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Research article

Polydispersity and surface energy strength in nematic colloids

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Abstract: We consider a Landau-de Gennes model for a polydisperse, inhomogeneous suspension of colloidal inclusions in a nematic host, in the dilute regime. We study the homogenised limit and compute the effective free energy of the composite material. By suitably choosing the shape of the inclusions and imposing a quadratic, Rapini-Papoular type surface anchoring energy density, we obtain an effective free energy functional with an additional linear term, which may be interpreted as an "effective field" induced by the inclusions. Moreover, we compute the effective free energy in a regime of "very strong anchoring", that is, when the surface energy effects dominate over the volume free energy.

Keywords: liquid crystals; nematic colloids; Landau-de Gennes; effective field; strong anchoring

1. Introduction

We consider a mixture of mesoscale size particles within an ambient fluid that contains locally aligned microscopic scale rod-like molecules, that is a nematic liquid crystals. This type of mixture, which is usually referred to as a *nematic colloid* material, has emerged in the recent years as the material of choice for testing a number of exciting hypothesis in the design of new materials. An overview of the field, and its applications, from the physical point of view is available in the reviews [17, 27].

The mathematical studies of such systems are still relatively few and focus on two extreme situations:

- the effect produced by one colloidal particle, particularly related to the so-called *'defect patterns'* that is the strong distortions produced at the interface between the particle and the ambient nematic fluid;
- the collective effects produced by the presence of many particle, fairly uniformly distributed, with a focus on the homogenised material.

In the first direction one should note that defects appear because of the anchoring conditions at the boundary of the particles, which generate topological obstructions [1, 2, 8–11, 28]. Indeed, there have been a number of works, identifying several physically relevant regimes [1] and the influence of external fields [2].

Our work focuses on the second direction, namely on long-scale effects produced by the effects of a large number of particles, namely on the homogenisation regime. There have been a couple of works in this direction, on which our builds, namely [5–7]. The main novelty of our approach, compared to those in [5–7], is that we allow for a much larger class of surface energy densities. We do not assume that the surface energy density is bounded from below and we do consider surface energy densities of quartic growth, which is the maximal growth compatible with the Sobolev embeddings. Surface energy densities of quartic growth have been proposed in the physical literature [3, 15, 24, 26].

We focus on a regime in which the total volume of the particles is much smaller than that of the ambient nematic environment, known as the dilute regime. Our aim is to provide a mathematical understanding of statements from the physical literature e.g., [19, 25] showing that in such a regime the colloidal nematics behave like a *homogenised, standard nematic material*, but with different (better) properties than those of the original nematic material.

In our previous work [12], we provided a first approach to these issues and we showed that using periodically distributed *identical* particles, one can design a suitable surface energy to obtain an apriori designed potential, that models the main physical properties of the material (in particular the nematic-isotropic transition temperature).

The purpose of these notes is two-fold: on the one hand, to extend the main results of [12] to *polydisperse* and *inhomogenoeus* nematic colloids; on the other hand, to explore a regime of parameters that differs from the one considered in [12].

Realistically, a set of colloidal inclusions will hardly be identical: The particles will differ in their size, shape, or charge. In order to account for polydispersity, we will consider several populations of colloidal inclusions, which may differ in their shape and properties. Moreover, we will not require the centres of mass of the inclusions to be homogeneously distributed in space. In mathematical terms, let $\mathscr{P}^1, \mathscr{P}^2, \ldots, \mathscr{P}^J$ be subsets of $\mathbb{R}^3$ (the reference shapes of the inclusions), and let $\Omega \subseteq \mathbb{R}^3$ be a bounded, smooth domain (the container). We define

$$\mathscr{P}^j_\varepsilon := \bigcup_{i=1}^{N^j_\varepsilon} \left(x^{i,j}_\varepsilon + \varepsilon^\alpha R^{i,j}_\varepsilon \mathscr{P}^j \right) \qquad \text{for } j \in \{1, \ldots, J\}, \tag{1.1}$$

where α is a positive number, the $x^{i,j}_\varepsilon$'s are points in Ω and the $R^{i,j}_\varepsilon$ are rotation matrices that satisfy suitable assumptions (see Section 2.1). As in [7, 12], we work in the *dilute regime*, namely we assume that $\alpha > 1$ so that the total volume occupied by the inclusions, $|\mathscr{P}_\varepsilon| \approx \varepsilon^{3\alpha-3}$, tends to zero as $\varepsilon \to 0$. However, we also assume that $\alpha < 3/2$ so that the total surface area of the inclusions, $\sigma(\partial\mathscr{P}_\varepsilon) \approx \varepsilon^{2\alpha-3}$, diverges as $\varepsilon \to 0$. We define $\mathscr{P}_\varepsilon := \cup_j \mathscr{P}_\varepsilon^j$ and $\Omega_\varepsilon := \Omega \setminus \mathscr{P}_\varepsilon$ (the space that is effectively occupied by the nematic liquid crystal). In accordance with the Landau-de Gennes theory, the nematic liquid crystal is described by a tensorial order parameter, that is, a symmetric, trace-free (3×3)-matrix field Q. We consider the free energy functional

$$\mathscr{F}_\varepsilon[Q] := \int_{\Omega_\varepsilon} (f_e(\nabla Q) + f_b(Q))\,\mathrm{d}x + \varepsilon^{3-2\alpha} \sum_{j=1}^{J} \int_{\partial\mathscr{P}_\varepsilon^j} f_s^j(Q, \nu)\,\mathrm{d}\sigma. \tag{1.2}$$

Here, f_e, f_b are suitable elastic and bulk energy densities (in the Landau-de Gennes theory, f_e is typically a positive definite, quadratic form in ∇Q and f_b is a quartic polynomial in Q; see Section 2.1), f_s^j is a surface anchoring energy densities (which may vary for different species of inclusions), and ν denotes the exterior unit normal to Ω. We prove a convergence result for *local* minimisers of $\mathscr{F}_\varepsilon$ to local minimsers of the effective free energy functional:

$$\mathscr{F}_0[Q] := \int_{\Omega} (f_e(\nabla Q) + f_b(Q) + f_{hom}(Q, x))\,\mathrm{d}x.$$

The "homogenised potential" f_{hom}, which keeps memory of the surface integral, is explicitly computable in terms of the f_s^j's, the distribution of the centres of mass $x_\varepsilon^{i,j}$ and the rotations $R_\varepsilon^{i,j}$. As an application of this result, we show that polydisperse inclusions may be used to mimic the effects of an applied electric field. More precisely, for a pre-assigned parameter $W \in \mathbb{R}$ and a pre-assigned symmetric matrix P, we may tune the shape $\mathscr{P}_\varepsilon^j$ of the inclusions and the surface energy densities, so to have in the limit

$$f_{hom}(Q) = W\mathrm{tr}(QP).$$

When P has the form $P = E \otimes E$ for some $E \in \mathbb{R}^3$, this expression may be interpreted as an electrostatic energy density induced by the "effective field" E, up to terms that do not depend on Q.

Moving beyond the issue of polydispersity we consider another physically restrictive assumption we made in [12], namely concerning the anchoring strength. In (1.2), the scaling of parameters is chosen so to have a factor of $\varepsilon^{3-2\alpha}$ in front of the surface integral, which compensates exactly the growth of the surface area, $\sigma(\partial\mathscr{P}_\varepsilon) \approx \varepsilon^{2\alpha-3}$. However, other choices of the scaling are possible, corresponding to different choices of the anchoring strength at the boundary of the inclusions. One can easily check that having a weaker anchoring, say of the type $\varepsilon^{2\alpha-3+\delta}$ with $\delta > 0$ will lead to a vanishing of the homogenized term, so the main interest is to understand what happens for stronger anchoring.To illustrate this possibility, we study the asymptotic behaviour, as $\varepsilon \to 0$, of minimisers of

$$\mathscr{F}_{\varepsilon,\gamma}[Q] := \int_{\Omega_\varepsilon} (f_e(\nabla Q) + f_b(Q))\,\mathrm{d}x + \varepsilon^{3-2\alpha-\gamma} \sum_{j=1}^{J} \int_{\partial\mathscr{P}_\varepsilon^j} f_s^j(Q, \nu)\,\mathrm{d}\sigma,$$

where γ is a positive parameter. This scaling corresponds to a much stronger surface anchoring and we expect the behaviour of minimisers to be dominated by the surface energy, as $\varepsilon \to 0$. Indeed, we will show that for γ small enough the functionals $\mathscr{F}_{\varepsilon,\gamma}$ Γ-converge to the constrained functional

$$\widetilde{\mathscr{F}}(Q) := \begin{cases} \int_\Omega (f_e(\nabla Q) + f_b(Q))\,\mathrm{d}x & \text{if } f_{hom}(Q(x),\, x) = 0 \text{ for a.e. } x \in \Omega \\ +\infty & \text{otherwise,} \end{cases}$$

as $\varepsilon \to 0$.

This result leaves a number of interesting of open problems, the most immediate ones being what is the optimal range of γ for which this holds and, directly related to this, if one gets a different limit for large values of γ.

The paper is organized as follows: in the following, in Section 2.1 we consider the polydisperse setting and the general homogenisation result. The main results of this section, namely Theorem 2.1 and Proposition 2.2 are presented after the introduction of the mathematical setting, in Subsection 2.1. The proof of the results is provided in Subsection 2.3 after a number of preliminary results, need in the proof, provided in Subsection 2.2.

In Section 3 we provide an application of the results in Section 2.1, namely showing that, in a polydispersive regime, one can obtain a linear term in the homogenised potential (Proposition 3.2).

Finally, in Section 4 we study the case when the scaling of the anchoring strength is $\varepsilon^{3-2\alpha-\gamma}$ with γ suitably small, and provide in Theorem 4.3 the Γ-converegence result mentioned above. Its proof is done at the end of the section after a number of preliminary results.

2. An homogenisation result for polydisperse, inhomogeneous nematic colloids in the dilute regime

2.1. Statement of the homogenisation result

The Landau-de Gennes Q-tensor. In the Landau-de Gennes theory, the local configuration of a nematic liquid crystal is represented by a symmetric, symmetric, trace-free, real (3×3)-matrix, known as the Q-tensor, which describes the anisotropic optical properties of the medium [13,21]. We denote by $\mathscr{S}_0$ the set of matrix as above. For $Q, P \in \mathscr{S}_0$, we denote $Q \cdot P := \mathrm{tr}(QP)$. This defines a scalar product on $\mathscr{S}_0$, and the corresponding norm will be denoted by $|Q| := (\mathrm{tr}(Q^2))^{1/2} = (\sum_{i,j} Q_{ij})^{1/2}$.

