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Research article

Saddle-shaped positive solutions for elliptic systems with bistable nonlinearity†

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Abstract: In this paper we prove the existence of infinitely many saddle-shaped positive solutions for non-cooperative nonlinear elliptic systems with bistable nonlinearities in the phase-separation regime. As an example, we prove that the system

$$\begin{cases} -\Delta u = u - u^3 - \Lambda u v^2 \\ -\Delta v = v - v^3 - \Lambda u^2 v \\ u, v > 0 \end{cases} \qquad \text{in } \mathbb{R}^N, \text{ with } \Lambda > 1,$$

has infinitely many saddle-shape solutions in dimension 2 or higher. This is in sharp contrast with the case $\Lambda \in (0, 1]$, for which, on the contrary, only constant solutions exist.

Keywords: elliptic systems; entire solutions; saddle solutions; bistable nonlinearity; variational methods

1. Introduction

This paper concerns existence of multiple positive solutions for certain non-cooperative nonlinear elliptic systems with bistable nonlinearities, whose prototype is

$$\begin{cases} -\Delta u = u - u^3 - \Lambda u v^2 \\ -\Delta v = v - v^3 - \Lambda u^2 v \\ u, v > 0 \end{cases} \qquad \text{in } \mathbb{R}^N, \text{ with } \Lambda > 1. \tag{1.1}$$

This system arises in the study of domain walls and interface layers for two-components Bose-Einstein condensates [4]. Domain walls solutions satisfying asymptotic conditions

$$\begin{cases} (u, v) \to (1, 0) & \text{as } x_N \to +\infty, \\ (u, v) \to (0, 1) & \text{as } x_N \to -\infty, \end{cases}, \tag{1.2}$$

in dimension $N = 1$ have been carefully studied in [2, 4], where in particular it is shown the existence of such a solution for every $\Lambda > 1$ [4], and its uniqueness in the class of solutions with one monotone component [2]. In fact, uniqueness holds also without such assumption, and even in higher dimension [9]; precisely, in [9] it is shown that a solution to (1.1)–(1.2) (with the limits being uniform in $x' \in \mathbb{R}^{N-1}$) in $\mathbb{R}^N$ with $\Lambda > 1$ is necessarily montone in both the components with respect to x_N, and 1-dimensional. The assumption $\Lambda > 1$ is natural, since (1.1)–(1.2) has no solution at all when $\Lambda \in (0, 1]$. Indeed, it is proved that (1.1) has only constant solutions for both $\Lambda \in (0, 1)$ [9], and $\Lambda = 1$ [9, 13].

We also refer to [1, 3] for recent results regarding a system obtained from (1.1) adding in each equation an additional term representing the spin coupling.

To sum up, up to now it is known that (1.1) has only constant solutions for $\Lambda \in (0, 1]$, and at least one 1-dimensional non-constant solution for $\Lambda > 1$. Moreover, solutions with uniform limits as in (1.2) are necessarily 1-dimensional, and unique modulo translations. In this paper we prove the existence of infinitely many geometrically distinct solutions to (1.1) in any dimension $N \geq 2$, for any $\Lambda > 1$. This result enlightens once more the dichotomy $\Lambda \in (0, 1]$ vs. $\Lambda > 1$. While for $\Lambda \in (0, 1]$ problem (1.1) is rigid in itself, and only possesses constant solutions, for $\Lambda > 1$ we have multiplicity of non-constant solutions, and rigidity results can be recovered only with some extra assumption, such as (1.2).

Our result is based upon variational methods, and strongly exploits the symmetry of the problem. Roughly speaking, we shall construct solutions to (1.1) such that $u - v$ "looks like" a sing changing solution of the Allen-Cahn equation $-\Delta w = w - w^3$, with $u \simeq w^+$, and $v \simeq w^-$. The building blocks w in our construction will be both the saddle-type planar solutions (also called "pizza solutions") [5], and the saddle solutions in $\mathbb{R}^{2m}$ [7, 8].

1.1. Statement of the main results

We consider the following general version of (1.1):

$$\begin{cases} -\Delta u = f(u) - \Lambda u^p v^{p+1} & \text{in } \mathbb{R}^N \\ -\Delta v = f(v) - \Lambda u^{p+1} v^p & \text{in } \mathbb{R}^N \\ u, v > 0 & \text{in } \mathbb{R}^N, \end{cases} \qquad \text{with } \Lambda > 0, \tag{1.3}$$

where $N \geq 2$, $p \geq 1$, and f is of bistable type; more precisely, let $f : \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz continuous and odd nonlinearity. For a value $M > 0$, we define the potential

$$F(t) = \int_t^M f(s)\, ds,$$

so that $F \in C^{1,1}(\mathbb{R})$, and $F' = -f$. We suppose that:

$$F \geq 0 = F(\pm M) \quad \text{in } \mathbb{R}, \quad \text{and} \quad F > 0 \quad \text{in } (-M, M). \tag{1.4}$$

Note that in this case $f(0) = f(\pm M) = 0$. F is often called a double well potential, and f is called bistable nonlinearity. A simple example is $f(t) = t - t^3$.

With the above notation, we introduce

$$W(s,t) = F(s) + F(t) + \frac{\Lambda}{p+1}|s|^{p+1}|t|^{p+1}, \qquad (s,t) \in \mathbb{R}^2.$$

The first of our main result concerns the existence of infinitely many geometrically distinct solutions for problem (1.3) in the plane. We consider polar coordinates $(r,\theta) \in [0,+\infty) \times [0,2\pi)$ in the plane. For any positive integer k, we define:

R_k, the rotation of angle π/k in counterclockwise sense;
R_k^i, the rotation of angle $i\pi/k$ in counterclockwise sense, with $i = 1, \dots, 2k$;
ℓ_0, the line of equation $x_2 = \tan(\pi/2k)x_1$ in $\mathbb{R}^2$;
ℓ_i, the line $R_k^i(\ell_1)$, $i = 1, \dots, k-1$;
T_i, the reflection with respect to ℓ_i;
$\alpha_k = \tan(\pi/(2k))$;
$\mathcal{S}_k$, the open circular sector $\{-\pi/(2k) < \theta < \pi/(2k)\} = \{\alpha_k x_1 > |x_2|\} \subset \mathbb{R}^2$.

