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Editorial

Inverse problems in imaging and engineering science†

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One of the most important tools in the theory of inverse problems is unique continuation of solutions to partial differential equations. In fact, five out of the eight articles in the present special issue "Inverse problems in imaging and engineering science" are closely related to unique continuation, giving applications of this mathematical technique in engineering sciences.

Two classical examples of unique continuation are analytic continuation of holomorphic functions and Holmgren's uniqueness theorem for linear partial differential equations with real analytic coefficients. In the context of inverse problems, it is often important to consider coefficients that are not real analytic, and in this case unique continuation goes back to [11] where Carleman proved uniqueness for the Cauchy problem for a large class of partial differential equations in two dimensions, with data on a non-characteristic curve.

Unique continuation problems are typically ill-posed. For instance, the classical example of Hadamard shows that the Cauchy problem for the Laplace equation can not be solved in general, and even when the solution exists, it does not depend continuously on the data. However, continuous dependence is recovered under an apriori bound for the solution. This type of continuity is called conditional stability and it was first systematically studied by Fritz John [14]. For further references on unique continuation problems in purely mathematical context, we direct the reader to the treatment of these problems using Carleman estimates by Hörmander [13], a modern classic.

The paper by Stefanov [8] in this special issue considers conditional stability estimates for unique continuation in the context of thermoacoustic tomography, an emerging technique in biomedical imaging. The technique is based on sending low frequency microwaves into a medium, causing rapid thermal expansion, which again generates acoustic wave propagating in the medium. The microwave absorption profile is then reconstructed from a measurement of the acoustic pressure. The reconstruction is based on solving an inverse initial source problem for the acoustic wave equation, a special type of unique continuation problem.

The inverse initial source problem for the wave equation is a starting point also in the paper by Alberti, Capdeboscq and Privat [1]. The problem can be unconditionally stable in favorable geometric settings, essentially characterized by Bardos, Lebeau and Rauch [9]. In [1] the authors begin by studying unconditional stability estimates when the initial source is a random variable, and then proceed to introduce an abstract framework for inverse problems with randomized stability constants.

The paper [3] by Chen, Cheng, Floridia, Wada and Yamamoto considers an inverse source problem different from that in [1, 8]. Their problem models the determination of air dose rates of radioactive substance at the human height level by high-altitude data, collected for example by a drone. From the mathematical point of view, the paper is based on unique continuation of a harmonic function along a line, and the corresponding conditional stability estimate.

Several elliptic partial differential equations are known to satisfy the strong unique continuation property saying that if a solution vanishes to infinite order at a point then it vanishes identically. García-Ferrero and Rüland [4] show that higher order fractional Laplace operators satisfy this property. The paper is aimed toward applications in Calderón type problems. The most classical example of such a problem arises as a mathematical model for electrical impedance tomography [10], an imaging technique in which the electrical conductivity of a body is inferred from surface electrode measurements.

In Calderón type problems the unknown quantities to be determined are spatially varying coefficients in an elliptic partial differential equation. Similar coefficient determination problems arise in engineering applications also for other types of equations. The paper [2] by Blåsten, Zouari, Louati and Ghidaoui considers a coefficient determination problem for a hyperbolic equation modeling the pressure and pipe cross-sectional discharge in a network of pipes, the application being blockage detection from remote measurements in the network. The method proposed in the paper uses unique continuation techniques to recover the cross-sectional pipe area.

It was shown by Greenleaf, Lassas and Uhlmann [12] that Calderón's problem does not have a unique solution when the electrical conductivity is allowed to be singular. Their proof was based on transformation optics and the same technique was later used to design cloaking devices [15, 16]. The paper [7] by Hoai-Minh Nguyen and Tu Nguyen gives a study of cloaking for the heat equation.

The paper by Lionheart [6], moving away from unique continuation and transformation optics, studies another set of important tools in the theory of inverse problems, namely ray transforms. While the most important application of ray transforms is computed tomography, the mathematical theory of which is based on the Radon transform [17], these transforms arise also in the theory of coefficient determination problems. Lionheart introduces in [6] a new concept of histogram ray transform and gives an application to analysis of the neutron transmission spectra near a Bragg edge.

The paper [5] by Li, Liu, Tsui and Wang does not consider an inverse problem per se, rather it applies techniques developed in the context of inverse scattering problems to shape generation. The work relies on machine learning that is becoming increasingly popular computational approach also in the field of inverse problems.

This collection of papers focuses on interesting inverse problems in engineering sciences. We hope that it can also act as an introduction to several of the most important mathematical techniques in inverse problems, for a reader not previously familiar with these.

References

1. Alberti G, Capdeboscq Y, Privat Y (2020) On the randomised stability constant for inverse problems. *Mathematics in Engineering* 2: 264–286.
2. Blåsten E, Zouari F, Louati M, et al. (2019) Blockage detection in networks: the area reconstruction method. *Mathematics in Engineering* 1: 849–880.
3. Chen Y, Cheng J, Floridia G, et al. (2020) Conditional stability for an inverse source problem and an application to the estimation of air dose rate radioactive substances by drone data. *Mathematics in Engineering* 2: 26–33.
4. García-Ferrero MÁ, Rüland A (2019) Strong unique continuation for the higher order fractional Laplacian. *Mathematics in Engineering* 1: 715–774.
5. Li J, Liu H, Tsui W-Y, et al. (2019) An inverse scattering approach for geometric body generation: a machine learning perspective. *Mathematics in Engineering* 1: 800–823.
6. Lionheart WRB (2020) Histogram tomography. *Mathematics in Engineering* 2: 55–74.
7. Nguyen H-M, Nguyen T (2019) Approximate cloaking for the heat equation via transformation optics. *Mathematics in Engineering* 1: 775–788.
8. Stefanov P (2020) Conditionally stable unique continuation and applications to thermoacoustic tomography. *Mathematics in Engineering* 2: 26–33.
9. Bardos C, Lebeau G, Rauch J (1992) Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary. *SIAM J Control Optim* 30: 1024–1065.
10. Calderón A-P (1980) On an inverse boundary value problem. *Soc Brasil Mat* 65–73.
11. Carleman T (1939) Sur un problème d'unicité pur les systèmes d'équations aux dérivées partielles à deux variables indépendantes. *Ark Mat Astr Fys* 26: 17.
12. Greenleaf A, Lassas M, Uhlmann G (2003) On nonuniqueness for Calderon's inverse problem. *Math Res Lett* 10: 685–693.
13. Hörmander L (1985) *The analysis of linear partial differential operators. IV*, Berlin: Springer-Verlag.
14. John F (1960) Continuous dependence on data for solutions of partial differential equations with a presribed bound. *Commun Pure Appl Math* 13: 551–585.
15. Leonhardt U (2006) Optical conformal mapping. *Science* 312: 1777–1780.
16. Pendry JB, Schurig D, Smith DR (2006) Controlling electromagnetic fields. *Science* 321: 1780–1782.
17. Radon J (1917) On the determination of functions from their integrals along certain manifolds. *Ber Verh Sachs Akad Wiss* 69: 262–277.

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