The domain. let $\mathscr{P}^1, \mathscr{P}^2, \ldots, \mathscr{P}^J$ be subsets of $\mathbb{R}^3$ (the reference shapes of the inclusions), and let $\Omega \subseteq \mathbb{R}^3$ be a bounded, smooth domain (the container). We define $\mathscr{P}^j_\varepsilon$ as in (1.1), where α, $x^{i,j}_\varepsilon$, $R^{i,j}_\varepsilon$ satisfy the following assumptions:

- H_1. $1 < \alpha < 3/2$.
- H_2. There exists a constant $\lambda_\Omega > 0$ such that

$$\mathrm{dist}(x^{i,j}_\varepsilon, \partial\Omega) + \frac{1}{2} \inf_{(h,k)\neq(i,j)} |x^{h,k}_\varepsilon - x^{i,j}_\varepsilon| \geq \lambda_\Omega \varepsilon$$

  for any $\varepsilon > 0$, any $j \in \{1, \ldots, J\}$ and any $i \in \{1, \ldots, N^j_\varepsilon\}$.
- H_3. For any $j \in \{1, \ldots, J\}$, there exists a non-negative function $\xi^j \in L^\infty(\Omega)$, such that

$$\mu^j_\varepsilon := \varepsilon^3 \sum_{i=1}^{N^j_\varepsilon} \delta_{x^{i,j}_\varepsilon} \rightharpoonup^* \xi^j\,\mathrm{d}x \qquad \text{as measures in } \mathbb{R}^3, \text{ as } \varepsilon \to 0.$$

H$_4$. For any $j \in \{1, \dots, J\}$, there exists a Lipschitz-continuous map $R^j_* \colon \overline{\Omega} \to \mathrm{SO}(3)$ such that $R^{i,j}_\varepsilon = R^j_*(x^{i,j}_\varepsilon)$ for any $\varepsilon > 0$ and any $i \in \{1, \dots, N^j_\varepsilon\}$.
H$_5$. For any $j \in \{1, \dots, J\}$, $\mathscr{P}^j \subseteq \mathbb{R}^3$ is a compact, convex set whose interior contains the origin.

The assumption (H$_2$) is a separation condition on the inclusions. As a consequence of (H$_2$), the number of the inclusions, for each population j, is $N^j_\varepsilon \lesssim \varepsilon^{-3}$. Therefore, the total volume of the inclusions in each population is bounded by $N^j_\varepsilon \varepsilon^3 \lesssim \varepsilon^{3\alpha-3} \to 0$, because of (H$_1$). Thus, we are in the diluted regime, as in [7, 12]. We define

$$\mathscr{P}_\varepsilon := \bigcup_{j=1}^{J} \mathscr{P}^j_\varepsilon, \qquad \Omega_\varepsilon := \Omega \setminus \mathscr{P}_\varepsilon.$$

The assumption (H$_5$) guarantees that Ω_ε is a Lipschitz domain.

The free energy functional. For $Q \in H^1(\Omega_\varepsilon, \mathscr{S}_0)$, we consider the free energy functional

$$\mathscr{F}_\varepsilon[Q] := \int_{\Omega_\varepsilon} \left(f_e(\nabla Q) + f_b(Q)\right) \mathrm{d}x + \varepsilon^{3-2\alpha} \sum_{j=1}^{J} \int_{\partial \mathscr{P}^j_\varepsilon} f^j_s(Q, \nu)\, \mathrm{d}\sigma. \tag{2.1}$$

The surface anchoring energy densities depend on j, as colloids that belong to different populations may have different surface properties. For the rest, our assumptions for the elastic energy density f_e, bulk energy density f_b, and surface energy densities f^j_s are the same as in [12]. We say that a function $f \colon \mathscr{S}_0 \otimes \mathbb{R}^3 \to \mathbb{R}$ is strongly convex if there exists $\theta > 0$ such that the function $\mathscr{S}_0 \otimes \mathbb{R}^3 \ni D \mapsto f(D) - \theta|D|^2$ is convex.

H$_6$. $f_e \colon \mathscr{S}_0 \otimes \mathbb{R}^3 \to [0, +\infty)$ is differentiable and strongly convex. Moreover, there exists a constant $\lambda_e > 0$ such that

$$\lambda_e^{-1}|D|^2 \le f_e(D) \le \lambda_e |D|^2, \qquad |(\nabla f_e)(D)| \le \lambda_e \left(|D| + 1\right)$$

for any $D \in \mathscr{S}_0 \otimes \mathbb{R}^3$.
H$_7$. $f_b \colon \mathscr{S}_0 \to \mathbb{R}$ is continuous, non-negative and there exists $\lambda_b > 0$ such that $0 \le f_b(Q) \le \lambda_b(|Q|^6 + 1)$ for any $Q \in \mathscr{S}_0$.
H$_8$. For any $j \in \{1, \dots, J\}$, the function $f^j_s \colon \mathscr{S}_0 \times \mathbb{S}^2 \to \mathbb{R}$ is locally Lipschitz-continuous. Moreover, there exists a constant $\lambda_s > 0$ such that

$$|f^j_s(Q_1, \nu_1) - f^j_s(Q_2, \nu_2)| \le \lambda_s \left(|Q_1|^3 + |Q_2|^3 + 1\right) \left(|Q_1 - Q_2| + |\nu_1 - \nu_2|\right)$$

for any $j \in \{1, \dots, J\}$ and any (Q_1, ν_1), (Q_2, ν_2) in $\mathscr{S}_0 \times \mathbb{S}^2$.

A physically relevant example of elastic energy density f_e that satisfies (H$_6$) is given by

$$f^{LdG}_e(\nabla Q) := L_1\, \partial_k Q_{ij}\, \partial_k Q_{ij} + L_2\, \partial_j Q_{ij}\, \partial_k Q_{ik} + L_3\, \partial_j Q_{ik}\, \partial_k Q_{ij} \tag{2.2}$$

(Einstein's summation convention is assumed), so long as the coefficients L_1, L_2, L_3 satisfy

$$L_1 > 0, \qquad -L_1 < L_3 < 2L_1, \qquad -\frac{3}{5}L_1 - \frac{1}{10}L_3 < L_2 \tag{2.3}$$

(see e.g., [13, 20]). The assumption (H_7) is satisfied by the quartic Landau-de Gennes bulk potential, given by

$$f_b^{LdG}(Q) := a\,\mathrm{tr}(Q^2) - b\,\mathrm{tr}(Q^3) + c\left(\mathrm{tr}(Q^2)\right)^2 + \kappa(a,\, b,\, c)$$

where $a \in \mathbb{R}$, $b > 0$, $c > 0$ are coefficients depending on the material and the temperature and $\kappa(a,\, b, c) \in \mathbb{R}$ is a constant, chosen in such a way that $\inf f_b^{LdG} = 0$. An example of surface energy density that satisfies (H_8) is the Rapini-Papoular type energy density:

$$f_s(Q,\, \nu) := W\,\mathrm{tr}(Q - Q_\nu)^2 \qquad \text{with } Q_\nu := \nu \otimes \nu - \frac{\mathrm{Id}}{3}$$

and W a (typically positive) parameter. However, (H_8) allows for much more general surface energy densities, which may not be positive and may have up to quartic growth in Q (for examples, see e.g., [3, 15, 24, 26] and the references therein). In addition to (H_8), physically relevant surface energy densities must satisfy symmetry properties (frame-indifference, invariance with respect to the sign of ν) but these will play no rôle in our analysis.

The homogenised potential. For any $j \in \{1,\, \dots,\, J\}$, let us define $f^j_{hom} \colon \mathscr{S}_0 \times \overline{\Omega} \to \mathbb{R}$ as

$$f^j_{hom}(Q,\, x) := \int_{\partial\mathscr{P}^j} f^j_s(Q,\, R^j_*(x)\nu_{\mathscr{P}^j})\,\mathrm{d}\sigma \qquad \text{for } (Q,\, x) \in \mathscr{S}_0 \times \overline{\Omega}, \tag{2.4}$$

where $\nu_{\mathscr{P}^j}$ denotes the *inward*-pointing unit normal to $\partial\mathscr{P}^j$, and $R^j_* \colon \overline{\Omega} \to \mathrm{SO}(3)$ is the map given by (H_4). Finally, let

$$f_{hom}(Q,\, x) := \sum_{j=1}^{J} \xi^j(x) f^j_{hom}(Q,\, x) \qquad \text{for } (Q,\, x) \in \mathscr{S}_0 \times \overline{\Omega}, \tag{2.5}$$

where $\xi^j \in L^\infty(\Omega)$ is the function given by (H_3). Our candidate homogenised functional is defined for any $Q \in H^1(\Omega, \mathscr{S}_0)$ as

$$\mathscr{F}_0[Q] := \int_\Omega \left(f_e(\nabla Q) + f_b(Q) + f_{hom}(Q,\, x)\right)\mathrm{d}x. \tag{2.6}$$

The convergence result. The assumptions (H_1)–(H_8) are *not* enough to guarantee that global minimisers of $\mathscr{F}_\varepsilon$ exist and actually, it may happen that $\mathscr{F}_\varepsilon$ is unbounded from below [12, Lemma 3.6]. Instead, our main result focus on the asymptotic behaviour of *local* minimisers. Given $g \in H^{1/2}(\partial\Omega,\, \mathscr{S}_0)$, we let $H^1_g(\Omega_\varepsilon, \mathscr{S}_0)$ — respectively, $H^1_g(\Omega,\, \mathscr{S}_0)$ — be the set of maps $Q \in H^1(\Omega_\varepsilon, \mathscr{S}_0)$ — respectively, $Q \in H^1(\Omega, \mathscr{S}_0)$ — that satisfy $Q = g$ on $\partial\Omega$, in the sense of traces. For each $Q \in H^1_g(\Omega_\varepsilon, \mathscr{S}_0)$, we define the map $E_\varepsilon Q \in H^1_g(\Omega,\, \mathscr{S}_0)$ by $E_\varepsilon Q := Q$ on Ω_ε and $E_\varepsilon Q := Q^{i,j}_\varepsilon$ on $\mathscr{P}^{i,j}_\varepsilon$, where $Q^{i,j}_\varepsilon$ is the unique solution of Laplace's problem

$$\begin{cases} -\Delta Q^{i,j}_\varepsilon = 0 & \text{in } \mathscr{P}^{i,j}_\varepsilon \\ Q^{i,j}_\varepsilon = Q & \text{on } \partial\mathscr{P}^{i}_\varepsilon. \end{cases} \tag{2.7}$$

Theorem 2.1. *Suppose that the assumptions* (H_1)–(H_8) *are satisfied. Suppose, moreover, that $Q_0 \in H^1_g(\Omega, \mathscr{S}_0)$ is an isolated H^1-local minimiser for $\mathscr{F}_0$ — that is, there exists $\delta_0 > 0$ such that*

$$\mathscr{F}_0[Q_0] < \mathscr{F}_0[Q]$$

for any $Q \in H^1_g(\Omega, \mathscr{S}_0)$ such that $Q \neq Q_0$ and $\|Q - Q_0\|_{H^1(\Omega)} \leq \delta_0$. Then, for any ε small enough, there exists an H^1-local minimiser Q_ε for $\mathscr{F}_\varepsilon$ such that $E_\varepsilon Q_\varepsilon \to Q_0$ strongly in $H^1(\Omega)$ as $\varepsilon \to 0$.

