{|x-y|} z_\xi^2(x) v(y) w(y)\, dx\, dy \\ &- \frac{1}{\pi} \int_{\mathbb{R}^2 \times \mathbb{R}^2} \log \frac{1}{|x-y|} z_\xi(x) v(x) z_\xi(y) w(y)\, dx\, dy. \end{aligned}$$

The second derivative is therefore the linear operator defined by

$$\mathcal{L}(z_\xi)\colon \varphi \mapsto -\Delta\varphi + (a - w)\,\varphi + 2 z_\xi \left(\frac{\log}{2\pi} \star (z_\xi \varphi) \right),$$

where

$$w(x) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \log|x-y||z_\xi(y)|^2\,dy, \quad x \in \mathbb{R}^2.$$

Following the proof in [13], Lemma 15, let $\{v_n\}_n$, $\{w_n\}_n$ be two sequences in X such that $\|v_n\| \le 1$, $\|w_n\| \le 1$, $v_n \rightharpoonup v_0$ and $w_n \rightharpoonup w_0$. Without loss of generality we can assume $v_0 = w_0 = 0$, so that

$$v_n \rightharpoonup 0, \quad w_n \rightharpoonup 0.$$

From the compact embedding of X in $L^s(\mathbb{R}^2)$ for $s \ge 2$, we can say that

$$v_n \to 0, \quad w_n \to 0 \tag{2.11}$$

in $L^s(\mathbb{R}^2)$, for every $s \ge 2$. We compute, using Theorem 2.4,

$$\int_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-y|} z_\xi(x)v_n(x)z_\xi(y)w_n(y)\,dx\,dy \le C\|z_\xi v_n\|_{L^{\frac{4}{3}}(\mathbb{R}^2)}\|z_\xi w_n\|_{L^{\frac{4}{3}}(\mathbb{R}^2)}.$$

Now, by Hölder's inequality,

$$\limsup_{n\to+\infty} \|z_\xi v_n\|^{\frac{4}{3}}_{L^{\frac{4}{3}}(\mathbb{R}^2)} \le \limsup_{n\to+\infty} \left(\int_{\mathbb{R}^2} |z_\xi(x)|^4\,dx\right)^{\frac{1}{3}} \left(\int_{\mathbb{R}^2} |v_n(x)|^2\,dx\right)^{\frac{2}{3}} = 0$$

thanks to Theorem 2.4 and (2.11).

Similarly,

$$\limsup_{n\to\infty} \left| \int_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-y|} z_\xi(x)v_n(x)z_\xi w_n(y)\,dx\,dy \right| = 0. \tag{2.12}$$

This proves that the linear operator

$$\varphi \mapsto 2z_\xi\left(\frac{\log}{2\pi} \star (z_\xi\varphi)\right)$$

is compact. At this point, we should notice that the linear operator

$$\varphi \mapsto -\Delta\varphi + (a-w)\,\varphi$$

is not invertible on X. To overcome this difficulty, we set

$$c^2 = \frac{1}{2\pi}\int_{\mathbb{R}^2} |z_\xi(x)|^2\,dx$$

and rewrite $\mathcal{L}(z_\xi)$ as follows:

$$\mathcal{L}(z_\xi)\colon \varphi \mapsto -\Delta\varphi + \left(a + c^2\log(1+|x|)\right)\varphi - \left(c^2\log(1+|x|) + w\right)\varphi + 2z_\xi\left(\frac{\log}{2\pi} \star (z_\xi\varphi)\right).$$

Since

$$\lim_{|x|\to+\infty} \left(w(x) + c^2\log(1+|x|)\right) = 0$$

by [8, Proposition 2.3], the multiplication operator

$$\varphi \mapsto \left(c^2 \log(1 + |x|) + w\right)\varphi$$

is compact. We may conclude that the functional $I_0''(z_\xi)$ is of the form $L - K$ where

$$L\varphi = -\Delta\varphi + \left(a + c^2 \log(1 + |x|)\right)\varphi$$

is a linear, continuous, invertible operator and K is a linear, continuous, compact operator. A different proof, based on a direct computation, appears in [**?**]. □

3. Existence results for equation (1.2)

3.1. Proof of Theorem 1.3

Since the properties **(ND)** and **(Fr)** hold we can say that, for $|\varepsilon|$ small, the reduced functional has the following form:

$$\Phi_\varepsilon(z_\xi) = I_\varepsilon(z_\xi + w_\varepsilon(\xi)) = c_0 + \varepsilon G(z_\xi) + o(\varepsilon),$$

with $c_0 = I_0(z_\xi)$.

Let $\Gamma\colon \mathbb{R}^2 \to \mathbb{R}$ be the function defined as

$$\Gamma(\xi) = G(z_\xi) = -\frac{1}{p+1}\int_{\mathbb{R}^2} h(x)|z_\xi|^{p+1}\,dx, \quad \xi \in \mathbb{R}^2.$$

Lemma 3.1. *Suppose that $h \in L^\infty(\mathbb{R}^2) \cap L^q(\mathbb{R}^2)$ for some $q > 1$. Then*

$$\lim_{|\xi|\to\infty} \Gamma(\xi) = 0.$$

Proof. By the Hölder inequality,

$$|\Gamma(\xi)| \le \frac{1}{p+1}\int_{\mathbb{R}^2} |h(x)||z_\xi|^{p+1}\,dx \le \frac{1}{p+1}\left(\int_{\mathbb{R}^2} |h(x)|^q\,dx\right)^{\frac{1}{q}}\left(\int_{\mathbb{R}^2} |z_\xi|^{(p+1)q'}\,dx\right)^{\frac{1}{q'}}$$

and since z_ξ decays to zero as $|\xi| \to \infty$ we have

$$|\Gamma(\xi)| \le C\left(\int_{\mathbb{R}^2} |z_\xi|^{(p+1)q'}\,dx\right)^{\frac{1}{q'}} \to 0 \quad \text{as } |\xi| \to +\infty.$$

The proof is completed. □

Thanks to the previous Lemma we can now prove the existence of local solutions for (2.1).

