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Research article

Griffith energies as small strain limit of nonlinear models for nonsimple brittle materials†

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Abstract: We consider a nonlinear, frame indifferent Griffith model for nonsimple brittle materials where the elastic energy also depends on the second gradient of the deformations. In the framework of free discontinuity and gradient discontinuity problems, we prove existence of minimizers for boundary value problems. We then pass to a small strain limit in terms of suitably rescaled displacement fields and show that the nonlinear energies can be identified with a linear Griffith model in the sense of Γ-convergence. This complements the study in [39] by providing a linearization result in arbitrary space dimensions.

Keywords: brittle materials; variational fracture; nonsimple materials; free discontinuity problems; Griffith energies; Γ-convergence; functions of bounded variation and deformation

1. Introduction

Mathematical models in solids mechanics typically do not predict the mechanical behavior correctly at every scale, but have a certain limited range of applicability. A central example in that direction are models for hyperelastic materials in nonlinear (finite) elasticity and their linear (infinitesimal) counterparts. The last decades have witnessed remarkable progress in providing a clear relationship between different models via Γ-convergence [30]. In their seminal work [33], Dal Maso, Negri and Percivale performed a nonlinear-to-linear analysis in terms of suitably rescaled displacement fields and proved the convergence of minimizers for corresponding boundary value problems. This study has been extended in various directions, including different growth assumptions on the stored energy densities [1], the passage from atomistic-to-continuum models [13, 55], multiwell energies [2, 54], plasticity [51], and viscoelasticity [43].

In the present contribution, we are interested in an analogous analysis for materials undergoing fracture. Based on the variational approach to quasistatic crack evolution by Francfort and Marigo [37], where the displacements and the (a priori unknown) crack paths are determined from an energy minimization principle, we consider an energy functional of Griffith-type. Such variational models of brittle fracture, which comprise an elastic energy stored in the uncracked region of the body and a surface contribution comparable to the size of the crack of codimension one, have been widely studied both at finite and infinitesimal strains, see [7, 18, 32, 34, 38, 45, 48] without claim of being exhaustive. We refer the reader to [11] for a general overview.

In this context, first results addressing the question of a nonlinear-to-linear analysis have been obtained in [52, 53] in a two-dimensional evolutionary setting for a fixed crack set or a restricted class of admissible cracks, respectively. Subsequently, the problem was studied in [44] from a different perspective. Here, a simultaneous discrete-to-continuum and nonlinear-to-linear analysis is performed for general crack geometries, but under the simplifying assumption that all deformations are close to the identity mapping.

Eventually, a result in dimension two without a priori assumptions on the crack paths and the deformations, in the general framework of free discontinuity problems (see [35]), has been derived in [39]. This analysis relies fundamentally on delicate geometric rigidity results in the spirit of [22, 46]. At this point, the geometry of crack paths in the plane is crucially exploited and higher dimensional analogs seem to be currently out of reach. In spite of the lack of rigidity estimates, the goal of this contribution is to perform a nonlinear-to-linear analysis for brittle materials in the spirit of [39] in higher space dimensions. This will be achieved by starting from a slightly different nonlinear model for so-called *nonsimple materials*.

Whereas the elastic properties of simple materials depend only on the first gradient, the notion of a nonsimple material refers to the fact that the elastic energy depends additionally on the second gradient of the deformation. This idea goes back to Toupin [57, 58] and has proved to be useful in modern mathematical elasticity, see e.g., [8, 9, 14, 36, 43, 50], since it brings additional compactness and rigidity to the problem. In a similar fashion, we consider here a Griffith model with an additional second gradient in the elastic part of the energy. This leads to a model in the framework of free discontinuity and gradient discontinuity problems.

The goal of this contribution is twofold. We first show that the regularization allows to prove existence of minimizers for boundary value problems without convexity properties for the stored elastic energy. In particular, we do not have to assume quasiconvexity [4]. Afterwards, we identify an effective linearized Griffith energy as the Γ-limit of the nonlinear and frame indifferent models for vanishing strains. In this context, it is important to mention that, in spite of the formulation of the nonlinear model in terms of nonsimple materials, the effective limit is a 'standard' Griffith functional in linearized elasticity depending only on the first gradient. A similar justification for the treatment of nonsimple materials has recently been discussed in [43] for a model in nonlinear viscoelasticity.

The existence result for boundary value problems at finite strains is formulated in the space $GSBV^2_2(\Omega;\mathbb{R}^d)$, see (2.2) below, consisting of the mappings for which both the function itself and its derivative are in the class of *generalized special functions of bounded variation* [6]. The relevant compactness and lower semicontinuity results stated in Theorem 3.3 essentially follow from a study on second order variational problems with free discontinuity and gradient discontinuity [16]. Another key ingredient is the recent work [42] which extends the classical compactness result due to

Ambrosio [3] to problems without a priori bounds on the functions.

Concerning the passage to the linearized system, the essential step is to establish a compactness result in terms of suitably rescaled displacement fields which measure the distance of the deformations from the identity. Whereas in [39] this is achieved by means of delicate geometric rigidity estimates, the main idea in our approach is to partition the domain into different regions in which the gradient is 'almost constant'. This construction relies on the coarea formula in BV and is the fundamental point where the presence of a second order term in the energy is used to pass rigorously to a linear theory. The linear limiting model is formulated on the space of *generalized special functions of bounded deformation* $GSBD^2$, which has been studied extensively over the last years, see e.g., [19–21, 23–28, 31, 40, 41, 45, 49].

The paper is organized as follows. In Section 2 we first introduce our nonlinear model for nonsimple brittle materials and state our main results: We first address the existence of minimizers for boundary value problems at finite strains. Then, we present a compactness and Γ-convergence result in the passage from the nonlinear to the linearized theory. Here, we also discuss the convergence of minima and minimzers under given boundary data. Section 3 is devoted to some preliminary results about the function spaces $GSBV$ and $GSBD$. In particular, we present a compactness result in $GSBV^2_2$ involving the second gradient (see Theorem 3.3). Finally, Section 4 contains the proofs of our results.

2. The model and main results

In this section we introduce our model and present the main results. We start with some basic notation. Throughout the paper, $\Omega \subset \mathbb{R}^d$ is an open and bounded set. The notations $\mathcal{L}^d$ and $\mathcal{H}^{d-1}$ are used for the Lebesgue measure and the $(d-1)$-dimensional Hausdorff measure in $\mathbb{R}^d$, respectively. We set $S^{d-1} = \{x \in \mathbb{R}^d : |x| = 1\}$. For an $\mathcal{L}^d$-measurable set $E \subset \mathbb{R}^d$, the symbol χ_E denotes its indicator function. For two sets $A, B \subset \mathbb{R}^d$, we define $A \triangle B = (A \setminus B) \cup (B \setminus A)$. The identity mapping on $\mathbb{R}^d$ is indicated by $\mathbf{id}$ and its derivative, the identity matrix, by $\mathbf{Id} \in \mathbb{R}^{d\times d}$. The sets of symmetric and skew symmetric matrices are denoted by $\mathbb{R}^{d\times d}_{\rm sym}$ and $\mathbb{R}^{d\times d}_{\rm skew}$, respectively. We set $\mathrm{sym}(F) = \frac{1}{2}(F^T + F)$ for $F \in \mathbb{R}^{d\times d}$ and define $SO(d) = \{R \in \mathbb{R}^{d\times d} : R^T R = \mathbf{Id},\ \det R = 1\}$.

2.1. A nonlinear model for nonsimple materials and boundary value problems

In this subsection we introduce our nonlinear model and discuss the existence of minimizers for boundary value problems.

Function spaces: To introduce our Griffith-type model for nonsimple materials, we first need to introduce the relevant spaces. We use standard notation for $GSBV$ functions, see [6, Section 4] and [32, Section 2]. In particular, we let

$$GSBV^2(\Omega;\mathbb{R}^d) = \{y \in GSBV(\Omega;\mathbb{R}^d) : \ \nabla y \in L^2(\Omega;\mathbb{R}^{d\times d}),\ \mathcal{H}^{d-1}(J_y) < +\infty\}, \tag{2.1}$$

where $\nabla y(x)$ denotes the approximate differential at $\mathcal{L}^d$-a.e. $x \in \Omega$ and J_y the jump set. We define the space

$$GSBV^2_2(\Omega;\mathbb{R}^d) := \{y \in GSBV^2(\Omega;\mathbb{R}^d) : \ \nabla y \in GSBV^2(\Omega;\mathbb{R}^{d\times d})\}. \tag{2.2}$$

The approximate differential and the jump set of ∇y will be denoted by $\nabla^2 y$ and $J_{\nabla y}$, respectively. (To avoid confusion, we point out that in the paper [32] the notation $GSBV^2_2(\Omega;\mathbb{R}^d)$ was used differently, namely for $GSBV^2(\Omega;\mathbb{R}^d) \cap L^2(\Omega;\mathbb{R}^d)$.)

A similar space has been considered in [15, 16] to treat second order free discontinuity functionals, e.g., a weak formulation of the Blake & Zissermann model [10] of image segmentation. We point out that the functions are allowed to exhibit discontinuities. Thus, the analysis is outside of the framework of the space of special functions with bounded Hessian $SBH(\Omega)$, considered in problems of second order energies for elastic-perfectly plastic plates, see e.g., [17].

Nonlinear Griffith energy for nonsimple materials: We let $W : \mathbb{R}^{d\times d} \to [0, +\infty)$ be a single well, frame indifferent stored energy functional. More precisely, we suppose that there exists $c > 0$ such that

$$\begin{aligned} &\text{(i)}\ \ W \text{ continuous and } C^3 \text{ in a neighborhood of } SO(d),\\ &\text{(ii)}\ \ \text{Frame indifference: } W(RF) = W(F) \text{ for all } F \in \mathbb{R}^{d\times d}, R \in SO(d),\\ &\text{(iii)}\ \ W(F) \geq c\,\mathrm{dist}^2(F, SO(d)) \ \text{ for all } F \in \mathbb{R}^{d\times d},\ W(F) = 0 \text{ iff } F \in SO(d). \end{aligned} \tag{2.3}$$

We briefly note that we can also treat inhomogeneous materials where the energy density has the form $W : \Omega \times \mathbb{R}^{d\times d} \to [0, +\infty)$. Moreover, it suffices to assume $W \in C^{2,\alpha}$, where $C^{2,\alpha}$ is the Hölder space with exponent $\alpha \in (0, 1]$, see Remark 4.2 for details.

Let $\kappa > 0$ and $\beta \in (\frac{2}{3}, 1)$. For $\varepsilon > 0$, define the energy $\mathcal{E}_\varepsilon(\cdot, \Omega) : GSBV^2_2(\Omega; \mathbb{R}^d) \to [0, +\infty]$ by

$$\mathcal{E}_\varepsilon(y, \Omega) = \begin{cases} \varepsilon^{-2} \int_\Omega W(\nabla y(x))\, dx + \varepsilon^{-2\beta} \int_\Omega |\nabla^2 y(x)|^2\, dx + \kappa \mathcal{H}^{d-1}(J_y) & \text{if } J_{\nabla y} \subset J_y, \\ +\infty & \text{else.} \end{cases} \tag{2.4}$$

Here and in the following, the inclusion $J_{\nabla y} \subset J_y$ has to be understood up to an $\mathcal{H}^{d-1}$-negligible set. Since W grows quadratically around $SO(d)$, the parameter ε corresponds to the typical scaling of strains for configurations with finite energy.

Due to the presence of the second term, we deal with a Griffith-type model for *nonsimple materials*. As explained in the introduction, elastic energies which depend additionally on the second gradient of the deformation were introduced by Toupin [57, 58] to enhance compactness and rigidity properties. In the present context, we add a second gradient term for a material undergoing fracture. This regularization effect acts on the entire *intact region* $\Omega \setminus J_y$ of the material. This is modeled by the condition $J_{\nabla y} \subset J_y$.

The goal of this contribution is twofold. We first show that the regularization allows to prove existence of minimizers for boundary value problems without convexity properties of W. The main result of the present work is then to identify a linearized Griffith energy in the small strain limit $\varepsilon \to 0$ which is related to the nonlinear energies $\mathcal{E}_\varepsilon$ through Γ-convergence. We point out that the effective limit is a 'standard' Griffith model in linearized elasticity depending only on the first gradient, see (2.14) below, although we start with a nonlinear model for nonsimple materials.

We observe that the condition $J_{\nabla y} \subset J_y$ is not closed under convergence in measure on Ω. In fact, consider, e.g., $\Omega = (-1, 1)^2$, $\Omega_1 = (-1, 0)\times(-1, 1)$, $\Omega_2 = (0, 1)\times(-1, 1)$, and for $\delta \geq 0$ the configurations

$$y_\delta(x_1, x_2) = (x_1, x_2)\chi_{\Omega_1} + (2x_1 + \delta, x_2)\chi_{\Omega_2} \qquad \text{for } (x_1, x_2) \in \Omega.$$

Then $J_{\nabla y_\delta} = J_{y_\delta} = \{0\} \times (-1, 1)$ for $\delta > 0$ and $y_\delta \to y_0$ in measure on Ω as $\delta \to 0$. However, there holds $\emptyset = J_{y_0} \subset J_{\nabla y_0} = \{0\} \times (-1, 1)$. Therefore, we need to pass to a relaxed formulation.

Proposition 2.1 (Relaxation)**.** *Let $\Omega \subset \mathbb{R}^d$ be open and bounded. Suppose that W satisfies* (2.3)*. Then the relaxed functional $\overline{\mathcal{E}}_\varepsilon(\cdot,\Omega) : GSBV^2_2(\Omega;\mathbb{R}^d) \to [0,+\infty]$ defined by*

$$\overline{\mathcal{E}}_\varepsilon(y,\Omega) = \inf\{\liminf_{n\to\infty} \mathcal{E}_\varepsilon(y_n,\Omega) : y_n \to y \text{ in measure on } \Omega\}$$

is given by

$$\overline{\mathcal{E}}_\varepsilon(y,\Omega) = \varepsilon^{-2}\int_\Omega W(\nabla y(x))\,dx + \varepsilon^{-2\beta}\int_\Omega |\nabla^2 y(x)|^2\,dx + \kappa\mathcal{H}^{d-1}(J_y \cup J_{\nabla y}). \tag{2.5}$$

The result is proved in Subsection 4.1. Clearly, $\overline{\mathcal{E}}_\varepsilon(\cdot,\Omega)$ is lower semicontinuous with respect to the convergence in measure. We point out that this latter property has essentially been shown in [16], cf. Theorem 3.2.