The proof of Theorem 2.1 follows a variational approach, and is based on the following fact:

Proposition 2.2. *Let $Q_\varepsilon \in H^1_g(\Omega_\varepsilon, \mathscr{S}_0)$ be such that $E_\varepsilon Q_\varepsilon \rightharpoonup Q$ weakly in $H^1(\Omega)$ as $\varepsilon \to 0$. Then, there holds*

$$\int_\Omega (f_e(\nabla Q) + f_b(Q))\,\mathrm{d}x \leq \liminf_{\varepsilon\to 0} \int_{\Omega_\varepsilon} (f_e(\nabla Q_\varepsilon) + f_b(Q_\varepsilon))\,\mathrm{d}x \tag{2.8}$$

$$\int_\Omega f_{hom}(Q,\, x)\,\mathrm{d}x = \lim_{\varepsilon\to 0} \varepsilon^{3-2\alpha} \sum_{j=1}^{J} \int_{\partial\mathscr{P}^j_\varepsilon} f^j_s(Q_\varepsilon,\, \nu)\,\mathrm{d}\sigma. \tag{2.9}$$

Proposition 2.2 can be reformulated as a Γ-convergence result. Indeed, from Proposition 2.2 we immediately have $\mathscr{F}_0 \leq \Gamma\text{-}\liminf_{\varepsilon\to 0} \mathscr{F}_\varepsilon$ (with respect to a suitable topology, induced by the operator E_ε). A trivial recovery sequence suffices to obtain the opposite Γ-lim sup inequality, thanks to (2.9) and the fact that in the dilute limit, $|\mathscr{P}_\varepsilon| \to 0$. Theorem 2.1 follows from Proposition 2.2 by general properties of the Γ-convergence.

Throughout the paper, we will write $A \lesssim B$ as a short-hand for $A \leq CB$, where C is a positive constant, depending only on the domain, the boundary datum and the free energy functional (2.1), but not on ε.

2.2. Preliminary results

The main technical tool is the following trace inequality, which is adapted from [7, Lemma 4.1].

Lemma 2.3 ([12, Lemma 3.1])**.** *Let $\mathscr{P} \subseteq \mathbb{R}^3$ be a compact, convex set whose interior contains the origin. Then, there exists a constant $C = C(\mathscr{P}) > 0$ such that, for any $a > 0$, $b \geq 2a$ and any $u \in H^1(b\mathscr{P} \setminus a\mathscr{P})$, there holds*

$$\int_{\partial(a\mathscr{P})} |u|^4\,\mathrm{d}\sigma \leq C \int_{b\mathscr{P}\setminus a\mathscr{P}} \left(|\nabla u|^2 + |u|^6\right)\mathrm{d}x + \frac{Ca^2}{b^3} \int_{b\mathscr{P}\setminus a\mathscr{P}} |u|^4\,\mathrm{d}x.$$

Given an inclusion $\mathscr{P}^{i,j}_\varepsilon = x^{i,j}_\varepsilon + \varepsilon^\alpha R^{i,j}_\varepsilon \mathscr{P}^j$, we consider $\widehat{\mathscr{P}}^{i,j}_\varepsilon := x^{i,j}_\varepsilon + \mu\varepsilon R^{i,j}_\varepsilon \mathscr{P}^j$, where $\mu > 0$ is a small (but fixed) parameter. By taking μ small enough, we can make sure that the $\widehat{\mathscr{P}}^{i,j}_\varepsilon$'s are pairwise disjoint. Then, by applying Lemma 2.3 component-wise on $\widehat{\mathscr{P}}^{i,j}_\varepsilon \setminus \mathscr{P}^{i,j}_\varepsilon$ and summing the corresponding inequalities over i and j, we deduce

Lemma 2.4. *For any $Q \in H^1(\Omega_\varepsilon, \mathscr{S}_0)$, there holds*

$$\varepsilon^{3-2\alpha} \int_{\partial\mathscr{P}_\varepsilon} |Q|^4\,\mathrm{d}\sigma \lesssim \varepsilon^{3-2\alpha} \int_{\Omega_\varepsilon} \left(|\nabla Q|^2 + |Q|^6\right)\mathrm{d}x + \int_{\Omega_\varepsilon} |Q|^4\,\mathrm{d}x.$$

Another tool is the harmonic extension operator, $E_\varepsilon \colon H^1(\Omega_\varepsilon, \mathscr{S}_0) \to H^1(\Omega, \mathscr{S}_0)$, defined by (2.7).

Lemma 2.5. *The operator* $E_\varepsilon \colon H^1(\Omega_\varepsilon, \mathscr{S}_0) \to H^1(\Omega, \mathscr{S}_0)$ *satisfies the following properties.*

(i). There exists a constant $C > 0$ *such that* $\|\nabla(E_\varepsilon Q)\|_{L^2(\Omega)} \leq C\|\nabla Q\|_{L^2(\Omega_\varepsilon)}$ *for any* $\varepsilon > 0$ *and any* $Q \in H^1(\Omega_\varepsilon, \mathscr{S}_0)$.

(ii). If the maps $Q_\varepsilon \in H^1(\Omega, \mathscr{S})$ *converge* $H^1(\Omega)$*-strongly to* Q *as* $\varepsilon \to 0$*, then* $E_\varepsilon(Q_{\varepsilon|\Omega_\varepsilon}) \to Q$ *strongly in* $H^1(\Omega)$ *as* $\varepsilon \to 0$*, too.*

Proof. For any i, j, consider the inclusion $\mathscr{P}_\varepsilon^{i,j} = x_\varepsilon^{i,j} + \varepsilon^\alpha R_\varepsilon^{i,j}\mathscr{P}^j$ and let $\mathscr{R}_\varepsilon^{i,j} := x_\varepsilon^{i,j} + 2\varepsilon^\alpha R_\varepsilon^{i,j}\mathscr{P}^j$. Let $\mathscr{R}_\varepsilon := \cup_{i,j}\mathscr{R}_\varepsilon^{i,j}$. The properties of Laplace equation, combined with a scaling argument (see, e.g., [12, Lemma 3.4]), imply that

$$\|\nabla(E_\varepsilon Q)\|_{L^2(\mathscr{P}_\varepsilon)} \lesssim \|\nabla Q\|_{L^2(\mathscr{R}_\varepsilon\setminus\mathscr{P}_\varepsilon)}. \tag{2.10}$$

Statement (i) then follows immediately. To prove Statement (ii), take a sequence $Q_\varepsilon \in H^1(\Omega, \mathscr{S}_0)$ that converges strongly to Q as $\varepsilon \to 0$. Then,

$$\begin{aligned}\left\|\nabla Q_\varepsilon - \nabla(E_\varepsilon(Q_{\varepsilon|\Omega_\varepsilon}))\right\|_{L^2(\Omega)} &\leq \|\nabla Q_\varepsilon\|_{L^2(\mathscr{P}_\varepsilon)} + \left\|\nabla(E_\varepsilon(Q_{\varepsilon|\Omega_\varepsilon}))\right\|_{L^2(\mathscr{P}_\varepsilon)} \\ &\overset{(2.10)}{\lesssim} \|\nabla Q_\varepsilon\|_{L^2(\mathscr{R}_\varepsilon)} \lesssim \|\nabla Q\|_{L^2(\mathscr{R}_\varepsilon)} + \|\nabla Q - \nabla Q_\varepsilon\|_{L^2(\Omega)}.\end{aligned}$$

Both terms in the right-hand side converge to 0 as $\varepsilon \to 0$, because $|\mathscr{R}_\varepsilon| \lesssim \varepsilon^{3\alpha-3} \to 0$, and Statement (ii) follows. □

2.3. Proof of Theorem 2.1

The proof of Theorem 2.1 is largely similar to that of [12, Theorem 1.1]. We reproduce here only some steps of the proof, either because there require a modification or because they will be useful in Section 4.

Remarks on the lower semi-continuity of $\mathscr{F}_\varepsilon$. Even before we address the asymptotic analysis as $\varepsilon \to 0$, we should make sure that, for fixed $\varepsilon > 0$, the functional $\mathscr{F}_\varepsilon$ is sequentially lower semi-continuous with respect to the weak topology on $H^1(\Omega_\varepsilon, \mathscr{S}_0)$. If the surface energy density f_s is bounded from below, then the surface integral is lower semi-continuous by Fatou lemma. If f_s has subcritical growth, that is $|f_s(Q)| \lesssim |Q|^p + 1$ for some $p < 4$, then the lower semi-continuity of the surface integral follows from the compact Sobolev embedding $H^{1/2}(\partial\Omega_\varepsilon, \mathscr{S}_0) \hookrightarrow L^p(\partial\Omega_\varepsilon, \mathscr{S}_0)$ and Lebesgue's dominated convergence theorem. However, our assumption (H_8) allows for surface energy densities that have quartic growth and are unbounded from below, e.g. $f_s(Q) := -|Q|^4$. In this case, the surface integral alone may *not* be sequentially weakly lower-semi continuous [12, Lemma 3.10]. However, lower semi-continuity may be restored at least on *bounded* subsets of $H^1(\Omega_\varepsilon, \mathscr{S}_0)$, when ε is small:

Proposition 2.6 ([12, Proposition 3.9])**.** *Suppose that the assumptions* (H_1)–(H_8) *are satisfied. For any* $M > 0$ *there exists* $\varepsilon_0(M) > 0$ *such that following statement holds: if* $0 < \varepsilon \leq \varepsilon_0(M)$, $Q_k \rightharpoonup Q$ *weakly in* $H^1_g(\Omega_\varepsilon, \mathscr{S}_0)$ *and if* $\|\nabla Q_k\|_{L^2(\Omega_\varepsilon)} \leq M$ *for any* k*, then*

$$\mathscr{F}_\varepsilon[Q] \leq \liminf_{k\to+\infty} \mathscr{F}_\varepsilon[Q_k].$$

The proof carries over from [12], almost word by word, using Lemma 2.4. Essentially, the loss of lower semi-continuity that may arise from the surface integral can be quantified, with the help of Lemma 2.4 and the bound on ∇Q_k. However, since the surface integral is multiplied by a *small* factor $\varepsilon^{3-2\alpha}$, this loss of lower semi-continuity is compensated by the strong convexity of the elastic term f_e, for ε sufficiently small.

Pointwise convergence of the surface energy terms. For ease of notation, let us define

$$J_\varepsilon[Q] := \varepsilon^{3-2\alpha} \sum_{j=1}^{J} \int_{\partial \mathscr{P}_\varepsilon^j} f_s^j(Q, \nu) \, \mathrm{d}\sigma \tag{2.11}$$

$$J_0[Q] := \int_\Omega f_{hom}(Q, x) \, \mathrm{d}x \qquad \text{for } Q \in H^1_g(\Omega, \mathscr{S}_0) \tag{2.12}$$

We state some properties of the functions $f^j_{hom} \colon \mathscr{S}_0 \times \overline{\Omega} \to \mathbb{R}$, $f_{hom} \colon \mathscr{S}_0 \times \overline{\Omega} \to \mathbb{R}$, defined by (2.5), (2.4) respectively.

Lemma 2.7. *For any $j \in \{1, \ldots, J\}$, the function f^j_{hom} is locally Lispchitz-continuous, and there holds*

$$\left| f^j_{hom}(Q, x) \right| \lesssim |Q|^4 + 1, \qquad \left| \nabla f^j_{hom}(Q, x) \right| \lesssim |Q|^3 + 1 \tag{2.13}$$

for any $(Q, x) \in \mathscr{S}_0 \times \overline{\Omega}$. Moroever, the function f_{hom} satisfies

$$|f_{hom}(Q_1, x) - f_{hom}(Q_2, x)| \lesssim \left(|Q_1|^3 + |Q_2|^3 + 1\right) |Q_1 - Q_2| \tag{2.14}$$

for any $Q_1, Q_2 \in \mathscr{S}_0$ and any $x \in \overline{\Omega}$.