Proof of Theorem 1.3. The hypothesis of the previous Lemma are satisfied, so we have that $\Gamma(\xi)$ goes to 0 as $|\xi| \to \infty$. From (h_1) follows that $\Gamma(0) = -\frac{1}{p+1}\int_{\mathbb{R}^2} h(x)|z_0|^{p+1}\,dx \neq 0$. Then Γ is not identically zero and follows that Γ has a maximum or a minimum on $\mathbb{R}^2$ and the existence of a solution follows from Theorem 2.16 in [4]. □

3.2. Proof of Theorem 1.4

As before, we call $P = P_\xi : X \to W_\xi$ the orthogonal projection onto $W_\xi = \left(T_{z_\xi}Z\right)^\perp$, $\widetilde{W_\xi} := \langle z_\xi\rangle \oplus \left(T_{z_\xi}Z\right)$ and $R_\xi(w) = I_0'(z_\xi + w) - I_0''(z_\xi)[w]$.

Remark 3.2. By the variational characterization of the Mountain-Pass solution u as in [4, Remark 4.2], the spectrum of $PI_0''(u)$ has exactly one negative simple eigenvalue with eigenspace spanned by u itself. Moreover, $\lambda = 0$ is an eigenvalue with multiplicity N and eigenspace spanned by $D_i u$, $i = 1, \dots, N$ and there exists $\kappa > 0$ such that

$$(PI_0''(u)[v]|v) \geq \kappa\|v\|^2, \quad \forall v \perp \langle u\rangle \oplus T_u Z,$$

and hence the rest of the spectrum is positive.

We prove the following

Theorem 3.3. *(i) There is $C > 0$ such that $\|\left(PI_0''(z_\xi)\right)^{-1}\|_{L(W_\xi,W_\xi)} \leq C$, for every $\xi \in \mathbb{R}^2$,*
(ii) $R_\xi(w) = o(\|w\|)$, uniformly with respect to $\xi \in \mathbb{R}^2$.

Proof. Since z_ξ is a Mountain-Pass solution, Remark 3.2 holds, hence it suffices to show that there exists $\kappa > 0$ such that

$$(PI_0''(z_\xi)[v] \mid v) \geq \kappa\|v\|^2, \quad \forall \xi \in \mathbb{R}^2, \quad \forall v \perp \widetilde{W_\xi}.$$

For any fixed $\xi \in \mathbb{R}^2$, say $\xi = 0$, $PI_0''(z_0) = PI_0''(u)$ is invertible and there exists $\kappa > 0$ such that

$$(PI_0''(u)[v] \mid v) \geq \kappa\|v\|^2, \quad \forall v \in \widetilde{W} := \langle u\rangle \oplus T_u Z.$$

Let $v^\xi(x) = v(x + \xi)$, then

$$\begin{aligned}(PI_0''(z_\xi)[v] \mid v) = P\Big[&\int_{\mathbb{R}^2} \left(|\nabla v|^2 + av^2\right) dx - \frac{1}{2\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-y|} z_\xi^2(x)v^2(y)\,dx\,dy \\ &- \frac{1}{\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-y|} z_\xi(x)v(x)z_\xi(y)v(y)\,dx\,dy\Big]\end{aligned}$$

and thanks to the change of variables $x = t + \xi$ and $y = s + \xi$ we obtain

$$\begin{aligned}(PI_0''(z_\xi)[v] \mid v) &= P\Big[\int_{\mathbb{R}^2} \left(|\nabla v(t+\xi)|^2 + av^2(t+\xi)\right) dt \\ &\quad - \frac{1}{2\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|t-s|} z_\xi^2(t+\xi)v^2(s+xi)dtds \\ &\quad - \frac{1}{\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|t-s|} z_\xi(t+\xi)v(t+\xi)z_\xi(s+\xi)v(s+\xi)dtds\Big] \\ &= P\Big[\int_{\mathbb{R}^2} \left(|\nabla v^\xi|^2 + a\left(v^\xi\right)^2\right) dt - \frac{1}{2\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-y|} z_0^2(t)|v^\xi(s)|^2\,dt\,ds \\ &\quad - \frac{1}{\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-y|} z_0(t)v^\xi(t)z_0(s)v^\xi(s)dtds\Big] = PI_0''(u)(v^\xi, v^\xi).\end{aligned}$$

Moreover, $v^\xi \perp \widetilde{W}$ whenever $v \perp \widetilde{W_\xi}$, hence

$$(PI_0''(z_\xi)[(v,v)]v] \mid v) = (PI_0''(u)[v^\xi] \mid v^\xi) \geq \kappa\|v^\xi\|^2 = \kappa\|v\|^2, \quad \forall \xi \in \mathbb{R}^2, \quad \forall v \perp \widetilde{W_\xi}$$

and so (i) si true.