In the following, our goal is to study boundary value problems. To this end, we suppose that there exist two bounded Lipschitz domains $\Omega' \supset \Omega$. We will impose Dirichlet boundary data on $\partial_D\Omega := \Omega' \cap \partial\Omega$. As usual for the weak formulation in the framework of free discontinuity problems, this will be done by requiring that configurations y satisfy $y = g$ on $\Omega' \setminus \overline{\Omega}$ for some $g \in W^{2,\infty}(\Omega';\mathbb{R}^d)$. From now on, we write $\mathcal{E}_\varepsilon(\cdot) = \mathcal{E}_\varepsilon(\cdot,\Omega')$ and $\overline{\mathcal{E}}_\varepsilon(\cdot) = \overline{\mathcal{E}}_\varepsilon(\cdot,\Omega')$ for notational convenience. The following result about existence of minimizers will be proved in Subsection 4.1.

Theorem 2.2 (Existence of minimizers)**.** *Let $\Omega \subset \Omega' \subset \mathbb{R}^d$ be bounded Lipschitz domains. Suppose that W satisfies* (2.3)*, and let $g \in W^{2,\infty}(\Omega';\mathbb{R}^d)$. Then the minimization problem*

$$\inf_{y\in GSBV^2_2(\Omega';\mathbb{R}^d)} \Big\{\overline{\mathcal{E}}_\varepsilon(y) :\ y = g \text{ on } \Omega' \setminus \overline{\Omega}\Big\} \tag{2.6}$$

admits solutions.

2.2. *Compactness of rescaled displacement fields*

The main goal of the present work is the identification of an effective linearized Griffith energy in the small strain limit. In this subsection, we formulate the relevant compactness result. Let $\Omega' \supset \Omega$ be bounded Lipschitz domains. The limiting energy is defined on the space of generalized special functions of bounded deformation $GSBD^2(\Omega')$. For basic properties of $GSBD^2(\Omega')$ we refer to [31] and Section 3.3 below. In particular, for $u \in GSBD^2(\Omega')$, we denote by $e(u) = \frac{1}{2}(\nabla u^T + \nabla u)$ the approximate symmetric differential and by J_u the jump set.

The general idea in linearization results in many different settings (see, e.g., [2,13,33,43,44,52,54,55]) is the following: Given a sequence $(y_\varepsilon)_\varepsilon$ with $\sup_\varepsilon \mathcal{E}_\varepsilon(y_\varepsilon) < +\infty$, define displacement fields which measure the distance of the deformations from the identity, rescaled by the small parameter ε, i.e.,

$$u_\varepsilon = \frac{1}{\varepsilon}(y_\varepsilon - \mathbf{id}). \tag{2.7}$$

It turns out, however, that in general no compactness can be expected if the body may undergo fracture. Consider, e.g., the functions $y_\varepsilon = \mathbf{id}\chi_{\Omega'\setminus B} + Rx\chi_B$, for a small ball $B \subset \Omega$ and a rotation $R \in SO(d)$, $R \neq \mathbf{Id}$. Then $|u_\varepsilon|, |\nabla u_\varepsilon| \to \infty$ on B as $\varepsilon \to 0$. The main idea in our approach is the observation that this phenomenon can be avoided if the deformation is *rotated back to the identity* on the set B. This will be made precise in Theorem 2.3(a) below where we pass to *piecewise rotated functions*. For such functions, we can control at least the symmetric part of ∇u_ε for the rescaled displacement fields defined in (2.7). This will allow us to derive a compactness result in the space $GSBD^2(\Omega')$, see Theorem 2.3(b).

Recall the definition of $GSBV^2_2(\Omega';\mathbb{R}^d)$ in (2.2). To account for boundary data $h \in W^{2,\infty}(\Omega';\mathbb{R}^d)$, we introduce the spaces

$$\begin{aligned} \mathcal{S}_{\varepsilon,h} &= \{y \in GSBV^2_2(\Omega';\mathbb{R}^d) :\ y = \mathbf{id} + \varepsilon h \text{ on } \Omega' \setminus \overline{\Omega}\}, \\ GSBD^2_h &= \{u \in GSBD^2(\Omega') :\ u = h \text{ on } \Omega' \setminus \overline{\Omega}\}. \end{aligned} \tag{2.8}$$

Recall $\beta \in (\frac{2}{3}, 1)$ and the definition of $\overline{\mathcal{E}}_\varepsilon = \overline{\mathcal{E}}_\varepsilon(\cdot, \Omega')$ in (2.5). For definition and basic properties of Caccioppoli partitions we refer to Section 3.1. In particular, for a set of finite perimeter $E \subset \Omega'$, we denote by $\partial^* E$ its essential boundary and by $(E)^1$ the points where E has density one, see [6, Definition 3.60].

Theorem 2.3 (Compactness)**.** *Let $\gamma \in (\frac{2}{3}, \beta)$. Assume that W satisfies* (2.3)*, and let $h \in W^{2,\infty}(\Omega';\mathbb{R}^d)$. Let $(y_\varepsilon)_\varepsilon$ be a sequence satisfying $y_\varepsilon \in \mathcal{S}_{\varepsilon,h}$ and $\sup_\varepsilon \overline{\mathcal{E}}_\varepsilon(y_\varepsilon) < +\infty$.*
(a) (Piecewise rotated functions) There exist Caccioppoli partitions $(P^\varepsilon_j)_j$ of Ω' and corresponding rotations $(R^\varepsilon_j)_j \subset SO(d)$ such that the piecewise rotated functions $y^{\mathrm{rot}}_\varepsilon \in GSBV^2_2(\Omega';\mathbb{R}^d)$ given by

$$y^{\mathrm{rot}}_\varepsilon := \sum\nolimits_{j=1}^\infty R^\varepsilon_j\, y_\varepsilon \chi_{P^\varepsilon_j} \tag{2.9}$$

satisfy

$$\begin{aligned} &\text{(i)}\ \ y^{\mathrm{rot}}_\varepsilon = \mathbf{id} + \varepsilon h \text{ on } \Omega' \setminus \overline{\Omega}, \\ &\text{(ii)}\ \ \mathcal{H}^{d-1}\big((J_{y^{\mathrm{rot}}_\varepsilon} \cup J_{\nabla y^{\mathrm{rot}}_\varepsilon}) \setminus (J_{y_\varepsilon} \cup J_{\nabla y_\varepsilon})\big) \le \mathcal{H}^{d-1}\Big(\Big(\Omega' \cap \bigcup\nolimits_{j=1}^\infty \partial^* P^\varepsilon_j\Big) \setminus J_{\nabla y_\varepsilon}\Big) \le C\varepsilon^{\beta-\gamma}, \\ &\text{(iii)}\ \ \|\mathrm{sym}(\nabla y^{\mathrm{rot}}_\varepsilon) - \mathbf{Id}\|_{L^2(\Omega')} \le C\varepsilon, \\ &\text{(iv)}\ \ \|\nabla y^{\mathrm{rot}}_\varepsilon - \mathbf{Id}\|_{L^2(\Omega')} \le C\varepsilon^\gamma \end{aligned} \tag{2.10}$$

for a constant $C > 0$ independent of ε.
(b) (Compactness of rescaled displacement fields) There exists a subsequence (not relabeled) and a function $u \in GSBD^2_h$ such that the rescaled displacement fields $u_\varepsilon \in GSBV^2_2(\Omega';\mathbb{R}^d)$ defined by

$$u_\varepsilon := \frac{1}{\varepsilon}(y^{\mathrm{rot}}_\varepsilon - \mathbf{id}) \tag{2.11}$$

satisfy

$$\begin{aligned} &\text{(i)}\ \ u_\varepsilon \to u \quad \text{a.e. in } \Omega' \setminus E_u, \\ &\text{(ii)}\ \ e(u_\varepsilon) \rightharpoonup e(u) \ \text{ weakly in } L^2(\Omega' \setminus E_u;\mathbb{R}^{d\times d}_{\mathrm{sym}}), \\ &\text{(iii)}\ \ \mathcal{H}^{d-1}(J_u) \le \liminf\nolimits_{\varepsilon\to 0} \mathcal{H}^{d-1}(J_{u_\varepsilon}) \le \liminf\nolimits_{\varepsilon\to 0} \mathcal{H}^{d-1}(J_{y_\varepsilon} \cup J_{\nabla y_\varepsilon}), \\ &\text{(iv)}\ \ e(u) = 0 \ \text{ on } E_u, \quad \mathcal{H}^{d-1}((\partial^* E_u \cap \Omega') \setminus J_u) = \mathcal{H}^{d-1}(J_u \cap (E_u)^1) = 0, \end{aligned} \tag{2.12}$$

where $E_u := \{x \in \Omega :\ |u_\varepsilon(x)| \to \infty\}$ is a set of finite perimeter.

Here and in the sequel, we follow the usual convention that convergence of the continuous parameter $\varepsilon \to 0$ stands for convergence of arbitrary sequences $\{\varepsilon_i\}_i$ with $\varepsilon_i \to 0$ as $i \to \infty$, see [12, Definition 1.45]. The compactness result will be proved in Subsection 4.2.

Note that (2.10)(i) implies $y_\varepsilon^{\rm rot} \in \mathcal{S}_{\varepsilon,h}$. In view of (2.10)(ii), the frame indifference of the elastic energy, and $\gamma < \beta$, one can show that the Griffith-type energy (2.5) of $y_\varepsilon^{\rm rot}$ is asymptotically not larger than the one of y_ε. The control on the symmetric part of the derivative (2.10)(iii) is essential to obtain compactness in $GSBD^2(\Omega')$ for the sequence $(u_\varepsilon)_\varepsilon$. Property (2.10)(iv) will be needed to control higher order terms in the passage to linearized elastic energies, see Theorem 2.7 below.

The presence of the set E_u is due to the compactness result in $GSBD^2(\Omega')$, see [26] and Theorem 3.4. In principle, the phenomenon that the sequence is unbounded on a set of positive measure can be avoided by generalizing the definition of (2.11): In [45, Theorem 6.1] and [39, Theorem 2.2] it has been shown that, by subtracting in (2.11) suitable translations on a Caccioppoli partition of Ω' related to y_ε, one can achieve $E_u = \emptyset$. This construction, however, is limited so far to dimension two. As discussed in [26], the presence of E_u is not an issue for minimization problems of Griffith energies since a minimizer can be recovered by choosing u affine on E_u with $e(u) = 0$, cf. (2.12)(iv). We also note that $E_u \subset \Omega$, i.e., $E_u \cap (\Omega' \setminus \overline{\Omega}) = \emptyset$.

Definition 2.4 (Asymptotic representation)**.** We say that a sequence $(y_\varepsilon)_\varepsilon$ with $y_\varepsilon \in \mathcal{S}_{\varepsilon,h}$ is *asymptotically represented* by a limiting displacement $u \in GSBD^2_h$, and write $y_\varepsilon \rightsquigarrow u$, if there exist sequences of Caccioppoli partitions $(P_j^\varepsilon)_j$ of Ω' and corresponding rotations $(R_j^\varepsilon)_j \subset SO(d)$ such that (2.10) and (2.12) hold for some fixed $\gamma \in (\frac{2}{3},\beta)$, where $y_\varepsilon^{\rm rot}$ and u_ε are defined in (2.9) and (2.11), respectively.

Theorem 2.3 shows that for each $(y_\varepsilon)_\varepsilon$ with $\sup_\varepsilon \overline{\mathcal{E}}_\varepsilon(y_\varepsilon) < +\infty$ there exists a subsequence $(y_{\varepsilon_k})_k$ and $u \in GSBD^2_h$ such that $y_{\varepsilon_k} \rightsquigarrow u$ as $k \to \infty$. We speak of asymptotic representation instead of convergence, and we use the symbol $\rightsquigarrow$, in order to emphasize that Definition 2.4 cannot be understood as a convergence with respect to a certain topology. In particular, the limiting function u for a given (sub-)sequence $(y_\varepsilon)_\varepsilon$ is not determined uniquely, but depends fundamentally on the choice of the sequences $(P_j^\varepsilon)_j$ and $(R_j^\varepsilon)_j$. To illustrate this phenomenon, we consider an example similar to [39, Example 2.4].

Example 2.5 (Nonuniqueness of limits)**.** Consider $\Omega' = (0,3) \times (0,1)$, $\Omega = (1,3) \times (0,1)$, $\Omega_1 = (0,2) \times (0,1)$, $\Omega_2 = (2,3) \times (0,1)$, $h \equiv 0$, and

$$y_\varepsilon(x) = x\chi_{\Omega_1}(x) + \bar{R}_\varepsilon\, x\chi_{\Omega_2}(x) \quad \text{for } x \in \Omega',$$

where $\bar{R}_\varepsilon \in SO(2)$ with $\bar{R}_\varepsilon = \mathbf{Id} + \varepsilon A + \mathrm{O}(\varepsilon^2)$ for some $A \in \mathbb{R}^{2\times 2}_{\rm skew}$. Then two possible alternatives are

$$\begin{aligned} &(1)\ \ P_1^\varepsilon = \Omega_1,\ \ P_2^\varepsilon = \Omega_2,\ \ R_1^\varepsilon = \mathbf{Id},\ \ R_2^\varepsilon = \bar{R}_\varepsilon^{-1},\\ &(2)\ \ \tilde{P}_1^\varepsilon = \Omega',\ \ \tilde{R}_1^\varepsilon = \mathbf{Id}. \end{aligned}$$

Letting $u_\varepsilon = \varepsilon^{-1}(\sum_{j=1}^2 R_j^\varepsilon y_\varepsilon \chi_{P_j^\varepsilon} - \mathbf{id})$ and $\tilde{u}_\varepsilon = \varepsilon^{-1}(y_\varepsilon - \mathbf{id})$, we find the limits $u \equiv 0$ and $\tilde{u}(x) = A\,x\chi_{\Omega_2}(x)$, respectively.