Proof. Using the definition (2.4) of f^j_{hom}, and the assumption (H$_8$), we obtain

$$\begin{aligned} \left| f^j_{hom}(Q_1, x_1) - f^j_{hom}(Q_2, x_2) \right| &\leq \int_{\partial \mathscr{P}^j} \left| f^j_s(Q_1, R^j_*(x_1)\nu_{\mathscr{P}^j}) - f^j_s(Q_2, R^j_*(x_2)\nu_{\mathscr{P}^j}) \right| \mathrm{d}\sigma \\ &\lesssim \int_{\partial \mathscr{P}^j} \left(|Q_1|^3 + |Q_2|^3 + 1\right) \left(|Q_1 - Q_2| + \left| \left(R^j_*(x_1) - R^j_*(x_2)\right) \nu_{\mathscr{P}^j} \right| \right) \mathrm{d}\sigma \end{aligned}$$

Since R^j_* is Lipschitz-continuous by (H$_4$), we deduce

$$\left| f^j_{hom}(Q_1, x_1) - f^j_{hom}(Q_2, x_2) \right| \lesssim \left(|Q_1|^3 + |Q_2|^3 + 1\right) (|Q_1 - Q_2| + |x_1 - x_2|) \tag{2.15}$$

and (2.14) follows. We multiply the previous inequality by ξ^j, where ξ^j is given by (H$_3$), take $x_1 = x_2 = x$, and sum over j. Since $\xi^j \in L^\infty(\Omega)$ by Assumption (H$_3$), we obtain

$$\begin{aligned} |f_{hom}(Q_1, x) - f_{hom}(Q_2, x)| &\overset{(2.5)}{\leq} \sum_{j=1}^{J} \left\| \xi^j \right\|_{L^\infty(\Omega)} \left| f^j_{hom}(Q_1, x) - f^j_{hom}(Q_2, x) \right| \\ &\lesssim \left(|Q_1|^3 + |Q_2|^3 + 1\right) |Q_1 - Q_2|. \end{aligned}$$

□

Let us introduce the auxiliary quantity

$$\tilde{J}_\varepsilon[Q] := \varepsilon^{3-2\alpha} \sum_{j=1}^{J} \sum_{i=1}^{N_\varepsilon^j} \int_{\partial \mathscr{P}_\varepsilon^{i,j}} f_s^j(Q(x_\varepsilon^{i,j}),\, \nu)\, \mathrm{d}\sigma. \tag{2.16}$$

Lemma 2.8. *For any bounded, Lipschitz map $Q \colon \overline{\Omega} \to \mathscr{S}_0$, there holds*

$$\tilde{J}_\varepsilon[Q] = \sum_{j=1}^{J} \int_{\mathbb{R}^3} f_{hom}^j(Q(x),\, x)\, \mathrm{d}\mu_\varepsilon^j(x) \tag{2.17}$$

(where the measures μ_ε^j are defined by (H$_3$)*), and*

$$\left| J_\varepsilon[Q] - \tilde{J}_\varepsilon[Q] \right| \lesssim \varepsilon^\alpha \left(\|Q\|_{L^\infty(\Omega)}^3 + 1 \right) \|\nabla Q\|_{L^\infty(\Omega)}. \tag{2.18}$$

Proof. For any i and j, consider the single inclusion $\mathscr{P}_\varepsilon^{i,j} := x_\varepsilon^{i,j} + \varepsilon^\alpha R_\varepsilon^{i,j} \mathscr{P}_\varepsilon^j$. Since $\nu(x) = R_\varepsilon^{i,j}\, \nu_{\mathscr{P}^j}(\varepsilon^{-\alpha}(R_\varepsilon^{i,j})^\mathsf{T}(x - x_\varepsilon^{i,j}))$ for any $x \in \partial \mathscr{P}_\varepsilon^{i,j}$, by a change of variable we obtain

$$\begin{aligned} \tilde{J}_\varepsilon[Q] &= \varepsilon^3 \sum_{j=1}^{J} \sum_{i=1}^{N_\varepsilon^j} \int_{\partial \mathscr{P}^j} f_s^j(Q(x_\varepsilon^{i,j}),\, R_\varepsilon^{i,j} \nu_{\mathscr{P}})\, \mathrm{d}\sigma \\ &\overset{(\mathrm{H}_4)}{=} \varepsilon^3 \sum_{j=1}^{J} \sum_{i=1}^{N_\varepsilon^j} \int_{\partial \mathscr{P}^j} f_s^j(Q(x_\varepsilon^{i,j}),\, R_*^j(x_\varepsilon^{i,j}) \nu_{\mathscr{P}})\, \mathrm{d}\sigma \\ &\overset{(2.5)}{=} \varepsilon^3 \sum_{j=1}^{J} \sum_{i=1}^{N_\varepsilon^j} f_{hom}^j(Q(x_\varepsilon^{i,j}),\, x_\varepsilon^{i,j}). \end{aligned}$$

Now, (2.17) follows from the definition of μ_ε^j, (H$_3$). On the other hand, by decomposing the integral on $\partial \mathscr{P}^j$ as a sum of integrals over the boundary of each inclusion, we obtain

$$\begin{aligned} \left| J_\varepsilon[Q] - \tilde{J}_\varepsilon[Q] \right| &\lesssim \varepsilon^{3-2\alpha} \sum_{j=1}^{J} \sum_{i=1}^{N_\varepsilon^j} \int_{\partial \mathscr{P}_\varepsilon^{i,j}} \left| f_s^j(Q(x),\, \nu) - f_s^j(Q(x_\varepsilon^{i,j}),\, \nu) \right| \mathrm{d}\sigma(x) \\ &\overset{(\mathrm{H}_8)}{\lesssim} \varepsilon^{3-2\alpha} \sum_{i=1}^{N_\varepsilon^j} \sum_{j=1}^{J} \int_{\partial \mathscr{P}_\varepsilon^{i,j}} \left(|Q(x)|^3 + \left| Q(x_\varepsilon^{i,j}) \right|^3 + 1 \right) \left| Q(x) - Q(x_\varepsilon^{i,j}) \right| \mathrm{d}\sigma(x) \end{aligned}$$

Since Q is assumed to be Lipschitz continuous and the diameter of $\mathscr{P}_\varepsilon^{i,j}$ is $\lesssim \varepsilon^\alpha$, we have $|Q(x) - Q(x_\varepsilon^{i,j})| \lesssim \varepsilon^\alpha \|\nabla Q\|_{L^\infty(\Omega)}$. This implies

$$\left| J_\varepsilon[Q] - \tilde{J}_\varepsilon[Q] \right| \lesssim \varepsilon^{3-\alpha} \sigma(\partial \mathscr{P}_\varepsilon) \left(\|Q\|_{L^\infty(\mathbb{R}^3)}^3 + 1 \right) \|\nabla Q\|_{L^\infty(\Omega)}.$$

Finally, we note that $\sigma(\partial \mathscr{P}_\varepsilon) \lesssim \varepsilon^{2\alpha-3}$, because there are $\mathrm{O}(\varepsilon^{-3})$ inclusions (as a consequence of (H$_2$)) and the surface area of each inclusion is $\mathrm{O}(\varepsilon^{2\alpha})$. Thus, the lemma follows. □

Since $\mu_\varepsilon^j \rightharpoonup^* \xi^j \mathrm{d}x$ by Assumption (H$_3$), as an immediate consequence of Lemma 2.8 and (2.5), (2.12) we obtain

Proposition 2.9. *For any* bounded, Lipschitz *map $Q \colon \overline{\Omega} \to \mathscr{S}_0$, there holds $J_\varepsilon[Q] \to J_0[Q]$ as $\varepsilon \to 0$.*

Once Proposition 2.9 is proved, the rest of the proof of Proposition 2.2 and Theorem 2.1 follows exactly as in [12].

3. Linear terms in the homogenised bulk potential

Proposition 3.1. *There exist (possibly disconnected) shapes* $\mathfrak{P}_k \subset \mathbb{R}^3$, $k \in \{1,2,3,4,5,6\}$ *such that taking as surface energy the Rapini-Papoular surface energy* $f_s(Q,\nu) = \mathrm{tr}(Q - Q_\nu)^2$ *with* $Q_\nu = \nu\otimes\nu - \frac{1}{3}\mathrm{Id}$ *where* ν *is the exterior unit-normal, we have:*

$$f_{hom}^{\mathfrak{P}_k}(Q) = \left(\frac{2}{3} + \mathrm{tr}(Q^2)\right)\sigma(\partial\mathfrak{P}_k) - 2\mathrm{tr}(QM_k) \tag{3.1}$$

where

$$M_k = \left(\frac{\pi}{3} + \frac{\pi}{2}\right)\mathrm{Id} - \frac{\pi}{2} e_k \otimes e_k, \qquad k \in \{1,2,3\}$$

$$M_4 = \left(\frac{\pi}{3} + \frac{\pi}{2}\right)\mathrm{Id} - \frac{\pi}{2} e_3 \otimes e_3 + \frac{2}{3}(e_1\otimes e_2 + e_2 \otimes e_1),$$

$$M_5 = \left(\frac{\pi}{3} + \frac{\pi}{2}\right)\mathrm{Id} - \frac{\pi}{2} e_2 \otimes e_2 + \frac{2}{3}(e_1\otimes e_3 + e_3 \otimes e_1),$$

$$M_6 = \left(\frac{\pi}{3} + \frac{\pi}{2}\right)\mathrm{Id} - \frac{\pi}{2} e_1 \otimes e_1 + \frac{2}{3}(e_2\otimes e_3 + e_3 \otimes e_2),$$

with M_k, $k \in \{1,2,3,4,5,6\}$ *a basis in the linear space of* 3×3 *symmetric matrices.*

Proof. By formula (2.4) we have:

$$f_{hom}^{\mathfrak{P}_k}(Q) = \int_{\partial\mathfrak{P}_k} \left(\mathrm{tr}(Q^2) - 2\mathrm{tr}(QQ_\nu) + \mathrm{tr}(Q_\nu)^2\right) \mathrm{d}\sigma \tag{3.2}$$

hence we readily get (3.1) with $M_k = \int_{\partial\mathfrak{P}_k} \nu(x)\otimes\nu(x)\,\mathrm{d}\sigma(x)$.

Let us take for $1 \le i, j \le 3$ with $i \ne j$ the *'potato wedges'* domains

$$\Omega_{ij}^+ := \{x = (x_1,x_2,x_3) \in \mathbb{R}^3 \colon |x| \le 1,\ x_i \ge 0,\ x_j \ge 0\} \tag{3.3}$$

as candidates for *'parts of'* our shapes $\mathfrak{P}_k$'s (see Figure 1).

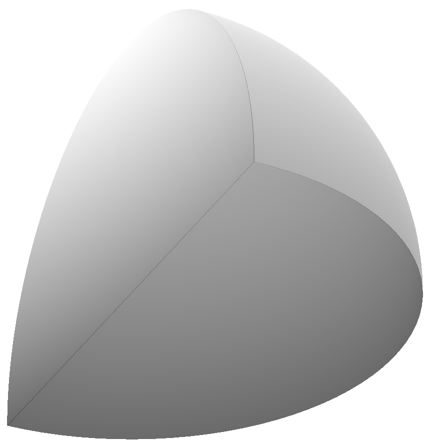

Figure 1. The 'potato wedge' domain Ω_{23}^+.

