http://www.aimspress.com/journal/mine

Mathematics in Engineering, 2(1): 119–140.
DOI:10.3934/mine.2020007
Received: 24 October 2019
Accepted: 23 November 2019
Published: 06 December 2019

Research article

Energy asymptotics in the Brezis–Nirenberg problem: The higher-dimensional case†

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† **This contribution is part of the Special Issue:** Qualitative Analysis and Spectral Theory for Partial Differential Equations
Guest Editor: Veronica Felli
Link: `https://www.aimspress.com/newsinfo/1371.html`

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Abstract: For dimensions $N \geq 4$, we consider the Brézis-Nirenberg variational problem of finding

$$S(\epsilon V) := \inf_{0 \not\equiv u \in H_0^1(\Omega)} \frac{\int_\Omega |\nabla u|^2 \, dx + \epsilon \int_\Omega V\, |u|^2 \, dx}{\left(\int_\Omega |u|^q \, dx\right)^{2/q}},$$

where $q = \frac{2N}{N-2}$ is the critical Sobolev exponent, $\Omega \subset \mathbb{R}^N$ is a bounded open set and $V : \overline{\Omega} \to \mathbb{R}$ is a continuous function. We compute the asymptotics of $S(0) - S(\epsilon V)$ to leading order as $\epsilon \to 0+$. We give a precise description of the blow-up profile of (almost) minimizing sequences and, in particular, we characterize the concentration points as being extrema of a quotient involving the Robin function. This complements the results from our recent paper in the case $N = 3$.

Keywords: Brezis–Nirenberg problem; energy asymptotic; minimizing sequences; blow-up

1. Introduction and main results

1.1. Setting of the problem

Let $N \geq 4$ and let $\Omega \subset \mathbb{R}^N$ be a bounded open set. For $\epsilon > 0$ and a function $V \in C(\overline{\Omega})$, Brézis and Nirenberg study in their famous paper [3] the quotient functional

$$\mathcal{S}_{\epsilon V}[u] := \frac{\int_\Omega |\nabla u|^2\,dx + \epsilon \int_\Omega V\,|u|^2\,dx}{\left(\int_\Omega |u|^q\,dx\right)^{2/q}}, \qquad q = \frac{2N}{N-2}, \tag{1.1}$$

and the corresponding variational problem of finding

$$S(\epsilon V) := \inf_{0 \not\equiv u \in H_0^1(\Omega)} \mathcal{S}_{\epsilon V}[u]. \tag{1.2}$$

This number is to be compared with

$$S_N = \pi N(N-2)\left(\frac{\Gamma(N/2)}{\Gamma(N)}\right)^{2/N},$$

the sharp constant [1, 11, 12, 14] in the Sobolev inequality. Indeed, in [3] it is shown that $S(\epsilon V) < S_N$ as soon as

$$\mathcal{N}(V) := \{x \in \Omega : V(x) < 0\} \tag{1.3}$$

is non-empty. This behavior is in stark contrast to the case $N = 3$ also treated in [3], where there is an $\epsilon_V > 0$ such that $S(\epsilon V) = S_N$ for all $\epsilon \in (0, \epsilon_V]$ even if $\mathcal{N}(V)$ is non-empty.

The purpose of this paper is, for $N \geq 4$, to describe the asymptotics of $S_N - S(\epsilon V)$ to leading order as $\epsilon \to 0$, as well as the asymptotic behavior of corresponding (almost) minimizing sequences and, in particular, their concentration behavior. This is the higher-dimensional complement to our recent paper [6], where analogous results are shown in the more difficult case $N = 3$.

Notation. To prepare the statement of our main results, we now introduce some key objects for the following analysis. An important role is played by the Green's function of the Dirichlet Laplacian on Ω, which, in the normalization of [10], satisfies in the sense of distributions

$$\begin{cases} -\Delta_x G(x,y) = (N-2)\,\omega_N\,\delta_y & \text{in } \Omega, \\ G(x,y) = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where ω_N is the surface of the unit sphere in $\mathbb{R}^N$, and δ_y denotes the Dirac delta function centered at y. We denote by

$$H(x,y) = \frac{1}{|x-y|^{N-2}} - G(x,y) \tag{1.5}$$

the regular part of G. The function $H(x,\cdot)$, defined on $\Omega \setminus \{x\}$, extends to a continuous function on Ω and we may define the *Robin function*

$$\phi(x) := H(x,x). \tag{1.6}$$

Using this function, we define the numbers

$$\sigma_N(\Omega, V) := \sup_{x\in\mathcal{N}(V)} \Big(\phi(x)^{-\frac{2}{N-4}}\, |V(x)|^{\frac{N-2}{N-4}}\Big), \qquad N \geq 5\,,$$
$$\sigma_4(\Omega, V) := \sup_{x\in\mathcal{N}(V)} \Big(\phi(x)^{-1}|V(x)|\Big), \qquad N = 4\,,$$

which will turn out to essentially be the coefficients of the leading order term in $S_N - S(\epsilon V)$.

Another central role is played by the family of functions

$$U_{x,\lambda}(y) = \frac{\lambda^{(N-2)/2}}{(1+\lambda^2|x-y|^2)^{(N-2)/2}} \qquad x \in \mathbb{R}^N,\ \lambda > 0. \tag{1.7}$$

It is well-known that the $U_{x,\lambda}$ are exactly the optimizers of the Sobolev inequality on $\mathbb{R}^N$.

Since (1.1) is a perturbation of the Sobolev quotient, it is reasonable to expect the $U_{x,\lambda}$ to be nearly optimal functions for (1.2). However, since (1.2) is set on $H^1_0(\Omega)$, we consider, as in [2], the functions $PU_{x,\lambda} \in H^1_0(\Omega)$ uniquely determined by the properties

$$\Delta PU_{x,\lambda} = \Delta U_{x,\lambda} \quad \text{in } \Omega, \qquad PU_{x,\lambda} = 0 \quad \text{on } \partial\Omega\,. \tag{1.8}$$

Moreover, let

$$T_{x,\lambda} := \operatorname{span}\{PU_{x,\lambda}, \partial_\lambda PU_{x,\lambda}, \partial_{x_i} PU_{x,\lambda}\,(i = 1,2,\dots,N)\}$$

and let $T^{\perp}_{x,\lambda}$ be the orthogonal complement of $T_{x,\lambda}$ in $H^1_0(\Omega)$ with respect to the inner product $\int_\Omega \nabla u \cdot \nabla v\, dy$.

In what follows we denote by $\|\cdot\|$ the L^2−norm on Ω. Finally, given a set X and two functions $f_1, f_2 : X \to \mathbb{R}$, we write $f_1 \lesssim f_2$ if there exists a numerical constant c such that $f_1(x) \leq c\, f_2(x)$ for all $x \in X$.

1.2. Main results

Throughout this paper and without further mention we assume that the following properties are satisfied.

Assumption 1.1. The set $\Omega \subset \mathbb{R}^N$, $N \geq 4$, is open and bounded and has a C^2 boundary. Moreover, $V \in C(\overline{\Omega})$ and $\mathcal{N}(V) \neq \emptyset$, with $\mathcal{N}(V)$ defined in (1.3).

Here is our first main result. It gives the asymptotics of $S_N - S(\epsilon V)$ to leading order in ϵ.

Theorem 1.2. *As $\epsilon \to 0+$, we have*

$$S(\epsilon V) = S_N - C_N\, \sigma_N(\Omega, V)\, \epsilon^{\frac{N-2}{N-4}} + o(\epsilon^{\frac{N-2}{N-4}}) \qquad \text{if } N \geq 5 \tag{1.9}$$

and

$$S(\epsilon V) = S_4 - \exp\Big(-\frac{4}{\epsilon}(1+o(1))\,\sigma_4(\Omega, V)^{-1}\Big) \qquad \text{if } N = 4. \tag{1.10}$$

The constants C_N are defined in equation (1.14) *below.*

Our second main result shows that the blow-up profile of an arbitrary almost minimizing sequence (u_ϵ) is given to leading order by the family of functions $PU_{x,\lambda}$. Moreover, we give a precise characterization of the blow-up speed $\lambda = \lambda_\epsilon$ and of the point x_0 around which the u_ϵ concentrate.

Theorem 1.3. *Let $(u_\epsilon) \subset H^1_0(\Omega)$ be a family of functions such that*

$$\lim_{\epsilon\to 0} \frac{\mathcal{S}_{\epsilon V}[u_\epsilon] - S(\epsilon V)}{S_N - S(\epsilon V)} = 0 \qquad \text{and} \qquad \int_\Omega |u_\epsilon|^q \, dx = \left(\frac{S_N}{N(N-2)} \right)^{\frac{q}{q-2}} . \tag{1.11}$$

Then there are $(x_\epsilon) \subset \Omega$, $(\lambda_\epsilon) \subset (0,\infty)$, $(\alpha_\epsilon) \subset \mathbb{R}$ and $(w_\epsilon) \subset H^1_0(\Omega)$ with $w_\epsilon \in T^\perp_{x_\epsilon,\lambda_\epsilon}$ such that

$$u_\epsilon = \alpha_\epsilon \left(PU_{x_\epsilon,\lambda_\epsilon} + w_\epsilon \right) \tag{1.12}$$

and, along a subsequence, $x_\epsilon \to x_0$ for some $x_0 \in \mathcal{N}(V)$. Moreover,

$$\begin{cases} \phi(x_0)^{-\frac{2}{N-4}} \, |V(x_0)|^{\frac{N-2}{N-4}} = \sigma_N(\Omega, V), & N \geq 5, \\ \phi(x_0)^{-1} |V(x_0)| = \sigma_4(\Omega, V) \,, & N = 4, \end{cases}$$

$$\begin{cases} \|\nabla w_\epsilon\| = o(\epsilon^{\frac{N-2}{2N-8}}), & N \geq 5, \\ \|\nabla w_\epsilon\| \leq \exp\Big(-\frac{2}{\epsilon} \, (1 + o(1)) \, \sigma_4(\Omega, V)^{-1} \Big), & N = 4, \end{cases}$$

$$\begin{cases} \lim_{\epsilon\to 0} \, \epsilon \, \lambda_\epsilon^{N-4} = \frac{N \, (N-2)^2 \, a_N \, \phi(x_0)}{2 \, b_N \, |V(x_0)|} \,, & N \geq 5, \\ \lim_{\epsilon\to 0} \, \epsilon \, \ln \lambda_\epsilon = \frac{2 \, \phi(x_0)}{|V(x_0)|} \,, & N = 4, \end{cases}$$

$$\begin{cases} \alpha_\epsilon = s \Big(1 + D_N \epsilon^{\frac{N-2}{N-4}} + o(\epsilon^{\frac{N-2}{N-4}}) \Big), & N \geq 5, \\ \alpha_\epsilon = s \Big(1 + \exp\Big(-\frac{4}{\epsilon} \, (1 + o(1)) \, \sigma_4(\Omega, V)^{-1} \Big) \Big), & N = 4, \end{cases} \qquad \text{for some } s \in \{\pm 1\} \,.$$