To prove (ii) we observe that

$$\begin{aligned}
R_\xi(w) = I_0'(z_\xi + w)(v) - I_0''(z_\xi)(w,v) &= \int_{\mathbb{R}^2} \Big[\nabla(z_\xi + w)\cdot\nabla v + a(z_\xi + w)v\Big]\,dx \\
&- \frac{1}{2\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-y|}(z_\xi + w)^2(x)(z_\xi + w)(y)v(y)\,dx\,dy \\
&- \int_{\mathbb{R}^2} (\nabla w\cdot\nabla v + awv)\,dx + \frac{1}{2\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-y|} z_\xi^2(x)w(y)v(y)\,dx\,dy \\
&+ \frac{1}{\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-y|} z_\xi(x)w(x)z_\xi(y)v(y)\,dx\,dy.
\end{aligned}$$

After some computations we obtain

$$\begin{aligned}
R_\xi(w) = &\int_{\mathbb{R}^2} \Big(\nabla z_\xi\cdot\nabla v + az_\xi v\Big)\,dx - \frac{1}{2\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-y|} z_\xi^2(x)z_\xi(y)v(y)\,dx\,dy \\
&- \frac{1}{2\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-y|} z_\xi^2(x)w(y)v(y)\,dx\,dy \\
&- \frac{1}{\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-y|} z_\xi(x)w(x)z_\xi(y)v(y)\,dx\,dy \\
&- \frac{1}{\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-y|} z_\xi(x)w(x)w(y)v(y)\,dx\,dy \\
&- \frac{1}{2\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-y|} w^2(x)z_\xi(y)v(y)\,dx\,dy \\
&- \frac{1}{2\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-y|} w^2(x)w(y)v(y)\,dx\,dy \\
&+ \frac{1}{2\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-y|} z_\xi^2(x)w(y)v(y)\,dx\,dy \\
&+ \frac{1}{\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-y|} z_\xi(x)w(x)z_\xi(y)v(y)\,dx\,dy
\end{aligned}$$

and since z_ξ is critical point we finally have

$$\begin{aligned}
R_\xi(w) = &-\frac{1}{\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-y|} z_\xi(x)w(x)w(y)v(y)\,dx\,dy \\
&- \frac{1}{2\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-y|} w^2(x)z_\xi(y)v(y)\,dx\,dy \\
&- \frac{1}{2\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-y|} w^2(x)w(y)v(y)\,dx\,dy.
\end{aligned}$$

By (2.7), (2.8), (2.4), Hölder's inequality we have

$$|R_\xi(w)| \leq \frac{1}{\pi}\Big(|z_\xi|_*|w|_*\|w\|_{L^2}\|v\|_{L^2} + \|z_\xi\|_{L^2}\|w\|_{L^2}|w|_*|v|_* + C_1\|z_\xi\|_{L^{\frac{4}{3}}}\|w\|^2_{L^{\frac{4}{3}}}\|v\|_{L^{\frac{4}{3}}}\Big)$$

$$+\frac{1}{2\pi}\left(|w|_*^2\|z_\xi\|_{L^2}\|v\|_{L^2}+\|w\|_{L^2}^2|z_\xi|_*|v|_*+C_2\|w\|_{L^{\frac{8}{3}}}^2\|z_\xi\|_{L^{\frac{4}{3}}}\|v\|_{L^{\frac{4}{3}}}\right)$$
$$+\frac{1}{2\pi}\left(|w|_*^2\|w\|_{L^2}\|v\|_{L^2}+\|w\|_{L^2}^2|w|_*|v|_*+C_3\|w\|_{L^{\frac{8}{3}}}^2\|w\|_{L^{\frac{4}{3}}}\|v\|_{L^{\frac{4}{3}}}\right),$$

by the compact embedding of X into $L^s(\mathbb{R}^2)$, $s \geq 2$

$$\begin{aligned}|R_\xi(w)| \leq{}& \frac{1}{\pi}\left(\|z_\xi\|_X\|w\|_X^2\|v\|_X+\|z_\xi\|_X\|w\|_X^2\|v\|_X+C_1\|z_\xi\|_X\|w\|_X^2\|v\|_X\right)\\&+\frac{1}{2\pi}\left(\|w\|_X^2\|z_\xi\|_X\|v\|_X+\|w\|_X^2\|z_\xi\|_X\|v\|_X+C_2\|w\|_X^2\|z_\xi\|_X\|v\|_X\right)\\&+\frac{1}{2\pi}\left(\|w\|_X^3\|v\|_X+\|w\|_X^3\|v\|_X+C_3\|w\|_X^3\|v\|_X\right)\end{aligned} \tag{3.1}$$

and finally

$$|R_\xi(w)| \leq C_4\|z_\xi\|_X\|w\|_X^2\|v\|_X+C_5\|w\|_X^2\|z_\xi\|_X\|v\|_X+C_6\|w\|_X^3\|v\|_X.$$

Hence,

$$\frac{|R_\xi(w)|}{\|w\|_X} \leq C_4\|z_\xi\|_X\|w\|_X\|v\|_X+C_5\|w\|_X\|z_\xi\|_X\|v\|_X+C_6\|w\|_X^2\|v\|_X$$

and this goes to 0 as $\|w\|_X \to 0$ uniformly with respect to $\xi \in \mathbb{R}^2$. □

This Lemma allows us to use Lemma 2.21 in [4], so there exists $\varepsilon_0 > 0$ such that for all $|\varepsilon| \leq \varepsilon_0$ and all $\xi \in \mathbb{R}^2$ the auxiliary equation $PI'_\varepsilon\left(z_\xi + w\right) = 0$ has a unique solution $w_\varepsilon(z_\xi)$ with

$$\lim_{\varepsilon\to 0}\|w_\varepsilon(z_\xi)\| = 0, \tag{3.2}$$

uniformly with respect to $\xi \in \mathbb{R}^2$.