Theorem 1.1 (Saddle-type solutions in the plane)**.** *Let $p \geq 1$, $f \in C^{0,1}(\mathbb{R})$ be odd, and suppose that its primitive F satisfies* (1.4)*. Suppose moreover that:*

$$\inf_{s\in[0,M]} W(s,s) > F(0). \tag{1.5}$$

Then, for every positive integer k, there exists a positive solution (u_k, v_k) to system (1.3) *in $\mathbb{R}^2$ having the following properties:*

(i) $0 < u_k, v_k < M$ *in* $\mathbb{R}^2$*;*
(ii) $v_k = u_k \circ T_i$ *for every* $i = 1, \dots, k$, *and* $u_k(x_1, -x_2) = u_k(x_1, x_2)$ *in* $\mathbb{R}^2$*;*
(iii) $u_k - v_k > 0$ *in* $\mathcal{S}_k$.

Notice in particular that $\{u_k - v_k = 0\} = \bigcup_{i=0}^{k-1} \ell_i$, which implies that $(u_k, v_k) \neq (u_j, v_j)$ if $j \neq k$. Regarding assumption (1.5), we stress that for any bistable f it is satisfied provided that $\Lambda > 0$ is sufficiently large, and can be explicitly checked in several concrete situations. In particular:

Corollary 1.2. *For every $\Lambda > 1$, problem* (1.1) *has infinitely many geometrically distinct non-constant solutions.*

The corollary follows from the theorem, observing that if $f(s) = s - s^3$ and $p = 1$, then

$$F(s) = \frac{(1-s^2)^2}{4}, \quad W(s,t) = F(s) + F(t) + \frac{\Lambda s^2 t^2}{2};$$

thus, condition (1.5) is satisfied if and only if

$$\inf_{s\in[0,1]} W(s,s) = W\left(\frac{1}{\sqrt{1+\Lambda}}, \frac{1}{\sqrt{1+\Lambda}}\right) = \frac{\Lambda}{2(1+\Lambda)} > \frac{1}{4} = F(0),$$

that is, if and only if $\Lambda > 1$. Notice that, if (1.5) is violated, we have have non-existence of non-constant solutions [9, 13], and hence (1.5) is sharp in this case.

The proof of Theorem 1.1 consists in a 2 steps procedure. At first, we construct a solution to (1.3) in a ball B_R with the desired symmetry properties, combining variational methods with an auxiliary parabolic problem. In a second step, we pass to the limit as $R \to +\infty$, obtaining convergence to an entire solution of (1.3). Assumption (1.5) enters in this second step in order to rule out the possibility that the limit profile (u, v) is a pair with $v = u$, with u possibly a constant. Roughly speaking, (1.5) makes the coexistence of u and v in the same region unfavorable with respect to the segregation, from the variational point of view.

This kind of construction is inspired by [6, 11, 12], where an analogue strategy was used to prove existence of solutions to

$$\Delta u = uv^2, \quad \Delta v = u^2 v, \quad u, v > 0 \text{ in } \mathbb{R}^N. \tag{1.6}$$

With respect to [6, 11, 12], however, the method has to be substantially modified. Solutions to (1.6) "look like" harmonic function in the same way as solutions to (1.3) "look like" solutions to the Allen-Cahn equation. Therefore, tools related with harmonic functions such as monotonicity formulae and blow-up analysis, which were crucially used in [6, 11, 12], are not available in our context, and have to be replaced by a direct inspection of the variational background. In such an inspection it emerges the role of the competition parameter Λ, which is not present in (1.6) (of course, Λ could be added in front of the coupling on the right hand side in (1.6); but it could be absorbed with a scaling, and hence it would not play any role).

Remark 1.3. Let us consider the scalar equation $\Delta w + f(w) = 0$. The existence of a *saddle-type* (or pizza) solution w_k with the properties

(i) $-M < w_k < M$ in $\mathbb{R}^2$;
(ii) $w_k \circ T_i = -w_k$ for every $i = 1, \dots, k$, and $w_k(x_1, -x_2) = w_k(x_1, x_2)$ in $\mathbb{R}^2$;
(iii) $w_k > 0$ in $\mathcal{S}_k$.

was established by Alessio, Calamai and Montecchiari in [5], under slightly stronger assumption on F with respect to those considered here. In some sense, $u_k - v_k$ looks like w_k, since they share the same symmetry properties, and for this reason we can call (u_k, v_k) saddle-type (or pizza) solution.

Our method for Theorem 1.1 can be easily adapted also in the scalar case, giving an alternative proof for the existence result in [5]. For the sake of completeness, we present the details in the appendix of this paper. The main advantage is that our construction easily gives the following energy estimate

$$\int_{B_R} \left(\frac{1}{2} |\nabla w_k|^2 + F(w_k) \right) \le CR, \tag{1.7}$$

with $C > 0$ depending only on k, but not on R. Such estimate seems to be unknown, expect for the case $k = 2$, where it was proved in [7].