The constants a_N, b_N and D_N are defined in equations (1.13) *and* (1.15) *below.*

The coefficients appearing in Theorems 1.2 and 1.3 are

$$a_N := \int_{\mathbb{R}^N} \frac{dz}{(1+z^2)^{(N+2)/2}}, \qquad b_N := \begin{cases} \int_{\mathbb{R}^N} \frac{dz}{(1+z^2)^{N-2}}, & N \geq 5, \\ \omega_4, & N = 4, \end{cases} \tag{1.13}$$

as well as

$$C_N := S_N^{\frac{2-N}{2}} \, (N(N-2))^{\frac{N-2}{2}} \, \frac{N-4}{N-2} \left(\frac{N(N-2)^2}{2} \right)^{\frac{2}{4-N}} \, a_N^{-\frac{2}{N-4}} \, b_N^{\frac{N-2}{N-4}}, \qquad N \geq 5, \tag{1.14}$$

and

$$D_N := a_N^{-\frac{2}{N-4}} b_N^{\frac{N-2}{N-4}} S_N^{-\frac{N}{2}} \, (N(N-2))^{\frac{N}{2} - \frac{N-2}{N-4}} \left(\frac{N-2}{2} \right)^{-\frac{N-2}{N-4}} , \qquad N \geq 5. \tag{1.15}$$

A simple computation using beta functions yields the numerical values

$$a_N = \frac{\omega_N}{N}, \quad N \geq 4, \qquad \text{and} \qquad b_N = \omega_N \, \frac{\Gamma\left(\frac{N}{2}\right) \Gamma\left(\frac{N}{2} - 2\right)}{2 \, \Gamma(N-2)}, \quad N \geq 5.$$

1.3. Discussion

Let us put our main results, Theorems 1.2 and 1.3, into perspective with respect to existing results in the literature.

Of course, minimizers of the variational problem (1.2) satisfy the corresponding Euler–Lagrange equation. It is natural to study general positive solutions of this equation, even if they do not arise as minimizers of (1.2). In the special case where V is a negative constant, Brézis and Peletier [4] discussed the concentration behavior of such general solutions and made some conjectures, which were later proved by Han [7] and Rey [9]. Probably one can use their precise concentration results to give an alternative proof of our main results in the special case where V is constant and probably one can even extend the analysis of Han and Rey to the case of non-constant V.

Our approach here is different and, we believe, simpler for the problem at hand. We work directly with the variational problem (1.2) and *not* with the Euler–Lagrange equation. Therefore, our concentration results are not only true for minimizers but even for 'almost minimizers' in the sense of (1.11). We believe that this is interesting in its own right. On the other hand, a disadvantage of our method compared to the Han–Rey method is that it gives concentration results only in H^1 norm and not in L^∞ norm and that it is restricted to energy minimizing solutions of the Euler–Lagrange equation.

In the special case where V is a negative constant, our results are very similar to results obtained by Takahashi [13], who combined elements from the Han–Rey analysis (see, e.g., [13, Equation (2.4) and Lemma 2.6]) with variational ideas adapted from Wei's treatment [15] of a closely related problem; see also [5]. Takahashi obtains the energy asymptotics in Theorem 1.2 as well as the characterization of the concentration point and the concentration scale in Theorem 1.3 under the assumption that u_ϵ is a minimizer for (1.2). Thus, in our paper we generalize Takahashi's results to non-constant V and to almost minimizing sequences and we give an alternative, self-contained proof which does not rely on the works of Han and Rey.

In dimensions $N \geq 5$, the function $\phi^{-2/(N-4)}|V|^{(N-2)/(N-4)}$, which enters into the definition of $\sigma_N(\Omega, V)$, has appeared earlier in the work [8] by Molle and Pistoia. Their setting, however, is different from ours. On the one hand, they consider general positive solutions of the corresponding Euler–Lagrange equation, not necessarily energy minimizers. On the other hand, they assume that the blow-up point lies in the interior and they seem to assume that the blow-up scale satisfies $\lambda_\epsilon \sim \epsilon^{-1/(N-4)}$ (see [8, Theorem 4.4]). In our energy minimizing setting we show that these assumptions are satisfied for minimizers and, moreover, that the blow-up point is not only a critical point, but a maximum point of the function $\phi^{-2/(N-4)}|V|^{(N-2)/(N-4)}$.

The present work is a companion paper to [6] relying on the techniques developed there in the three dimensional case. In particular, Theorems 1.2 and 1.3 should be compared with [6, Theorems 1.3 and 1.7], respectively. Although the expansions for $N \geq 4$ have the same structure as in the case $N = 3$, the latter case is more involved. In fact, when $N = 3$, the coefficient of the leading order term, namely the term of order ϵ, vanishes and one has to expand the energy to the next order, namely ϵ^2.

Besides the extensions of known results that we achieve here, we also think it is worthwhile from a methodological point of view to present our arguments again in the conceptually easier case $N \geq 4$. In the three-dimensional case the basic technique is iterated twice, which to some extent obscures the underlying simple idea. Moreover, we hope our work sheds some new light on the similarities and differencies between the two cases.

The structure of this paper is as follows. In Section 2 we prove the upper bound from Theorem 1.2 by inserting the $PU_{x,\lambda}$ as test functions. The proof of the corresponding lower bound is prepared in Sections 3 and 4, where we derive a crude asymptotic expansion for a general almost minimizing sequence (u_ϵ) and the corresponding expansion of $\mathcal{S}_{\epsilon V}[u_\epsilon]$. Section 5 contains the proof of Theorems 1.2 and 1.3. A crucial ingredient there is the coercivity inequality (5.1) from [10], which allows us to estimate the remainder terms and to refine the aforementioned expansion of u_ϵ. Finally, an appendix contains two auxiliary technical results.

2. Upper bound

The computation of the upper bound to $S(\epsilon V)$ uses the functions $PU_{x,\lambda}$, with suitably chosen x and λ, as test functions. The following theorem gives a precise expansion of the value $\mathcal{S}_{\epsilon V}[PU_{x,\lambda}]$. To state it, we introduce the distance to the boundary of Ω as

$$d(x) = \operatorname{dist}(x, \partial\Omega), \qquad x \in \Omega.$$

Theorem 2.1. *Let $x = x_\lambda$ be a sequence of points such that $d(x)\lambda \to \infty$. Then as $\lambda \to \infty$, we have*

$$\int_\Omega |\nabla PU_{x,\lambda}|^2 \, dy = N(N-2)\left(\frac{S_N}{N(N-2)}\right)^{\frac{q}{q-2}} + N(N-2)\, a_N\, \lambda^{2-N}\, \phi(x) + O((d(x)\lambda)^{\frac{4}{3}-N}), \tag{2.1}$$

$$\int_\Omega V PU_{x,\lambda}^2 \, dy = \begin{cases} \lambda^{-2}\, b_N\, V(x) + O\Big((d(x)\lambda)^{2-N}\Big) + o(\lambda^{-2}), & N \geq 5, \\ \frac{\log\lambda}{\lambda^2}\, b_4\, V(x) + O\Big((d(x)\lambda)^{-2}\Big) + o\Big(\frac{\log\lambda}{\lambda^2}\Big) & N = 4, \end{cases} \tag{2.2}$$

and

$$\int_\Omega |PU_{x,\lambda}|^q \, dy = \left(\frac{S_N}{N(N-2)}\right)^{\frac{q}{q-2}} - q\, a_N\, \lambda^{2-N}\, \phi(x) + o((d(x)\lambda)^{2-N}). \tag{2.3}$$

In particular, as $\lambda \to \infty$,

$$\mathcal{S}_{\epsilon V}[PU_{x,\lambda}] = \begin{cases} S_N + \Big(\frac{S_N}{N(N-2)}\Big)^{\frac{2}{2-q}}\Big(\frac{N(N-2)\, a_N\, \phi(x)}{\lambda^{N-2}} + b_N\, \epsilon\, \frac{V(x)}{\lambda^2}\Big) + o((d(x)\lambda)^{2-N}) + o(\epsilon\lambda^{-2}), & N \geq 5, \\ S_4 + \frac{8}{S_4}\Big(\frac{8\, a_4\, \phi(x)}{\lambda^2} + b_4\, \epsilon\, \frac{V(x)\, \log\lambda}{\lambda^2}\Big) + o((d(x)\lambda)^{-2}) + o(\epsilon \frac{\log\lambda}{\lambda^2}), & N = 4. \end{cases} \tag{2.4}$$

In view of Proposition 3.1 below, the assumption $d(x)\lambda \to \infty$ in Theorem 2.1 is no restriction, even when dealing with general almost minimizing sequences.