We refer to [39, Section 2.3] for a further discussion about different choices of the involved partitions and rigid motions. Here, we show that it is possible to identify uniquely the relevant notions $e(u)$ and J_u of the limit. This is content of the following lemma.

Lemma 2.6 (Characterization of limiting displacements)**.** *Suppose that a sequence $(y_\varepsilon)_\varepsilon$ satisfies $y_\varepsilon \rightsquigarrow u_1$ and $y_\varepsilon \rightsquigarrow u_2$, where $u_1, u_2 \in GSBD^2_h$, $u_1 \neq u_2$. Let $E_{u_1}, E_{u_2} \subset \Omega$ be the sets given in* (2.12)*. Then*

(a) $e(u_1) = e(u_2)$ *$\mathcal{L}^d$-a.e. on* $\Omega' \setminus (E_{u_1} \cup E_{u_2})$.

(b) If additionally $(y_\varepsilon)_\varepsilon$ is a minimizing sequence, i.e.,

$$\overline{\mathcal{E}}_\varepsilon(y_\varepsilon) \leq \inf_{\bar{y} \in \mathcal{S}_{\varepsilon,h}} \overline{\mathcal{E}}_\varepsilon(\bar{y}) + \rho_\varepsilon \quad \textit{with } \rho_\varepsilon \to 0 \textit{ as } \varepsilon \to 0, \tag{2.13}$$

then $e(u_1) = e(u_2)$ $\mathcal{L}^d$-a.e. on Ω', and $J_{u_1} = J_{u_2}$ up to an $\mathcal{H}^{d-1}$-negligible set.

Note that property (a) is consistent with Example 2.5. Example 2.5 also shows that the property $J_{u_1} = J_{u_2}$ is not satisfied in general but some extra condition, e.g., the one in (2.13), is necessary. We refer to Example 4.3 below for an illustration that in case (a) the strains are not necessarily the same inside $E_{u_1} \cup E_{u_2}$. The result will be proved in Subsection 4.4.

2.3. Passage from the nonlinear to a linearized Griffith model

We now show that the nonlinear energies of Griffith-type can be related to a linearized Griffith model in the small strain limit by Γ-convergence. We also discuss the convergence of minimizers for boundary value problems. Given bounded Lipschitz domains $\Omega \subset \Omega'$, we define the energy $\mathcal{E} : GSBD^2(\Omega') \to [0, +\infty)$ by

$$\mathcal{E}(u) = \int_{\Omega'} \frac{1}{2} Q(e(u)) + \kappa \mathcal{H}^{d-1}(J_u), \tag{2.14}$$

where $\kappa > 0$, and $Q : \mathbb{R}^{d\times d} \to [0, +\infty)$ is the quadratic form $Q(F) = D^2W(\mathbf{Id})F : F$ for all $F \in \mathbb{R}^{d\times d}$. In view of (2.3), Q is positive definite on $\mathbb{R}^{d\times d}_{\rm sym}$ and vanishes on $\mathbb{R}^{d\times d}_{\rm skew}$.

For the Γ-limsup inequality, more precisely for the application of the density result stated in Theorem 3.6, we make the following geometrical assumption on the Dirichlet boundary $\partial_D\Omega = \Omega' \cap \partial\Omega$: there exists a decomposition $\partial\Omega = \partial_D\Omega \cup \partial_N\Omega \cup N$ with

$$\partial_D\Omega, \partial_N\Omega \text{ relatively open}, \quad \mathcal{H}^{d-1}(N) = 0, \quad \partial_D\Omega \cap \partial_N\Omega = \emptyset, \quad \partial(\partial_D\Omega) = \partial(\partial_N\Omega), \tag{2.15}$$

and there exist $\bar{\delta} > 0$ small and $x_0 \in \mathbb{R}^d$ such that for all $\delta \in (0, \bar{\delta})$ there holds

$$O_{\delta,x_0}(\partial_D\Omega) \subset \Omega, \tag{2.16}$$

where $O_{\delta,x_0}(x) := x_0 + (1-\delta)(x - x_0)$.

We now present our main Γ-convergence result. Recall Definition 2.4, as well as the definition of the nonlinear energies in (2.4) and (2.5). Moreover, recall the spaces $\mathcal{S}_{\varepsilon,h}$ and $GSBD^2_h$ in (2.8) for $h \in W^{2,\infty}(\Omega'; \mathbb{R}^d)$.

Theorem 2.7 (Passage to linearized model)**.** *Let $\Omega \subset \Omega' \subset \mathbb{R}^d$ be bounded Lipschitz domains. Suppose that W satisfies* (2.3) *and that* (2.15)–(2.16) *hold. Let $h \in W^{2,\infty}(\Omega'; \mathbb{R}^d)$.*

(a) (Compactness) For each sequence $(y_\varepsilon)_\varepsilon$ with $y_\varepsilon \in \mathcal{S}_{\varepsilon,h}$ and $\sup_\varepsilon \mathcal{E}_\varepsilon(y_\varepsilon) < +\infty$, there exists a subsequence (not relabeled) and $u \in GSBD^2_h$ such that $y_\varepsilon \rightsquigarrow u$.

(b) (Γ*-liminf inequality) For each sequence* $(y_\varepsilon)_\varepsilon$*,* $y_\varepsilon \in \mathcal{S}_{\varepsilon,h}$*, with* $y_\varepsilon \rightsquigarrow u$ *for some* $u \in GSBD^2_h$ *we have*

$$\liminf_{\varepsilon\to 0} \mathcal{E}_\varepsilon(y_\varepsilon) \geq \mathcal{E}(u).$$

(c) (Γ*-limsup inequality) For each* $u \in GSBD^2_h$ *there exists a sequence* $(y_\varepsilon)_\varepsilon$*,* $y_\varepsilon \in \mathcal{S}_{\varepsilon,h}$*, such that* $y_\varepsilon \rightsquigarrow u$ *and*

$$\lim_{\varepsilon\to 0} \mathcal{E}_\varepsilon(y_\varepsilon) = \mathcal{E}(u).$$

The same statements hold with $\overline{\mathcal{E}}_\varepsilon$ *in place of* $\mathcal{E}_\varepsilon$*.*

We point out that we identify a 'standard' Griffith energy in linearized elasticity although we departed from a nonlinear model for nonsimple materials. As a corollary, we obtain the convergence of minimizers for boundary value problems.

Corollary 2.8 (Minimization problems)**.** *Consider the setting of Theorem 2.7. Then*

$$\inf_{\bar{y}\in\mathcal{S}_{\varepsilon,h}} \mathcal{E}_\varepsilon(\bar{y}) \ \to \ \min_{u\in GSBD^2_h} \mathcal{E}(u) \tag{2.17}$$

as $\varepsilon \to 0$*. Moreover, for each sequence* $(y_\varepsilon)_\varepsilon$ *with* $y_\varepsilon \in \mathcal{S}_{\varepsilon,h}$ *satisfying*

$$\mathcal{E}_\varepsilon(y_\varepsilon) \leq \inf_{\bar{y}\in\mathcal{S}_{\varepsilon,h}} \mathcal{E}_\varepsilon(\bar{y}) + \rho_\varepsilon \quad \text{ with } \rho_\varepsilon \to 0 \text{ as } \varepsilon \to 0, \tag{2.18}$$

there exist a subsequence (not relabeled) and $u \in GSBD^2_h$ *with* $\mathcal{E}(u) = \min_{v\in GSBD^2_h} \mathcal{E}(v)$ *such that* $y_\varepsilon \rightsquigarrow u$*.*

The results announced in this subsection will be proved in Subsection 4.3.

3. Preliminaries

In this section we collect some fundamental properties about (generalized) special functions of bounded variation and deformation. In particular, we recall and prove some results for $GSBV^2_2$ and $GSBD^2$ that will be needed for the proofs in Section 4.

3.1. Caccioppoli partitions

We say that a partition $(P_j)_j$ of an open set $\Omega \subset \mathbb{R}^d$ is a *Caccioppoli partition* of Ω if $\sum_j \mathcal{H}^{d-1}(\partial^* P_j) < +\infty$, where $\partial^* P_j$ denotes the *essential boundary* of P_j (see [6, Definition 3.60]). The local structure of Caccioppoli partitions can be characterized as follows (see [6, Theorem 4.17]).

Theorem 3.1. *Let* $(P_j)_j$ *be a Caccioppoli partition of* Ω*. Then*

$$\bigcup\nolimits_j (P_j)^1 \cup \bigcup\nolimits_{i\neq j} (\partial^* P_i \cap \partial^* P_j)$$

contains $\mathcal{H}^{d-1}$*-almost all of* Ω*.*

Here, $(P)^1$ denote the points where P has density one (see again [6, Definition 3.60]). Essentially, the theorem states that $\mathcal{H}^{d-1}$-a.e. point of Ω either belongs to exactly one element of the partition or to the intersection of exactly two sets $\partial^* P_i$, $\partial^* P_j$.

3.2. $GSBV^2$ and $GSBV_2^2$ functions

For the general notions on SBV and $GSBV$ functions and their properties we refer to [6, Section 4]. For $\Omega \subset \mathbb{R}^d$ open and $m \in \mathbb{N}$, we define $GSBV^2(\Omega;\mathbb{R}^m)$ as in (2.1), for general m. We denote by ∇y the approximate differential and by J_y the set of approximate jump points of y, which is an $\mathcal{H}^{d-1}$-rectifiable set. We recall that $GSBV^2(\Omega;\mathbb{R}^m)$ is a vector space, see [32, Proposition 2.3]. In a similar fashion, we say $y \in SBV^2(\Omega;\mathbb{R}^m)$ if $y \in SBV(\Omega;\mathbb{R}^m)$, $\nabla y \in L^2(\Omega;\mathbb{R}^{m\times d})$, and $\mathcal{H}^{d-1}(J_y) < +\infty$.

We define $GSBV_2^2(\Omega;\mathbb{R}^m)$ as in (2.2), for general m. For $m = 1$ we write $GSBV_2^2(\Omega)$. By definition, $\nabla y \in GSBV^2(\Omega;\mathbb{R}^{m\times d})$, and we use the notation $\nabla^2 y$ and $J_{\nabla y}$ for the approximate differential and the jump set of ∇y, respectively. Applying [32, Proposition 2.3] on y and ∇y, we find that $GSBV_2^2(\Omega;\mathbb{R}^m)$ is a vector space. The following result is the key ingredient for the proof of Proposition 2.1.

Theorem 3.2 (Compactness in $GSBV_2^2$). *Let $\Omega \subset \mathbb{R}^d$ be open and bounded, and let $m \in \mathbb{N}$. Let $(y_n)_n$ be a sequence in $GSBV_2^2(\Omega;\mathbb{R}^m)$. Suppose that there exists a continuous, increasing function $\psi : [0,\infty) \to [0,\infty)$ with $\lim_{t\to\infty}\psi(t) = +\infty$ such that*

$$\sup_{n\in\mathbb{N}} \Big(\int_\Omega \psi(|y_n|)\,dx + \int_\Omega |\nabla^2 y_n|^2\,dx + \mathcal{H}^{d-1}(J_{y_n} \cup J_{\nabla y_n})\Big) < +\infty.$$

Then there exist a subsequence, still denoted by $(y_n)_n$, and a function $y \in [GSBV(\Omega)]^m$ with $\nabla y \in GSBV^2(\Omega;\mathbb{R}^{m\times d})$ such that for all $0 < \gamma_2 \le \gamma_1 \le 2\gamma_2$ there holds

(i) $y_n \to y$ *a.e. in* Ω,

(ii) $\nabla y_n \to \nabla y$ *a.e.* Ω,

(iii) $\nabla^2 y_n \rightharpoonup \nabla^2 y$ *weakly in* $L^2(\Omega;\mathbb{R}^{m\times d\times d})$,

(iv) $$\gamma_1\mathcal{H}^{d-1}(J_y) + \gamma_2\mathcal{H}^{d-1}(J_{\nabla y}\setminus J_y) \le \liminf_{n\to\infty}\,(\gamma_1\mathcal{H}^{d-1}(J_{y_n}) + \gamma_2\mathcal{H}^{d-1}(J_{\nabla y_n}\setminus J_{y_n})). \tag{3.1}$$

If in addition $\sup_{n\in\mathbb{N}}\|\nabla y_n\|_{L^2(\Omega)} < +\infty$, then $y \in GSBV_2^2(\Omega;\mathbb{R}^m)$.

Proof. First, we observe that it suffices to treat the case $m = 1$ since otherwise one may argue componentwise, see particularly [38, Lemma 3.1] how to deal with property (iv). The result has been proved in [16, Theorem 4.4, Theorem 5.13, Remark 5.14] with the only difference that we just assume $\sup_{n\in\mathbb{N}}\int_\Omega \psi(|y_n|)\,dx < +\infty$ here instead of $\sup_{n\in\mathbb{N}}\|y_n\|_{L^2(\Omega)} < +\infty$. We briefly indicate the necessary adaptions in the proof of [16, Theorem 4.4] for $m = 1$. To ease comparison with [16], we point out that in that paper the notation $GSBV^2(\Omega)$ is used for functions u with $u \in GSBV(\Omega)$ and $\nabla u \in [GSBV(\Omega)]^d$.