Theorem 1.1 establishes the existence of infinitely many positive solutions to (1.3) in the plane. These can be regarded as solutions also in higher dimension $N \ge 3$, but it is natural to ask whether there exist solutions to (1.3) in $\mathbb{R}^N$ not coming from solutions in $\mathbb{R}^{N-1}$. We can give a positive answer to this question in any even dimension. Let $N = 2m$, and let us consider the *Simons cone*

$$C = \left\{ x \in \mathbb{R}^{2m} : \ x_1^2 + \cdots + x_m^2 = x_{m+1}^2 + \cdots + x_{2m}^2 \right\}.$$

We define two radial variables s and t by

$$s = \sqrt{x_1^2 + \cdots + x_m^2} \geq 0, \qquad t = \sqrt{x_{m+1}^2 + \cdots + x_{2m}^2} \geq 0. \tag{1.8}$$

Theorem 1.4 (Saddle solutions in $\mathbb{R}^{2m}$)**.** *Let $m \geq 2$ be a positive integer, $p \geq 1$, $f \in C^{0,1}(\mathbb{R})$ be odd, and suppose that its primitive F satisfies* (1.4)*. Suppose moreover that* (1.5) *holds. Then, for every positive integer m, there exists a positive solution (u, v) to system* (1.3) *in $\mathbb{R}^{2m}$ having the following properties:*

(i) $0 < u, v < M$ *in* $\mathbb{R}^{2m}$*;*
(ii) $v(s, t) = u(t, s)$*;*
(iii) $u - v > 0$ *in* $O = \{s > t\}$.

Notice that $\{u - v = 0\} = C$, and that (u, v) looks like the saddle solution of the scalar Allen-Cahn equation found in [7]. The strategy of the proof is the same as the one of Theorem 1.4. However, the proof of Theorem 1.4 is a bit simpler, since we can take advantage of an energy estimate like (1.7), which is known to hold for saddle solutions in $\mathbb{R}^{2m}$ (see formula (1.15) in [7]) but, as already observed, was unknown for saddle-type solutions in the plane.

Structure of the paper. In Section 2 we prove Theorem 1.1. In Section 3 we prove Theorem 1.4. In the appendix, we give an alternative proof with respect to [5] of the existence of saddle-type solutions for the scalar equation in the plane, yielding to the energy estimate (1.7).

2. Saddle-type solutions for bistable systems in the plane

In this section we prove Theorem 1.1. The existence of a solution in the whole plane $\mathbb{R}^2$ will be obtained by approximation with solutions in B_R.

Throughout this section, the positive integer k (index of symmetry) will always be fixed, and hence the dependence of the quantities with respect to k will often be omitted.

In the sector $\mathcal{S} = \mathcal{S}_k$, we define

$$w_k = \min\left\{M, \frac{\alpha x_1 - |x_2|}{\sqrt{2}}\right\},$$

where $\alpha = \alpha_k = \tan(\pi/(2k))$. Notice that $w_k > 0$ in $\mathcal{S}_k$ and $w_k = 0$ on $\partial\mathcal{S}_k$. Thus, we can extend w_k in the whole of $\mathbb{R}^2$ by iterated odd reflections with respect to the lines ℓ_i. In this way, we obtain a function, still denoted by w_k, defined in $\mathbb{R}^2$, with

(i) $-M \leq w_k \leq M$ in $\mathbb{R}^2$;
(ii) $w_k \circ T_i = -w_k$ for every $i = 1, \ldots, k$, and $w_k(x_1, -x_2) = w_k(x_1, x_2)$ in $\mathbb{R}^2$;
(iii) $w_k > 0$ in $\mathcal{S}_k$,

that is, w_k has the same symmetry properties of the saddle-type solutions in [5].

Now, for any $\Omega \subset \mathbb{R}^2$ open, and for every $(u, v) \in H^1(\Omega, \mathbb{R}^2)$, we introduce the functional

$$J((u, v), \Omega) := \int_\Omega \left(\frac{1}{2}|\nabla u|^2 + \frac{1}{2}|\nabla v|^2 + W(u, v)\right). \tag{2.1}$$

Moreover, for $R > 0$, we let $\mathcal{S}_R = \mathcal{S}_k \cap B_R$ and consider the set

$$A_R := \left\{ (u,v) \in H^1(B_R,\mathbb{R}^2) \;\middle|\; \begin{array}{l} (u,v) = (w_k^+, w_k^-) \text{ on } \partial B_R,\ 0 \le u,v \le M \quad \text{in } B_R \\ v = u \circ T_i \text{ for every } i = 1,\dots,k, \\ u(x_1,-x_2) = u(x_1,x_2) \text{ in } B_R,\ u \ge v \quad \text{in } \mathcal{S}_R, \end{array} \right\}.$$

Notice that $A_R \neq \emptyset$, since for instance $(w_k^+, w_k^-) \in A_R$.

Lemma 2.1. *For every $R > 0$, there exists a solution $(u_R, v_R) \in A_R$ to*

$$\begin{cases} -\Delta u = f(u) - \Lambda |u|^{p-1} u |v|^{p+1} & \text{in} \quad B_R \\ -\Delta v = f(v) - \Lambda |u|^{p+1} |v|^{p-1} v & \text{in} \quad B_R \\ u = w_k^+, \quad v = w_k^- & \text{on } \partial B_R. \end{cases} \tag{2.2}$$

Proof. The proof of the lemma is inspired by [6, Theorem 4.1]. Since the weak convergence in H^1 implies the almost everywhere convergence, up to a subsequence, the set A_R is weakly closed in H^1. Moreover, the functional $J(\cdot\,, B_R)$ is clearly bounded from below and weakly lower semi-continuous. Therefore, there exists a minimizer (u_R, v_R) of $J(\cdot\,, B_R)$ in A_R. To show that such a minimizer is a solution to (3.2), we consider the auxiliary parabolic problem

$$\begin{cases} \partial_t U - \Delta U = \tilde{f}(U) - \Lambda |U|^{p-1} U |V|^{p+1} & \text{in } (0,+\infty) \times B_R \\ \partial_t V - \Delta V = \tilde{f}(V) - \Lambda |U|^{p+1} |V|^{p-1} V & \text{in } (0,+\infty) \times B_R \\ U = w_k^+, \quad V = w_k^- & \text{on } (0,+\infty) \times \partial B_R \\ (U(0,\cdot), V(0,\cdot)) \in A_R, \end{cases} \tag{2.3}$$

where $\tilde{f} : \mathbb{R} \to \mathbb{R}$ is a globally Lipschitz continuous odd function such that $\tilde{f}(s) = f(s)$ for $s \in [-M-1, M+1]$. The existence and uniqueness of a local solution, defined on a maximal time interval $[0,T)$, follow by standard parabolic theory. Notice that

$$\partial_t U - \Delta U = c_1(t,x) U,$$

for

$$c_1(t,x) = \begin{cases} -\Lambda |U(t,x)|^{p-1} |V(t,x)|^{p+1} + \frac{\tilde{f}(U(t,x))}{U(t,x)} & \text{if } U(t,x) \neq 0 \\ 0 & \text{if } U(t,x) = 0. \end{cases}$$