Corollary 2.2. *As $\epsilon \to 0+$, we have*

$$S(\epsilon V) \leq S_N - C_N\, \sigma_N(\Omega, V)\, \epsilon^{\frac{N-2}{N-4}} + o(\epsilon^{\frac{N-2}{N-4}}) \qquad \text{if } N \geq 5 \tag{2.5}$$

and

$$S(\epsilon V) \leq S_4 - \exp\Big(-\frac{4}{\epsilon}(1 + o(1))\, \sigma_4(\Omega, V)^{-1}\Big) \qquad \text{if } N = 4. \tag{2.6}$$

Proof of Corollary 2.2. By [10, (2.8)], we have

$$d(x)^{2-N} \lesssim \phi(x) \lesssim d(x)^{2-N}. \tag{2.7}$$

(Note that this bound uses the C^2 assumption on Ω.) First, let $N \geq 5$. Since, moreover, $V = 0$ on $\partial\mathcal{N}(V) \setminus \partial\Omega$, the function $\phi^{-\frac{2}{N-4}}\,|V|^{\frac{N-2}{N-4}}$ can be extended to a continuous function on $\overline{\mathcal{N}(V)}$ which vanishes on $\partial\mathcal{N}(V)$. Thus there is $z_0 \in \mathcal{N}(V)$ such that

$$\sigma_N(\Omega, V) = \phi(z_0)^{-\frac{2}{N-4}}\,|V(z_0)|^{\frac{N-2}{N-4}}, \qquad N \geq 5. \tag{2.8}$$

The corollary for $N \geq 5$ now follows by choosing $x = z_0$ in (2.4) and optimizing the quantity $\frac{N(N-2)\,a_N\,\phi(z_0)}{\lambda^{N-2}} + b_N\,\epsilon\,\frac{V(z_0)}{\lambda^2}$ in λ. The optimal choice is

$$\lambda(\epsilon) = \left(\frac{N\,(N-2)^2\,a_N\,\phi(z_0)}{2\,b_N\,|V(z_0)|}\right)^{\frac{1}{N-4}} \epsilon^{-\frac{1}{N-4}}\,, \tag{2.9}$$

and (2.5) follows from a straightforward computation.

Similarly, if $N = 4$, since $\frac{|V(y)|}{\phi(y)}$ is a positive continuous function on $\mathcal{N}(V)$ which goes to 0 as $y \to \partial\mathcal{N}(V)$, we find some $z_0 \in \mathcal{N}(V)$ such that

$$\sigma_4(\Omega, V) = \frac{|V(z_0)|}{\phi(z_0)}\,. \tag{2.10}$$

Thus we may choose $x = z_0$ in (2.4) and optimize the quantity $A\lambda^{-2} - B\epsilon\lambda^{-2}\log\lambda$ in $\lambda > 0$, where $A = 8\,a_4\,\phi(z_0) + o(1)$ and $B = b_4\,|V(z_0)| + o(1)$. The optimal choice is

$$\lambda(\epsilon) = \sqrt{e}\exp\Big(\frac{A}{B\epsilon}\Big). \tag{2.11}$$

Inserting this into (2.4), we get

$$\begin{aligned} S(\epsilon V) \leq \mathcal{S}_{\epsilon V}[PU_{x,\lambda(\epsilon)}] &= S_4 - \frac{4b_4}{eS_4}\epsilon|V(z_0)|\exp\left(-\frac{16\,a_4\,(\phi(z_0)+o(1))}{b_4\,\epsilon\,|V(z_0)|+o(1)}\right) \\ &= S_4 - \exp\Big(-\frac{4}{\epsilon}\,(1+o(1))\inf_{x\in\mathcal{N}(V)}\frac{\phi(x)}{|V(x)|}\Big), \end{aligned}$$

where we have used the fact that

$$\epsilon\,b\exp\Big(-\frac{a}{\epsilon}\Big) = \exp\Big(-\frac{a}{\epsilon} + o\Big(\frac{1}{\epsilon}\Big)\Big), \qquad \epsilon \to 0+, \tag{2.12}$$

holds for all $a \geq 0$ and all $b > 0$. This completes the proof of (2.6), and thus of Corollary 2.2. □

Proof of Theorem 2.1. We prove Eqs. (2.1)–(2.3) separately. Then expansion (2.4) follows by a straightforward Taylor expansion of the quotient functional $\mathcal{S}_{\epsilon V}[PU_{x,\lambda}]$.

Proof of (2.1). Since the $U_{x,\lambda}$ satisfy the equation

$$-\Delta_y U_{x,\lambda}(y) = N(N-2)\,U_{x,\lambda}(y)^{q-1}, \quad y \in \mathbb{R}^N, \tag{2.13}$$

it follows using integration by parts that

$$\int_\Omega |\nabla PU_{x,\lambda}|^2\,dy = N(N-2)\int_\Omega U_{x,\lambda}^{q-1} PU_{x,\lambda}\,dy.$$

On the other hand, by [10, Prop. 1] we know that

$$PU_{x,\lambda} = U_{x,\lambda} - \varphi_{x,\lambda}, \qquad \varphi_{x,\lambda} = \frac{H(x,\cdot)}{\lambda^{(N-2)/2}} + f_{x,\lambda}, \tag{2.14}$$

where

$$\|f_{x,\lambda}\|_{L^\infty(\Omega)} = O\left(\lambda^{-(N+2)/2}\, d(x)^{-N}\right), \qquad \lambda \to \infty. \tag{2.15}$$

By putting the above equations together we obtain

$$\int_\Omega |\nabla PU_{x,\lambda}|^2\, dy = N(N-2)\left(\int_\Omega U_{x,\lambda}^q\, dy - \lambda^{\frac{2-N}{2}} \int_\Omega U_{x,\lambda}^{q-1}\, H(x,\cdot)\, dy - \int_\Omega U_{x,\lambda}^{q-1}\, f_{x,\lambda}\, dy\right). \tag{2.16}$$

A direct calculation shows that

$$\int_\Omega U_{x,\lambda}^q\, dy = \int_{\mathbb{R}^N} U_{x,\lambda}^q\, dy + O((d(x)\lambda)^{-N}) = \left(\frac{S_N}{N(N-2)}\right)^{\frac{q}{q-2}} + O((d(x)\lambda)^{-N}). \tag{2.17}$$

Moreover, for any $x \in \Omega$ we have

$$d(x)^{2-N} \lesssim \|H(x,\cdot)\|_{L^\infty(\Omega)} \lesssim d(x)^{2-N} \tag{2.18}$$

and

$$\sup_{y\in\Omega} |\nabla_y H(x,y)| \lesssim d(x)^{1-N}, \tag{2.19}$$

by the maximum principle, compare [10, Sec. 2 and Appendix]. Now let $\rho \in (0, \frac{d(x)}{2})$. A direct calculation using (1.7), (2.18) and (2.19) shows that

$$\begin{aligned}\int_{B_\rho(x)} U_{x,\lambda}^{q-1}\, H(x,\cdot)\, dy &= \lambda^{1+\frac{N}{2}}\, (\phi(x) + O(\rho\, d(x)^{1-N})) \int_{B_\rho(x)} \frac{dy}{(1+\lambda^2|x-y|^2)^{(N+2)/2}} \\ &= \lambda^{1-\frac{N}{2}}\, a_N\, \left(\phi(x) + O(\rho\, d(x)^{1-N})\right)(1 + O((\lambda\rho)^{-2}))\end{aligned}$$

and

$$\begin{aligned}\int_{\Omega\setminus B_\rho(x)} U_{x,\lambda}^{q-1}\, H(x,\cdot)\, dy &= \lambda^{1+\frac{N}{2}}\, O(d(x)^{2-N}) \int_\rho^\infty \frac{r^{N-1}\, dr}{(1+\lambda^2 r^2)^{\frac{N+2}{2}}} \\ &= \lambda^{1-\frac{N}{2}}\, O(d(x)^{2-N}) \int_{\rho\lambda}^\infty \frac{t^{N-1}\, dt}{(1+t^2)^{\frac{N+2}{2}}} \\ &= \lambda^{1-\frac{N}{2}}\, O\left(d(x)^{2-N}\, (\lambda\rho)^{-2}\right).\end{aligned}$$

Hence for the second term on the right hand side of (2.16) we get

$$\lambda^{\frac{2-N}{2}} \int_\Omega U_{x,\lambda}^{q-1}\, H(x,\cdot)\, dy = \lambda^{2-N}\, a_N\, \phi(x) + \lambda^{2-N}\, O\left(\rho\, d(x)^{1-N}\right) + \lambda^{2-N}\, O\left(d(x)^{2-N}\, (\lambda\rho)^{-2}\right). \tag{2.20}$$

As for the last term on the right hand side of (2.16), we note that in view of (2.15)

$$\left|\int_\Omega U_{x,\lambda}^{q-1}\, f_{x,\lambda}\, dy\right| \le \|f_{x,\lambda}\|_{L^\infty(\Omega)} \int_{\mathbb{R}^N} U_{x,\lambda}^{q-1}\, dy = \|f_{x,\lambda}\|_{L^\infty(\Omega)}\, a_N\, \lambda^{1-\frac{N}{2}} = O\left((\lambda\, d(x))^{-N}\right).$$

The claim thus follows from (2.16) by choosing $\rho = d(x)^{1/3}\lambda^{-2/3}$ in (2.20). (Notice that $\rho = d(x)(d(x)\lambda)^{-2/3} \le \frac{d(x)}{2}$ for λ large enough.)