We calculate the term

$$\int_{\partial\Omega_{12}^+} \nu\otimes\nu\,\mathrm{d}\sigma = \underbrace{\int_{\Omega_{12}^+\cap\{x_1=0\}} \nu\otimes\nu\,\mathrm{d}\sigma}_{:=\mathcal{I}_1} + \underbrace{\int_{\Omega_{12}^+\cap\{x_2=0\}} \nu\otimes\nu\,\mathrm{d}\sigma}_{:=\mathcal{I}_2} + \underbrace{\int_{\partial\Omega_{12}^+\cap\{x_1\cdot x_2>0\}} \nu\otimes\nu\,\mathrm{d}\sigma}_{:=\mathcal{I}_3}$$

Then:

$$\mathcal{I}_1 = e_1\otimes e_1 \int_{\{x_2^2+x_3^2\le 1,\, x_1=0\}} \mathrm{d}\sigma = \frac{\pi}{2} e_1\otimes e_1 \tag{3.4}$$

where $e_1 := (1,0,0)$. Similarly we get $\mathcal{I}_2 = \frac{\pi}{2} e_2\otimes e_2$ with $e_2 := (0,1,0)$. Finally:

$$(\mathcal{I}_3)_{ij} = \int_{\partial\Omega_{12}^+\cap\{x_1\cdot x_2>0\}} x_i x_j\,\mathrm{d}\sigma(x), \qquad \forall\, i,j\in\{1,2,3\}. \tag{3.5}$$

Because x_1x_3, respectively x_2x_3 are odd functions in the variable x_3 along the domain $\Omega_{12}^+\cap\{x_1\cdot x_2>0\}$ we have that $(\mathcal{I}_3)_{13} = (\mathcal{I}_3)_{31} = (\mathcal{I}_3)_{23} = (\mathcal{I}_3)_{32} = 0$. Furthermore:

$$\begin{aligned}(\mathcal{I}_3)_{12} = (\mathcal{I}_3)_{21} &= \int_{\partial\Omega_{12}^+\cap\{x_1\cdot x_2>0\}} x_1x_2\,\mathrm{d}\sigma_x = \int_0^\pi \left(\int_0^{\frac{\pi}{2}} \sin^3\theta\cos\varphi\sin\varphi\ \mathrm{d}\varphi\right)\mathrm{d}\theta \\ &= \int_0^\pi \sin^3\theta\,\mathrm{d}\theta\cdot\int_0^{\frac{\pi}{2}}\cos\varphi\sin\varphi\ \mathrm{d}\varphi = \frac{4}{3}\cdot\frac{1}{2} = \frac{2}{3}\end{aligned} \tag{3.6}$$

Similarly we get: $(\mathcal{I}_3)_{11} = (\mathcal{I}_3)_{22} = (\mathcal{I}_3)_{33} = \frac{\pi}{3}$. Summarizing the last calculations, we get:

$$\int_{\partial\Omega_{12}^+} \nu\otimes\nu\,\mathrm{d}\sigma = \begin{pmatrix} \frac{\pi}{3}+\frac{\pi}{2} & \frac{2}{3} & 0 \\ \frac{2}{3} & \frac{\pi}{3}+\frac{\pi}{2} & 0 \\ 0 & 0 & \frac{\pi}{3} \end{pmatrix} \tag{3.7}$$

Analogous calculations provide

$$\int_{\partial\Omega_{13}^+} \nu\otimes\nu\,\mathrm{d}\sigma = \begin{pmatrix} \frac{\pi}{3}+\frac{\pi}{2} & 0 & \frac{2}{3} \\ 0 & \frac{\pi}{3} & 0 \\ \frac{2}{3} & 0 & \frac{\pi}{3}+\frac{\pi}{2} \end{pmatrix} \tag{3.8}$$

$$\int_{\partial\Omega_{23}^+} \nu\otimes\nu\,\mathrm{d}\sigma = \begin{pmatrix} \frac{\pi}{3} & 0 & 0 \\ 0 & \frac{\pi}{3}+\frac{\pi}{2} & \frac{2}{3} \\ 0 & \frac{2}{3} & \frac{\pi}{3}+\frac{\pi}{2} \end{pmatrix} \tag{3.9}$$

Similarly we define, for $1\le i<j\le 3$,

$$\Omega_{ij}^- := \{x = (x_1,x_2,x_3)\in\mathbb{R}^3\colon |x|\le 1,\ x_i\le 0,\ x_j\ge 0\} \tag{3.10}$$

and we have:

$$\int_{\partial\Omega_{12}^-} \nu\otimes\nu\,\mathrm{d}\sigma = \begin{pmatrix} \frac{\pi}{3}+\frac{\pi}{2} & -\frac{2}{3} & 0 \\ -\frac{2}{3} & \frac{\pi}{3}+\frac{\pi}{2} & 0 \\ 0 & 0 & \frac{\pi}{3} \end{pmatrix} \tag{3.11}$$

respectively

$$\int_{\partial\Omega_{13}^{-}} \nu \otimes \nu \, \mathrm{d}\sigma = \begin{pmatrix} \frac{\pi}{3}+\frac{\pi}{2} & 0 & -\frac{2}{3} \\ 0 & \frac{\pi}{3} & 0 \\ -\frac{2}{3} & 0 & \frac{\pi}{3}+\frac{\pi}{2} \end{pmatrix} \tag{3.12}$$

$$\int_{\partial\Omega_{23}^{-}} \nu \otimes \nu \, \mathrm{d}\sigma = \begin{pmatrix} \frac{\pi}{3} & 0 & 0 \\ 0 & \frac{\pi}{3}+\frac{\pi}{2} & -\frac{2}{3} \\ 0 & -\frac{2}{3} & \frac{\pi}{3}+\frac{\pi}{2} \end{pmatrix} \tag{3.13}$$

We take then:

$$\mathfrak{P}_1 := \Omega_{23}^{+} \cup (\Omega_{23}^{-} - (0,1,0)), \quad \mathfrak{P}_2 := \Omega_{13}^{+} \cup (\Omega_{13}^{-} - (1,0,0)),$$
$$\mathfrak{P}_3 := \Omega_{12}^{+} \cup (\Omega_{12}^{-} - (1,0,0))$$

and, respectively

$$\mathfrak{P}_4 := \Omega_{12}^{+}, \qquad \mathfrak{P}_5 := \Omega_{13}^{+}, \qquad \mathfrak{P}_6 := \Omega_{23}^{+}.$$ □

Proposition 3.2. *Let P be a 3×3 symmetric matrix, not necessarily traceless, and $W \in \mathbb{R}$. There exists a family of $J_P \in \mathbb{N}$ shapes $\mathscr{P}^j$ and corresponding surface energy strengths i_j, $j \in \{1, \ldots, J_P\}$ such that, taking for each shape the Rapini-Papoular surface energy with corresponding intensity i_j, i.e.,*

$$f_s^j(Q,\nu) = W\, i_j \,\mathrm{tr}(Q - Q_\nu)^2 \tag{3.14}$$

with $Q_\nu := \nu \otimes \nu - \frac{1}{3}\mathrm{Id}$ and ν is the exterior unit-normal, the corresponding homogenised potential is:

$$f_{hom}^P(Q) = -W\alpha_P\left(\frac{1}{3} + \frac{1}{2}\mathrm{tr}(Q^2)\right) + W\mathrm{tr}(QP) \tag{3.15}$$

with $\alpha_P \in \mathbb{R}$ explicitly computable in terms of the shapes volumes and the surface energy strengths.

Proof. We take M_k, $k \in \{1,\ldots,6\}$, as provided in Proposition 3.1, to be a linear basis in the spaces of 3×3 symmetric matrices. Then there exists $a_k := P \cdot M_k$, $k \in \{1,\ldots,6\}$ such that

$$P = \sum_{k=1}^{6} a_k M_k$$

Let $\mathscr{P}^1, \ldots, \mathscr{P}^{J_P}$ be the connected components of $\mathfrak{P}^1, \ldots, \mathfrak{P}^k$. Each $\mathscr{P}^j$ is a compact, convex set of the form (3.3) or (3.10) (see Figure 1). For $j \in \{1, \ldots, J_P\}$, we define the corresponding intensities as $i_j = -\frac{1}{2}a_k$ where $k = k(j)$ is such that $\mathscr{P}^j \subseteq \mathfrak{P}^k$. Then, noting that the homogenised potentials corresponding to each species will add together to provide the homogenised porential corresponding to all the species, we get:

$$f_{hom}^P(Q) = -\frac{1}{2}\sum_{k=1}^{7} a_k f_{hom}^{\mathfrak{P}^k}(Q) = -W\left(\frac{1}{3} + \frac{1}{2}\mathrm{tr}(Q^2)\right)\sum_{k=1}^{6} i_k\sigma(\partial\mathfrak{P}_k) + W\mathrm{tr}(QP) \tag{3.16}$$

hence we obtain the claimed (3.15) with $\alpha_P := \sum_{k=1}^{6} i_k\sigma(\partial\mathfrak{P}_k)$. □

Remark 3.3. *We can, without loss of generality, drop the constant term in a bulk potential, since adding a constant to an energy functional does not change the minimiser. In particular in f^P_{hom} we can ignore the term $-\frac{W}{3}\alpha_P$.*

We wish now to choose the surface energy densities f_s^j of Rapini-Papoular type and the shapes of the colloidal particles, in such a way that given the symmetric 3×3 matrix P and $W \in \mathbb{R}$, local minimisers of the Landau-de Gennes functional

$$\begin{aligned}\mathscr{F}_\varepsilon[Q] = \int_{\Omega_\varepsilon} \left(f_e^{LdG}(\nabla Q) + a\,\mathrm{tr}(Q^2) - b\,\mathrm{tr}(Q^3) + c\left(\mathrm{tr}(Q^2)\right)^2 \right) \mathrm{d}x \\ + \varepsilon^{3-2\alpha} \sum_{j=1}^{J} \int_{\partial \mathscr{P}_\varepsilon^j} f_s^j(Q,\, \nu)\, \mathrm{d}\sigma\end{aligned} \tag{3.17}$$

(with f_e^{LdG} given by (2.2)) converge to local minimisers of the homogenised functional

$$\mathscr{F}_0[Q] = \int_\Omega \left(f_e^{LdG}(\nabla Q) + a'\,\mathrm{tr}(Q^2) - b\,\mathrm{tr}(Q^3) + c\left(\mathrm{tr}(Q^2)\right)^2 + W\mathrm{tr}(PQ) \right) \mathrm{d}x. \tag{3.18}$$

We will assume that $1 < \alpha < 3/2$ and the centres of the inclusions, $x_\varepsilon^{i,j}$, satisfy (H_2) and that they are uniformly distributed, i.e., they satify (H_3) with $\xi^j = 1$. We also assume that all inclusions of the same family are parallel to each other, that is, we take $R_\varepsilon^{i,j} = \mathrm{Id}$ for any $i,\, j,\, \varepsilon$ (in particular, (H_4) is satisfied with $R_*^j = \mathrm{Id}$).

Remark 3.4. *One could also choose colloidal particles and corresponding surface energies that modify the b and c coefficients, but for this it would not suffice to use Rapini-Papoular type of surface energies (see for instance Section* 2.2 *in [12]).*

Corollary 3.5. *Let $(a,\, b,\, c) \in \mathbb{R}^3$ with $c > 0$. Let $a' \in \mathbb{R}$, $W > 0$, and let P be a symmetric, 3×3 matrix. Suppose that the inequalities* (2.3) *are satisfied. Then, there exists a family of shapes $\mathscr{P}^j$ and a corresponding surface energy f_s^j for each of them, such that for any isolated local minimiser Q_0 of the functional $\mathscr{F}_0$ defined by* (3.18)*, and for $\varepsilon > 0$ small enough, there exists a local minimiser Q_ε of the functional $\mathscr{F}_\varepsilon$, defined by* (3.17)*, such that $E_\varepsilon Q_\varepsilon \to Q_0$ strongly in $H^1(\Omega,\, \mathscr{S}_0)$.*

Proof. This statement is a particular case of our main result, Theorem 2.1. If (2.3) holds and $c > 0$, $c' > 0$, then the conditions (H_6)–(H_7) are satisfied.