We now prove

Lemma 3.4. *There exists $\varepsilon_1 > 0$ such that for all $|\varepsilon| \leq \varepsilon_1$, the following result holds:*

$$\lim_{|\xi|\to\infty} w_\xi = 0, \quad \textit{strongly in } X.$$

Proof. We first show two preliminaries results:

(a) w_ξ weakly converges in X to some $w_{\varepsilon,\infty} \in X$ as $|\xi| \to \infty$. Moreover, the weak limit $w_{\varepsilon,\infty}$ is a weak solution of

$$-\Delta w_{\varepsilon,\infty} + a w_{\varepsilon,\infty} - \frac{1}{2\pi}\left[\log\frac{1}{|\cdot|} \star w_{\varepsilon,\infty}^2\right] w_{\varepsilon,\infty} = \varepsilon h(x)|w_{\varepsilon,\infty}|^{p-1}w_{\varepsilon,\infty}; \tag{3.3}$$

(b) $w_{\varepsilon,\infty} = 0$.

As a consequence of (3.2) we have that $w_\varepsilon(z_\xi)$ weakly converges in X to some $w_{\varepsilon,\infty} \in X$, as $|\xi| \to \infty$. Recall that $w_\varepsilon(z_\xi)$ is a solution of the auxialiary equation $PI'_\varepsilon\left(z_\xi + w_\xi(z_\xi)\right) = 0$, namely

$$
\begin{aligned}
-\Delta w_{\varepsilon,\xi} + a w_{\varepsilon,\xi} - \frac{1}{2\pi}\left[\log\frac{1}{|\cdot|} \star w_{\varepsilon,\xi}^2\right] w_{\varepsilon,\xi} - \frac{1}{2\pi}\left[\log\frac{1}{|\cdot|} \star z_\xi^2\right] w_{\varepsilon,\xi} \\
- \frac{1}{\pi}\left[\log\frac{1}{|\cdot|} \star z_\xi^2\right]\left(z_\xi + w_{\varepsilon,\xi}\right) - \frac{1}{2\pi}\left[\log\frac{1}{|\cdot|} \star w_{\varepsilon,\xi}^2\right] z_\xi \\
= \varepsilon h(x)|z_\xi + w_{\varepsilon,\xi}|^{p-1}\left(z_\xi + w_{\varepsilon,\xi}\right) - z_\xi^{p-1} - \sum_{i=1}^{2} a_i D_i z_\xi,
\end{aligned}
$$

where

$$
a_i = \int_{\mathbb{R}^2}\left(\varepsilon h(x)|z_\xi + w_{\varepsilon,\xi}|^{p-1} - z_\xi^{p-1}\right) D_i z_\xi \, dx,
$$

and D_i denotes the partial derivative with respect to x_i.

Let v be any test function, then

$$
\begin{aligned}
&\int_{\mathbb{R}^2}\left(\nabla w_{\varepsilon,\xi}\cdot\nabla v + a w_{\varepsilon,\xi} v\right) dx - \frac{1}{2\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2}\log\frac{1}{|x-y|} w_{\varepsilon,\xi}^2(x) w_{\varepsilon,\xi}(y) v(y)\, dx\, dy \\
&= \frac{1}{2\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2}\log\frac{1}{|x-y|} z_\xi^2(x) w_{\varepsilon,\xi}(y) v(y)\, dx\, dy \\
&\quad + \frac{1}{\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2}\log\frac{1}{|x-y|} z_\xi^2(x)\left(z_\xi + w_{\varepsilon,\xi}\right)(y) v(y)\, dx\, dy \\
&\quad + \frac{1}{2\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2}\log\frac{1}{|x-y|} w_{\varepsilon,\xi}^2(x) z_\xi(y) v(y)\, dx\, dy + \int_{\mathbb{R}^2}\varepsilon h(x)|z_\xi + w_{\varepsilon,\xi}|^{p-1} v(x)\, dx \\
&\quad - \int_{\mathbb{R}^2} z_\xi^{p-1}(x) v(x)\, dx - \sum_{i=1}^{2} a_i \int_{\mathbb{R}^2} D_i z_\xi(x) v(x)\, dx.
\end{aligned}
$$

Following the computations in the proof of Proposition 2.5, in particular (2.12), and by Theorem 2.4 we can say that

$$
\begin{aligned}
&\frac{1}{2\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2}\log\frac{1}{|x-y|} z_\xi^2(x) w_{\varepsilon,\xi}(y) v(y)\, dx\, dy \to 0, \\
&\frac{1}{\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2}\log\frac{1}{|x-y|} z_\xi^2(x)\left(z_\xi + w_{\varepsilon,\xi}\right)(y) v(y)\, dx\, dy \to 0, \\
&\frac{1}{2\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2}\log\frac{1}{|x-y|} w_{\varepsilon,\xi}^2(x) z_\xi(y) v(y)\, dx\, dy \to 0,
\end{aligned}
$$

as $|\xi| \to \infty$.

Now, we need to pass to the limit in

$$
\int_{\mathbb{R}^2}\varepsilon h(x)|z_\xi + w_{\varepsilon,\xi}|^{p-1} v(x)\, dx.
$$

In order to do that, we show that

$$\lim_{|\xi|\to 0}\int_{\mathbb{R}^2} z_\xi^{p-1-k} w_{\varepsilon,\xi}^k v\,dx = 0, \quad \forall k\in[0,p-1). \tag{3.4}$$