Since $\tilde{f}(0) = 0$, we have that c_1 is bounded from above by the Lipschitz constant L of $\tilde{f}$, and it is not difficult to check that $U(t,\cdot) \ge 0$ in B_R for every $t \in [0,T)$: indeed, taking into account the boundary conditions,

$$\begin{aligned} \frac{d}{dt}\left(\frac{1}{2} \int_{B_R} (U^-)^2 \right) &= -\int_{B_R} U^- (\partial_t U) = -\int_{B_R} U^- (\Delta U + c_1(t,x) U) \\ &\le -\int_{B_R} |\nabla U^-|^2 + L \int_{B_R} (U^-)^2 \le L \int_{B_R} (U^-)^2, \end{aligned}$$

whence it follows that

$$\frac{d}{dt}\left(e^{-2Lt} \int_{B_R} (U^-)^2 \right) \le 0.$$

Therefore, the non-negativity of $U(t,\cdot)$ for $t \in (0,T)$ follows from the non-negativity of $U(0,\cdot)$. The same argument also shows that $V(t,\cdot) \geq 0$ for every such t. Using the positivity of U, it is not difficult to prove that U is also uniformly bounded from above: since $\tilde{f}(M) = 0$, we have

$$\partial_t(M-U) - \Delta(M-U) \geq c_2(t,x)(M-U),$$

where c_2 is the bounded function

$$c_2(t,x) = \begin{cases} \frac{\tilde{f}(U(t,x))-\tilde{f}(M)}{U(t,x)-M} & \text{if } U(t,x) \neq 0 \\ 0 & \text{if } U(t,x) = M, \end{cases}$$

and the same argument used above implies that $0 \leq U \leq M$ on $(0,T) \times B_R$. Similarly, $0 \leq V \leq M$. As a consequence, the solution (U,V) can be globally continued in time on $(0,+\infty)$. Furthermore, in (2.3) we can replace $\tilde{f}$ with f, since they coincide on $[-M-1, M+1]$.

We also observe that, since U is constant in time on ∂B_R, the energy of the solution is non-increasing:

$$\begin{aligned} \frac{d}{dt} J((U(t,\cdot),V(t,\cdot));B_R) &= \int_{B_R} \nabla U \cdot \nabla U_t + \nabla V \cdot \nabla V_t + \partial_1 W(U,V) U_t + \partial_2 W(U,V) V_t \\ &= \int_{B_R} (-\Delta U + \partial_1 W(U,V))\, U_t + (-\Delta V + \partial_2 W(U,V))\, V_t \\ &= -\int_{B_R} U_t^2 + V_t^2 \leq 0. \end{aligned} \tag{2.4}$$

As in [6], we can now show that A_R is positively invariant under the parabolic flow. Let (U,V) be a solution with initial datum in A_R. By the symmetry of (2.3), we have that $(V(t,T_i x), U(t,T_i x))$ is still a solution. By the symmetry of initial and boundary data, and by uniqueness, such solution must coincide with $(U(t,\cdot),V(t,\cdot))$. This means in particular that $V(t,x) = U(t,T_i x)$. Likewise, $U(t,x_1,-x_2) = U(t,x_1,x_2)$. Notice that the symmetries imply that $U - V = 0$ on $\partial\mathcal{S}_k$. Thus, recalling that $\mathcal{S}_R = \mathcal{S}_k \cap B_R$, we have

$$\begin{cases} \partial_t(U-V) - \Delta(U-V) = c(t,x)(U-V) & \text{in } (0,+\infty) \times \mathcal{S}_R \\ U - V \geq 0 & \text{on } (0,+\infty) \times \partial\mathcal{S}_R \\ U - V \geq 0 & \text{on } \{0\} \times \mathcal{S}_R, \end{cases} \tag{2.5}$$

where the bounded function c is defined by

$$c(t,x) = \begin{cases} \frac{f(U(t,x))-f(V(t,x))}{U(t,x)-V(t,x)} + \Lambda U^p(t,x)V^p(t,x) & \text{if } U(t,x) \neq V(t,x) \\ \Lambda U^p(t,x)V^p(t,x) & \text{if } U(t,x) = V(t,x). \end{cases}$$

The parabolic maximum principle implies that $U \geq V$ in $\mathcal{S}_R$ globally in time, and, in turn, this gives the invariance of A_R.

At this point we consider the solution (U_R, V_R) to (2.3) with initial datum (u_R, v_R), minimizer of $J(\cdot\,, B_R)$ in A_R. By minimality in A_R and by (2.4), we have

$$J((u_R,v_R);B_R) \leq J((U_R(t,\cdot),V_R(t,\cdot));B_R) \leq J((u_R,v_R);B_R) \qquad \Longrightarrow \qquad U_t^2 + V_t^2 \equiv 0,$$

and hence $U_R \equiv u_R$ and $V_R \equiv v_R$. But then $(u_R(x), v_R(x))$ is a (stationary) solution of the parabolic problem (2.3), that is, it solves the stationary problem (2.2), and in addition $(u_R, v_R) \in A_R$. This completes the proof. □

We are ready to complete the:

Proof of Theorem 1.1. First, of all, we discuss the convergence of $\{(u_R, v_R) : R > 1\}$. Let $\rho > 1$. Since $0 \le u_R, v_R \le M$, we have that

$$|\Delta u_R(x)| \le \max_{s\in[0,M]} |f(s)| + \Lambda M^{2p+1}, \quad |\Delta v_R(x)| \le \max_{s\in[0,M]} |f(s)| + \Lambda M^{2p+1}.$$