We take J_P species $\mathscr{P}^j$, $j \in \{1, \ldots, J_P\}$ and surface energies given by (3.14), as in Proposition 3.1. Each $\mathscr{P}_j$ is a compact, convex set of the form (3.3) or (3.10), so (H_5) is satisfied (up to translations) and (H_8) is satisfied too. The homogenised potential corresponding to these is:

$$f^P_{hom}(Q) = -W\alpha_P \left(\frac{1}{3} + \frac{1}{2}\mathrm{tr}(Q^2) \right) + W\mathrm{tr}(QP) \tag{3.19}$$

where $\alpha_P := \sum_{k=1}^{6} \frac{P \cdot M_k}{2} \sigma(\partial \mathfrak{P}_k)$ (and the $M_k, k \in \{1, \ldots, 6\}$ are those from Proposition 3.1). We further take one more species, of spherical colloids $\mathscr{P}^{J_P+1} := B_1$, and define the surface energy density

$$f_s^{J_P+1}(Q,\, \nu) := \frac{1}{4\pi} \left(a' + \frac{W\alpha_P}{2} - a \right) (\nu \cdot Q^2 \nu). \tag{3.20}$$

This produces (see also for instance Remark 2.9 in [12]) a homogenised potential

$$f_{hom}^{sph}(Q) := \left(a' + \frac{W\alpha_P}{2} - a\right)\mathrm{tr}(Q^2)$$

Then the homogenised potential for all the $J_P + 1$ species is

$$\begin{aligned} f_{hom}(Q) &= f_{hom}^{sph}(Q) + f_{hom}^{P}(Q) \\ &= (a' - a)\,\mathrm{tr}(Q^2) + b\,\mathrm{tr}(Q^3) + c\,(\mathrm{tr}(Q^2))^2 + W\mathrm{tr}(PQ) - \frac{W}{3}\alpha_P \end{aligned}$$

and, since we can, without loss of generality, see Remark 3.3 drop the constant term $-\frac{W}{3}\alpha_P$, the corollary follows from Theorem 2.1. □

4. The limit functional in the case of stronger anchoring strength

The purpose of this section is to study the asymptotic behaviour, as $\varepsilon \to 0$, of minimisers of a functional with a different choice of the scaling for the surface anchoring strength. We consider the free energy functional:

$$\mathscr{F}_{\varepsilon,\gamma}[Q] := \int_{\Omega_\varepsilon} (f_e(\nabla Q) + f_b(Q))\,\mathrm{d}x + \varepsilon^{3-2\alpha-\gamma} \sum_{j=1}^{J} \int_{\partial\mathscr{P}_\varepsilon^j} f_s^j(Q, \nu)\,\mathrm{d}\sigma. \tag{4.1}$$

(where $\nu(x)$ denotes as usually the exterior normal at the point x on the boundary), with $\alpha \in (1, \frac{3}{2})$ and

K$_1$. $0 < \gamma < 1/4$.

Due to the extra factor $\varepsilon^{-\gamma}$ in front of the surface integral, we *cannot* apply Proposition 2.6 to obtain the lower semi-continuity of $\mathscr{F}_{\varepsilon,\gamma}$ for fixed ε. Therefore, in contrast with the previous sections, we assume boundedness from below on the surface term.

K$_2$. $f_s \geq 0$.

Remark 4.1. *Under the assumption* (K$_2$)*, the sequential weak lower semi-continuity of $\mathscr{F}_\varepsilon$ (for fixed ε) follows from the compact embedding $H^{1/2}(\partial\Omega_\varepsilon) \hookrightarrow L^2(\partial\Omega_\varepsilon)$ and Fatou's lemma. Therefore, a routine application of the direct method of the Calculus of Variations shows that minimisers of $\mathscr{F}_\varepsilon$ exist, for any $\varepsilon > 0$.*

As a consequence of (K$_2$) and of (H$_3$), the function f_{hom} is non-negative, too. In fact, we will also assume that

K$_3$. $\inf\{f_{hom}(Q, x) \colon Q \in \mathscr{S}_0\} = 0$ for any $x \in \overline{\Omega}$.

Recall that, for any $j \in \{1, \ldots, J\}$, the measures $\mu_\varepsilon^j := \varepsilon^{-3} \sum_i \delta_{x_\varepsilon^{i,j}}$ are supposed to converge weakly* to a non-negative function $\xi^j \in L^\infty(\Omega)$. We need to prescribe a rate of convergence. We express the rate of convergence in terms of the $W^{-1,1}$-norm (that is, the dual Lipschitz norm, also known as flat norm in some contexts):

$$\begin{aligned} \mathbb{F}_\varepsilon := \max_{j=1,2,\ldots,J} \sup\Big\{ &\int_\Omega \varphi\,\mathrm{d}\mu_\varepsilon^j - \int_\Omega \varphi\,\xi^j\,\mathrm{d}x \colon \\ &\varphi \in W^{1,\infty}(\Omega), \quad \|\nabla\varphi\|_{L^\infty(\Omega)} + \|\varphi\|_{L^\infty(\Omega)} \leq 1 \Big\}. \end{aligned} \tag{4.2}$$

K_4. There exists a constant $\lambda_{\mathrm{flat}} > 0$ such that $\mathbb{F}_\varepsilon \le \lambda_{\mathrm{flat}}\varepsilon$ for any ε.

Remark 4.2. *The assumption* (K_4) *is satisfied if the inclusions are periodically distributed. Consider, for simplicity, the case* $J = 1$*, and suppose that the centres of the inclusions,* x^i_ε*, are exactly the points* $y \in (\varepsilon\mathbb{Z})^3$ *such that* $y + [-\varepsilon/2, \varepsilon/2]^3 \subseteq \Omega$*. Let* $\Omega_\varepsilon := \cup_i(x^i_\varepsilon + [-\varepsilon/2, \varepsilon/2]^3)$*. Then, for any* $\varphi \in W^{1,\infty}(\Omega)$*, we have*

$$\left|\int_\Omega \varphi \,\mathrm{d}\mu_\varepsilon - \int_\Omega \varphi \,\mathrm{d}x\right| \le \sum_{i=1}^{N_\varepsilon} \int_{x^i_\varepsilon + [-\varepsilon/2,\, \varepsilon/2]^3} \left|\varphi - \varphi(x^i_\varepsilon)\right| + \int_{\Omega\setminus\Omega_\varepsilon} |\varphi|$$
$$\le \frac{\sqrt{3}\,\varepsilon}{2} \|\nabla\varphi\|_{L^\infty(\Omega)} |\Omega_\varepsilon| + \|\varphi\|_{L^\infty(\Omega)} |\Omega \setminus \Omega_\varepsilon| \,.$$

Moreover, $|\Omega \setminus \Omega_\varepsilon| \lesssim \varepsilon$*, because* $\Omega \setminus \Omega_\varepsilon \subseteq \{y \in \Omega \colon \mathrm{dist}(y, \partial\Omega) \le \sqrt{3}\,\varepsilon\}$*. Therefore,* (K_4) *holds.*

Finally, we assume some regularity on the boundary datum $g \colon \partial\Omega \to \mathscr{S}_0$.

K_5. g is bounded and Lipschitz.

Γ-convergence to a constrained problem. We can now define the candidate limit functional. Let

$$\mathscr{A} := \left\{Q \in H^1_g(\Omega, \mathscr{S}_0) \colon f_{hom}(x, Q(x)) = 0 \ \text{ for a.e. } x \in \Omega\right\}, \tag{4.3}$$

and $\widetilde{\mathscr{F}} \colon L^2(\Omega, \mathscr{S}_0) \to (-\infty, +\infty]$,

$$\widetilde{\mathscr{F}}(Q) := \begin{cases} \displaystyle\int_\Omega (f_e(\nabla Q) + f_b(Q))\,\mathrm{d}x & \text{if } Q \in \mathscr{A} \\ +\infty & \text{otherwise.} \end{cases}$$

Theorem 4.3. *Suppose that the assumptions* (H_1)–(H_8)*,* (K_1)–(K_5) *are satisfied. Then, the following statements hold.*

i. Given a family of maps $Q_\varepsilon \in H^1_g(\Omega_\varepsilon, \mathscr{S}_0)$ *such that* $\sup_\varepsilon \mathscr{F}_{\varepsilon,\gamma}(Q) < +\infty$*, there exists a non-relabelled sequence and* $Q_0 \in \mathscr{A}$ *such that* $E_\varepsilon Q_\varepsilon \rightharpoonup Q_0$ *weakly in* $H^1(\Omega)$*,*

$$\widetilde{\mathscr{F}}(Q_0) \le \liminf_{\varepsilon\to 0} \mathscr{F}_{\varepsilon,\gamma}(Q_\varepsilon).$$

ii. For any $Q_0 \in \mathscr{A}$*, there exists a sequence of maps* $Q_\varepsilon \in H^1_g(\Omega_\varepsilon, \mathscr{S}_0)$ *such that* $E_\varepsilon Q_\varepsilon \to Q_0$ *strongly in* $H^1(\Omega)$ *and*

$$\limsup_{\varepsilon\to 0} \mathscr{F}_{\varepsilon,\gamma}(Q_\varepsilon) \le \widetilde{\mathscr{F}}(Q_0).$$

Remark 4.4. *The theorem is only meaningful when* $\mathscr{A}$ *is non-empty, and it may happen that* $\mathscr{A}$ *is empty even if* $f_{hom}(g(x), x) = 0$ *for any* $x \in \partial\Omega$*.*

4.1. Proof of Theorem 4.3

Before we give the proof of Theorem 4.3, we state some auxiliary results. We first recall some properties of the convolution, which will be useful in constructing the recovery sequence.