We split the integral as

$$\int_{\mathbb{R}^2} z_\xi^{p-1-k} w_{\varepsilon,\xi}^k v\,dx = \int_{|x|\le\rho} z_\xi^{p-1-k} w_{\varepsilon,\xi}^k v\,dx + \int_{|x|>\rho} z_\xi^{p-1-k} w_{\varepsilon,\xi}^k v\,dx$$

where $\rho>0$. Using Hölder's inequality with $p=k+1$, so $p'=\frac{p}{p-1-k}$, we obtain

$$\begin{aligned}\left|\int_{|x|\le\rho} z_\xi^{p-1-k} w_{\varepsilon,\xi}^k v\,dx\right| &\le \left(\int_{|x|\le\rho} |z_\xi^{(p-1-k)p'}\,dx\right)^{\frac{1}{p'}}\left(\int_{|x|\le\rho} |w_{\varepsilon,\xi}|^{kp}|v|^p\,dx\right)^{\frac{1}{p}}\\ &\le C\left(\int_{|x|\le\rho} |z_\xi|^p\,dx\right)^{\frac{1}{p'}}\end{aligned}$$

and this goes to 0 as ρ goes to ∞. On the other hand,

$$\left|\int_{|x|>\rho} z_\xi^{p-1-k} w_{\varepsilon,\xi}^k v\,dx\right| \le \left(\int_{|x|>\rho} |z_\xi^{(p-1-k)p'}|w_{\varepsilon,\xi}|^{kp'}\,dx\right)^{\frac{1}{p'}}\left(\int_{|x|>\rho} |v|^p\,dx\right)^{\frac{1}{p}},$$

and since v is a test function this integral goes to 0 as ρ goes to ∞.

Hence, (3.4) holds and then

$$\int_{\mathbb{R}^2} \varepsilon h(x)|z_\xi + w_{\varepsilon,\xi}|^{p-1} v(x)\,dx \to \int_{\mathbb{R}^2} \varepsilon h(x)|w_{\varepsilon,\infty}|^{p-1} v(x)\,dx,.$$

Moreover,

$$\begin{aligned}&\int_{\mathbb{R}^2} |w_{\varepsilon,\xi}|^{p-1} v(x)\,dx \to \int_{\mathbb{R}^2} |w_\infty|^{p-1} v(x)\,dx,\\ &\int_{\mathbb{R}^2} \varepsilon h(x)|w_{\varepsilon,\xi}|^{p-1} v(x)\,dx \to \int_{\mathbb{R}^2} \varepsilon h(x)|w_\infty|^{p-1} v(x)\,dx,\end{aligned}$$

as $|\xi|\to\infty$ and again by Theorem 2.4,

$$\begin{aligned}&\int_{\mathbb{R}^2} z_\xi^{p-1}(x)v(x)\,dx \to 0,\\ &\int_{\mathbb{R}^2} D_i z_\xi(x)v(x)\,dx \to 0,\end{aligned}$$

as $|\xi|\to\infty$.

Finally, we obtain

$$\int_{\mathbb{R}^2} (\nabla w_{\varepsilon,\infty}\cdot\nabla v + a w_{\varepsilon,\infty} v)\,dx - \frac{1}{2\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-y|} w_{\varepsilon,\infty}^2(x) w_{\varepsilon,\infty}(y)v(y)\,dx\,dy$$

$$= \int_{\mathbb{R}^2} \varepsilon h(x)|w_{\varepsilon,\infty}|^{p-1} v(x)\,dx,$$

thus $w_{\varepsilon,\infty}$ is a weak solution of (3.3), namely **(a)** holds.

By (3.2) we have that $\lim_{|\varepsilon|\to 0} w_{\varepsilon,\infty} = 0$. Since the unique solution $w \in X$ of

$$-\Delta w + aw - \frac{1}{2\pi}\left[\log\frac{1}{|\cdot|} \star w^2\right] w = \varepsilon h(x)|w|^{p-1} w$$

with small norm is $w = 0$. To show that, we need to prove that there exists a constant $C > 0$ such that $\|w\|_X \geq C$.

Consider the first Gâteaux derivative, computed in (2.9), evaluated at w along w, namely

$$\begin{aligned} 0 &= I'_\varepsilon(w)w \\ &= \int_{\mathbb{R}^2} \left(|\nabla w|^2 + aw^2\right) dx - \frac{1}{2\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-y|} w^2(x)w^2(y)\,dx\,dy - \varepsilon\int_{\mathbb{R}^2} h(x)|w|^{p+1}\,dx, \end{aligned}$$

thus

$$\|w\|^2_{H^1(\mathbb{R}^2)} = \frac{1}{2\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-y|} w^2(x)w^2(y)\,dx\,dy + \varepsilon\int_{\mathbb{R}^2} h(x)|w|^{p+1}\,dx,$$

and thus

$$\|w\|^2_{H^1(\mathbb{R}^2)} \leq \frac{1}{2\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-y|} w^2(x)w^2(y)\,dx\,dy + \varepsilon\int_{\mathbb{R}^2} |h(x)||w|^{p+1}\,dx. \tag{3.5}$$

Now, observe that from (2.3) we have

$$\begin{aligned} &-\int_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-y|}|w(x)|^2|w(y)|^2\,dx\,dy = \\ &\int_{\mathbb{R}^2\times\mathbb{R}^2} \log(1+|x-y|)\,|w(x)|^2|w(y)|^2\,dx\,dy - \int_{\mathbb{R}^2\times\mathbb{R}^2} \log\left(1+\frac{1}{|x-y|}\right) w^2(x)w^2(y)\,dx\,dy, \end{aligned}$$

hence

$$\begin{aligned} &\int_{\mathbb{R}^2\times\mathbb{R}^2} \log\left(1+\frac{1}{|x-y|}\right) w^2(x)w^2(y)\,dx\,dy = \\ &\int_{\mathbb{R}^2\times\mathbb{R}^2} \log(1+|x-y|)\,w^2(x)w^2(y)\,dx\,dy + \int_{\mathbb{R}^2\times\mathbb{R}^2} \log\left(\frac{1}{|x-y|}\right) w^2(x)w^2(y)\,dx\,dy \end{aligned}$$

and finally

$$\int_{\mathbb{R}^2\times\mathbb{R}^2} \log\left(1+\frac{1}{|x-y|}\right) w^2(x)w^2(y)\,dx\,dy \leq \int_{\mathbb{R}^2\times\mathbb{R}^2} \log\left(\frac{1}{|x-y|}\right) w^2(x)w^2(y)\,dx\,dy.$$