Thus interior L^p estimates (see e.g. [10, Chapter 9]), applied in balls of radius 2 centered in points of $\overline{B_\rho}$ with $p > N$, and the Morrey embedding theorem, imply that there exists $C > 0$ depending only on M and Λ (but independent of R and ρ) such that

$$\|u_R\|_{C^{1,\alpha}(\overline{B_\rho})} + \|v_R\|_{C^{1,\alpha}(\overline{B_\rho})} \le C \qquad \text{in } B_\rho, \text{ for all } R > \rho + 2 \tag{2.6}$$

(for every $0 < \alpha < 1$). By the Ascoli-Arzelà theorem, up to a subsequence $\{(u_R, v_R)\}$ converges in $C^{1,\alpha}(\overline{B_\rho})$ to a solution in B_ρ, for every $0 < \alpha < 1$. Taking a sequence $\rho \to +\infty$, a diagonal selection finally gives $(u_R, v_R) \to (u, v)$ in $C^{1,\alpha}_{\text{loc}}(\mathbb{R}^2)$, up to a subsequence, with (u, v) solution to

$$\begin{cases} -\Delta u = f(u) - \Lambda u^p v^{p+1} & \text{in } \mathbb{R}^2 \\ -\Delta v = f(v) - \Lambda u^{p+1} v^p & \text{in } \mathbb{R}^2 \\ 0 \le u, v \le M & \text{in } \mathbb{R}^2. \end{cases}$$

Notice that, by convergence, (u, v) satisfies the symmetry property (*ii*) in Theorem 1.1, and moreover $u - v \ge 0$ in $\mathcal{S}_k$. As in (2.5), for any $\rho > 0$

$$\begin{cases} -\Delta(u - v) = c(x)(u - v) & \text{in } \mathcal{S}_\rho \\ u - v \ge 0 & \text{in } \mathcal{S}_\rho, \end{cases}$$

for a bounded function c. Thus, the strong maximum principle implies that either $u > v$ in $\mathcal{S}_\rho$, of $u \equiv v$ in $\mathcal{S}_\rho$. Since $\rho > 0$ is arbitrarily chosen, we have that either $u > v$ in $\mathcal{S}$, of $u \equiv v$ in $\mathcal{S}$. We claim that the latter alternative cannot take place. To prove this claim, we use a comparison argument similar as the one by Cabré and Terra in [7] for the construction of the saddle solution for scalar bystable equations. First of all, we observe that, by symmetry, any function in A_R is determined only by its restriction on $\mathcal{S}_k$. Thus the minimality of (u_R, v_R) can be read as

$$J((u_R, v_R), \mathcal{S}_R) \le J((u, v), \mathcal{S}_R) \qquad \forall (u, v) \in A_R.$$

Let $1 < \rho < R - 2$, and let ξ be a radial smooth cut-off function with $\xi \equiv 1$ in $B_{\rho-1}$, $\xi \equiv 0$ in B_ρ^c, $0 \le \xi \le 1$. We define

$$\varphi_R(x) = \xi(x) w_k^+(x) + (1 - \xi(x)) u_R(x) = \xi(x) \min\left\{M, \frac{\alpha x_1 - |x_2|}{\sqrt{2}}\right\} + (1 - \xi(x)) u_R(x),$$

and

$$\psi_R(x) = \xi(x)w_k^-(x) + (1-\xi(x))v_R(x) = (1-\xi(x))v_R(x),$$

where we recall that $\alpha = \tan(\pi/(2k))$. It is immediate to verify that (φ_R, ψ_R) is an admissible competitor for (u_R, v_R) on $\mathcal{S}_R$. Moreover, by (2.6), there exists $C > 0$ such that

$$\|\varphi_R\|_{W^{1,\infty}(B_\rho)} + \|\psi_R\|_{W^{1,\infty}(B_\rho)} \le C \qquad \forall R > \rho + 2. \tag{2.7}$$

By minimality

$$J((u_R, v_R), \mathcal{S}_R) \le J((\varphi_R, \psi_R), \mathcal{S}_R),$$

and since $(\varphi_R, \psi_R) = (u_R, v_R)$ in $\mathcal{S}_R \setminus \mathcal{S}_\rho$ we deduce that

$$J((u_R, v_R), \mathcal{S}_\rho) \le J((\varphi_R, \psi_R), \mathcal{S}_\rho) \le J((\varphi_R, 0), \mathcal{S}_{\rho-1}) + C|\mathcal{S}_\rho \setminus \mathcal{S}_{\rho-1}| \tag{2.8}$$

where we used the global boundedness of $\{(\varphi_R, \psi_R)\}$ in $W^{1,\infty}(B_\rho)$, see (2.7). The last term can be easily computed as

$$|\mathcal{S}_\rho \setminus \mathcal{S}_{\rho-1}| = \frac{\pi}{k}(\rho^2 - (\rho-1)^2) \le \frac{2\pi}{k}\rho.$$

For the first term, recalling that $F(M) = 0$, $\xi \equiv 1$ in $B_{\rho-1}$, and $w_k > 0$ in $\mathcal{S}_k$, we have

$$\begin{aligned}
&\int_{\mathcal{S}_{\rho-1}} \left(\frac{1}{2}|\nabla\varphi_R|^2 + F(\varphi_R) + F(0)\right) \\
&\qquad = \int_{\mathcal{S}_{\rho-1}\cap\left\{\alpha x_1 - |x_2| < \sqrt{2}M\right\}} \left(\frac{1}{2}|\nabla w_k|^2 + F(w_k)\right) + \int_{\mathcal{S}_{\rho-1}} F(0) \\
&\qquad \le C\left|\mathcal{S}_{\rho-1} \cap \left\{\alpha x_1 - |x_2| < \sqrt{2}M\right\}\right| + F(0)|\mathcal{S}_{\rho-1}|.
\end{aligned}$$