Proof of (2.2). We have

$$\int_\Omega V\, PU_{x,\lambda}^2\, dy = \int_\Omega V\, U_{x,\lambda}^2\, dy + \int_\Omega V\, (\varphi_{x,\lambda}^2 - 2\, U_{x,\lambda}\, \varphi_{x,\lambda})\, dy\,. \tag{2.21}$$

Since by [10, Prop. 1],

$$0 \le \varphi_{x,\lambda}(y) \le U_{x,\lambda}(y) \qquad \forall\, y \in \Omega, \tag{2.22}$$

together with (2.14), (2.15) and (2.18) we obtain the following upper bound on the last integral in (2.21),

$$\Big| \int_\Omega V\, (\varphi_{x,\lambda}^2 - 2\, U_{x,\lambda}\, \varphi_{x,\lambda})\, dy \Big| \le 2\, \|V\|_{L^\infty(\Omega)}\, \|\varphi_{x,\lambda}\|_{L^\infty(\Omega)} \int_\Omega U_{x,\lambda}\, dy = O\Big((d(x)\lambda)^{2-N}\Big)\,.$$

To treat the first term on the right hand side of (2.21), first assume $N \ge 5$. Choose a sequence $\rho = \rho_\lambda$ such that $\rho \le d(x)$, $\rho \to 0$ and $\rho\lambda \to \infty$ as $\lambda \to \infty$. (This is always possible, whether or not $d(x) \to 0$.) Then, by continuity of V,

$$\begin{aligned}
\int_\Omega V\, U_{x,\lambda}^2\, dy &= (V(x) + o(1)) \int_{B_\rho(x)} U_{x,\lambda}^2\, dy + \int_{\Omega\setminus B_\rho(x)} V\, U_{x,\lambda}^2\, dy \\
&= \lambda^{-2}\, b_N\, V(x) + o(\lambda^{-2}) + O\Big(\int_{\Omega\setminus B_\rho(x)} U_{x,\lambda}^2\, dy\Big) \\
&= \lambda^{-2}\, b_N\, V(x) + o(\lambda^{-2}) + O\Big(\lambda^{-2}(\rho\lambda)^{-N+4}\Big) = \lambda^{-2}\, b_N\, V(x) + o(\lambda^{-2}).
\end{aligned}$$

Similarly, in the case $N = 4$ we let $B_\tau(x)$ and $B_R(x)$ be two balls centered at x with radii τ and R chosen such that $B_\tau(x) \subset \Omega \subset B_R(x)$ and split the last integration in two parts as follows. Extending V by zero to $B_R(x) \setminus \Omega$ we get

$$\begin{aligned}
\int_{\Omega\setminus B_\tau(x)} V\, U_{x,\lambda}^2\, dy &= \int_{B_R(x)\setminus B_\tau(x)} V\, U_{x,\lambda}^2\, dy \le \omega_4\, \|V\|_{L^\infty(\Omega)} \int_\tau^R \frac{\lambda^2}{(1+\lambda^2|x-y|^2)^2}\, r^3\, dr \\
&= \omega_4\, \|V\|_{L^\infty(\Omega)}\, \lambda^{-2} \int_{\tau\lambda}^{R\lambda} \frac{t^3}{(1+t^2)^2}\, dt = O(\lambda^{-2}\, \log(R/\tau)).
\end{aligned} \tag{2.23}$$

On the other hand, denoting by $o_\tau(1)$ a quantity that vanishes as $\tau \to 0$ and assuming that $\tau\lambda \to \infty$ we get

$$\begin{aligned}
\int_{B_\tau(x)} V\, U_{x,\lambda}^2\, dy &= b_4\, V(x) \int_0^\tau \frac{\lambda^2\, r^3\, dr}{(1+\lambda^2|x-y|^2)^2} + o_\tau(1) \int_0^\tau \frac{\lambda^2\, r^3\, dr}{(1+\lambda^2|x-y|^2)^2} \\
&= b_4\, \lambda^{-2}\, V(x) \int_0^{\tau\lambda} \frac{t^3\, dt}{(1+t^2)^2} + \lambda^{-2}\, o_\tau(1) \int_0^{\tau\lambda} \frac{t^3\, dt}{(1+t^2)^2} \\
&= b_4\, \frac{\log\lambda}{\lambda^2}\, V(x) + o_\tau(1)\, O\Big(\frac{\log\lambda}{\lambda^2}\Big) + O\Big(\frac{\log\tau}{\lambda^2}\Big).
\end{aligned}$$

By choosing $\tau = \frac{1}{\log\lambda}$ and taking into account (2.23) we arrive at (2.2) in case $N = 4$.

Proof of (2.3). Recall that $q > 2$. Hence from the Taylor expansion of the function $t \mapsto t^q$ on an interval $[0, b]$ it follows that for any $a \in [0, b]$ we have

$$|\, b^q - (b-a)^q - q\, b^{q-1}\, a\,| \le \frac{q(q-1)}{2}\, b^{q-2}\, a^2. \tag{2.24}$$

Because of (2.22) and (2.14) we can apply (2.24) with $b = U_{x,\lambda}(y)$ and $a = \varphi_{x,\lambda}(y)$ to obtain the following point-wise upper bound:

$$\left| PU_{x,\lambda}^q - U_{x,\lambda}^q + q\, U_{x,\lambda}^{q-1}\, \varphi_{x,\lambda} \right| \le \frac{q(q-1)}{2}\, U_{x,\lambda}^{q-2}\, \varphi_{x,\lambda}^2 . \tag{2.25}$$

Together with estimate (A.2) this gives

$$\left| \int_\Omega \left(PU_{x,\lambda}^q - U_{x,\lambda}^q + q\, U_{x,\lambda}^{q-1}\, \varphi_{x,\lambda} \right) dy \right| = O\left((d(x)\,\lambda)^{-N} \right) . \tag{2.26}$$

On the other hand, the calculations in the proof of (2.1) show that

$$\int_\Omega U_{x,\lambda}^{q-1}\, \varphi_{x,\lambda}\, dy = \lambda^{2-N}\, a_N\, \phi(x) + O\left((d(x)\lambda)^{\frac{4}{3}-N} \right) = \lambda^{2-N}\, a_N\, \phi(x) + o\left((d(x)\lambda)^{2-N} \right).$$

In view of (2.17) and (2.26) this completes the proof. □

3. Lower bound. Preliminaries

As a starting point for the proof of the lower bound on $S(\epsilon V)$, we derive a crude asymptotic form of almost minimizers of $\mathcal{S}_{\epsilon V}$. The following result is essentially well-known. We have recalled the proof in [6, Appendix B] in the case $N = 3$, but the same argument carries over to $N \ge 4$.

Proposition 3.1. *Let $(u_\epsilon) \subset H_0^1(\Omega)$ be a sequence of functions satisfying*

$$\mathcal{S}_{\epsilon V}[u_\epsilon] = S_N + o(1) , \qquad \int_\Omega |u_\epsilon|^q\, dx = \left(\frac{S_N}{N(N-2)} \right)^{\frac{q}{q-2}} . \tag{3.1}$$

Then, along a subsequence,

$$u_\epsilon = \alpha_\epsilon \left(PU_{x_\epsilon,\lambda_\epsilon} + w_\epsilon \right) , \tag{3.2}$$

where

$$\begin{aligned} \alpha_\epsilon &\to s \qquad \textit{for some } s \in \{-1, +1\} , \\ x_\epsilon &\to x_0 \qquad \textit{for some } x_0 \in \overline{\Omega} , \\ \lambda_\epsilon d_\epsilon &\to \infty , \\ \|\nabla w_\epsilon\| &\to 0 \qquad \textit{and} \qquad w_\epsilon \in T_{x_\epsilon,\lambda_\epsilon}^\perp . \end{aligned} \tag{3.3}$$

Here $d_\epsilon = dist\,(x_\epsilon, \partial\Omega)$.

Convention:

From now on we will assume that (u_ϵ) satisfies (1.11). In particular, assumption (3.1) is satisfied. We will always work with a sequence of ϵ's for which the conclusions of Proposition 3.1 hold. To enhance readability, we will drop the index ϵ from α_ϵ, x_ϵ, λ_ϵ, d_ϵ and w_ϵ.

4. Lower bound. The main expansion

In this section we expand $\mathcal{S}_{\epsilon V}[u_\epsilon]$ by using the decomposition (3.2) of u_ϵ. We shall show the following result.

Proposition 4.1. *Let* $(u_\epsilon) \subset H_0^1(\Omega)$ *satisfy* (3.2) *and* (3.3). *Then*

$$|\alpha|^{-2} \int_\Omega |\nabla u_\epsilon|^2\, dy = \int_\Omega |\nabla PU_{x,\lambda}|^2\, dy + \int_\Omega |\nabla w|^2\, dy\,, \tag{4.1}$$

$$|\alpha|^{-q} \int_\Omega |u_\epsilon|^q\, dy = \int_\Omega PU_{x,\lambda}^q\, dy + \frac{q(q-1)}{2} \int_\Omega U_{x,\lambda}^{q-2}\, w^2\, dy + o\left(\int_\Omega |\nabla w|^2 + (\lambda d)^{2-N}\right), \tag{4.2}$$

$$|\alpha|^{-2}\epsilon \int_\Omega V u_\epsilon^2\, dy = \epsilon \int_\Omega V PU_{x,\lambda}^2\, dy + \mathcal{O}\left(\epsilon \int_\Omega |\nabla w|^2\, dy + \epsilon \sqrt{\int_\Omega |\nabla w|^2\, dy}\, \sqrt{\int_\Omega |V|\, PU_{x,\lambda}^2\, dy}\right). \tag{4.3}$$

In particular,

$$\mathcal{S}_{\epsilon V}[u_\epsilon] = \mathcal{S}_{\epsilon V}[PU_{x,\lambda}] + I[w] + \mathcal{O}\left(\epsilon \sqrt{\int_\Omega |\nabla w|^2\, dy}\, \sqrt{\int_\Omega |V| PU_{x,\lambda}^2\, dy}\right) + o\left(\int_\Omega |\nabla w|^2\, dy + (\lambda d)^{2-N}\right), \tag{4.4}$$

where

$$I[w] := \left(\int_\Omega U_{x,\lambda}^q\, dy\right)^{-\frac{2}{q}} \left(\int_\Omega |\nabla w|^2\, dy - N(N+2) \int_\Omega U_{x,\lambda}^{q-2}\, w^2\, dy\right). \tag{4.5}$$

Proof. We prove Eqs. (4.1)–(4.3) separately. Then the expansion (4.4) follows by a straightforward Taylor expansion of the quotient functional $\mathcal{S}_{\epsilon V}$, using $\mathcal{S}_{\epsilon V}[u_\epsilon] = \mathcal{S}_{\epsilon V}[|\alpha|^{-1} u_\epsilon]$.

In the sequel we denote by $c_1, c_2, \ldots$ various positive constants which are independent of ϵ.

Proof of (4.1). This follows by (3.2) and $w \in T_{x,\lambda}^\perp$.