Lemma 4.5. *For any $P \in H^1(\mathbb{R}^3, \mathscr{S}_0)$ and $\sigma > 0$, there exists a smooth map $P_\sigma\colon \mathbb{R}^3 \to \mathscr{S}_0$ that satisfies the following properties:*

$$\|P_\sigma\|_{L^\infty(\mathbb{R}^3)} \lesssim \sigma^{-1/2} \|P\|_{L^6(\mathbb{R}^3)}, \quad \|\nabla P_\sigma\|_{L^\infty(\mathbb{R}^3)} \lesssim \sigma^{-3/2} \|\nabla P\|_{L^2(\mathbb{R}^3)} \tag{4.4}$$

$$\|P - P_\sigma\|_{L^2(\mathbb{R}^3)} \lesssim \sigma \|\nabla P\|_{L^2(\mathbb{R}^3)} \tag{4.5}$$

$$\|\nabla P - \nabla P_\sigma\|_{L^2(\mathbb{R}^3)} \to 0 \qquad \text{as } \sigma \to 0. \tag{4.6}$$

Moreover, if $U \subseteq U'$ are Borel subsets of $\mathbb{R}^3$ such that $\mathrm{dist}(U, \mathbb{R}^3 \setminus U') > \sigma$, then

$$\|P_\sigma\|_{L^2(U)} \leq \|P\|_{L^2(U')}. \tag{4.7}$$

Proof. Let us take a non-negative, even function $\varphi \in C^\infty_c(\mathbb{R}^3)$, supported in the unit ball B_1, such that $\|\varphi\|_{L^1(\mathbb{R}^3)} = 1$. Let $\varphi_\sigma(x) := \sigma^{-3}\varphi(x/\sigma)$. By a change of variable, we see that

$$\|\varphi_\sigma\|_{L^p(\mathbb{R}^3)} = \sigma^{3/p-3} \|\varphi\|_{L^p(\mathbb{R}^3)} \qquad \text{for any } p \in [1, +\infty). \tag{4.8}$$

Let P_σ be defined as the convolution $P_\sigma := P * \varphi_\sigma$. Then, by Young's inequality, we have

$$\|\nabla P_\sigma\|_{L^\infty(\mathbb{R}^3)} = \|(\nabla P) * \varphi_\sigma\|_{L^\infty(\mathbb{R}^3)} \leq \|\nabla P\|_{L^2(\mathbb{R}^3)} \|\varphi_\sigma\|_{L^2(\mathbb{R}^3)} \overset{(4.8)}{\lesssim} \sigma^{-3/2} \|\nabla P\|_{L^2(\mathbb{R}^3)}.$$

The other inequality in (4.4) is obtained in a similar way. The condition (4.6) is a well-known property of convolutions.

Let us prove (4.5). Let ψ be the Fourier transform* of φ. Then, ψ is smooth and rapidly decaying (that is, it belongs to the Schwartz space $\mathscr{S}(\mathbb{R}^3)$) and, in particular, it is Lipschitz continuous. Moreover, $\psi(0) = \int_{\mathbb{R}^3} \varphi(x)\,\mathrm{d}x = 1$. By the properties of the Fourier transform, we have $\widehat{\varphi_\sigma}(\xi) = \psi(\sigma\xi)$. By applying Plancherel theorem, we obtain

$$\begin{aligned} \|P - P_\sigma\|^2_{L^2(\mathbb{R}^3)} &= \int_{\mathbb{R}^3} |\hat{P}(\xi)|^2 (1 - \psi(\sigma\xi))^2 \,\mathrm{d}\xi \\ &= \int_{\mathbb{R}^3} |\hat{P}(\xi)|^2 (\psi(0) - \psi(\sigma\xi))^2 \,\mathrm{d}\xi \\ &\leq \sigma^2 \|\nabla\psi\|^2_{L^\infty(\mathbb{R}^3)} \int_{\mathbb{R}^3} |\xi|^2 \, |\hat{P}(\xi)|^2 \,\mathrm{d}\xi \\ &= \frac{\sigma^2}{4\pi^2} \|\nabla\psi\|^2_{L^\infty(\mathbb{R}^3)} \|\nabla P\|^2_{L^2(\mathbb{R}^3)}. \end{aligned}$$

It only remains to prove (4.7). Let χ be the indicator function of U' (i.e., $\chi = 1$ on U' and $\chi = 0$ elsewhere). Observe that $P_\sigma = (\chi P) * \varphi_\sigma$ on U, because φ_σ is supported on the ball B_σ of radius σ and, by assumption, $\mathrm{dist}(U, \mathbb{R}^3 \setminus U') > \sigma$. Then, Young inequality implies

$$\|P\|_{L^2(U)} \leq \|\chi P\|_{L^2(\mathbb{R}^3)} \|\varphi_\sigma\|_{L^1(\mathbb{R}^3)} = \|P\|_{L^2(U')}. \qquad \square$$

*We adopt the convention $\widehat{\varphi}(\xi) = \int_{\mathbb{R}^3} \varphi(x) \exp(-2\pi i x \cdot \xi)\,\mathrm{d}x$.

Lemma 4.6. *Let $\Omega \subseteq \mathbb{R}^3$ a bounded, smooth domain, and let $g\colon \Omega \to \mathscr{S}_0$ be a bounded, Lipschitz map. For any $Q \in H^1_g(\Omega, \mathscr{S}_0)$ and $\sigma \in (0, 1)$, there exists a bounded, Lipschitz map $Q_\sigma\colon \overline{\Omega} \to \mathscr{S}_0$ that satisfies the following properties:*

$$Q_\sigma = g \qquad \text{on } \partial\Omega \tag{4.9}$$

$$\|Q_\sigma\|_{L^\infty(\Omega)} \lesssim \sigma^{-1/2}\left(\|Q\|_{H^1(\Omega)} + \|g\|_{L^\infty(\Omega)}\right) \tag{4.10}$$

$$\|\nabla Q_\sigma\|_{L^\infty(\Omega)} \lesssim \sigma^{-3/2}\left(\|Q\|_{H^1(\Omega)} + \|g\|_{W^{1,\infty}(\Omega)}\right) \tag{4.11}$$

$$\|Q - Q_\sigma\|_{L^2(\Omega)} \lesssim \sigma\,\|Q\|_{H^1(\Omega)} \tag{4.12}$$

$$\|\nabla Q - \nabla Q_\sigma\|_{L^2(\Omega)} \to 0 \qquad \text{as } \sigma \to 0. \tag{4.13}$$

Proof. Since Ω is bounded and smooth, we can extend g to a bounded, Lipschitz map $\mathbb{R}^3 \to \mathscr{S}_0$, still denoted g for simplicity, in such a way that $\|g\|_{L^\infty(\mathbb{R}^3)} \lesssim \|g\|_{L^\infty(\Omega)}$, $\|\nabla g\|_{L^\infty(\mathbb{R}^3)} \lesssim \|\nabla g\|_{L^\infty(\Omega)}$. Let $P := Q - g$. Then, $P \in H^1_0(\Omega, \mathscr{S}_0)$, and we extend P to a new map $P \in H^1(\mathbb{R}^3, \mathscr{S}_0)$ by setting $P := 0$ on $\mathbb{R}^3 \setminus \Omega$. By applying Lemma 4.5 to P, we construct a family of smooth maps $(P_\sigma)_{\sigma>0}$ that satisfies (4.4)–(4.7). We define

$$\tilde{P}_\sigma(x) := \min\left(1,\, \sigma^{-1}\mathrm{dist}(x, \partial\Omega)\right) P_\sigma(x) \qquad \text{for } x \in \Omega.$$

The map $\tilde{P}_\sigma\colon \overline{\Omega} \to \mathscr{S}_0$ is bounded, Lipschitz, and $\tilde{P}_\sigma = 0$ on $\partial\Omega$. We claim that $\tilde{P}_\sigma$ satisfies (4.10)–(4.13) with $g \equiv 0$; the lemma will follow by taking $Q_\sigma := \tilde{P}_\sigma + g$. First, we note that (4.10) is a consequence of the extension of P to the whole $\mathbb{R}^3$ and (4.4) together with the Gagliardo-Nirenberg-Soboleve inequality. Then we check (4.11). Clearly $|\tilde{P}_\sigma| \leq |P_\sigma|$. Using the chain rule, and keeping in mind that the function $\mathrm{dist}(\cdot, \partial\Omega)$ is 1-Lipschitz, we see that

$$|\nabla\tilde{P}_\sigma| \leq |\nabla P_\sigma| + \sigma^{-1}|P_\sigma| \qquad \text{a.e. on } \Omega \tag{4.14}$$

and (4.11) follows, with the help of (4.4).

We pass to the proof of (4.12). Let $\Gamma_\sigma := \{x \in \mathbb{R}^3\colon \mathrm{dist}(x, \partial\Omega) < \sigma\}$. Since $\partial\Omega$ is a compact, smooth manifold, for sufficiently small σ the set Γ_σ is diffeomorphic to the product $\partial\Omega \times (-\sigma, \sigma)$. We identify $\Gamma_\sigma \simeq \partial\Omega \times (-\sigma, \sigma)$ and denote the variable in Γ_σ as $x = (y, t) \in \partial\Omega \times (-\sigma, \sigma)$. We apply Poincaré inequality to the map P on each slice $\{y\} \times (-\sigma, \sigma)$:

$$\int_{-\sigma}^{\sigma} |P(y, t)|^2\,\mathrm{d}t = \int_0^{\sigma} |P(y, t) - P(y, 0)|^2\,\mathrm{d}t \lesssim \sigma^2 \int_0^{\sigma} |\partial_t P(y, t)|^2\,\mathrm{d}t.$$

By integrating with respect to $y \in \partial\Omega$, we obtain

$$\|P\|_{L^2(\Gamma_\sigma)} \lesssim \sigma\,\|\nabla P\|_{L^2(\Gamma_\sigma)} \tag{4.15}$$

and hence,

$$\|\tilde{P}_\sigma - P_\sigma\|_{L^2(\Omega)} \lesssim \|P_\sigma\|_{L^2(\Gamma_\sigma)} \lesssim \|P\|_{L^2(\Gamma_\sigma)} + \|P - P_\sigma\|_{L^2(\Omega)} \overset{(4.5),(4.15)}{\lesssim} \sigma\,\|\nabla P\|_{L^2(\Omega)}.$$

Finally, let us prove (4.13). Combining (4.7) and (4.15), we deduce

$$\|P_\sigma\|_{L^2(\Gamma_\sigma)} \leq \|P\|_{L^2(\Gamma_{2\sigma})} \lesssim \sigma\,\|\nabla P\|_{L^2(\Gamma_{2\sigma})}. \tag{4.16}$$

Therefore, we have

$$\begin{aligned}\|\nabla\tilde{P}_\sigma - \nabla P_\sigma\|_{L^2(\Omega)} &\leq \|\nabla\tilde{P}_\sigma\|_{L^2(\Gamma_\sigma)} + \|\nabla P_\sigma\|_{L^2(\Gamma_\sigma)} \\ &\overset{(4.14)}{\lesssim} \|\nabla P_\sigma\|_{L^2(\Gamma_\sigma)} + \sigma^{-1}\|P_\sigma\|_{L^2(\Gamma_\sigma)} \\ &\overset{(4.16)}{\lesssim} \|\nabla P\|_{L^2(\Gamma_{2\sigma})} + \|\nabla P_\sigma - \nabla P\|_{L^2(\Omega)}\end{aligned}$$

and both terms in the right-hand side converge to zero as $\sigma \to 0$, due to (4.6). □

Lemma 4.7. *For any $Q \in H^1_g(\Omega, \mathscr{S}_0)$, there exists a sequence $(Q_\varepsilon)_{\varepsilon>0}$ in $H^1_g(\Omega, \mathscr{S}_0)$ that converges $H^1(\Omega)$-strongly to Q and satisfies*

$$|J_\varepsilon[Q_\varepsilon] - J_0[Q]| \lesssim \varepsilon^{1/4}\left(\|Q\|^4_{H^1(\Omega)} + 1\right)$$

(the functionals J_ε, J_0 are defined in (2.11), (2.12) *respectively). The constant implied in front of the right-hand side depends on the $L^\infty(\partial\Omega)$-norms of g and ∇g, as well as Ω, f^j_s, $\mathscr{P}^j$, R^j_* with $j \in \{1, \ldots, J\}$.*

Proof. Let us fix a small $\varepsilon > 0$. Let β be a positive parameter, to be chosen later, and let $Q_\varepsilon := Q_{\varepsilon^\beta} \in H^1_g(\Omega, \mathscr{S}_0)$ be the Lipschitz map given by Lemma 4.6. We have

$$|J_\varepsilon[Q_\varepsilon] - J_0[Q]| \leq |J_\varepsilon[Q_\varepsilon] - \tilde{J}_\varepsilon[Q_\varepsilon]| + |\tilde{J}_\varepsilon[Q_\varepsilon] - J_0[Q_\varepsilon]| + |J_0[Q_\varepsilon] - J_0[Q]| \tag{4.17}$$

where $\tilde{J}_\varepsilon$ is defined by (2.16). We will estimate separately all the terms in the right-hand side.