Moreover,

$$\varepsilon\int_{\mathbb{R}^2} |h(x)||w|^{p+1}\,dx \leq C_1\|w\|^{p+1}_{L^{p+1}(\mathbb{R}^2)} \leq C_2\|w\|^{p+1}_{H^1(\mathbb{R}^2)}.$$

By (3.5) we obtain

$$\|w\|^2_{H^1(\mathbb{R}^2)} \leq \int_{\mathbb{R}^2\times\mathbb{R}^2} \log\left(\frac{1}{|x-y|}\right) w^2(x)w^2(y)\,dx\,dy + \varepsilon \int_{\mathbb{R}^2} |h(x)||w|^{p+1}\,dx$$
$$\leq C\|w\|^4_{H^1(\mathbb{R}^2)} + C_2\|w\|^{p+1}_{H^1(\mathbb{R}^2)} \leq C_3\|w\|^{\eta}_{H^1(\mathbb{R}^2)},$$

where $\eta = \max\{p+1, 4\} > 2$ since $p > 1$; recalling that $\|w\|^{\eta-2}_{H^1(\mathbb{R}^2)} \leq C_4\|w\|^{\eta-2}_X$ we obtain

$$\|w\|^{\eta-2}_X \geq \frac{1}{C_5} > 0.$$

From this we infer that $w_{\varepsilon,\infty} = 0$, provided that $|\varepsilon| \ll 1$. Hence **(b)** is true.

We recall that $w_{\varepsilon,\xi}$ satisfies

$$w_{\varepsilon,\xi} = \left(PI_0''\left(z_\xi\right)\right)^{-1}\left[\varepsilon PG'\left(z_\xi + w_{\varepsilon,\xi}\right) - PR_\xi\left(w_{\varepsilon,\xi}\right)\right], \tag{3.6}$$

where

$$G'\left(z_\xi + w_{\varepsilon,\xi}\right) = -\int_{\mathbb{R}^2} h(x)|z_\xi + w_{\varepsilon,\xi}|^{p-1}\left(z_\xi + w_{\varepsilon,\xi}\right)\,dx$$

and

$$R_\xi(w_{\varepsilon,\xi}) = -\frac{1}{\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-y|} z_\xi(x)w_{\varepsilon,\xi}(x)w_{\varepsilon,\xi}(y)v(y)\,dx\,dy$$
$$-\frac{1}{2\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-y|} w^2_{\varepsilon,\xi}(x)z_\xi(y)v(y)\,dx\,dy$$
$$-\frac{1}{2\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-y|} w^2_{\varepsilon,\xi}(x)w_{\varepsilon,\xi}(y)v(y)\,dx\,dy.$$

From (3.6) and Theorem 3.3 it follows

$$\|w_{\varepsilon,\xi}\|^2 \leq C\left[|\varepsilon|\left|\left(G'\left(z_\xi + w_{\varepsilon,\xi}|w_{\varepsilon,\xi}\right)\right)\right| + \left|\left(R_\xi(w_{\varepsilon,\xi})|w_{\varepsilon,\xi}\right)\right|\right]. \tag{3.7}$$

We infer that

$$\left|\left(G'\left(z_\xi + w_{\varepsilon,\xi}|w_{\varepsilon,\xi}\right)\right)\right| \leq \int_{\mathbb{R}^2} |h(x)||z_\xi + w_{\varepsilon,\xi}|^p|w_{\varepsilon,\xi}(x)|\,dx$$
$$\leq \|h\|_\infty\left(\int_{\mathbb{R}^2} |z_\xi|^p|w_{\varepsilon,\xi}|\,dx + \int_{\mathbb{R}^2} |w_{\varepsilon,\xi}|^{p+1}\,dx\right)$$

and this goes to 0 as $|\xi| \to \infty$ by the compact embedding of X in $L^s(\mathbb{R}^2)$ for all $s \geq 2$, **(a)** and **(b)** proved above.

Then, by (3.1) we find that

$$\left|\left(R_\xi(w_{\varepsilon,\xi})|w_{\varepsilon,\xi}\right)\right| \leq$$

$$\leq \frac{1}{\pi}\Big(\|z_\xi\|_X\|w_{\varepsilon,\xi}\|_X^2\|w_{\varepsilon,\xi}\|_X + \|z_\xi\|_X\|w_{\varepsilon,\xi}\|_X^2\|w_{\varepsilon,\xi}\|_X + C_1\|z_\xi\|_X\|w_{\varepsilon,\xi}\|_X^2\|w_{\varepsilon,\xi}\|_X\Big)$$
$$+ \frac{1}{2\pi}\Big(\|w_{\varepsilon,\xi}\|_X^2\|z_\xi\|_X\|w_{\varepsilon,\xi}\|_X + \|w_{\varepsilon,\xi}\|_X^2\|z_\xi\|_X\|w_{\varepsilon,\xi}\|_X + C_2\|w_{\varepsilon,\xi}\|_X^2\|z_\xi\|_X\|w_{\varepsilon,\xi}\|_X\Big)$$
$$+ \frac{1}{2\pi}\Big(\|w_{\varepsilon,\xi}\|_X^3\|w_{\varepsilon,\xi}\|_X + \|w_{\varepsilon,\xi}\|_X^3\|w_{\varepsilon,\xi}\|_X + C_3\|w_{\varepsilon,\xi}\|_X^3\|w_{\varepsilon,\xi}\|_X\Big)$$

thus,

$$\Big|\Big(R_\xi(w_{\varepsilon,\xi})|w_{\varepsilon,\xi}\Big)\Big| \leq C_4\|w_{\varepsilon,\xi}\|_X^4 + o(\varepsilon).$$