Proof of (4.2). Recall that $\alpha^{-1} u_\epsilon = U_{x,\lambda} + (w - \varphi_{x,\lambda})$ by (2.14) and (3.2). We use the associated pointwise estimate

$$\begin{aligned}&\left| |\alpha|^{-q} |u_\epsilon|^q - U_{x,\lambda}^q - q\, U_{x,\lambda}^{q-1} (w - \varphi_{x,\lambda}) - \frac{q(q-1)}{2}\, U_{x,\lambda}^{q-2} (w - \varphi_{x,\lambda})^2 \right| \\ &\quad \le c_1 \left(|w - \varphi_{x,\lambda}|^q + |w - \varphi_{x,\lambda}|^{q-(q-3)_+} U_{x,\lambda}^{(q-3)_+} \right),\end{aligned}$$

where $(q-3)_+ = \max\{q-3, 0\}$. Using (2.25), it follows that

$$\begin{aligned}&\left| |\alpha|^{-q} |u_\epsilon|^q - PU_{x,\lambda}^q - q\, U_{x,\lambda}^{q-1} w - \frac{q(q-1)}{2}\, U_{x,\lambda}^{q-2} w^2 \right| \\ &\le c_2 \left(|w - \varphi_{x,\lambda}|^q + |w - \varphi_{x,\lambda}|^{q-(q-3)_+} U_{x,\lambda}^{(q-3)_+} + U_{x,\lambda}^{q-2} \varphi_{x,\lambda} |w| + U_{x,\lambda}^{q-2} \varphi_{x,\lambda}^2 \right) \\ &\le c_3 \left(|w|^q + \varphi_{x,\lambda}^q + |w|^{q-(q-3)_+} U_{x,\lambda}^{(q-3)_+} + \varphi_{x,\lambda}^{q-(q-3)_+} U_{x,\lambda}^{(q-3)_+} + U_{x,\lambda}^{q-2} \varphi_{x,\lambda} |w| + U_{x,\lambda}^{q-2} \varphi_{x,\lambda}^2 \right) \\ &\le c_4 \left(|w|^q + |w|^{q-(q-3)_+} U_{x,\lambda}^{(q-3)_+} + U_{x,\lambda}^{q-2} \varphi_{x,\lambda} |w| + U_{x,\lambda}^{q-2} \varphi_{x,\lambda}^2 \right).\end{aligned}$$

In the last inequality we used (2.22) to simplify the form of the remainder terms. Now we use the identity

$$N(N-2)\int_\Omega U_{x,\lambda}^{q-1} w\,dy = \int_\Omega \nabla U_{x,\lambda}\cdot\nabla w\,dy = \int_\Omega \nabla PU_{x,\lambda}\cdot\nabla w\,dy = 0,$$

which follows from (1.8), (2.13) and $w\in T_{x,\lambda}^{\perp}$. Therefore, with the help of the Hölder inequality, we find

$$\begin{aligned}
&\left|\int_\Omega\left(|\alpha|^{-q}|u_\epsilon|^q - PU_{x,\lambda}^q - \frac{q(q-1)}{2}\,U_{x,\lambda}^{q-2}w^2\right)dy\right| \\
&\le\ c_4\Bigg[\int_\Omega |w|^q\,dy + \left(\int_\Omega |w|^q\,dy\right)^{\frac{q-(q-3)_+}{q}}\left(\int_\Omega U_{x,\lambda}^q\,dy\right)^{\frac{(q-3)_+}{q}} \\
&\qquad + \left(\int_\Omega U_{x,\lambda}^{\frac{q(q-2)}{q-1}}\,\varphi_{x,\lambda}^{\frac{q}{q-1}}\,dy\right)^{\frac{q-1}{q}}\left(\int_\Omega |w|^q\,dy\right)^{\frac{1}{q}} + \int_\Omega U_{x,\lambda}^{q-2}\,\varphi_{x,\lambda}^2\,dy\Bigg] \\
&\le\ c_5\Bigg[\left(\int_\Omega |\nabla w|^2\,dy\right)^{\frac{q-(q-3)_+}{2}} + \left(\int_\Omega U_{x,\lambda}^{\frac{q(q-2)}{q-1}}\,\varphi_{x,\lambda}^{\frac{q}{q-1}}\,dy\right)^{\frac{q-1}{q}}\left(\int_\Omega |\nabla w|^2\,dy\right)^{\frac{1}{2}} + \int_\Omega U_{x,\lambda}^{q-2}\,\varphi_{x,\lambda}^2\,dy\Bigg].
\end{aligned}$$

In the last step, we used the Sobolev inequality and the equation (3.3) for w, together with

$$\int_\Omega U_{x,\lambda}^q\,dy\ \le\ \int_{\mathbb{R}^N} U_{x,\lambda}^q\,dy\ =\ \left(\frac{S_N}{N(N-2)}\right)^{\frac{q}{q-2}}.$$

It follows from Lemma A.1 and (3.3) that

$$\begin{aligned}
\left(\int_\Omega U_{x,\lambda}^{\frac{q(q-2)}{q-1}}\,\varphi_{x,\lambda}^{\frac{q}{q-1}}\,dy\right)^{\frac{q-1}{q}}\ &=\ o\left((d\lambda)^{\frac{2-N}{2}}\right), \\
\int_\Omega U_{x,\lambda}^{q-2}\,\varphi_{x,\lambda}^2\,dy\ &=\ o\left((d\,\lambda)^{2-N}\right).
\end{aligned}$$

Thus, we conclude that, as $\epsilon\to 0$,

$$\left|\int_\Omega\left(|\alpha|^{-q}|u_\epsilon|^q - PU_{x,\lambda}^q - \frac{q(q-1)}{2}\,U_{x,\lambda}^{q-2}w^2\right)dy\right| = o\left(\int_\Omega |\nabla w|^2\,dy + (\lambda d)^{2-N}\right).$$

Proof of (4.3). We write

$$|\alpha|^{-2}\int_\Omega V u_\epsilon^2\,dy = \int_\Omega V\,PU_{x,\lambda}^2\,dy + 2\int_\Omega V\,PU_{x,\lambda}\,w\,dy + \int_\Omega V w^2\,dy. \tag{4.6}$$

By the Hölder and Sobolev inequalities we have

$$\left|\int_\Omega V w^2\,dy\right| \le \left(\int_\Omega |V|^{\frac{N}{2}}\,dy\right)^{\frac{2}{N}}\left(\int_\Omega |w|^q\,dy\right)^{\frac{2}{q}} \le S_N^{-1}\left(\int_\Omega |V|^{\frac{N}{2}}\,dy\right)^{\frac{2}{N}}\int_\Omega |\nabla w|^2\,dy,$$

and

$$\begin{aligned}
\left|\int_\Omega V PU_{x,\lambda}\,w\,dy\right| &\le \left(\int_\Omega |V|\,PU_{x,\lambda}^2\,dy\right)^{\frac{1}{2}}\left(\int_\Omega |V|\,w^2\,dy\right)^{\frac{1}{2}} \\
&\le S_N^{-1/2}\left(\int_\Omega |V|\,PU_{x,\lambda}^2\,dy\right)^{\frac{1}{2}}\left(\int_\Omega |V|^{\frac{N}{2}}\,dy\right)^{\frac{1}{n}}\left(\int_\Omega |\nabla w|^2\,dy\right)^{\frac{1}{2}}.
\end{aligned}$$

Hence (4.3) follows by inserting these estimates into (4.6). □

5. Proof of the main results

We now deduce Theorems 1.2 and 1.3 from Proposition 4.1. To do so, we make crucial use of the following coercivity bound proved in [10, Appendix D].

Proposition 5.1. *For all $x \in \Omega$, $\lambda > 0$ and $v \in T_{x,\lambda}^{\perp}$, one has*

$$\int_\Omega |\nabla v|^2\,dy - N(N+2)\int_\Omega U_{x,\lambda}^{q-2}\,v^2\,dy \;\geq\; \frac{4}{N+4}\int_\Omega |\nabla v|^2\,dy\,. \tag{5.1}$$

Corollary 5.2. *For all $\epsilon > 0$ small enough, we have, if $N \geq 5$,*

$$\begin{aligned} 0 \geq (1+o(1))(S_N - S(\epsilon V)) + \left(\frac{S_N}{N(N-2)}\right)^{\frac{2}{2-q}} \left(\frac{N(N-2)\,a_N\,\phi(x)}{\lambda^{N-2}} + b_N\,\epsilon\,\frac{V(x)}{\lambda^2}\right) \\ + c\int_\Omega |\nabla w|^2\,dy + o((\lambda d)^{2-N}) + o(\epsilon\lambda^{-2}) \end{aligned} \tag{5.2}$$

and, if $N = 4$,

$$\begin{aligned} 0 \geq (1+o(1))(S_4 - S(\epsilon V)) + \frac{8}{S_4}\left(\frac{8a_4\phi(x)}{\lambda^2} + b_4 V(x)\frac{\epsilon\log\lambda}{\lambda^2}\right) \\ + c\int_\Omega |\nabla w|^2\,dy + o((\lambda d)^{-2}) + o(\epsilon\lambda^{-2}\log\lambda)\,. \end{aligned} \tag{5.3}$$

Proof. Firstly, it follows directly from (5.1) and the definition of $I[w]$ in (4.5) that there is a $c > 0$ such that for all $\epsilon > 0$ small enough, we have

$$I[w] \geq 4c\int_\Omega |\nabla w|^2\,dy\,. \tag{5.4}$$

Using Proposition 4.1 and (5.4) it follows that for ϵ small enough one has

$$\mathcal{S}_{\epsilon V}[u_\epsilon] \geq \mathcal{S}_{\epsilon V}[PU_{x,\lambda}] + 2c\int_\Omega |\nabla w|^2\,dy + O\left(\epsilon\sqrt{\int_\Omega |\nabla w|^2\,dy}\,\sqrt{\int_\Omega |V|\,PU_{x,\lambda}^2\,dy}\right) + o\left((\lambda d)^{2-N}\right).$$