For $k \in \mathbb{N}$, we define some $\varphi_k \in C^2(\mathbb{R})$ by $\varphi_k(t) = t$ for $t \in [-k+1, k-1]$, $|\varphi_k(t)| = k$ for $|t| > k+1$, and $0 \le \varphi_k' \le 1$. By $\|\varphi_k \circ y_n\|_{L^1(\Omega)} \le k\mathcal{L}^d(\Omega)$ and by using an interpolation inequality one can check that $(\varphi_k \circ y_n)_n$ is bounded in $BV_{\rm loc}(\Omega)$, see [16, (4.8)]. Therefore, by a diagonal argument there exist a subsequence of $(y_n)_n$ and functions $w_k \in BV_{\rm loc}(\Omega)$ for all $k \in \mathbb{N}$ such that

$$\varphi_k \circ y_n \to w_k \quad \text{a.e. in } \Omega \quad \text{for all } k \in \mathbb{N}. \tag{3.2}$$

Since ψ is continuous and increasing, and $|\varphi_k(t)| \le |t|$ for all $t \in \mathbb{R}$, we also get by Fatou's lemma

$$\|\psi(|w_k|)\|_{L^1(\Omega)} \le \liminf_{n\to\infty}\|\psi(|\varphi_k \circ y_n|)\|_{L^1(\Omega)} \le \sup_{n\in\mathbb{N}}\int_\Omega \psi(|y_n|)\,dx < +\infty. \tag{3.3}$$

Let $E_k = \{|w_k| < k-1\}$. The properties of φ_k along with (3.2) imply

$$y_n \to w_k \quad \text{a.e. in } E_k \text{ for all } k \in \mathbb{N}, \qquad w_k = w_l \quad \text{on } E_k \text{ for all } k \le l. \tag{3.4}$$

By using (3.3) we observe that $\mathcal{L}^d(\Omega \setminus E_k) \to 0$ as $k \to \infty$ since $\lim_{t\to\infty} \psi(t) = +\infty$. This together with (3.4) shows that the measurable function $y : \Omega \to \mathbb{R}$ defined by $y := \lim_{k\to\infty} w_k$ satisfies $y = w_k$ on E_k for all $k \in \mathbb{N}$ and therefore

$$y_n \to y \quad \text{a.e. in } \Omega.$$

The rest of the proof starting with [16, (4.10)] remains unchanged. In [16], it has been shown that $y \in GSBV(\Omega)$ and $\nabla y \in [GSBV(\Omega)]^d$. Since $\nabla^2 y \in L^2(\Omega;\mathbb{R}^{d\times d})$ and $\mathcal{H}^{d-1}(J_{\nabla y}) < +\infty$, we actually get $\nabla y \in GSBV^2(\Omega;\mathbb{R}^d)$. Finally, given an additional control on $(\nabla y_n)_n$ in L^2, we also find $\nabla y \in L^2(\Omega;\mathbb{R}^d)$ and $\mathcal{H}^{d-1}(J_y) < +\infty$. This implies $y \in GSBV^2_2(\Omega)$, see (2.2). □

We now proceed with a version of Theorem 3.2 without a priori bounds on the functions. We also take boundary data into account. The result relies on Theorem 3.2 and [42].

Theorem 3.3 (Compactness in $GSBV^2_2$ without a priori bounds)**.** *Let $\Omega \subset \Omega' \subset \mathbb{R}^d$ be bounded Lipschitz domains, and let $m \in \mathbb{N}$. Let $g \in W^{2,\infty}(\Omega';\mathbb{R}^m)$. Consider $(y_n)_n \subset GSBV^2_2(\Omega';\mathbb{R}^m)$ with $y_n = g$ on $\Omega' \setminus \overline{\Omega}$ and*

$$\sup_{n\in\mathbb{N}} \Big(\int_{\Omega'} (|\nabla y_n|^2 + |\nabla^2 y_n|^2)\, dx + \mathcal{H}^{d-1}(J_{y_n} \cup J_{\nabla y_n}) \Big) < +\infty.$$

Then we find a subsequence (not relabeled), modifications $(z_n)_n \subset GSBV^2_2(\Omega';\mathbb{R}^m)$ satisfying $z_n = g$ on $\Omega' \setminus \overline{\Omega}$ and

$$\begin{aligned} &\text{(i)} \ \ z_n = g \text{ on } S_n := \{\nabla z_n \neq \nabla y_n\} \cup \{\nabla^2 z_n \neq \nabla^2 y_n\}, \quad \text{where } \mathcal{L}^d(S_n) \to 0 \text{ as } n \to \infty, \\ &\text{(ii)} \ \lim_{n\to\infty} \mathcal{H}^{d-1}((J_{z_n} \cup J_{\nabla z_n}) \setminus (J_{y_n} \cup J_{\nabla y_n})) = 0, \end{aligned} \tag{3.5}$$

as well as a limiting function $y \in GSBV^2_2(\Omega';\mathbb{R}^m)$ with $y = g$ on $\Omega' \setminus \overline{\Omega}$ such that

$$\begin{aligned} &\text{(i)} \ \ z_n \to y \text{ in measure on } \Omega', \\ &\text{(ii)} \ \nabla z_n \to \nabla y \text{ a.e. } \Omega' \text{ and } \nabla z_n \rightharpoonup \nabla y \text{ weakly in } L^2(\Omega';\mathbb{R}^{m\times d}) \\ &\text{(iii)} \ \nabla^2 z_n \rightharpoonup \nabla^2 y \text{ weakly in } L^2(\Omega';\mathbb{R}^{m\times d\times d}) \\ &\text{(iv)} \ \mathcal{H}^{d-1}(J_y \cup J_{\nabla y}) \le \liminf_{n\to\infty} \mathcal{H}^{d-1}(J_{z_n} \cup J_{\nabla z_n}). \end{aligned} \tag{3.6}$$

In general, it is indispensable to pass to modifications. Consider, e.g., the sequence $y_n = n\chi_U$ for some set $U \subset \Omega$ of finite perimeter. The idea in [42, Theorem 3.1], where this result is proved in the space $GSBV^2(\Omega;\mathbb{R}^m)$, relies on constructing modifications $(z_n)_n$ by (cf. [42, (37)–(38)])

$$z_n = g\chi_{R_n} + \sum\nolimits_{j\ge1} (y_n - t^n_j)\chi_{P^n_j} \tag{3.7}$$

for Caccioppoli partitions $\Omega' = \bigcup_{j\ge1} P^n_j \cup R_n$, and suitable translations $(t^n_j)_{j\ge1} \subset \mathbb{R}^m$, where

$$\begin{aligned} &\text{(i)} \ \ \lim_{n\to\infty} \mathcal{L}^d(R_n) = 0, \\ &\text{(ii)} \ \lim_{n\to\infty} \mathcal{H}^{d-1}(J_{z_n} \setminus J_{y_n}) = \lim_{n\to\infty} \mathcal{H}^{d-1}((\partial^* R_n \cap \Omega') \setminus J_{y_n}) = 0. \end{aligned} \tag{3.8}$$

Proof of Theorem 3.3. We briefly indicate the necessary adaptions with respect to [42, Theorem 3.1] to obtain the result in the frame of $GSBV^2_2(\Omega';\mathbb{R}^m)$ involving second derivatives. First, by [42, Theorem 3.1] we find modifications $(z_n)_n$ as in (3.7) satisfying $z_n = g$ on $\Omega' \setminus \overline{\Omega}$ and $y \in GSBV^2(\Omega';\mathbb{R}^m)$ such that $z_n \to y$ in measure on Ω', up to passing to a subsequence. By (3.8) we get (3.5).

As $z_n \to y$ in measure on Ω', [45, Remark 2.2] implies that there exists a continuous, increasing function $\psi : [0,\infty) \to [0,\infty)$ with $\lim_{t\to\infty} \psi(t) = +\infty$ such that up to subsequence (not relabeled) $\sup_{n\in\mathbb{N}} \int_{\Omega'} \psi(|z_n|)\,dx < +\infty$. Moreover, by the assumptions on y_n, (3.5), and the fact that $g \in W^{2,\infty}(\Omega';\mathbb{R}^m)$ we get that ∇z_n and $\nabla^2 z_n$ are uniformly controlled in L^2, as well as $\sup_{n\in\mathbb{N}} \mathcal{H}^{d-1}(J_{z_n} \cup J_{\nabla z_n}) < +\infty$. Then Theorem 3.2 yields $y \in GSBV^2_2(\Omega';\mathbb{R}^m)$. Along with (3.1) for $\gamma_1 = \gamma_2$ we also get (3.6), apart from the weak convergence of $(\nabla z_n)_n$. The weak convergence readily follows from $\sup_{n\in\mathbb{N}} \|\nabla z_n\|_{L^2(\Omega')} \le \sup_{n\in\mathbb{N}} \|\nabla y_n\|_{L^2(\Omega')} + \|\nabla g\|_{L^2(\Omega')} < +\infty$. $\square$

3.3. *$GSBD^2$ functions*

We refer the reader to [5] and [31] for the definition, notations, and basic properties of SBD and $GSBD$ functions, respectively. Here, we only recall briefly some relevant notions which can be defined for generalized functions of bounded deformation: let $\Omega \subset \mathbb{R}^d$ open and bounded. In [31, Theorem 6.2 and Theorem 9.1] it is shown that for $u \in GSBD(\Omega)$ the jump set J_u is $\mathcal{H}^{d-1}$-rectifiable and that an approximate symmetric differential $e(u)(x)$ exists at $\mathcal{L}^d$-a.e. $x \in \Omega$. We define the space $GSBD^2(\Omega)$ by

$$GSBD^2(\Omega) := \{u \in GSBD(\Omega) : e(u) \in L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}}),\, \mathcal{H}^{d-1}(J_u) < +\infty\}\,.$$

The space $GSBD^2(\Omega)$ is a vector subspace of the vector space of $\mathcal{L}^d$-measurable function, see [31, Remark 4.6]. Moreover, there holds $GSBV^2(\Omega;\mathbb{R}^d) \subset GSBD^2(\Omega)$. The following compactness result in $GSBD^2$ has been proved in [26].

Theorem 3.4 ($GSBD^2$ compactness)**.** *Let $\Omega \subset \mathbb{R}^d$ be open, bounded. Let $(u_n)_n \subset GSBD^2(\Omega)$ be a sequence satisfying*

$$\sup_{n\in\mathbb{N}} \left(\|e(u_n)\|_{L^2(\Omega)} + \mathcal{H}^{d-1}(J_{u_n})\right) < +\infty.$$

Then there exists a subsequence (not relabeled) such that the set $A := \{x \in \Omega : |u_n(x)| \to \infty\}$ has finite perimeter, and there exists $u \in GSBD^2(\Omega)$ such that

$$\begin{aligned} &\text{(i)} \ \ u_n \to u \quad \textit{in measure on } \Omega \setminus A,\\ &\text{(ii)} \ \ e(u_n) \rightharpoonup e(u) \quad \textit{weakly in } L^2(\Omega \setminus A;\mathbb{R}^{d\times d}_{\mathrm{sym}}),\\ &\text{(iii)} \ \ \liminf_{n\to\infty} \mathcal{H}^{d-1}(J_{u_n}) \ge \mathcal{H}^{d-1}(J_u \cup (\partial^* A \cap \Omega)). \end{aligned} \tag{3.9}$$

We briefly remark that (3.9)(i) is slightly weaker with respect to (3.6)(i) in Theorem 3.3 (or the corresponding version in $GSBV$, see [42]) in the sense that there might be a set A where the sequence $(u_n)_n$ is *unbounded*, cf. the example below Theorem 3.3. This phenomenon is avoided in Theorem 3.3 by passing to suitable modifications which consists in subtracting piecewise constant functions, see (3.7). We point out that an analogous result in $GSBD^2$ is so far only available in dimension two, see [45, Theorem 6.1]. We now state two density results.

Theorem 3.5 (Density). *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Let $u \in GSBD^2(\Omega)$. Then there exists a sequence $(u_n)_n \subset SBV^2(\Omega;\mathbb{R}^d) \cap L^\infty(\Omega;\mathbb{R}^d)$ such that each J_{u_n} is closed and included in a finite union of closed connected pieces of C^1 hypersurfaces, each u_n belongs to $C^\infty(\overline{\Omega} \setminus J_{u_n};\mathbb{R}^d) \cap W^{m,\infty}(\Omega \setminus J_{u_n};\mathbb{R}^d)$ for every $m \in \mathbb{N}$, and the following properties hold:*

$$\begin{aligned} &\text{(i)} \;\; u_n \to u \text{ in measure on } \Omega,\\ &\text{(ii)} \;\; \|e(u_n) - e(u)\|_{L^2(\Omega)} \to 0,\\ &\text{(iii)} \;\; \mathcal{H}^{d-1}(J_{u_n} \triangle J_u) \to 0. \end{aligned}$$

Proof. The result follows by combining [25, Theorem 1.1] and [28, Theorem 1.1]. First, [25, Theorem 1.1] yields an approximation u_n satisfying $u_n \in SBV^2(\Omega;\mathbb{R}^d) \cap W^{1,\infty}(\Omega \setminus J_{u_n};\mathbb{R}^d)$, and then [28, Theorem 1.1] gives the higher regularity. □

An adaption of the proof allows to impose boundary conditions on the approximating sequence. Suppose that the Lipschitz domains $\Omega \subset \Omega'$ satisfy the conditions introduced in (2.15)–(2.16). By $\mathcal{W}(\Omega;\mathbb{R}^d)$ we denote the space of all functions $u \in SBV(\Omega;\mathbb{R}^d)$ such that J_u is a finite union of disjoint $(d-1)$-simplices and $u \in W^{k,\infty}(\Omega \setminus J_u;\mathbb{R}^d)$ for every $k \in \mathbb{N}$.

Theorem 3.6 (Density with boundary data). *Let $\Omega \subset \Omega' \subset \mathbb{R}^d$ be bounded Lipschitz domains satisfying (2.15)–(2.16). Let $g \in W^{r,\infty}(\Omega')$ for $r \in \mathbb{N}$. Let $u \in GSBD^2(\Omega')$ with $u = g$ on $\Omega' \setminus \overline{\Omega}$. Then there exists a sequence of functions $(u_n)_n \subset SBV^2(\Omega;\mathbb{R}^d)$, a sequence of neighborhoods $(U_n)_n \subset \Omega'$ of $\Omega' \setminus \Omega$, and a sequence of neighborhoods $(\Omega_n)_n \subset \Omega$ of $\Omega \setminus U_n$ such that $u_n = g$ on $\Omega' \setminus \overline{\Omega}$, $u_n|_{U_n} \in W^{r,\infty}(U_n;\mathbb{R}^d)$, and $u_n|_{\Omega_n} \in \mathcal{W}(\Omega_n;\mathbb{R}^d)$, and the following properties hold:*

$$\begin{aligned} &\text{(i)} \;\; u_n \to u \text{ in measure on } \Omega',\\ &\text{(ii)} \;\; \|e(u_n) - e(u)\|_{L^2(\Omega')} \to 0,\\ &\text{(iii)} \;\; \mathcal{H}^{d-1}(J_{u_n}) \to \mathcal{H}^{d-1}(J_u). \end{aligned} \tag{3.10}$$

In particular, $u_n \in W^{r,\infty}(\Omega \setminus J_{u_n};\mathbb{R}^d)$.