The set

$$\mathcal{S}_k \cap \left\{\alpha x_1 - |x_2| < \sqrt{2}M\right\}$$

is contained in the (non-disjoint) union of the two strips

$$\left\{\alpha x_1 - \sqrt{2}M < x_2 < \alpha x_1,\ x_1 > 0\right\} \cup \left\{-\alpha x_1 < x_2 < 2\sqrt{M} - \alpha x_1,\ x_1 > 0\right\} = S_1 \cup S_2.$$

Therefore,

$$\begin{aligned}
\left|\mathcal{S}_{\rho-1} \cap \left\{\alpha x_1 - |x_2| < \sqrt{2}M\right\}\right| &\le |S_1 \cap \{0 < x_1 < \rho\}| + |S_2 \cap \{0 < x_1 < \rho\}| \\
&= 2\int_0^\rho \left(\int_{-\sqrt{2}M+\alpha x_1}^{\alpha x_1} 1\, dx_2\right) dx_1 = 2\sqrt{2}M\rho.
\end{aligned}$$

Coming back to (2.8), we conclude that there exists a constant $C > 0$ such that, for every $\rho > 1$ and $R > \rho + 2$,

$$J((u_R, v_R), \mathcal{S}_\rho) \le C\rho + F(0)|\mathcal{S}_{\rho-1}|$$

for every $1 < \rho < R - 2$, where $C > 0$ is a positive constant independent of both ρ and R. Passing to the limit as $R \to +\infty$, we infer by C^1_{loc}-convergence that

$$J((u, v), \mathcal{S}_\rho) \le C\rho + F(0)|\mathcal{S}_{\rho-1}| \tag{2.9}$$

for every $\rho > 1$. Notice that, in this estimate, the leading term as $\rho \to +\infty$ is

$$F(0)|\mathcal{S}_{\rho-1}| \sim \frac{\pi}{k} F(0)\rho^2.$$

Suppose now by contradiction that $u \equiv v$ in $\mathcal{S}_k$. Recalling that $0 \le u, v \le M$, we have that

$$\begin{aligned} J((u,v),\mathcal{S}_\rho) &= \int_{\mathcal{S}_\rho} |\nabla u|^2 + W(u,u) \ge \int_{\mathcal{S}_\rho} \min_{s\in[0,M]} W(s,s) \\ &= \min_{s\in[0,M]} W(s,s)|\mathcal{S}_\rho| \sim \frac{\pi}{k} \min_{s\in[0,M]} W(s,s)\rho^2 \end{aligned}$$

as $\rho \to +\infty$. Comparing with (2.9), we obtain a contradiction for large ρ, thanks to assumption (1.5). Therefore, $u > v$ in $\mathcal{S}_k$. Since $u = v$ on $\partial\mathcal{S}_k$, we also infer that both u and v cannot be constant. The maximum principle implies then that $u, v > 0$ in $\mathbb{R}^2$, and from this it is not difficult to deduce that $u, v < M$: indeed, if $u(x_0) = M$, then x_0 is a strict maximum point for u with

$$\Delta u(x_0) = -f(M) + \Lambda M^p v(x_0)^{p+1} = \Lambda M^p v(x_0)^{p+1} > 0,$$

which is not possible. This completes the proof. □

3. Existence of saddle solutions in higher dimension

The proof of Theorem 1.4 follows the same strategy as the one of Theorem 1.1, being actually a bit simpler. Let $m \ge 2$ be a positive integer. By [7, Theorem 1.3]*, under our assumption (1.4) on F the Allen-Cahn equation $\Delta w + f(w) = 0$ in $\mathbb{R}^{2m}$ admits a saddle solution w_m, that is a solution satisfying:

(*i*) w depends only on the variables s and t defined in (1.8);
(*ii*) $w_m(s,t) = -w_m(t,s)$;
(*iii*) $w_m > 0$ in $\mathcal{O} = \{s > t\}$.

In addition, $|w_m| < M$ in $\mathbb{R}^{2m}$, and

$$\int_{B_R} \frac{1}{2}|\nabla w_m|^2 + F(w_m) \le CR^{2m-1} \qquad \text{for all } R > 1, \tag{3.1}$$

Now, as in Section 2, we consider the energy functional $J((u,v),\Omega)$ defined in (2.1) (in this section $\Omega \subset \mathbb{R}^{2m}$), and the set

$$A_R := \left\{ (u,v) \in H^1(B_R,\mathbb{R}^2) \;\middle|\; \begin{array}{l} (u,v) = (w_m^+, w_m^-) \text{ on } \partial B_R, \\ v(s,t) = u(t,s), \\ u \ge v \quad \text{in } \mathcal{O}_R,\ 0 \le u,v \le M \quad \text{in } B_R \end{array} \right\},$$

where $\mathcal{O}_R = \mathcal{O} \cap B_R$.

Lemma 3.1. *For every $R > 0$, there exists a solution $(u_R, v_R) \in A_R$ to*

$$\begin{cases} -\Delta u = f(u) - \Lambda |u|^{p-1}u|v|^{p+1} & \text{in } \quad B_R \\ -\Delta v = f(v) - \Lambda |u|^{p+1}|v|^{p-1}v & \text{in } \quad B_R \\ u = w_m^+, \quad v = w_m^- & \text{on } \partial B_R. \end{cases} \tag{3.2}$$

*For the existence and the energy estimate in the theorem, it is sufficient that f is locally Lipschitz, rather than C^1

The proof is analogue to the one of Lemma 2.1, and is omitted.