Since

$$\epsilon\sqrt{\int_\Omega |\nabla w|^2\,dy}\,\sqrt{\int_\Omega |V|\,PU_{x,\lambda}^2\,dy} \;\leq\; c\int_\Omega |\nabla w|^2\,dy + \frac{\epsilon^2}{4c}\int_\Omega |V|\,PU_{x,\lambda}^2\,dy\,,$$

this further implies that for $\epsilon > 0$ small enough

$$\mathcal{S}_{\epsilon V}[u_\epsilon] \geq \mathcal{S}_{\epsilon V}[PU_{x,\lambda}] + c\int_\Omega |\nabla w|^2\,dy + O\left(\epsilon^2\int_\Omega |V|\,PU_{x,\lambda}^2\,dy\right) + o\left((\lambda d)^{2-N}\right).$$

Using (2.2) for the potential term and recalling (3.3), we obtain

$$\mathcal{S}_{\epsilon V}[u_\epsilon] \geq \begin{cases} \mathcal{S}_{\epsilon V}[PU_{x,\lambda}] + c\int_\Omega |\nabla w|^2\,dy + o(\epsilon\lambda^{-2}) + o((\lambda d)^{2-N}), & N \geq 5, \\ \mathcal{S}_{\epsilon V}[PU_{x,\lambda}] + c\int_\Omega |\nabla w|^2\,dy + o(\epsilon\lambda^{-2}\log\lambda) + o((\lambda d)^{-2}), & N = 4. \end{cases}$$

Now the fact that $S_N - \mathcal{S}_{\epsilon V}[u_\epsilon] = (1+o(1))(S_N - S(\epsilon V))$ by (1.11), together with the expansion of $\mathcal{S}_{\epsilon V}[PU_{x,\lambda}]$ from Theorem 2.1, implies the claimed bounds (5.2) and (5.3). □

In the next lemma, we prove that the limit point x_0 lies in the set $\mathcal{N}(V)$.

Lemma 5.3. *We have $x_0 \in \mathcal{N}(V)$. In particular, $d^{-1} = O(1)$ as $\epsilon \to 0$ and $x \in \mathcal{N}(V)$ for ϵ small enough.*

Proof. We first treat the case $N \geq 5$. In (5.2), we drop the non-negative gradient term and write the remaining lower order terms as

$$\left(\frac{S_N}{N(N-2)}\right)^{\frac{2}{2-q}} \left(\frac{N(N-2)\, a_N\, \phi(x)}{\lambda^{N-2}} + b_N\, \epsilon\, \frac{V(x)}{\lambda^2}\right) + o((\lambda d)^{2-N}) + o(\epsilon \lambda^{-2})$$
$$= \left(\frac{S_N}{N(N-2)}\right)^{\frac{2}{2-q}} \left(A(d\lambda)^{2-N} - B\epsilon(d\lambda)^{-2}\right),$$

where

$$A = N(N-2)\, a_N\, \phi(x) d^{N-2} + o(1), \qquad B = -b_N V(x_0) d^2 + o(1). \tag{5.5}$$

Notice that since $\phi(x) \gtrsim d^{2-N}$ by (2.7), the quantity A is positive and bounded away from zero. Moreover, by (5.2) and the fact that $S(\epsilon V) < S_N$, which follows from Corollary 2.2, we must have $B > 0$. Optimizing in $d\lambda$ yields the lower bound

$$A(d\lambda)^{2-N} - B\epsilon(d\lambda)^{-2} \geq -cA^{-\frac{2}{N-4}} B^{\frac{N-2}{N-4}} \epsilon^{\frac{N-2}{N-4}}, \tag{5.6}$$

for some explicit constant $c > 0$ independent of ϵ. On the other hand, by Corollary 2.2, there is $\rho > 0$ such that the leading term in (5.2) is bounded by

$$(1 + o(1))(S_N - S(\epsilon V)) \geq \rho\, \epsilon^{\frac{N-2}{N-4}} \tag{5.7}$$

for all $\epsilon > 0$ small enough. Plugging (5.6) and (5.7) into (5.2) and rearranging terms, we thus deduce that

$$B \geq \rho^{\frac{N-4}{N-2}} A^{\frac{2}{N-2}} c^{-\frac{N-4}{N-2}}. \tag{5.8}$$

As observed above, the quantity A is bounded away from zero and therefore (5.8) implies that B is bounded away from zero. Hence, in view of (5.5), d is bounded away from zero and $V(x_0) < 0$. The fact that $x \in \mathcal{N}(V)$ for ϵ small enough is a consequence of the continuity of V. This completes the proof in case $N \geq 5$.

Now we consider the case $N = 4$ in a similar way. In (5.3), we drop the non-negative gradient term and write the remaining lower order terms as

$$\frac{8}{S_4}\left(\frac{8a_4\phi(x)}{\lambda^2} + b_4 V(x)\frac{\epsilon \log \lambda}{\lambda^2}\right) + o((\lambda d)^{-2}) + o(\epsilon\lambda^{-2}\log\lambda)$$
$$= \frac{8}{S_4}\left(A(d\lambda)^{-2} - B\epsilon(d\lambda)^{-2}\log(d\lambda)\right), \tag{5.9}$$

where

$$A = 8a_4\phi(x)d^2 + o(1), \qquad B = -b_4(V(x_0) + o(1))d^2(1 - \frac{\log d}{\log d\lambda}). \tag{5.10}$$

Since $\phi(x) \gtrsim d(x)^{-2}$ by (2.7), the quantity A is positive and bounded away from zero. Moreover, by (5.3) and the fact that $S(\epsilon V) < S_4$, we must have $B > 0$. Optimizing (5.9) in $d\lambda$ yields the lower bound

$$A(d\lambda)^{-2} - B\epsilon(d\lambda)^{-2}\log(d\lambda) \geq -\frac{B\epsilon}{2e}\exp\left(-\frac{2A}{B\epsilon}\right) = -\exp\left(-\frac{2A}{B\epsilon} + \log(\frac{B\epsilon}{2e})\right). \tag{5.11}$$

On the other hand, by Corollary 2.2, there is $\rho > 0$ such that the leading term in (5.3) is bounded by

$$(1 + o(1))(S_4 - S(\epsilon V)) \geq \exp(-\frac{\rho}{\epsilon}). \tag{5.12}$$

Plugging (5.11) and (5.12) into (5.3), we thus deduce that

$$0 \geq \exp(-\frac{\rho}{\epsilon}) - \exp\left(-\frac{2A}{B\epsilon} + \log(\frac{B\epsilon}{2e})\right),$$

which leads to

$$-\frac{2A}{B} + \epsilon\log(\frac{B\epsilon}{2e}) \geq -\rho. \tag{5.13}$$

Since $\phi(x) \gtrsim d^{-2}$ by (2.7), the quantity A is bounded away from zero and moreover B is bounded. Using this fact, the left hand side of (5.13) can be written as

$$-\frac{2A}{B}(1 - \frac{B\epsilon\log B}{2A}) + \epsilon\log\frac{\epsilon}{2e} = -\frac{2A}{B}(1 + o(1)) + o(1).$$

Together with (5.13), this easily implies, if $\epsilon > 0$ is small enough, that

$$B \geq \frac{A}{\rho}.$$

As before, in view of (5.10), we deduce that d is bounded away from zero and that $V(x_0) < 0$. The fact that $x \in \mathcal{N}(V)$ for ϵ small enough is again a consequence of the continuity of V. □

Proof of Theorem 1.2. We first treat the case $N \geq 5$. In view of Lemma 5.3, the lower bound (5.2) can be written as (upon dropping the non-negative gradient term)

$$\begin{aligned}0 &\geq (1 + o(1))(S_N - S(\epsilon V)) + \left(\frac{S_N}{N(N-2)}\right)^{\frac{2}{2-q}}\left(\frac{N(N-2)\,a_N\,(\phi(x_0) + o(1))}{\lambda^{N-2}} + b_N\,\epsilon\,\frac{V(x_0) + o(1)}{\lambda^2}\right)\\ &\geq (1 + o(1))(S_N - S(\epsilon V)) - C_N(\phi(x_0) + o(1))^{-\frac{2}{N-4}}\,|V(x_0) + o(1)|^{\frac{N-2}{N-4}}\epsilon^{\frac{N-2}{N-4}}\end{aligned}$$

by optimization in λ. Therefore

$$S(\epsilon V) \geq S_N - C_N\phi(x_0)^{-\frac{2}{N-4}}\,|V(x_0)|^{\frac{N-2}{N-4}}\epsilon^{\frac{N-2}{N-4}} + o(\epsilon^{\frac{N-2}{N-4}}) \geq S_N - C_N\sigma_N(\Omega, V)\epsilon^{\frac{N-2}{N-4}} + o(\epsilon^{\frac{N-2}{N-4}}),$$

where the last inequality uses the fact that $x_0 \in \mathcal{N}(V)$ by Lemma 5.3. Since the matching upper bound has already been proved in Theorem 2.1, the proof in case $N \geq 5$ is complete.

Similarly, we can handle the case $N = 4$. In view of Lemma 5.3, the lower bound (5.3) can be written as (upon dropping the non-negative gradient term)

$$0 \geq (1 + o(1))(S_4 - S(\epsilon V)) + \frac{8}{S_4}\left(\frac{8a_4(\phi(x_0) + o(1))}{\lambda^2} + b_4(V(x_0) + o(1))\frac{\epsilon\log\lambda}{\lambda^2}\right)$$

$$\geq (1+o(1))(S_4 - S(\epsilon V)) - \frac{4b_4}{eS_4}\epsilon|V(x_0)+o(1)|\exp\left(-\frac{4(\phi(x_0)+o(1))}{\epsilon|V(x_0)+o(1)|}\right)$$

by optimization in λ. Therefore

$$S(\epsilon V) \geq S_4 - \exp\left(-\frac{4}{\epsilon}(1+o(1))\frac{\phi(x_0)}{|V(x_0)|}\right) \geq S_4 - \exp\left(-\frac{4}{\epsilon}(1+o(1))\,\sigma_4(\Omega,V)^{-1}\right),$$

where the last inequality uses the fact that $x_0 \in \mathcal{N}(V)$ by Lemma 5.3. Since the matching upper bound has already been proved in Theorem 2.1, the proof in case $N = 4$ is complete. □

Proof of Theorem 1.3. We start again with the bounds from Corollary 5.2, but this time we need to take into account the various nonnegative remainder terms more carefully.