Proof. The fact that u can be approximated by a sequence $(u_n)_n \subset SBV^2(\Omega';\mathbb{R}^d) \cap L^\infty(\Omega;\mathbb{R}^d)$ satisfying (3.10) and $u_n = g$ in a neighborhood of $\Omega' \setminus \Omega$ has been addressed in [25, Proof of Theorem 5.4]. Here, also the necessity of the geometric assumptions (2.15)–(2.16) is discussed, see [25, Remark 5.6]. The fact that the approximating sequence can be chosen as in the statement then follows by applying on each u_n a construction very similar to the one of [47, Proposition 2.5] along with a diagonal argument. This construction consists in a suitable cut-off construction and the application of the density result [29]. We also refer to [56, Theorem 3.5] for a similar statement. □

4. Proofs

This section contains the proofs of our results.

4.1. Relaxation and existence of minimizers for the nonlinear model

In this subsection we prove Proposition 2.1 and Theorem 2.2.

Proof of Proposition 2.1. For $y \in GSBV_2^2(\Omega;\mathbb{R}^d)$ we define

$$\mathcal{E}'_\varepsilon(y) = \inf\{\liminf_{n\to\infty}\mathcal{E}_\varepsilon(y_n,\Omega) : y_n \to y \text{ in measure on } \Omega\}, \tag{4.1}$$

and define $\overline{\mathcal{E}}_\varepsilon(\cdot,\Omega)$ as in (2.5). We need to check that $\mathcal{E}'_\varepsilon = \overline{\mathcal{E}}_\varepsilon(\cdot,\Omega)$. In the proof, we write $\tilde{\subset}$ and $\tilde{=}$ for brevity if the inclusion or the identity holds up to an $\mathcal{H}^{d-1}$-negligible set, respectively.

Step 1: $\mathcal{E}'_\varepsilon \geq \overline{\mathcal{E}}_\varepsilon(\cdot,\Omega)$. Since by definition $\overline{\mathcal{E}}_\varepsilon(y,\Omega) \leq \mathcal{E}_\varepsilon(y,\Omega)$ for all $y \in GSBV_2^2(\Omega;\mathbb{R}^d)$, see (2.4), it suffices to confirm that $\overline{\mathcal{E}}_\varepsilon(\cdot,\Omega)$ is lower semicontinous with respect to the convergence in measure. To see this, consider $(y_n)_n \subset GSBV_2^2(\Omega;\mathbb{R}^d)$ with $y_n \to y$ in measure Ω and $\sup_{n\in\mathbb{N}}\overline{\mathcal{E}}_\varepsilon(y_n,\Omega) < +\infty$. By using [45, Remark 2.2], there exists a continuous, increasing function $\psi : [0,\infty) \to [0,\infty)$ with $\lim_{t\to\infty}\psi(t) = +\infty$ such that up to subsequence (not relabeled) $\sup_{n\in\mathbb{N}}\int_\Omega \psi(|y_n|)\,dx < +\infty$. Then from Theorem 3.2 we obtain

$$\overline{\mathcal{E}}_\varepsilon(y,\Omega) \leq \liminf_{n\to\infty}\overline{\mathcal{E}}_\varepsilon(y_n,\Omega).$$

In fact, for the second and the third term in (2.5) we use (3.1)(iii) and (iv) for $\gamma_1 = \gamma_2$, respectively. The first term in (2.5) is lower semicontinuous by the continuity of W, (3.1)(ii), and Fatou's lemma. This shows that $\overline{\mathcal{E}}_\varepsilon(\cdot,\Omega)$ is lower semicontinous and concludes the proof of $\mathcal{E}'_\varepsilon \geq \overline{\mathcal{E}}_\varepsilon(\cdot,\Omega)$.

Step 2: $\mathcal{E}'_\varepsilon \leq \overline{\mathcal{E}}_\varepsilon(\cdot,\Omega)$. In the proof, we will use the following argument several times: If $y_1, y_2 \in GSBV^2(\Omega;\mathbb{R}^d)$, then for a.e. $t \in \mathbb{R}$ there holds that $z := y_1 + ty_2 \in GSBV^2(\Omega;\mathbb{R}^d)$ satisfies $J_z = J_{y_1} \cup J_{y_2}$, see [38, Proof of Lemma 3.1] or [32, Proof of Lemma 4.5] for such an argument. We point out that here we exploit the fact that $GSBV^2(\Omega;\mathbb{R}^d)$ is a vector space.

Observe that for each $y \in GSBV_2^2(\Omega;\mathbb{R}^d)$ and each $\nu \in S^{d-1}$, the function $v := \nabla y \cdot \nu$ lies in $GSBV^2(\Omega;\mathbb{R}^d) \subset GSBD^2(\Omega)$. We can choose $\nu \in S^{d-1}$ such that there holds $\mathcal{H}^{d-1}(J_v \triangle J_{\nabla y}) = 0$. We apply Theorem 3.5 to approximate $v \in GSBD^2(\Omega)$ by a sequence $(v_n)_n \subset SBV^2(\Omega;\mathbb{R}^d)$ such that $v_n \in W^{2,\infty}(\Omega \setminus J_{v_n};\mathbb{R}^d)$ and

$$\mathcal{H}^{d-1}(J_{v_n}\triangle J_{\nabla y}) = \mathcal{H}^{d-1}(J_{v_n}\triangle J_v) \to 0 \tag{4.2}$$

as $n \to \infty$. We point out that $J_{\nabla v_n}\tilde{\subset}J_{v_n}$ since $v_n \in W^{2,\infty}(\Omega \setminus J_{v_n};\mathbb{R}^d)$. Using $v_n \in W^{2,\infty}(\Omega \setminus J_{v_n};\mathbb{R}^d)$ we can choose a sequence $(\eta_n)_n$ with $\eta_n \to 0$ such that $z_n := y + \eta_n v_n \in GSBV_2^2(\Omega;\mathbb{R}^d)$ satisfies $J_{z_n}\tilde{=}J_y \cup J_{v_n}$ and there holds $z_n \to y$ in measure on Ω. By (4.2), the continuity of W, $J_{z_n}\tilde{=}J_y \cup J_{v_n}$, and $J_{\nabla z_n}\tilde{\subset}J_{\nabla y} \cup J_{v_n}$ we get

$$\limsup_{n\to\infty}\overline{\mathcal{E}}_\varepsilon(z_n,\Omega) \leq \overline{\mathcal{E}}_\varepsilon(y,\Omega). \tag{4.3}$$

As $J_{z_n}\tilde{=}J_y \cup J_{v_n}$, $J_{\nabla y}\tilde{=}J_v$, and $J_{\nabla v_n}\tilde{\subset}J_{v_n}$, we also get

$$J_{\nabla z_n} \setminus J_{z_n} \,\tilde{\subset}\, (J_{\nabla y} \cup J_{\nabla v_n}) \setminus (J_y \cup J_{v_n}) \,\tilde{\subset}\, J_v \setminus J_{v_n}. \tag{4.4}$$

In view of (4.2), by a Besicovitch covering argument we can cover the rectifiable sets $J_v \setminus J_{v_n}$ by sets of finite perimeter $(E_n)_n \subset\subset \Omega$, each of which being a countable union of balls with radii smaller than $\frac{1}{n}$, such that

$$\mathcal{L}^d(E_n) + \mathcal{H}^{d-1}(\partial^* E_n) \to 0. \tag{4.5}$$

We finally define the sequence $y_n \in GSBV_2^2(\Omega;\mathbb{R}^d)$ by $y_n = z_n\chi_{\Omega\setminus E_n} + (\mathbf{id} + b_n)\chi_{E_n}$ for suitable constants $(b_n)_n \subset \mathbb{R}^d$ which are chosen such that $J_{y_n}\tilde{=}(J_{z_n}\setminus E_n)\cup\partial^* E_n$. Now in view of (4.4) and $J_v \setminus J_{v_n}\tilde{\subset}E_n$, we get

$J_{\nabla y_n} \tilde{\subset} J_{y_n}$. By (4.5) and $z_n \to y$ in measure on Ω we get $y_n \to y$ in measure on Ω. By (2.3)(iii) we obtain $W(\nabla y_n) = 0$, $\nabla^2 y_n = 0$ on E_n. Then by (2.5), (4.3), (4.5), and the fact that $J_{\nabla y_n} \tilde{\subset} J_{y_n} \tilde{=} (J_{z_n} \setminus E_n) \cup \partial^* E_n$ we get

$$\limsup_{n\to\infty} \overline{\mathcal{E}}_\varepsilon(y_n, \Omega) \le \limsup_{n\to\infty} \big(\overline{\mathcal{E}}_\varepsilon(z_n, \Omega) + \kappa \mathcal{H}^{d-1}(\partial^* E_n)\big) \le \overline{\mathcal{E}}_\varepsilon(y, \Omega).$$

Since $\overline{\mathcal{E}}_\varepsilon(y_n, \Omega) = \mathcal{E}_\varepsilon(y_n, \Omega)$ for all $n \in \mathbb{N}$ by $J_{\nabla y_n} \tilde{\subset} J_{y_n}$, (4.1) implies $\mathcal{E}'_\varepsilon(y) \le \overline{\mathcal{E}}_\varepsilon(y, \Omega)$. This concludes the proof. □

Proof of Theorem 2.2. We prove the existence of minimizers via the direct method. Let $(y_n)_n \subset GSBV^2_2(\Omega'; \mathbb{R}^d)$ with $y_n = g$ on $\Omega' \setminus \overline{\Omega}$ be a minimizing sequence for the minimization problem (2.6). By (2.3) we find $W(F) \ge c_1|F|^2 - c_2$ for $c_1, c_2 > 0$. Thus, $\sup_{n\in\mathbb{N}} \overline{\mathcal{E}}_\varepsilon(y_n) < +\infty$ also implies $\sup_{n\in\mathbb{N}} \|\nabla y_n\|_{L^2(\Omega')} < +\infty$, and we can apply Theorem 3.3. We obtain a sequence $(z_n)_n \subset GSBV^2_2(\Omega'; \mathbb{R}^d)$ satisfying $z_n = g$ on $\Omega' \setminus \overline{\Omega}$ and a limiting function $y \in GSBV^2_2(\Omega'; \mathbb{R}^d)$ with $y = g$ on $\Omega' \setminus \overline{\Omega}$ such that $z_n \to y$ in measure on Ω'. Using (2.5), (3.5), and $g \in W^{2,\infty}(\Omega'; \mathbb{R}^d)$ we calculate

$$\begin{aligned}\limsup_{n\to\infty} \big(\overline{\mathcal{E}}_\varepsilon(z_n) - \overline{\mathcal{E}}_\varepsilon(y_n)\big) \le \limsup_{n\to\infty} \Big(&\varepsilon^{-2} C_{W,g} \mathcal{L}^d(S_n) + \varepsilon^{-2\beta} \|\nabla^2 g\|^2_{L^2(S_n)} \\ &+ \kappa\big(\mathcal{H}^{d-1}(J_{z_n} \cup J_{\nabla z_n}) - \mathcal{H}^{d-1}(J_{y_n} \cup J_{\nabla y_n})\big)\Big) \le 0,\end{aligned}$$

where the constant $C_{W,g}$ depends on W and $\|\nabla g\|_{L^\infty(\Omega')}$. I.e., $(z_n)_n$ is also a minimizing sequence. By $z_n \to y$ in measure on Ω' and the fact that $\overline{\mathcal{E}}_\varepsilon$ is lower semicontinuous with respect to the convergence in measure on Ω', see Proposition 2.1, we get

$$\overline{\mathcal{E}}_\varepsilon(y) \le \liminf_{n\to\infty} \overline{\mathcal{E}}_\varepsilon(z_n) \le \liminf_{n\to\infty} \overline{\mathcal{E}}_\varepsilon(y_n) = \inf_{\bar{y} \in GSBV^2_2(\Omega';\mathbb{R}^d)} \Big\{\overline{\mathcal{E}}_\varepsilon(\bar{y}) : \ \bar{y} = g \text{ on } \Omega' \setminus \overline{\Omega}\Big\}.$$

This shows that y is a minimizer. □

4.2. Compactness

This subsection is devoted to the proof of Theorem 2.3.

Proof of Theorem 2.3(a). Consider a sequence $(y_\varepsilon)_\varepsilon$ with $y_\varepsilon \in \mathcal{S}_{\varepsilon,h}$, i.e., $y_\varepsilon = \mathbf{id} + \varepsilon h$ on $\Omega' \setminus \overline{\Omega}$. Suppose that $M := \sup_\varepsilon \overline{\mathcal{E}}_\varepsilon(y_\varepsilon) < +\infty$. We first construct Caccioppoli partitions (Step 1) and the corresponding rotations (Step 2) in order to define $y^{\mathrm{rot}}_\varepsilon$. Then we confirm (2.10) (Step 3).