Proof of Theorem 1.4. As in the 2-dimensional case, we can prove that up to a subsequence $(u_R, v_R) \to (u, v)$ in $C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^2)$ as $R \to +\infty$, with (u, v) solution to

$$\begin{cases} -\Delta u = f(u) - \Lambda u^p v^{p+1} \\ -\Delta v = f(v) - \Lambda u^{p+1} v^p & \text{in } \mathbb{R}^{2m}. \\ 0 \leq u, v \leq M \end{cases}$$

By convergence, $v(s,t) = u(t,s)$, $u - v \geq 0$ in O, and $0 \leq u, v \leq M$ in $\mathbb{R}^{2m}$. Also, for every $\rho > 0$

$$\begin{cases} -\Delta(u - v) = c(x)(u - v) & \text{in } O_\rho \\ u - v \geq 0 & \text{in } O_\rho, \end{cases}$$

for a bounded function c. Thus, the strong maximum principle implies that either $u > v$ in O, of $u \equiv v$ in O. We claim that the latter alternative cannot take place. Let $1 < \rho < R - 2$, and let ξ be a radial smooth cut-off function with $\xi \equiv 1$ in $B_{\rho-1}$, $\xi \equiv 0$ in B^c_ρ, $0 \leq \xi \leq 1$. We define

$$\varphi_R = \xi w_m^+ + (1 - \xi) u_R, \quad \psi_R = \xi w_m^- + (1 - \xi) v_R.$$

This is an admissible competitor in A_R, which coincides with (u_R, v_R) on $B_R \setminus B_\rho$. Therefore, by minimality and recalling (3.1), we have

$$\begin{aligned} J((u_R, v_R), B_\rho) &\leq J((w_m^+, w_m^-), B_{\rho-1}) + C|B_\rho \setminus B_{\rho-1}| \\ &\leq E(w_m, B_{\rho-1}) + \int_{B_{\rho-1}} F(0) + C\rho^{2m-1} \leq C\rho^{2m-1} + F(0)|B_{\rho-1}|. \end{aligned} \tag{3.3}$$

If, by contradiction, $u \equiv v$ in O, then we have that

$$J((u, v), B_\rho) = \int_{B_\rho} |\nabla u|^2 + W(u, u) \geq \int_{B_\rho} \min_{\sigma \in [0,M]} W(\sigma, \sigma) = \min_{\sigma \in [0,M]} W(\sigma, \sigma)|B_\rho|.$$

Comparing with (2.9), we obtain a contradiction for large ρ, thanks to assumption (1.5). Thus, $u > v$ in O, and the conclusion of the proof is straightforward. □

A. Alternative construction of saddle-type planar solutions

In this appendix we consider the scalar Allen-Cahn equation

$$-\Delta w = f(w) \qquad \text{in } \mathbb{R}^2, \tag{A.1}$$

and we prove the following result:

Theorem A.1. *Let $f \in C^{0,\alpha}(\mathbb{R})$ be odd, and suppose that its anti-primitive F satisfies* (1.4). *Then, for every $k \in \mathbb{N}$, there exists a solution w_k having the following properties:*

(i) $-M < w_k < M$ in $\mathbb{R}^2$;

(ii) $w_k \circ T_i = -w_k$ *for every* $i = 1, \ldots, k$, *and* $w_k(x_1, -x_2) = w_k(x_1, x_2)$ *in* $\mathbb{R}^2$*;*
(iii) $w_k > 0$ *in* $\mathcal{S}_k$.

Moreover, there exists a constant $C > 0$ *(possibly depending on k) such that*

$$\int_{B_R} \left(\frac{1}{2}|\nabla w|^2 + F(w)\right) \leq CR \qquad \textit{for every } R > 0. \tag{A.2}$$

Remark A.2. The existence of a solution w_k with the properties (i)–(iii) was established by Alessio, Calamai and Montecchiari in [5]. In [5] the authors also obtained a more precise description of the asymptotic behavior of w_k at infinity. On the other hand, the validity of the estimate (A.2) was unknown.

In order to show that w_k fulfills (A.2), we provide an alternative existence proof with respect to the one in [5]. It is tempting to conjecture that the solutions given by Theorem A.1, and those found in [5], coincide.

Our alternative proof is strongly inspired by [7], where Cabré and Terra proved existence of solutions in $\mathbb{R}^{2m}$ to (A.1) vanishing on the Simon's cone (when restricted to the case $m = 1$ - i.e., when we consider (A.1) in the plane - their result establishes the existence of the solution w_2). We first prove the existence of a solution $w_R = w_{k,R}$ to (A.1) in the ball B_R, for every $R > 0$, by variational argument. Passing in a suitable way to the limit as $R \to +\infty$, we shall obtain a solution in the whole plane $\mathbb{R}^2$ having the desired energy estimate.

The main simplification with respect to the proof of Theorem 1.1 stays in the fact that, dealing with a single equation, we will not need an auxiliary parabolic problem, but we will be able to prove the existence of a solution in B_R with the desired symmetry properties directly by variational methods.

The proof of Theorem A.1 takes the rest of this appendix. Let us fix k. For any $\Omega \subset \mathbb{R}^2$ open, and for every $w \in H^1(\Omega)$, we define

$$E(w, \Omega) := \int_\Omega \left(\frac{1}{2}|\nabla w|^2 + F(w)\right).$$

For $R > 0$, we consider $\mathcal{S}_R := B_R \cap \mathcal{S}_k$ and the set

$$H_R := \left\{w \in H_0^1(S_R) : w(x_1, -x_2) = w(x_1, x_2) \text{ a.e. in } \mathcal{S}_R\right\}.$$

Lemma A.3. *For every* $R > 0$, *problem*

$$\begin{cases} -\Delta w = f(w) & \textit{in } B_R \\ w(x_1, -x_2) = w(x_1, x_2) & \textit{in } B_R \\ w = 0 & \textit{on } \partial B_R, \end{cases} \tag{A.3}$$

has a solution w_R*, satisfying (ii) in Theorem A.1. Moreover,* $-M \leq w_R \leq M$ *in* B_R*,* $w_R \geq 0$ *in* $\mathcal{S}_R$*, and*

$$E(w_R, S_R) = \min\{E(w, S_R) : \ w \in H_R\}.$$

Proof. At first, we search a solution to the auxiliary problem

$$\begin{cases} -\Delta w = f(w) & \text{in } \mathcal{S}_R \\ w \in H_0^1(S_R), \quad w \geq 0 & \text{in } \mathcal{S}_R \\ w(x_1, x_2) = w(x_1, -x_2) & \text{in } \mathcal{S}_R, \end{cases} \tag{A.4}$$