Inserting the above inequality in (3.7) we obtain that

$$\|w_{\varepsilon,\xi}\|_X^2 \leq C_4\|w_{\varepsilon,\xi}\|_X^4 + o(\varepsilon), \qquad \text{as } |\xi| \to \infty$$

and passing to the limit we find that

$$\lim_{|\xi|\to\infty}\|w_{\varepsilon,\xi}\|_X^2 \leq C_4\lim_{|\xi|\to\infty}\|w_{\varepsilon,\xi}\|_X^4.$$

Since $w_{\varepsilon,\xi}$ is small in X as $|\varepsilon| \to 0$ we conclude that

$$\lim_{|\xi|\to\infty}\|w_{\varepsilon,\xi}\|_X = 0 \quad \text{ provided } |\varepsilon| \ll 1.$$

□

Now we are ready to prove the existence of a global solution for the problem (2.1).

Proof of Theorem 1.4. Consider the reduced function $\Phi_\varepsilon(\xi) = I_\varepsilon\left(z_\xi + w_{\varepsilon,\xi}\right)$, namely

$$\Phi_\varepsilon(\xi) = \frac{1}{2}\|z_\xi + w_{\varepsilon,\xi}\|_{H^1(\mathbb{R}^2)}^2 - \frac{1}{8\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2}\log\frac{1}{|x-y|}\left(z_\xi + w_{\varepsilon,\xi}\right)^2(x)\left(z_\xi + w_{\varepsilon,\xi}\right)^2(y)\,dx\,dy$$
$$- \frac{\varepsilon}{p+1}\int_{\mathbb{R}^2} h(x)|z_\xi + w_{\varepsilon,\xi}|^{p+1}\,dx. \tag{3.8}$$

From $I_0(z_\xi) = \frac{1}{2}\|z_\xi\|_{H^1(\mathbb{R}^2)}^2 - \frac{1}{8\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2}\log\frac{1}{|x-y|}z_\xi^2(x)z_\xi^2(y)\,dx\,dy$ and setting $c_0 = I_0(z_\xi)$ we have that

$$\frac{1}{2}\|z_\xi\|_{H^1(\mathbb{R}^2)}^2 = c_0 + \frac{1}{8\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2}\log\frac{1}{|x-y|}z_\xi^2(x)z_\xi^2(y)\,dx\,dy.$$

Moreover, $-\Delta z_\xi + az_\xi - \frac{1}{2\pi}\left[\log\frac{1}{|\cdot|} * z_\xi^2\right]z_\xi = 0$ implies

$$\left(z_\xi|w_{\varepsilon,\xi}\right)_{H^1(\mathbb{R}^2)} = \frac{1}{2\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2}\log\frac{1}{|x-y|}z_\xi^2(x)z_\xi(y)w_{\varepsilon,\xi}(y)\,dx\,dy.$$

Substituting into (3.8) we obtain

$$\Phi_\varepsilon(\xi) = \frac{1}{2}\|z_\xi\|_{H^1(\mathbb{R}^2)}^2 + \frac{1}{2}\|w_{\varepsilon,\xi}\|_{H^1(\mathbb{R}^2)}^2 + \left(z_\xi|w_{\varepsilon,\xi}\right)_{H^1(\mathbb{R}^2)}$$

$$
\begin{aligned}
&- \frac{1}{8\pi} \int_{\mathbb{R}^2\times\mathbb{R}^2} \log \frac{1}{|x-y|} \left(z_\xi + w_{\varepsilon,\xi}\right)^2 (x) \left(z_\xi + w_{\varepsilon,\xi}\right)^2 (y)\, dx\, dy \\
&- \frac{1}{p+1} \int_{\mathbb{R}^2} h(x) |z_\xi + w_{\varepsilon,\xi}|^{p+1}\, dx \\
&= c_0 + \frac{1}{8\pi} \int_{\mathbb{R}^2\times\mathbb{R}^2} \log \frac{1}{|x-y|} z_\xi^2(x) z_\xi^2(y)\, dx\, dy + \frac{1}{2} \|w_{\varepsilon,\xi}\|^2_{H^1(\mathbb{R}^2)} \\
&+ \frac{1}{2\pi} \int_{\mathbb{R}^2\times\mathbb{R}^2} \log \frac{1}{|x-y|} z_\xi^2(x) z_\xi(y) w_{\varepsilon,\xi}(y)\, dx\, dy \\
&- \frac{1}{8\pi} \int_{\mathbb{R}^2\times\mathbb{R}^2} \log \frac{1}{|x-y|} \left(z_\xi + w_{\varepsilon,\xi}\right)^2 (x) \left(z_\xi + w_{\varepsilon,\xi}\right)^2 (y)\, dx\, dy \\
&- \frac{\varepsilon}{p+1} \int_{\mathbb{R}^2} h(x) |z_\xi + w_{\varepsilon,\xi}|^{p+1}\, dx,
\end{aligned}
$$