Step 1: Definition of the Caccioppoli partitions. First, we apply the *BV* coarea formula (see [6, Theorem 3.40 or Theorem 4.34]) on each component $(\nabla y_\varepsilon)_{ij} \in GSBV^2(\Omega')$, $1 \le i, j \le d$, to write

$$\int_{-\infty}^{\infty} \mathcal{H}^{d-1}\big((\Omega' \setminus J_{\nabla y_\varepsilon}) \cap \partial^*\{(\nabla y_\varepsilon)_{ij} > t\}\big)\, dt = |D(\nabla y_\varepsilon)_{ij}|(\Omega' \setminus J_{\nabla y_\varepsilon}) \le \|\nabla^2 y_\varepsilon\|_{L^1(\Omega')}.$$

Using Hölder's inequality and (2.5) along with $\overline{\mathcal{E}}_\varepsilon(y_\varepsilon) \le M$, we then get

$$\int_{-\infty}^{\infty} \mathcal{H}^{d-1}\big((\Omega' \setminus J_{\nabla y_\varepsilon}) \cap \partial^*\{(\nabla y_\varepsilon)_{ij} > t\}\big)\, dt \le (\mathcal{L}^d(\Omega'))^{1/2} \|\nabla^2 y_\varepsilon\|_{L^2(\Omega')} \le (\mathcal{L}^d(\Omega') M)^{1/2} \varepsilon^\beta. \tag{4.6}$$

Fix $\gamma \in (\frac{2}{3}, \beta)$ and define $T_\varepsilon = \varepsilon^\gamma$. For all $k \in \mathbb{Z}$ we find $t_k^{ij} \in (kT_\varepsilon, (k+1)T_\varepsilon]$ such that

$$\mathcal{H}^{d-1}((\Omega' \setminus J_{\nabla y_\varepsilon}) \cap \partial^*\{(\nabla y_\varepsilon)_{ij} > t_k^{ij}\}) \le \frac{1}{T_\varepsilon}\int_{kT_\varepsilon}^{(k+1)T_\varepsilon} \mathcal{H}^{d-1}((\Omega' \setminus J_{\nabla y_\varepsilon}) \cap \partial^*\{(\nabla y_\varepsilon)_{ij} > t\})\, dt. \tag{4.7}$$

Let $G_k^{\varepsilon,ij} = \{(\nabla y_\varepsilon)_{ij} > t_k^{ij}\} \setminus \{(\nabla y_\varepsilon)_{ij} > t_{k+1}^{ij}\}$ and note that each set has finite perimeter in Ω' since it is the difference of two sets of finite perimeter. Now (4.6) and (4.7) imply

$$\sum\nolimits_{k\in\mathbb{Z}} \mathcal{H}^{d-1}((\Omega' \setminus J_{\nabla y_\varepsilon}) \cap \partial^* G_k^{\varepsilon,ij}) \le 2T_\varepsilon^{-1}(\mathcal{L}^d(\Omega')M)^{1/2}\varepsilon^\beta \le C\varepsilon^{\beta-\gamma} \tag{4.8}$$

for a sufficiently large constant $C > 0$ independent of ε. Since $\mathcal{L}^d(\Omega' \setminus \bigcup_{k\in\mathbb{Z}} G_k^{\varepsilon,ij}) = 0$, $(G_k^{\varepsilon,ij})_{k\in\mathbb{Z}}$ is a Caccioppoli partition of Ω'. We let $(P_j^\varepsilon)_{j\in\mathbb{N}}$ be the Caccioppoli partition of Ω' consisting of the nonempty sets of

$$\{G_{k_{11}}^{\varepsilon,11} \cap G_{k_{12}}^{\varepsilon,12} \cap \ldots \cap G_{k_{dd}}^{\varepsilon,dd} :\ k_{ij} \in \mathbb{Z} \text{ for } i,j = 1,\ldots,d\}.$$

Then (4.8) implies

$$\sum\nolimits_{j=1}^{\infty} \mathcal{H}^{d-1}(\partial^* P_j^\varepsilon \cap (\Omega' \setminus J_{\nabla y_\varepsilon})) \le C\varepsilon^{\beta-\gamma} \tag{4.9}$$

for a constant $C > 0$ independent of ε.

Step 2: Definition of the rotations. We now define corresponding rotations. Recalling $T_\varepsilon = \varepsilon^\gamma$ we get $|t_k^{ij} - t_{k+1}^{ij}| \le 2T_\varepsilon = 2\varepsilon^\gamma$ for all $k \in \mathbb{Z}$, $i, j = 1,\ldots,n$. Then by the definition of $G_k^{\varepsilon,ij}$, for each component P_j^ε of the Caccioppoli partition, we find a matrix $F_j^\varepsilon \in \mathbb{R}^{d\times d}$ such that

$$\|\nabla y_\varepsilon - F_j^\varepsilon\|_{L^\infty(P_j^\varepsilon)} \le c\varepsilon^\gamma, \tag{4.10}$$

where c depends only on d. For each $j \in \mathbb{N}$ with $P_j^\varepsilon \subset \Omega$ up to an $\mathcal{L}^d$-negligible set, we denote by $\bar{R}_j^\varepsilon$ the nearest point projection of F_j^ε onto $SO(d)$. For all other components P_j^ε, i.e., the components intersecting $\Omega' \setminus \overline{\Omega}$, we set $\bar{R}_j^\varepsilon = \mathbf{Id}$. We now show that for all $j \in \mathbb{N}$ and for $\mathcal{L}^d$-a.e. $x \in P_j^\varepsilon$ there holds

$$|\nabla y_\varepsilon(x) - \bar{R}_j^\varepsilon| \le \max\{C\varepsilon^\gamma,\ 2\,\mathrm{dist}(\nabla y_\varepsilon(x), SO(d))\} \tag{4.11}$$

for a constant $C > 0$ independent of ε.

First, we consider components P_j^ε which are contained in Ω up to an $\mathcal{L}^d$-negligible set. Recall that $\bar{R}_j^\varepsilon$ is defined as the nearest point projection of F_j^ε onto $SO(d)$. If $|\bar{R}_j^\varepsilon - F_j^\varepsilon| \le 3c\varepsilon^\gamma$, where c is the constant of (4.10), (4.11) follows from (4.10) and the triangle inequality. Otherwise, by (4.10) we get for $\mathcal{L}^d$-a.e. $x \in P_j^\varepsilon$

$$\begin{aligned}\mathrm{dist}(\nabla y_\varepsilon(x), SO(d)) &\ge \mathrm{dist}(F_j^\varepsilon, SO(d)) - c\varepsilon^\gamma = |\bar{R}_j^\varepsilon - F_j^\varepsilon| - c\varepsilon^\gamma \\ &\ge \tfrac{1}{2}(|\bar{R}_j^\varepsilon - F_j^\varepsilon| + c\varepsilon^\gamma) \ge \tfrac{1}{2}|\bar{R}_j^\varepsilon - \nabla y_\varepsilon(x)|.\end{aligned}$$

This implies (4.11). Now consider a component P_j^ε which intersects $\Omega' \setminus \overline{\Omega}$. Then by (4.10) and the fact that $y_\varepsilon = \mathbf{id} + \varepsilon h$ on $\Omega' \setminus \overline{\Omega}$ there holds

$$\|\mathbf{Id} + \varepsilon\nabla h - F_j^\varepsilon\|_{L^\infty(P_j^\varepsilon\setminus\Omega)} \le \|\nabla y_\varepsilon - F_j^\varepsilon\|_{L^\infty(P_j^\varepsilon)} \le c\varepsilon^\gamma.$$

Since $0 < \gamma < 1$, this yields $|F_j^\varepsilon - \mathbf{Id}| \le C\varepsilon^\gamma$ for a constant C depending also on $\|\nabla h\|_{L^\infty(\Omega')}$. This along with (4.10) implies (4.11) (for $\bar{R}_j^\varepsilon = \mathbf{Id}$). We define the rotations in the statement by $R_j^\varepsilon := (\bar{R}_j^\varepsilon)^{-1}$.

Step 3: Proof of (2.10). We are now in a position to prove (2.10). We define $y_\varepsilon^{\rm rot}$ as in (2.9), i.e., $y_\varepsilon^{\rm rot} = \sum_{j=1}^\infty R_j^\varepsilon y_\varepsilon \chi_{P_j^\varepsilon}$. Then (2.10)(i) follows from the fact that $y_\varepsilon = \mathbf{id} + \varepsilon h$ on $\Omega' \setminus \overline{\Omega}$ and $y_\varepsilon^{\rm rot} = y_\varepsilon$ on $\Omega' \setminus \overline{\Omega}$, where the latter holds due to $R_j^\varepsilon = \mathbf{Id}$ for all P_j^ε intersecting $\Omega' \setminus \overline{\Omega}$. Property (2.10)(ii) is a direct consequence of the definition of $y_\varepsilon^{\rm rot}$ and (4.9). To see (2.10)(iv), we use (4.11) and $R_j^\varepsilon = (\bar{R}_j^\varepsilon)^{-1}$ to get

$$\begin{aligned}\|\nabla y_\varepsilon^{\rm rot} - \mathbf{Id}\|^2_{L^2(\Omega')} &= \sum\nolimits_{j=1}^\infty \|\nabla y_\varepsilon - \bar{R}_j^\varepsilon\|^2_{L^2(P_j^\varepsilon)} \le C\varepsilon^{2\gamma}\mathcal{L}^d(\Omega') + 4\|\operatorname{dist}(\nabla y_\varepsilon, SO(d))\|^2_{L^2(\Omega')} \\ &\le C(\varepsilon^{2\gamma} + \varepsilon^2)\end{aligned}$$

for a constant depending on M, where the last step follows from (2.3)(iii), (2.5), and $\overline{\mathcal{E}}_\varepsilon(y_\varepsilon) \le M$. Since $0 < \gamma < 1$, (2.10)(iv) is proved. It remains to show (2.10)(iii). We recall the linearization formula (see [46, (3.20)])

$$|\mathrm{sym}(F - \mathbf{Id})| = \operatorname{dist}(F, SO(d)) + \mathrm{O}(|F - \mathbf{Id}|^2) \tag{4.12}$$

for $F \in \mathbb{R}^{d\times d}$. By Young's inequality and $|\mathrm{sym}(F - \mathbf{Id})| \le |F - \mathbf{Id}|$ this implies

$$\begin{aligned}|\mathrm{sym}(F - \mathbf{Id})|^2 &\le \min\{|F - \mathbf{Id}|^2,\ C\operatorname{dist}^2(F, SO(d)) + C|F - \mathbf{Id}|^4\} \\ &\le C\operatorname{dist}^2(F, SO(d)) + C\min\{|F - \mathbf{Id}|^2,\ |F - \mathbf{Id}|^4\}.\end{aligned}$$

Then we calculate

$$\begin{aligned}\int_{\Omega'} |\mathrm{sym}(\nabla y_\varepsilon^{\rm rot} - \mathbf{Id})|^2 &\le C\int_{\Omega'}\Big(\operatorname{dist}^2(\nabla y_\varepsilon^{\rm rot}, SO(d)) + \min\{|\nabla y_\varepsilon^{\rm rot} - \mathbf{Id}|^2,\ |\nabla y_\varepsilon^{\rm rot} - \mathbf{Id}|^4\}\Big) \\ &\le C\sum\nolimits_{j=1}^\infty \int_{P_j^\varepsilon}\Big(\operatorname{dist}^2(\nabla y_\varepsilon, SO(d)) + |\nabla y_\varepsilon - \bar{R}_j^\varepsilon|^2 \min\{1,\ |\nabla y_\varepsilon - \bar{R}_j^\varepsilon|^2\}\Big).\end{aligned}$$

By (4.11) we note that for a.e. $x \in P_j^\varepsilon$ there holds

$$|\nabla y_\varepsilon(x) - \bar{R}_j^\varepsilon|^2 \min\{1,\ |\nabla y_\varepsilon(x) - \bar{R}_j^\varepsilon|^2\} \le C\varepsilon^{4\gamma} + C\operatorname{dist}^2(\nabla y_\varepsilon(x), SO(d)).$$

Here, we used that, if $|\nabla y_\varepsilon(x) - \bar{R}_j^\varepsilon|^2 > 1$, the maximum in (4.11) is attained for $\operatorname{dist}(\nabla y_\varepsilon(x), SO(d))$, provided that ε is small enough. Therefore, we get

$$\int_{\Omega'} |\mathrm{sym}(\nabla y_\varepsilon^{\rm rot} - \mathbf{Id})|^2 \le C\int_{\Omega'} \operatorname{dist}^2(\nabla y_\varepsilon, SO(d)) + C\mathcal{L}^d(\Omega')\varepsilon^{4\gamma} \le C\varepsilon^2 + C\varepsilon^{4\gamma},$$

where in the last step we have again used (2.3)(iii), (2.5), and $\overline{\mathcal{E}}_\varepsilon(y_\varepsilon) \le M$. Since $\gamma > \frac{2}{3} \ge \frac{1}{2}$, we obtain (2.10)(iii). This concludes the proof of Theorem 2.3(a). □

Remark 4.1. *For later purposes, we point out that the construction shows* $y_\varepsilon^{\rm rot} = y_\varepsilon$ *on all* P_j^ε *intersecting* $\Omega' \setminus \overline{\Omega}$.

Proof of Theorem 2.3(b). We define the rescaled displacment fields $u_\varepsilon := \frac{1}{\varepsilon}(y^{\rm rot}_\varepsilon - \mathbf{id})$ as in (2.11). Clearly, there holds $u_\varepsilon \in GSBV^2(\Omega';\mathbb{R}^d) \subset GSBD^2(\Omega')$. Note that by (2.10)(iii) we obtain $\sup_\varepsilon \|e(u_\varepsilon)\|_{L^2(\Omega')} < +\infty$, where for shorthand we again write $e(u_\varepsilon) = \frac{1}{2}(\nabla u_\varepsilon^T + \nabla u_\varepsilon)$. Moreover, in view of (2.10)(ii) and $\beta > \gamma$, we get

$$\limsup\nolimits_{\varepsilon\to 0} \mathcal{H}^{d-1}(J_{u_\varepsilon}) \le \limsup\nolimits_{\varepsilon\to 0} \mathcal{H}^{d-1}(J_{y_\varepsilon} \cup J_{\nabla y_\varepsilon}) < +\infty. \tag{4.13}$$

Therefore, we can apply Theorem 3.4 on the sequence $(u_\varepsilon)_\varepsilon$ to obtain A and $u' \in GSBD^2(\Omega')$ such that (3.9) holds (up to passing to a subsequence). We first observe that $E_u = A$, where $E_u := \{x \in \Omega : |u_\varepsilon(x)| \to \infty\}$ and $A := \{x \in \Omega' : |u_\varepsilon(x)| \to \infty\}$. To see this, we have to check that $A \subset \Omega$. This follows from the fact that $u_\varepsilon = h$ on $\Omega' \setminus \overline{\Omega}$ for all ε, see (2.10)(i) and (2.11).