Proof for $N \geq 5$. We rewrite (5.2), using Lemma 5.3, as

$$0 \geq (1+o(1))(S_N - S(\epsilon V)) - C_N(\phi(x_0)+o(1))^{-\frac{2}{N-4}}\,|V(x_0)+o(1)|^{\frac{N-2}{N-4}}\epsilon^{\frac{N-2}{N-4}} + \mathcal{R} \tag{5.14}$$

with

$$\mathcal{R} = \left(\frac{A_\epsilon}{\lambda^{N-2}} - B_\epsilon\frac{\epsilon}{\lambda^2} + C_N A_\epsilon^{-\frac{2}{N-4}} B_\epsilon^{\frac{N-2}{N-4}}\epsilon^{\frac{N-2}{N-4}}\right) + c\int_\Omega |\nabla w|^2\,dy,$$

where we have set

$$A_\epsilon = \left(\frac{S_N}{N(N-2)}\right)^{\frac{2}{2-q}}(N(N-2)\,a_N\,(\phi(x_0)+o(1))), \quad B_\epsilon = \left(\frac{S_N}{N(N-2)}\right)^{\frac{2}{2-q}} b_N\,(V(x_0)+o(1)).$$

Notice that both summands of $\mathcal{R}$ are separately nonnegative. Inserting the upper bound from Corollary 2.2 into (5.14), we get

$$0 \geq C_N\left(\sigma_N(\Omega,V) - \phi(x_0)^{-\frac{2}{N-4}}\,|V(x_0)|^{\frac{N-2}{N-4}}\right)\epsilon^{\frac{N-2}{N-4}} + \mathcal{R} + o(\epsilon^{\frac{N-2}{N-4}}).$$

Since each one of the first two summands on the right hand side is nonnegative, we deduce that

$$\phi(x_0)^{-\frac{2}{N-4}}\,|V(x_0)|^{\frac{N-2}{N-4}} = \sup_{x\in\mathcal{N}(V)}\phi(x)^{-\frac{2}{N-4}}\,|V(x)|^{\frac{N-2}{N-4}} = \sigma_N(\Omega,V)$$

and

$$\mathcal{R} = o(\epsilon^{\frac{N-2}{N-4}}). \tag{5.15}$$

In particular, (5.15) implies that

$$\|\nabla w\|_2^2 = o(\epsilon^{\frac{N-2}{N-4}}). \tag{5.16}$$

Denote by

$$\lambda_0(\epsilon) = \left(\frac{(N-2)A_\epsilon}{2B_\epsilon}\right)^{\frac{1}{N-4}}\epsilon^{\frac{1}{4-N}}$$

the unique value of λ for which the first summand of $\mathcal{R}$ vanishes. Using Lemma A.2, the bound (5.15) implies that

$$\epsilon(\lambda^{-1} - \lambda_0(\epsilon)^{-1})^2 = o(\epsilon^{\frac{N-2}{N-4}}),$$

which is equivalent to

$$\lambda = \lambda_0(\epsilon) + o(\epsilon^{-\frac{1}{N-4}}) = \left(\frac{N\,(N-2)^2\,a_N\,\phi(x_0)}{2\,b_N\,|V(x_0)|}\right)^{\frac{1}{N-4}} \epsilon^{-\frac{1}{N-4}} + o(\epsilon^{-\frac{1}{N-4}}). \tag{5.17}$$

Finally, to obtain the asymptotics of α, by (4.2), (1.11), (2.3) and (5.16), we have that

$$|\alpha|^{-q}\left(\frac{S_N}{N(N-2)}\right)^{\frac{q}{q-2}} = \left(\frac{S_N}{N(N-2)}\right)^{\frac{q}{q-2}} - qa_N\lambda^{2-N}\phi(x_0) + \frac{q(q-1)}{2}\int_\Omega U_{x,\lambda}^{q-2}\,w^2\,dy + o(\lambda^{2-N})\,. \tag{5.18}$$

Moreover, by Hölder and Sobolev inequalities,

$$\int_\Omega U_{x,\lambda}^{q-2} w^2\,dy \lesssim \|\nabla w\|^2. \tag{5.19}$$

We easily conclude from (5.16)–(5.19) that

$$|\alpha| = 1 + D_N\sigma_N(\Omega, V)\epsilon^{\frac{N-2}{N-4}} + o(\epsilon^{\frac{N-2}{N-4}})$$

with D_N given in (1.15). This completes the proof of Theorem 1.3 in the case $N \geq 5$.

Proof for $N = 4$. We rewrite (5.3), using Lemma 5.3, as

$$0 \geq (1 + o(1))(S_4 - S(\epsilon V)) - \frac{B_\epsilon \epsilon}{2e}\exp\left(-\frac{2A_\epsilon}{B_\epsilon \epsilon}\right) + \mathcal{R} \tag{5.20}$$

with

$$\mathcal{R} = \left(\frac{A_\epsilon}{\lambda^2} - B_\epsilon\frac{\epsilon\log\lambda}{\lambda^2} + \frac{B_\epsilon\epsilon}{2e}\exp\left(-\frac{2A_\epsilon}{B_\epsilon\epsilon}\right)\right) + c\int_\Omega |\nabla w|^2\,dy,$$

where we have set

$$A_\epsilon = \frac{64}{S_4}a_4(\phi(x_0) + o(1)), \qquad B_\epsilon = \frac{8}{S_4}b_4|V(x_0) + o(1)|\,.$$

Notice that both summands of $\mathcal{R}$ are separately nonnegative. Inserting the upper bound from Corollary 2.2 into (5.20), we get

$$0 \geq (1 + o(1))\exp\left(-\frac{4}{\epsilon}\,(1 + o(1))\,\sigma_4(\Omega, V)^{-1}\right) - \frac{B_\epsilon\epsilon}{2e}\exp\left(-\frac{2A_\epsilon}{B_\epsilon\epsilon}\right) + \mathcal{R}\,. \tag{5.21}$$

Dropping the nonnegative term $\mathcal{R}$ from the right side and taking the logarithm of the resulting inequality, we obtain

$$-\frac{2A_\epsilon}{B_\epsilon\epsilon} + \log\frac{B_\epsilon\epsilon}{2e} \geq -\frac{4}{\epsilon}\,(1 + o(1))\,\sigma_4(\Omega, V)^{-1} + \log(1 + o(1))\,.$$

Multiplying by ϵ and passing to the limit we infer, since $a_4/b_4 = 1/4$,

$$-\frac{\phi(x_0)}{|V(x_0)|} \geq -\sigma_4(\Omega, V)^{-1}\,.$$

By definition of $\sigma_4(\Omega, V)$, this implies

$$\frac{|V(x_0)|}{\phi(x_0)} = \sigma_4(\Omega, V)\,, \tag{5.22}$$

as claimed. With this information at hand, we return to (5.21) and drop the nonnegative first term on the right side to infer that

$$\mathcal{R} \leq \frac{B_\epsilon \epsilon}{2e} \exp\left(-\frac{2A_\epsilon}{B_\epsilon \epsilon}\right).$$

Keeping only the second term in the definition of $\mathcal{R}$ and using (5.22) we deduce, in particular, that

$$\|\nabla w\|_2^2 \leq \exp\left(-\frac{4}{\epsilon}\,(1+o(1))\,\sigma_4(\Omega, V)^{-1}\right). \tag{5.23}$$

We now keep only the first term in the definition of $\mathcal{R}$ and obtain from (5.21), multiplied by $(2e/(B_\epsilon \epsilon))\exp(2A_\epsilon/(B_\epsilon \epsilon))$,

$$\begin{aligned} 1-(1+o(1))\frac{2e}{B_\epsilon \epsilon}\exp\left(\frac{2A_\epsilon}{B_\epsilon \epsilon}-\frac{4}{\epsilon}\,(1+o(1))\,\sigma_4(\Omega, V)^{-1}\right) &\geq \frac{2e}{B_\epsilon \epsilon}\exp\left(\frac{2A_\epsilon}{B_\epsilon \epsilon}\right)\mathcal{R} \\ &\geq \frac{2e}{B_\epsilon \epsilon}\exp\left(\frac{2A_\epsilon}{B_\epsilon \epsilon}\right)\left(\frac{A_\epsilon}{\lambda^2}-B_\epsilon\frac{\epsilon\log\lambda}{\lambda^2}\right)+1 \\ &= 1+y\,e^{y+1} \end{aligned}$$

with $y = \frac{2}{B_\epsilon \epsilon}(A_\epsilon - \epsilon B_\epsilon \log\lambda)$. In view of (5.22) and (2.12) we have

$$(1+o(1))\frac{2e}{B_\epsilon \epsilon}\exp\left(\frac{2A_\epsilon}{B_\epsilon \epsilon}-\frac{4}{\epsilon}\,(1+o(1))\,\sigma_4(\Omega, V)^{-1}\right) = \exp\left(o\left(\frac{1}{\epsilon}\right)\right),$$

and therefore

$$-\exp\left(o\left(\frac{1}{\epsilon}\right)\right) \geq y\,e^{y+1}\,.$$

This implies

$$0 < -y \leq o\left(\frac{1}{\epsilon}\right),$$

which is the same as

$$\frac{A_\epsilon}{B_\epsilon \epsilon} < \log\lambda \leq \frac{A_\epsilon}{B_\epsilon \epsilon} + o\left(\frac{1}{\epsilon}\right).$$

Recalling (5.22) we obtain

$$\lambda = \exp\left(-\frac{2}{\epsilon}\,(1+o(1))\,\sigma_4(\Omega, V)^{-1}\right), \tag{5.24}$$

as claimed. Finally, to obtain the asymptotics of α, we deduce from (5.18) and (5.19), together with the bounds (5.23) and (5.24), that

$$|\alpha| = 1 + \exp\left(-\frac{4}{\epsilon}\,(1+o(1))\,\sigma_4(\Omega, V)^{-1}\right).$$

This completes the proof of Theorem 1.3 in the case $N = 4$. □

Acknowledgments

Partial support through US National Science Foundation grant DMS-1363432 (R. L. F.) and Studienstiftung des deutschen Volkes (T. K.) is acknowledged. H. K. has been partially supported by Gruppo Nazionale per Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).