First, let us estimate the difference $J_\varepsilon[Q_\varepsilon] - \tilde{J}_\varepsilon[Q_\varepsilon]$. This can be achieved with the help of Lemma 2.8:

$$\begin{aligned}|J_\varepsilon[Q_\varepsilon] - \tilde{J}_\varepsilon[Q_\varepsilon]| &\overset{(2.18)}{\lesssim} \varepsilon^\alpha\left(\|Q_\varepsilon\|^3_{L^\infty(\Omega)} + 1\right)\|\nabla Q_\varepsilon\|_{L^\infty(\Omega)} \\ &\overset{(4.10),(4.11)}{\lesssim} \varepsilon^{\alpha-3\beta}\left(\|Q\|^4_{H^1(\Omega)} + 1\right)\end{aligned} \tag{4.18}$$

(here and througout the rest of the proof, the constant implied in front of the right-hand side may depend on the L^∞-norms of g and ∇g).

As for the second term, $\tilde{J}_\varepsilon[Q_\varepsilon] - J_0[Q_\varepsilon]$, we write $\tilde{J}_\varepsilon$ in the form (2.17) and we re-write J_0 using (2.5), (2.12):

$$\begin{aligned}|\tilde{J}_\varepsilon[Q_\varepsilon] - J_0[Q_\varepsilon]| &\leq \sum_{j=1}^J \left|\int_\Omega f^j_{hom}(Q_\varepsilon, x)\,\mathrm{d}\mu^j_\varepsilon - \int_\Omega f^j_{hom}(Q_\varepsilon, x)\,\xi^j\,\mathrm{d}x\right| \\ &\overset{(4.2)}{\leq} \mathbb{F}_\varepsilon \sum_{j=1}^J \left(\|\nabla(f^j_{hom}(Q_\varepsilon, \cdot))\|_{L^\infty(\Omega)} + \|f^j_{hom}(Q_\varepsilon, \cdot)\|_{L^\infty(\Omega)}\right).\end{aligned} \tag{4.19}$$

To estimate the terms at the right-hand side, we apply Lemma 2.7 and Lemma 4.6:

$$\begin{aligned}\|\nabla(f^j_{hom}(Q_\varepsilon, \cdot))\|_{L^\infty(\Omega)} &\overset{(2.13)}{\lesssim} \left(\|Q_\varepsilon\|^3_{L^\infty(\Omega)} + 1\right)\|\nabla Q_\varepsilon\|_{L^\infty(\Omega)} \\ &\overset{(4.11)}{\lesssim} \varepsilon^{-3\beta}\left(\|Q\|^4_{H^1(\Omega)} + 1\right),\end{aligned}$$

and

$$\|f^j_{hom}(Q_\varepsilon, \cdot)\|_{L^\infty(\Omega)} \overset{(2.13)}{\lesssim} \|Q_\varepsilon\|^4_{L^\infty(\Omega)} + 1 \overset{(4.11)}{\lesssim} \varepsilon^{-2\beta}\left(\|Q\|^4_{H^1(\Omega)} + 1\right).$$

Injecting these inequalities into (4.19), and using that $\mathbb{F}_\varepsilon \lesssim \varepsilon$ by Assumption (4.2), we obtain

$$|\tilde{J}_\varepsilon[Q_\varepsilon] - J_0[Q_\varepsilon]| \lesssim \varepsilon^{1-3\beta}\left(\|Q\|^4_{H^1(\Omega)} + 1\right). \tag{4.20}$$

Finally, the term $J_0[Q_\varepsilon] - J_0[Q]$. We apply Lemma 2.7 and the Hölder inequality:

$$\begin{aligned} |J_0[Q_\varepsilon] - J_0[Q]| &\leq \int_\Omega |f_{hom}(Q_\varepsilon, \cdot) - f_{hom}(Q, \cdot)| \\ &\overset{(2.14)}{\leq} \int_\Omega \left(|Q|^3 + |Q_\varepsilon|^3 + 1\right)|Q - Q_\varepsilon| \\ &\lesssim \left(\|Q\|^3_{L^6(\Omega)} + \|Q_\varepsilon\|^3_{L^6(\Omega)} + 1\right)\|Q - Q_\varepsilon\|_{L^2(\Omega)} \end{aligned}$$

The sequence Q_ε is bounded in $L^6(\Omega)$, thanks to Sobolev embedding and to Lemma 4.6. Therefore,

$$|J_0[Q_\varepsilon] - J_0[Q]| \overset{(4.12)}{\lesssim} \varepsilon^\beta \|Q\|^4_{H^1(\Omega)} + \varepsilon^\beta \|Q\|_{H^1(\Omega)}. \tag{4.21}$$

Combining (4.17), (4.18), (4.20) and (4.21), we deduce

$$|J_\varepsilon[Q_\varepsilon] - J_0[Q]| \lesssim \varepsilon^{\min(\alpha-3\beta,\, 1-3\beta, \beta)}\left(\|Q\|^4_{H^1(\Omega)} + 1\right).$$

Keeping into account that $\alpha > 1$, we see that the optimal choice of β is $\beta = 1/4$, and the lemma follows. □

Lemma 4.8. *Let $Q_\varepsilon \in H^1(\Omega_\varepsilon, \mathscr{S}_0)$ be a family of maps, such that $E_\varepsilon Q_\varepsilon \to Q$ strongly in $H^1(\Omega)$, as $\varepsilon \to 0$. Then,*

$$\limsup_{\varepsilon\to 0} \int_{\Omega_\varepsilon} (f_e(\nabla Q_\varepsilon) + f_b(Q_\varepsilon))\,\mathrm{d}x \leq \int_\Omega (f_e(\nabla Q) + f_b(Q))\,\mathrm{d}x.$$

Proof. By Sobolev embedding, we have $E_\varepsilon Q_\varepsilon \to Q$ strongly in $L^6(\Omega)$. Then, up to extraction of a non-relabelled subsequence, we find functions $h_e \in L^2(\Omega)$, $h_b \in L^6(\Omega)$ such that

$$|\nabla(E_\varepsilon Q_\varepsilon)| \leq h_e, \quad |E_\varepsilon Q_\varepsilon| \leq h_b \qquad \text{a.e. on } \Omega, \text{ for any } \varepsilon. \tag{4.22}$$

Let χ_ε be the the indicator function of Ω_ε (i.e., $\chi_\varepsilon := 1$ on Ω_ε and $\chi_\varepsilon := 0$ elsewhere). Thanks to (H_6), (H_7) and to (4.22), we have

$$(f_e(\nabla Q_\varepsilon) + f_b(Q_\varepsilon))\chi_\varepsilon \lesssim h_e^2 + h_b^6 + 1 \in L^1(\Omega) \qquad \text{a.e. on } \Omega, \text{ for any } \varepsilon.$$

Moreover, since $|\mathscr{P}_\varepsilon| \lesssim \varepsilon^{3\alpha-3} \to 0$, χ_ε converges to 1 strongly in $L^1(\Omega)$ and we may extract a further subsequence so to have $\chi_\varepsilon \to 1$ a.e. Then, the lemma follows from Lebesgue's dominated converge theorem. □

Proof of Theorem 4.3. Let $Q_\varepsilon \in H^1_g(\Omega_\varepsilon, \mathscr{S}_0)$, for $\varepsilon > 0$, be a family of maps such that $\sup_\varepsilon \mathscr{F}_{\varepsilon,\gamma}(Q) < +\infty$. We first extract a (non-relabelled) subsequence $\varepsilon \to 0$, so that $\limsup_{\varepsilon\to 0} \mathscr{F}_{\varepsilon,\gamma}(Q_\varepsilon)$ is achieved as a limit; this allows us to pass freely to subsequences, in what follows. Thanks to (H_6), (H_7), (K_2), we have $\sup_\varepsilon \|Q_\varepsilon\|_{L^2(\Omega_\varepsilon)} < +\infty$. By Lemma 2.5, there is a (non-relabelled) subsequence and $Q_0 \in H^1_g(\Omega, \mathscr{S}_0)$ such that $E_\varepsilon Q_\varepsilon \rightharpoonup Q_0$ weakly in $H^1(\Omega)$. By Proposition 2.2, there holds

$$\widetilde{\mathscr{F}}(Q) \leq \liminf_{\varepsilon\to 0} \int_{\Omega_\varepsilon} (f_e(\nabla Q_\varepsilon) + f_b(Q_\varepsilon)) \leq \liminf_{\varepsilon\to 0} \mathscr{F}_{\varepsilon,\gamma}(Q_\varepsilon)$$

and

$$J_0[Q_0] = \lim_{\varepsilon\to 0} J_\varepsilon[Q_\varepsilon] \leq \limsup_{\varepsilon\to 0} \varepsilon^\gamma \mathscr{F}_{\varepsilon,\gamma}(Q_\varepsilon) = 0,$$

so Q_0 belongs to the class $\mathscr{A}$ defined by (4.3). Thus, Statement (i) is proved.

We now prove Statement (ii). Let $Q_0 \in H^1_g(\Omega, \Omega)$ be fixed. We can suppose without loss of generality that $Q \in \mathscr{A}$, otherwise the statement is trivial. Due to Lemma 4.7, there is a sequence $\tilde{Q}_\varepsilon \in H^1_g(\Omega, \mathscr{S}_0)$ such that $\tilde{Q}_\varepsilon \to Q_0$ strongly in $H^1(\Omega)$ and

$$\left|J_\varepsilon[\tilde{Q}_\varepsilon]\right| = \varepsilon^{3-2\alpha} \left| \sum_{j=1}^{J} \int_{\partial\mathscr{P}^j} f_s^j(\tilde{Q}_\varepsilon, \nu)\, \mathrm{d}\sigma \right| \lesssim \varepsilon^{1/4} \left(\|Q\|^4_{L^4(\Omega)} + 1 \right). \tag{4.23}$$

Let $Q_\varepsilon := \tilde{Q}_{\varepsilon|\Omega_\varepsilon}$. By Lemma 2.5, $E_\varepsilon Q_\varepsilon \to Q_0$ strongly in $H^1(\Omega)$. Using Lemma 4.8 and (4.23), and recalling that $\gamma < 1/4$, we conclude that

$$\limsup_{\varepsilon\to 0} \mathscr{F}_{\varepsilon,\gamma}(Q_\varepsilon) = \limsup_{\varepsilon\to 0} \int_{\Omega_\varepsilon} (f_e(\nabla Q_\varepsilon) + f_b(Q_\varepsilon)) + \underbrace{\limsup_{\varepsilon\to 0} \varepsilon^{-\gamma} J_\varepsilon[Q_\varepsilon]}_{=0, \text{ by } (4.23)} \leq \widetilde{\mathscr{F}}(Q_0),$$

so the proof is complete. □

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Conflict of interest

The authors declare that there is no conflict of interest regarding the publication of this article.

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