We define $u := u'\chi_{\Omega'\setminus E_u} + \lambda\chi_{E_u}$ for some $\lambda \in \mathbb{R}^d$ such that $\partial^* E_u \cap \Omega' \subset J_u$ up to an $\mathcal{H}^{d-1}$-negligible set. Since $J_u \subset J_{u'} \cup (\partial^* E_u \cap \Omega')$, (3.9) then implies (2.12), where the last inequality in (2.12)(iii) follows from (4.13). Finally, $u \in GSBD^2_h$ follows from $u_\varepsilon = h$ on $\Omega' \setminus \overline{\Omega}$ and (2.12)(i). □

4.3. *Passage to linearized model by* Γ*-convergence*

We now give the proof of Theorem 2.7.

Proof of Theorem 2.7. Since $\overline{\mathcal{E}}_\varepsilon \le \mathcal{E}_\varepsilon$, see (2.4) and (2.5), the compactness result follows immediately from Theorem 2.3. It suffices to show the Γ-liminf inequality for $\overline{\mathcal{E}}_\varepsilon$ and the Γ-limsup inequality for $\mathcal{E}_\varepsilon$.

Step 1: Γ*-liminf inequality.* Consider $u \in GSBD^2_h$ and $(y_\varepsilon)_\varepsilon$, $y_\varepsilon \in \mathcal{S}_{\varepsilon,h}$, such that $y_\varepsilon \rightsquigarrow u$, i.e., by Definition 2.4 there exist $y^{\rm rot}_\varepsilon = \sum_{j=1}^\infty R^\varepsilon_j\, y_\varepsilon \chi_{P^\varepsilon_j}$ and $u_\varepsilon := \frac{1}{\varepsilon}(y^{\rm rot}_\varepsilon - \mathbf{id})$ such that (2.10) and (2.12) hold for some fixed $\gamma \in (\frac{2}{3}, \beta)$. The essential step is to prove

$$\liminf_{\varepsilon\to 0} \frac{1}{\varepsilon^2}\int_{\Omega'} W(\nabla y_\varepsilon) \ge \int_{\Omega'} \frac{1}{2} Q(e(u)). \tag{4.14}$$

Once (4.14) is shown, we conclude by (2.5) and (2.12)(iii) that

$$\liminf_{\varepsilon\to 0} \overline{\mathcal{E}}_\varepsilon(y_\varepsilon) \ge \liminf_{\varepsilon\to 0} \Big(\frac{1}{\varepsilon^2}\int_{\Omega'} W(\nabla y_\varepsilon) + \kappa\mathcal{H}^{d-1}(J_{y_\varepsilon} \cup J_{\nabla y_\varepsilon})\Big) \ge \int_{\Omega'} \frac{1}{2} Q(e(u)) + \kappa\mathcal{H}^{d-1}(J_u).$$

In view of (2.14), this shows $\liminf_{\varepsilon\to 0} \overline{\mathcal{E}}_\varepsilon(y_\varepsilon) \ge \mathcal{E}(u)$. To see (4.14), we first note that the frame indifference of W (see (2.3)(ii)) and the definitions of $y^{\rm rot}_\varepsilon$ and u_ε (see (2.9) and (2.11)) imply

$$W(\nabla y_\varepsilon) = W(\nabla y^{\rm rot}_\varepsilon) = W(\mathbf{Id} + \varepsilon\nabla u_\varepsilon). \tag{4.15}$$

In view of $\gamma > 2/3$, we can choose $\eta_\varepsilon \to +\infty$ such that

$$\varepsilon^{1-\gamma}\eta_\varepsilon \to +\infty \quad \text{and} \quad \varepsilon\eta_\varepsilon^3 \to 0. \tag{4.16}$$

We define $\chi_\varepsilon \in L^\infty(\Omega')$ by $\chi_\varepsilon(x) = \chi_{[0,\eta_\varepsilon)}(|\nabla u_\varepsilon(x)|)$. Note that $\mathcal{L}^d(\{|\nabla u_\varepsilon(x)| > \eta_\varepsilon\}) \le C(\varepsilon^{\gamma-1}/\eta_\varepsilon)^2$ by (2.10)(iv) and the fact that $u_\varepsilon = \frac{1}{\varepsilon}(y^{\rm rot}_\varepsilon - \mathbf{id})$. Thus, (4.16) implies $\chi_\varepsilon \to 1$ boundedly in measure on Ω'. The regularity of W implies $W(\mathbf{Id} + F) = \frac{1}{2}Q(\mathrm{sym}(F)) + \omega(F)$, where Q is defined in (2.14) and

$\omega : \mathbb{R}^{d\times d} \to \mathbb{R}$ is a function satisfying $|\omega(F)| \le C|F|^3$ for all $F \in \mathbb{R}^{d\times d}$ with $|F| \le 1$. Then by (4.15) and $W \ge 0$ we get

$$\begin{aligned}\liminf_{\varepsilon\to 0} \frac{1}{\varepsilon^2}\int_{\Omega'} W(\nabla y_\varepsilon) &\ge \liminf_{\varepsilon\to 0} \frac{1}{\varepsilon^2}\int_{\Omega'} \chi_\varepsilon W(\mathbf{Id} + \varepsilon\nabla u_\varepsilon)\\ &= \liminf_{\varepsilon\to 0} \int_{\Omega'} \chi_\varepsilon\Big(\frac{1}{2}Q(e(u_\varepsilon)) + \frac{1}{\varepsilon^2}\omega(\varepsilon\nabla u_\varepsilon)\Big)\\ &\ge \liminf_{\varepsilon\to 0}\Big(\int_{\Omega'\setminus E_u} \chi_\varepsilon \frac{1}{2}Q(e(u_\varepsilon)) + \int_{\Omega'} \chi_\varepsilon|\nabla u_\varepsilon|^3\varepsilon\frac{\omega(\varepsilon\nabla u_\varepsilon)}{|\varepsilon\nabla u_\varepsilon|^3}\Big),\end{aligned}$$

where $E_u = \{x \in \Omega : |u_\varepsilon(x)| \to \infty\}$. The second term converges to zero. Indeed, $\chi_\varepsilon\frac{|\omega(\varepsilon\nabla u_\varepsilon)|}{|\varepsilon\nabla u_\varepsilon|^3}$ is uniformly controlled by C and $\chi_\varepsilon|\nabla u_\varepsilon|^3\varepsilon$ is uniformly controlled by $\eta_\varepsilon^3\varepsilon$, where $\eta_\varepsilon^3\varepsilon \to 0$ by (4.16). As $e(u_\varepsilon) \rightharpoonup e(u)$ weakly in $L^2(\Omega' \setminus E_u, \mathbb{R}^{d\times d}_{\rm sym})$ by (2.12)(ii), Q is convex, and χ_ε converges to 1 boundedly in measure on $\Omega' \setminus E_u$, we conclude

$$\liminf_{\varepsilon\to 0} \frac{1}{\varepsilon^2}\int_{\Omega'} W(\nabla y_\varepsilon) \ge \int_{\Omega'\setminus E_u} \frac{1}{2}Q(e(u)) = \int_{\Omega'} \frac{1}{2}Q(e(u)),$$

where the last step follows from the fact that $e(u) = 0$ on E_u, see (2.12)(iv). This shows (4.14) and concludes the proof of the Γ-liminf inequality.

Step 2: Γ-limsup inequality. Consider $u \in GSBD^2_h$ with $h \in W^{2,\infty}(\Omega';\mathbb{R}^d)$. Let $\gamma \in (\frac{2}{3},\beta)$. By Theorem 3.6 we can find a sequence $(v_\varepsilon)_\varepsilon \in GSBV^2_2(\Omega';\mathbb{R}^d)$ with $v_\varepsilon = h$ on $\Omega' \setminus \overline{\Omega}$, $v_\varepsilon \in W^{2,\infty}(\Omega' \setminus J_{v_\varepsilon};\mathbb{R}^d)$, and

$$\begin{aligned}&\text{(i)}\ \ v_\varepsilon \to u \text{ in measure on } \Omega',\\ &\text{(ii)}\ \ \|e(v_\varepsilon) - e(u)\|_{L^2(\Omega')} \to 0,\\ &\text{(iii)}\ \ \mathcal{H}^{d-1}(J_{v_\varepsilon}) \to \mathcal{H}^{d-1}(J_u),\\ &\text{(iv)}\ \ \|\nabla v_\varepsilon\|_{L^\infty(\Omega')} + \|\nabla^2 v_\varepsilon\|_{L^\infty(\Omega')} \le \varepsilon^{(\beta-1)/2} \le \varepsilon^{\gamma-1}.\end{aligned} \tag{4.17}$$

Note that property (iv) can be achieved since the approximations satisfy $v_\varepsilon \in W^{2,\infty}(\Omega'\setminus J_{v_\varepsilon};\mathbb{R}^d)$. (Recall $\gamma < \beta < 1$.) Moreover, $v_\varepsilon \in W^{2,\infty}(\Omega' \setminus J_{v_\varepsilon};\mathbb{R}^d)$ also implies $J_{\nabla v_\varepsilon} \subset J_{v_\varepsilon}$.

We define the sequence $y_\varepsilon = \mathbf{id}+\varepsilon v_\varepsilon$. As $v_\varepsilon \in GSBV^2_2(\Omega';\mathbb{R}^d)$ and $v_\varepsilon = h$ on $\Omega'\setminus\overline{\Omega}$, we get $y_\varepsilon \in \mathcal{S}_{\varepsilon,h}$, see (2.8). We now check that $y_\varepsilon \rightsquigarrow u$ in the sense of Definition 2.4.

We define $y_\varepsilon^{\rm rot} = y_\varepsilon$, i.e., the Caccioppoli partition in (2.9) consists of the set Ω' only with corresponding rotation $\mathbf{Id}$. Then (2.10)(i),(ii) are trivially satisfied. As $\nabla y_\varepsilon^{\rm rot} - \mathbf{Id} = \varepsilon\nabla v_\varepsilon$, (2.10)(iii),(iv) follow from (4.17)(ii),(iv). The rescaled displacement fields u_ε defined in (2.11) satisfy $u_\varepsilon = v_\varepsilon$. Then (2.12) for $E_u = \emptyset$ follows from (4.17)(i)–(iii) and $J_{y_\varepsilon} = J_{v_\varepsilon}$.

Finally, we confirm $\lim_{\varepsilon\to 0}\mathcal{E}_\varepsilon(y_\varepsilon) = \mathcal{E}(u)$. In view of $J_{y_\varepsilon} = J_{v_\varepsilon}$, $J_{\nabla y_\varepsilon} \subset J_{y_\varepsilon}$, (4.17)(iii), and the definition of the energies in (2.4), (2.14), it suffices to show

$$\lim_{\varepsilon\to 0}\Big(\frac{1}{\varepsilon^2}\int_{\Omega'} W(\nabla y_\varepsilon) + \frac{1}{\varepsilon^{2\beta}}\int_{\Omega'} |\nabla^2 y_\varepsilon|^2\Big) = \int_{\Omega'} \frac{1}{2}Q(e(u)).$$

The second term vanishes by (4.17)(iv), $\beta < 1$, and the fact that $\nabla^2 y_\varepsilon = \varepsilon\nabla^2 v_\varepsilon$. For the first term, we again use that $W(\mathbf{Id} + F) = \frac{1}{2}Q(\mathrm{sym}(F)) + \omega(F)$ with $|\omega(F)| \le C|F|^3$ for $|F| \le 1$, and compute by

(4.17)(ii),(iv)

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\int_{\Omega'} W(\nabla y_\varepsilon) = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\int_{\Omega'} W(\mathbf{Id}+\varepsilon\nabla v_\varepsilon) = \lim_{\varepsilon\to 0}\int_{\Omega'}\Big(\frac{1}{2}Q(e(v_\varepsilon)) + \frac{1}{\varepsilon^2}\omega(\varepsilon\nabla v_\varepsilon)\Big)$$
$$= \int_{\Omega'}\frac{1}{2}Q(e(u)) + \lim_{\varepsilon\to 0}\int_{\Omega'} \mathrm{O}(\varepsilon|\nabla v_\varepsilon|^3) = \int_{\Omega'}\frac{1}{2}Q(e(u)),$$

where in the last step we have used that $\|\nabla v_\varepsilon\|_{L^\infty(\Omega')} \le C\varepsilon^{\gamma-1}$ for some $\gamma > 2/3$. This concludes the proof. □

Remark 4.2. *The proof shows that one can readily incorporate a dependence on the material point x in the density W, as long as* (2.3) *still holds. We also point out that it suffices to suppose that W is $C^{2,\alpha}$ in a neighborhood of $SO(d)$, provided that $1 > \beta > \gamma > \frac{2}{2+\alpha}$. In fact, in that case, one has $|\omega(F)| \le C|F|^{2+\alpha}$ for all $|F| \le 1$,and all estimates remain true, where in* (4.16) *one chooses η_ε with $\varepsilon^{1-\gamma}\eta_\varepsilon \to +\infty$ and $\varepsilon^\alpha\eta_\varepsilon^{2+\alpha} \to 0$.*

We close this subsection with the proof of Corollary 2.8.