thus

$$
\begin{aligned}
\Phi_\varepsilon(\xi) = c_0 &+ \frac{1}{8\pi} \int_{\mathbb{R}^2\times\mathbb{R}^2} \log \frac{1}{|x-y|} z_\xi^2(x) z_\xi^2(y)\, dx\, dy \\
&+ \frac{1}{2} \|w_{\varepsilon,\xi}\|^2_{H^1(\mathbb{R}^2)} + \frac{1}{2\pi} \int_{\mathbb{R}^2\times\mathbb{R}^2} \log \frac{1}{|x-y|} z_\xi^2(x) z_\xi(y) w_{\varepsilon,\xi}(y)\, dx\, dy \\
&- \frac{1}{8\pi} \int_{\mathbb{R}^2\times\mathbb{R}^2} \log \frac{1}{|x-y|} z_\xi^2(x) z_\xi^2(y)\, dx\, dy \\
&- \frac{1}{4\pi} \int_{\mathbb{R}^2\times\mathbb{R}^2} \log \frac{1}{|x-y|} z_\xi^2(x) z_\xi(y) w_{\varepsilon,\xi}(y)\, dx\, dy \\
&- \frac{1}{8\pi} \int_{\mathbb{R}^2\times\mathbb{R}^2} \log \frac{1}{|x-y|} z_\xi^2(x) w_{\varepsilon,\xi}^2(y)\, dx\, dy \\
&- \frac{1}{4\pi} \int_{\mathbb{R}^2\times\mathbb{R}^2} \log \frac{1}{|x-y|} z_\xi(x) w_{\varepsilon,\xi}(x) z_\xi^2(y)\, dx\, dy \\
&- \frac{1}{2\pi} \int_{\mathbb{R}^2\times\mathbb{R}^2} \log \frac{1}{|x-y|} z_\xi(x) w_{\varepsilon,\xi}(x) z_\xi(y) w_{\varepsilon,\xi}(y)\, dx\, dy \\
&- \frac{1}{4\pi} \int_{\mathbb{R}^2\times\mathbb{R}^2} \log \frac{1}{|x-y|} z_\xi(x) w_{\varepsilon,\xi}(x) w_{\varepsilon,\xi}^2(y)\, dx\, dy \\
&- \frac{1}{8\pi} \int_{\mathbb{R}^2\times\mathbb{R}^2} \log \frac{1}{|x-y|} w_{\varepsilon,\xi}^2(x) z_\xi^2(y)\, dx\, dy \\
&- \frac{1}{4\pi} \int_{\mathbb{R}^2\times\mathbb{R}^2} \log \frac{1}{|x-y|} w_{\varepsilon,\xi}^2(x) z_\xi(y) w_{\varepsilon,\xi}(y)\, dx\, dy \\
&- \frac{1}{8\pi} \int_{\mathbb{R}^2\times\mathbb{R}^2} \log \frac{1}{|x-y|} w_{\varepsilon,\xi}^2(x) w_{\varepsilon,\xi}^2(y)\, dx\, dy - \frac{\varepsilon}{p+1} \int_{\mathbb{R}^2} h(x) |z_\xi + w_{\varepsilon,\xi}|^{p+1}\, dx,
\end{aligned}
$$

and after a short computation

$$
\begin{aligned}
\Phi_\varepsilon(\xi) = c_0 &+ \frac{1}{2} \|w_{\varepsilon,\xi}\|^2_{H^1(\mathbb{R}^2)} + \frac{1}{2\pi} \int_{\mathbb{R}^2\times\mathbb{R}^2} \log \frac{1}{|x-y|} z_\xi^2(x) z_\xi(y) w_{\varepsilon,\xi}(y)\, dx\, dy \\
&- \frac{1}{4\pi} \int_{\mathbb{R}^2\times\mathbb{R}^2} \log \frac{1}{|x-y|} z_\xi^2(x) z_\xi(y) w_{\varepsilon,\xi}(y)\, dx\, dy
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{8\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2}\log\frac{1}{|x-y|}z_\xi^2(x)w_{\varepsilon,\xi}^2(y)\,dx\,dy\\
&-\frac{1}{4\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2}\log\frac{1}{|x-y|}z_\xi(x)w_{\varepsilon,\xi}(x)z_\xi^2(y)\,dx\,dy\\
&-\frac{1}{2\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2}\log\frac{1}{|x-y|}z_\xi(x)w_{\varepsilon,\xi}(x)z_\xi(y)w_{\varepsilon,\xi}(y)\,dx\,dy\\
&-\frac{1}{4\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2}\log\frac{1}{|x-y|}z_\xi(x)w_{\varepsilon,\xi}(x)w_{\varepsilon,\xi}^2(y)\,dx\,dy\\
&-\frac{1}{8\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2}\log\frac{1}{|x-y|}w_{\varepsilon,\xi}^2(x)z_\xi^2(y)\,dx\,dy\\
&-\frac{1}{4\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2}\log\frac{1}{|x-y|}w_{\varepsilon,\xi}^2(x)z_\xi(y)w_{\varepsilon,\xi}(y)\,dx\,dy\\
&-\frac{1}{8\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2}\log\frac{1}{|x-y|}w_{\varepsilon,\xi}^2(x)w_{\varepsilon,\xi}^2(y)\,dx\,dy\\
&-\frac{\varepsilon}{p+1}\int_{\mathbb{R}^2}h(x)|z_\xi+w_{\varepsilon,\xi}|^{p+1}\,dx.
\end{aligned}
$$

Now, by repating the same arguments used in Proposition 2.5 all the integrals over $\mathbb{R}^2\times\mathbb{R}^2$ converge to 0.

Moreover, by Lemma 3.4 we have $||w_{\varepsilon,\xi}||^2_{H^1(\mathbb{R}^2)}\to 0$ as $|\xi|\to+\infty$ and by Minkowski's inequality, Proposition 2.4, Lemma 3.4 and $(\mathbf{h_2})$:

$$\lim_{|\xi|\to+\infty}\int_{\mathbb{R}^2}h(x)|z_\xi+w_{\varepsilon,\xi}|^{p+1}\,dx=0.$$

Hence,

$$\lim_{|\xi|\to+\infty}\Phi_\varepsilon(\xi)=c_0.$$

This means that either Φ_ε is constant, or it has a maximum or minimum. In any case Φ_ε has a critical point and we can apply Theorem 2.23 in [4] to find a solution for problem (2.1). □

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Conflict of interest

The authors declare no conflict of interest.

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