Conflict of interest

The authors declare no conflict of interest.

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A. Auxiliary results

The proof of the following lemma is similar to the computation in [10, Appendix A]. We provide here details for the sake of completeness.

Lemma A.1. *Let $x = x_\lambda$ be a sequence of points in Ω such that $d(x)\lambda \to \infty$. Then*

$$\left(\int_\Omega U_{x,\lambda}^{\frac{q(q-2)}{q-1}} \varphi_{x,\lambda}^{\frac{q}{q-1}}\, dy\right)^{\frac{q-1}{q}} = \begin{cases} O\Big((d(x)\,\lambda)^{\frac{-2-N}{2}}\Big) & \text{if } N > 6, \\ O\Big((d(x)\,\lambda)^{-4}\log(d(x)\lambda)\Big) & \text{if } N = 6, \\ O\Big((d(x)\,\lambda)^{2-N}\Big) & \text{if } N = 4, 5 \end{cases} \tag{A.1}$$

and

$$\int_\Omega U_{x,\lambda}^{q-2}\, \varphi_{x,\lambda}^2\, dy = O\Big((d(x)\,\lambda)^{-N}\Big)\,. \tag{A.2}$$

Proof. We write $d = d(x)$ for short in the following proof.

Proof of (A.1). By Eqs. (2.14), (2.15) and (2.18),

$$\int_{B_d(x)} U_{x,\lambda}^{\frac{q(q-2)}{q-1}} \varphi_{x,\lambda}^{\frac{q}{q-1}}\, dy \le \|\varphi_{x,\lambda}\|_{L^\infty(\Omega)}^{\frac{q}{q-1}} \int_{B_d(x)} U_{x,\lambda}^{\frac{q(q-2)}{q-1}}\, dy = O\Big((d^{2-N}\,\lambda^{\frac{2-N}{2}})^{\frac{q}{q-1}}\Big) \int_{B_d(x)} U_{x,\lambda}^{\frac{q(q-2)}{q-1}}\, dy\,. \tag{A.3}$$

Moreover, since $\frac{q(q-2)}{q-1}\,\frac{N-2}{2} = \frac{4N}{N+2}$, from (1.7) we obtain

$$\begin{aligned} \int_{B_d(x)} U_{x,\lambda}^{\frac{q(q-2)}{q-1}}\, dy &= O\Big(\lambda^{\frac{4N}{N+2}}\Big) \int_0^d \frac{r^{N-1}\, dr}{(1+\lambda^2 r^2)^{\frac{4N}{N+2}}} = O\Big(\lambda^{\frac{2N-N^2}{N+2}}\Big) \int_0^{\lambda d} \frac{t^{N-1}\, dr}{(1+t^2)^{\frac{4N}{N+2}}} \\ &= O\Big(\lambda^{\frac{2N-N^2}{N+2}}\Big)\left(\int_1^{\lambda d} t^{\frac{N(N-6)}{N+2}}\, t^{-1}\, dt + O(1)\right). \end{aligned} \tag{A.4}$$

If $N > 6$, then

$$\int_1^{\lambda d} t^{\frac{N(N-6)}{N+2}}\, t^{-1}\, dt = O\Big((d\,\lambda)^{\frac{N(N-6)}{N+2}}\Big).$$

If $N = 6$, then

$$\int_1^{\lambda d} t^{\frac{N(N-6)}{N+2}}\, t^{-1}\, dt = O\left(\log(d\,\lambda)\right)$$

and if $N = 4, 5$, then

$$\int_1^{\lambda d} t^{\frac{N(N-6)}{N+2}}\, t^{-1}\, dt = O\left(1\right)$$

This gives the bound claimed in (A.1) in each case, provided we can bound the integral on the complement $\Omega \setminus B_d(x)$. On this region, we have by Hölder

$$\left(\int_{\Omega\setminus B_d(x)} U_{x,\lambda}^{\frac{q(q-2)}{q-1}} \varphi_{x,\lambda}^{\frac{q}{q-1}}\, dy\right)^{\frac{q-1}{q}} \le \left(\int_\Omega \varphi_{x,\lambda}^{\frac{2N}{N-2}}\, dy\right)^{\frac{N-2}{2N}} \left(\int_{\mathbb{R}^N\setminus B_d(x)} U_{x,\lambda}^{\frac{2N}{N-2}}\, dy\right)^{\frac{2}{N}}$$

$$= \mathcal{O}\Big((d\,\lambda)^{\frac{2-N}{2}}\Big)\left(\int_{\mathbb{R}^N\setminus B_d(x)} U_{x,\lambda}^{\frac{2N}{N-2}}\,dy\right)^{\frac{2}{N}}$$

$$= \mathcal{O}\Big((d\,\lambda)^{\frac{2-N}{2}}\Big)\left(\int_{d\lambda}^{\infty} \frac{dt}{t^{N+1}}\right)^{\frac{2}{N}}$$

$$= \mathcal{O}\Big((d\,\lambda)^{\frac{2-N}{2}}\Big)\,\mathcal{O}\Big((d\,\lambda)^{-2}\Big),$$

where we have used (1.7) and the fact that

$$\left(\int_{\Omega} \varphi_{x,\lambda}^{\frac{2N}{N-2}}\,dy\right)^{\frac{N-2}{2N}} = \mathcal{O}\Big((d\,\lambda)^{\frac{2-N}{2}}\Big) \tag{A.5}$$

by [10, Prop. 1(c)]. Combining all the estimates, we deduce (A.1).

Proof of (A.2). We split the domain of integration Ω again into $B_d(x)$ and $\Omega\setminus B_d(x)$. On $B_d(x)$, by (2.14),

$$\int_{B_d(x)} U_{x,\lambda}^{q-2}\,\varphi_{x,\lambda}^2\,dy \le \|\varphi_{x,\lambda}\|_{L^\infty(\Omega)}^2\left(\int_{B_d(x)} U_{x,\lambda}^{q-2}\,dy\right)$$

$$= \mathcal{O}\Big(d(x)^{4-2N}\,\lambda^{2-N}\Big)\left(\lambda^{2-N}\int_0^{d\lambda}\frac{t^{N-1}\,dt}{(1+t^2)^2}\right) = \mathcal{O}((d\lambda)^{-N}). \tag{A.6}$$

On $\Omega\setminus B_d(x)$, by Hölder and (A.5),

$$\int_{\Omega\setminus B_d(x)} U_{x,\lambda}^{q-2}\,\varphi_{x,\lambda}^2\,dy \le \left(\int_{\Omega}\varphi_{x,\lambda}^q\,dy\right)^{\frac{2}{q}}\left(\int_{\mathbb{R}^N\setminus B_d(x)} U_{x,\lambda}^q\,dy\right)^{\frac{q-2}{q}} = \mathcal{O}\Big((d(x)\,\lambda)^{2-N}\Big)\,\mathcal{O}\Big((d\,\lambda)^{-2}\Big). \tag{A.7}$$

Combining (A.6) and (A.7), we obtain (A.2). □

Lemma A.2. *Let* $f_\epsilon : (0,\infty)\to\mathbb{R}$ *be given by*

$$f_\epsilon(\lambda) = \frac{A_\epsilon}{\lambda^{N-2}} - B_\epsilon\frac{\epsilon}{\lambda^2}$$

with $A_\epsilon, B_\epsilon > 0$ *uniformly bounded away from 0 and* ∞. *Denote by*

$$\lambda_0 = \lambda_0(\epsilon) = \left(\frac{(N-2)A_\epsilon}{2B_\epsilon}\right)^{\frac{1}{N-4}}\epsilon^{\frac{1}{4-N}}$$

the unique global minimum of f_ϵ. *Then there is a* $c_0 > 0$ *such that for all* $\epsilon > 0$ *we have*

$$f_\epsilon(\lambda) - f_\epsilon(\lambda_0) \ge \begin{cases} c_0\epsilon\Big(\lambda^{-1}-\lambda_0(\epsilon)^{-1}\Big)^2 & \textit{if}\quad (\frac{A_\epsilon}{B_\epsilon})^{\frac{1}{N-4}}\epsilon^{-\frac{1}{N-4}}\lambda^{-1} \le 2(\frac{2}{N-2})^{\frac{1}{N-4}}, \\ c_0\epsilon^{\frac{N-2}{N-4}} & \textit{if}\quad (\frac{A_\epsilon}{B_\epsilon})^{\frac{1}{N-4}}\epsilon^{-\frac{1}{N-4}}\lambda^{-1} > 2(\frac{2}{N-2})^{\frac{1}{N-4}}. \end{cases}$$

Proof. Let $F(t) := t^{N-2} - t^2$ and denote by $t_0 := (\frac{2}{N-2})^{\frac{1}{N-4}}$ the unique global minimum on $(0,\infty)$ of F. Then it is easy to see that there is $c > 0$ such that

$$F(t) - F(t_0) \ge \begin{cases} c(t-t_0)^2 & \text{if}\quad 0 < t \le 2t_0, \\ ct_0^{N-2} & \text{if}\quad t > 2t_0. \end{cases}$$

The assertion of the lemma now follows by rescaling. Indeed, it suffices to observe that

$$f_\epsilon(\lambda) = A_\epsilon^{-\frac{2}{N-4}} B_\epsilon^{\frac{N-2}{N-4}} \epsilon^{\frac{N-2}{N-4}} F\left((\frac{A_\epsilon}{B_\epsilon})^{\frac{1}{N-4}} \epsilon^{-\frac{1}{N-4}} \lambda^{-1} \right)$$

and to use the boundedness of A_ϵ and B_ϵ. □

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