by minimizing the function $E(w, \mathcal{S}_R)$ in H. The existence of a minimizer follows easily by the direct method of the calculus of variations. Since $E(w, \mathcal{S}_R) = E(|w|, \mathcal{S}_R)$, it is not restrictive to suppose that $w_R \geq 0$. Also, since $E(\min\{w, M\}, \mathcal{S}_R) \leq E(w, \mathcal{S}_R)$ by assumption (1.4), we can suppose that $w_R \leq M$. Clearly, w_R solves the first equation in (A.4) in the set $\mathcal{S}_R \setminus \{\theta = 0\}$. The fact that w_R is also a solution across $\mathcal{S}_R \cap \{\theta = 0\}$ (thus a solution in $\mathcal{S}_R$) follows by the principle of symmetric criticality (see e.g., [14, Theorem 1.28] for a simple proof of this result, sufficient to our purposes).

Notice that $w_R \in C^1(\{0 < r < R, -\pi/2k \leq \theta \leq \pi/2k\})$ by standard elliptic regularity. Thus, we can reflect w_R $2k$ times in an odd way across $\ell_1, \ldots, \ell_k$, obtaining a solution in $B_R \setminus \{0\}$. To see that w_R is in fact a solution in B_R, we take a smooth function $\eta_\delta \in C^\infty(\overline{B_R})$ with $\eta_\delta \equiv 0$ in B_δ, $\eta_\delta \equiv 1$ in $B_{2\delta} \setminus B_\delta$, and $|\nabla \eta_\delta| \leq C/\delta$ in B_R. Then, for every $\varphi \in C_c^\infty(B_R)$, we have

$$\int_{B_R} \nabla w_R \cdot \nabla(\varphi \eta_\delta) - \int_{B_R} f(w_R) \varphi \eta_\delta = 0,$$

since $\varphi \eta_\delta$ is an admissible test function in $B_R \setminus \{0\}$. Passing to the limit as $\delta \to 0^+$, we deduce that

$$\int_{B_R} \nabla w_R \cdot \nabla \varphi - \int_{B_R} f(w_R) \varphi = 0 \qquad \forall \varphi \in C_c^\infty(B_R),$$

that is, w_R is a weak solution to (A.1) in B_R. This completes the proof. □

Proof of Theorem A.1. We wish to pass to the limit as $R \to +\infty$ and obtain a solution in the whole plane $\mathbb{R}^2$ as limit of the family $\{w_R\}$. As in the previous sections, by elliptic estimates we have that, up to a subsequence $w_R \to w$ in $C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^2)$ as $R \to \infty$, for every $\alpha \in (0, 1)$. The limit w inherits by w_R the symmetry property (*ii*) in Theorem A.1. Moreover, $w \geq 0$ in the sector $\mathcal{S} = \mathcal{S}_k$, and $|w| \leq M$ in the whole plane $\mathbb{R}^2$. Actually, the strict inequality $|w| < M$ holds, by the strong maximum principle. To complete the proof of the theorem, it remains then to show that w satisfies estimate (A.2), and that $w \not\equiv 0$.

As in the proof of Theorem 1.1, $\{w_R\}$ has a uniform gradient bound: there exists $C > 0$ (independent of R) such that

$$\|\nabla w_R\|_{L^\infty(B_{R-1})} \leq C \qquad \forall R > 1. \tag{A.5}$$

For an arbitrary $\rho > 1$, let now $R > \rho + 2$, and let $\xi \in C_c^\infty(B_\rho)$, with $\xi \equiv 1$ in $B_{\rho-1}$. We consider the following competitor for w_R:

$$\varphi_R(x) = \xi(x) \min\left\{\frac{\alpha x_1 - |x_2|}{\sqrt{2}}, M\right\} + (1 - \xi(x)) w_R(x).$$

Notice that this is the same type of competitor we used in Theorem 1.1. By minimality

$$E(w_R, S_R) \leq E(\varphi_R, S_R),$$

and since $w_R = \varphi_R$ in $\mathcal{S}_R \setminus \mathcal{S}_\rho$ we deduce that

$$\begin{aligned}\int_{\mathcal{S}_\rho}\left(\frac{1}{2}|\nabla w_R|^2 + F(w_R)\right) &\le \int_{\mathcal{S}_\rho}\left(\frac{1}{2}|\nabla \varphi_R|^2 + F(\varphi_R)\right)\\ &\le \int_{\mathcal{S}_{\rho-1}}\left(\frac{1}{2}|\nabla \varphi_R|^2 + F(\varphi_R)\right) + C|\mathcal{S}_\rho \setminus \mathcal{S}_{\rho-1}|,\end{aligned} \tag{A.6}$$

where we used the global boundedness of $\{\varphi_R\}$ in $W^{1,\infty}(B_\rho)$, see (A.5). At this point we can proceed as in the conclusion of the proof of Theorem 1.1: the right hand side in (A.6) can be estimated by $C\rho$, with C independent of ρ. Thus, we conclude that there exists a constant $C > 0$ such that, for every $\rho > 1$ and $R > \rho + 2$,

$$E(w_R, \mathcal{S}_\rho) = \int_{\mathcal{S}_\rho}\left(\frac{1}{2}|\nabla w_R|^2 + F(w_R)\right) \le C\rho,$$

Passing to the limit as $R \to +\infty$, we infer by C^1_{loc}-convergence that

$$E(w, \mathcal{S}_\rho) \le C\rho,$$

which implies, by symmetry, the estimate (A.2).

Suppose finally that $w \equiv 0$. Then the energy estimate (A.2) would give for every $\rho > 1$

$$\pi F(0)\rho^2 = E(0, B_\rho) \le C\rho,$$

which is not possible if ρ is sufficiently large. This proves that $w \not\equiv 0$, and completes the proof of Theorem A.1. □

Conflict of interest.

The author declares no conflict of interest.

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