Proof of Corollary 2.8. The statement follows in the spirit of the fundamental theorem of Γ-convergence, see, e.g., [12, Theorem 1.21]. We repeat the argument here for the reader's convenience. We observe that $\inf_{\bar y\in\mathcal{S}_{\varepsilon,h}}\mathcal{E}_\varepsilon(\bar y)$ is uniformly bounded by choosing $\mathbf{id}+\varepsilon h$ as competitor. Given $(y_\varepsilon)_\varepsilon$, $y_\varepsilon \in \mathcal{S}_{\varepsilon,h}$, satisfying (2.18), we apply Theorem 2.7(a) to find a subsequence (not relabeled), and $u \in GSBD^2_h$ such that $y_\varepsilon \rightsquigarrow u$ in the sense of Definition 2.4. Thus, by Theorem 2.7(b) we obtain

$$\mathcal{E}(u) \le \liminf_{\varepsilon\to 0}\mathcal{E}_\varepsilon(y_\varepsilon) \le \liminf_{\varepsilon\to 0}\inf_{\bar y\in\mathcal{S}_{\varepsilon,h}}\mathcal{E}_\varepsilon(\bar y). \tag{4.18}$$

By Theorem 2.7(c), for each $v \in GSBD^2_h$, there exists a sequence $(w_\varepsilon)_\varepsilon$ with $w_\varepsilon \rightsquigarrow v$ and $\lim_{\varepsilon\to 0}\mathcal{E}_\varepsilon(w_\varepsilon) = \mathcal{E}(v)$. This implies

$$\limsup_{\varepsilon\to 0}\inf_{\bar y\in\mathcal{S}_{\varepsilon,h}}\mathcal{E}_\varepsilon(\bar y) \le \lim_{\varepsilon\to 0}\mathcal{E}_\varepsilon(w_\varepsilon) = \mathcal{E}(v). \tag{4.19}$$

By combining (4.18)–(4.19) we find

$$\mathcal{E}(u) \le \liminf_{\varepsilon\to 0}\inf_{\bar y\in\mathcal{S}_{\varepsilon,h}}\mathcal{E}_\varepsilon(\bar y) \le \limsup_{\varepsilon\to 0}\inf_{\bar y\in\mathcal{S}_{\varepsilon,h}}\mathcal{E}_\varepsilon(\bar y) \le \mathcal{E}(v). \tag{4.20}$$

Since $v \in GSBD^2_h$ was arbitrary, we get that u is a minimizer of $\mathcal{E}$. Property (2.17) follows from (4.20) with $v = u$. In particular, the limit in (2.17) does not depend on the specific choice of the subsequence and thus (2.17) holds for the whole sequence. □

4.4. Characterization of limiting displacements

This final subsection is devoted to the proof of Lemma 2.6.

Proof of Lemma 2.6. Proof of (a). As a preparation, we observe that for two given rotations $R_1, R_2 \in SO(d)$ there holds

$$|\mathrm{sym}(R_2 R_1^T - \mathbf{Id})| \le C|R_1 - R_2|^2. \tag{4.21}$$

This follows from formula (4.12) applied for $F = R_2 R_1^T$.

Consider a sequence $(y_\varepsilon)_\varepsilon$. Let

$$y_\varepsilon^{\mathrm{rot},i} = \sum\nolimits_{j=1}^\infty R_j^{\varepsilon,i}\, y_\varepsilon \chi_{P_j^{\varepsilon,i}}, \quad i = 1,2, \tag{4.22}$$

be two sequences such that the corresponding rescaled displacement fields $u_\varepsilon^i = \varepsilon^{-1}(y_\varepsilon^{\mathrm{rot},i} - \mathbf{id})$, $i = 1,2$, converge to u_1 and u_2, respectively, in the sense of (2.12), where the exceptional sets are denoted by E_{u_1} and E_{u_2}, respectively. In particular, pointwise $\mathcal{L}^d$-a.e. in Ω' there holds

$$\begin{aligned} e(u_\varepsilon^1) - e(u_\varepsilon^2) &= \varepsilon^{-1}\mathrm{sym}\Big(\sum\nolimits_j R_j^{\varepsilon,1}\,\nabla y_\varepsilon \chi_{P_j^{\varepsilon,1}} - \sum\nolimits_j R_j^{\varepsilon,2}\,\nabla y_\varepsilon \chi_{P_j^{\varepsilon,2}}\Big) \\ &= \varepsilon^{-1}\mathrm{sym}\Big(\sum\nolimits_{j,k} (R_j^{\varepsilon,1} - R_k^{\varepsilon,2})\chi_{P_j^{\varepsilon,1}\cap P_k^{\varepsilon,2}}\,\nabla y_\varepsilon\Big) \\ &= \varepsilon^{-1}\mathrm{sym}\Big(\sum\nolimits_{j,k} (\mathbf{Id} - R_k^{\varepsilon,2}(R_j^{\varepsilon,1})^T)\chi_{P_j^{\varepsilon,1}\cap P_k^{\varepsilon,2}}\,\nabla y_\varepsilon^{\mathrm{rot},1}\Big). \end{aligned} \tag{4.23}$$

For brevity, we define $Z_\varepsilon \in L^\infty(\Omega';\mathbb{R}^{d\times d})$ by

$$Z_\varepsilon := \sum\nolimits_{j,k} (\mathbf{Id} - R_k^{\varepsilon,2}(R_j^{\varepsilon,1})^T)\chi_{P_j^{\varepsilon,1}\cap P_k^{\varepsilon,2}}. \tag{4.24}$$

By (2.10)(iv) and the triangle inequality we get

$$\begin{aligned} \sum_{j,k} \big\|R_j^{\varepsilon,1} - R_k^{\varepsilon,2}\big\|^2_{L^2(P_j^{\varepsilon,1}\cap P_k^{\varepsilon,2})} &\le C\sum_{j=1}^\infty \|(\nabla y_\varepsilon)^T - R_j^{\varepsilon,1}\|^2_{L^2(P_j^{\varepsilon,1})} + C\sum_{k=1}^\infty \|(\nabla y_\varepsilon)^T - R_k^{\varepsilon,2}\|^2_{L^2(P_k^{\varepsilon,2})} \\ &= C\|\nabla y_\varepsilon^{\mathrm{rot},1} - \mathbf{Id}\|^2_{L^2(\Omega')} + C\|\nabla y_\varepsilon^{\mathrm{rot},2} - \mathbf{Id}\|^2_{L^2(\Omega')} \le C\varepsilon^{2\gamma} \end{aligned}$$

for some given $\gamma \in (\frac{2}{3},\beta)$, and $C > 0$ independent of ε. Equivalently, this means

$$\sum\nolimits_{j,k} \mathcal{L}^d(P_j^{\varepsilon,1} \cap P_k^{\varepsilon,2})\big|R_j^{\varepsilon,1} - R_k^{\varepsilon,2}\big|^2 \le C\varepsilon^{2\gamma}.$$

By recalling (4.21) and (4.24) we then get

$$\|\mathrm{sym}(Z_\varepsilon)\|_{L^1(\Omega')} \le C\varepsilon^{2\gamma}, \qquad \|Z_\varepsilon\|_{L^2(\Omega')} \le C\varepsilon^\gamma.$$

This along with Hölder's inequality, (2.10)(iv) for $y_\varepsilon^{\mathrm{rot},1}$, and (4.23) yields

$$\begin{aligned} \|e(u_\varepsilon^1) - e(u_\varepsilon^2)\|_{L^1(\Omega')} &= \frac{1}{\varepsilon}\|\mathrm{sym}(Z_\varepsilon\,\nabla y_\varepsilon^{\mathrm{rot},1})\|_{L^1(\Omega')} \\ &\le \frac{1}{\varepsilon}\|\mathrm{sym}(Z_\varepsilon\,(\nabla y_\varepsilon^{\mathrm{rot},1} - \mathbf{Id}))\|_{L^1(\Omega')} + \frac{1}{\varepsilon}\|\mathrm{sym}(Z_\varepsilon)\|_{L^1(\Omega')} \\ &\le \frac{1}{\varepsilon}\|Z_\varepsilon\|_{L^2(\Omega')}\|\nabla y_\varepsilon^{\mathrm{rot},1} - \mathbf{Id}\|_{L^2(\Omega')} + \frac{1}{\varepsilon}\|\mathrm{sym}(Z_\varepsilon)\|_{L^1(\Omega')} \le C\varepsilon^{2\gamma-1}. \end{aligned} \tag{4.25}$$

We have that $e(u_\varepsilon^1) - e(u_\varepsilon^2)$ converges to $e(u_1) - e(u_2)$ weakly in $L^2(\Omega' \setminus (E_{u_1} \cup E_{u_2}); \mathbb{R}^{d\times d}_{\rm sym})$, see (2.12)(ii). Then (4.25) and the fact that $\gamma > \frac{2}{3} > \frac{1}{2}$ imply that $e(u_1) - e(u_2) = 0$ on $\Omega' \setminus (E_{u_1} \cup E_{u_2})$. This shows part (a) of the statement.

Proof of (b). Let $(y_\varepsilon)_\varepsilon$ be a sequence satisfying (2.13). Consider two piecewise rotated functions $y_\varepsilon^{{\rm rot},i}$ as given in (4.22) and let u_1, u_2 be the limits identified in (2.12), where the corresponding exceptional sets are denoted by E_{u_1}, E_{u_2}. We let $\mathcal{J}^i = \{j \in \mathbb{N} : P_j^{\varepsilon,i} \subset \Omega$ up to an $\mathcal{L}^d$-negligible set$\}$ for $i = 1, 2$, and set $D_\varepsilon := \bigcup_{i=1,2} \bigcup_{j\in\mathcal{J}^i} P_j^{\varepsilon,i}$. By (2.10)(ii) and $\gamma < \beta$ we obtain

$$\limsup_{\varepsilon\to 0} \mathcal{H}^{d-1}((\partial^* D_\varepsilon \cap \Omega') \setminus (J_{y_\varepsilon} \cup J_{\nabla y_\varepsilon})) = 0. \tag{4.26}$$

As also $\sup_\varepsilon \mathcal{H}^{d-1}(J_{y_\varepsilon} \cup J_{\nabla y_\varepsilon}) < +\infty$, we get that $\mathcal{H}^{d-1}(\partial^* D_\varepsilon)$ is uniformly controlled. Therefore, we may suppose that $D_\varepsilon \to D$ in measure for a set of finite perimter $D \subset \Omega$, see [6, Theorem 3.39]. We observe that $y_\varepsilon^{{\rm rot},i} = y_\varepsilon$ on $\Omega' \setminus D_\varepsilon$ for $i = 1, 2$ by Remark 4.1. Therefore, (2.11) implies that $E_{u_1} \setminus D = E_{u_2} \setminus D$. In the following, we denote this set by $\hat{E}$. Then, (2.11) and (2.12)(i) also yield

$$u_1 = u_2 \quad \text{a.e. on } \Omega' \setminus (D \cup \hat{E}). \tag{4.27}$$

To compare u_1 and u_2 inside $D \cup \hat{E}$, we introduce modifications: For $i = 1, 2$ and sequences $(\lambda_\varepsilon)_\varepsilon \subset \mathbb{R}^d$, let

$$y_\varepsilon^{\lambda_\varepsilon,i} := y_\varepsilon^{{\rm rot},i} + \lambda_\varepsilon \chi_{D_\varepsilon}. \tag{4.28}$$

By definition, D_ε does not intersect $\Omega' \setminus \overline{\Omega}$ and has finite perimeter by (4.26). Thus, we get $y_\varepsilon^{\lambda_\varepsilon,i} \in \mathcal{S}_{\varepsilon,h}$, see (2.8) and (2.10)(i). By (2.10)(ii), (4.26), and the fact that the elastic energy is frame indifferent we also observe that $(y_\varepsilon^{\lambda_\varepsilon,i})_\varepsilon$ is a minimizing sequence for $i = 1, 2$ and all $(\lambda_\varepsilon)_\varepsilon \subset \mathbb{R}^d$. We obtain

$$y_\varepsilon = y_\varepsilon^{{\rm rot},i} = y_\varepsilon^{\lambda_\varepsilon,i} \quad \text{ on } \Omega' \setminus D_\varepsilon \text{ for all } (\lambda_\varepsilon)_\varepsilon \subset \mathbb{R}^d,\ i = 1, 2. \tag{4.29}$$

This follows from (4.28) and $y_\varepsilon^{{\rm rot},i} = y_\varepsilon$ on $\Omega' \setminus D_\varepsilon$ for $i = 1, 2$, see Remark 4.1. We now consider two different cases:

(1) Fix $i = 1, 2$, $\lambda \in \mathbb{R}^d$, and consider $\lambda_\varepsilon = \lambda\varepsilon$. In view of (2.11), (2.12)(i), and (4.28), we get that $\varepsilon^{-1}(y_\varepsilon^{\lambda_\varepsilon,i} - \mathbf{id}) \to u_i + \lambda\chi_D$ in measure on $\Omega' \setminus E_{u_i}$. Thus, one can check that $y_\varepsilon^{\lambda_\varepsilon,i} \rightsquigarrow u_i^\lambda$ for some $u_i^\lambda \in GSBD^2_h$ satisfying

$$u_i^\lambda = u_i + \lambda\chi_D \text{ on } \Omega' \setminus E_{u_i}. \tag{4.30}$$

(2) Recall that $\hat{E} = E_{u_1} \setminus D = E_{u_2} \setminus D = \{x \in \Omega \setminus D : |\varepsilon^{-1}(y_\varepsilon^{{\rm rot},i} - \mathbf{id})| \to \infty\}$ for $i = 1, 2$. In view of (4.28), we can choose a suitable sequence $(\lambda_\varepsilon)_\varepsilon$ such that $|\varepsilon^{-1}(y_\varepsilon^{\lambda_\varepsilon,i} - \mathbf{id})| \to \infty$ on $\hat{E} \cup D$ for $i = 1, 2$. This along with (4.29) and (2.12)(i),(iv) implies that for $i = 1, 2$ we have $y_\varepsilon^{\lambda_\varepsilon,i} \rightsquigarrow \hat{u}$ for some $\hat{u} \in GSBD^2_h$ satisfying

$$\begin{aligned} &\text{(i)}\ \ \hat{u} = u_1 = u_2 \quad \text{a.e. on } \Omega' \setminus (\hat{E} \cup D),\\ &\text{(ii)}\ \ e(\hat{u}) = 0 \quad \text{a.e. on } \hat{E} \cup D, \quad \mathcal{H}^{d-1}(J_{\hat{u}} \cap (\hat{E} \cup D)^1) = 0, \end{aligned} \tag{4.31}$$

where $(\cdot)^1$ denotes the set of points with density 1.

We now combine the cases (1) and (2) to obtain the statement: since $(y_\varepsilon^{\lambda_\varepsilon,i})_\varepsilon$ are minimizing sequences, Corollary 2.8 implies that each u_i^λ, $\lambda \in \mathbb{R}^d$, $i = 1,2$, and $\hat{u}$ are minimizers of the problem $\min_{v \in GSBD_h^2} \mathcal{E}(v)$. In particular, as $e(u_i^\lambda) = e(u_i)$ for all $\lambda \in \mathbb{R}^d$ for both $i = 1,2$, the jump sets of u_1^λ, u_2^λ have to be independent of λ, i.e., $\mathcal{H}^{d-1}(J_{u_i} \triangle J_{u_i^\lambda}) = 0$ for all $\lambda \in \mathbb{R}^d$ and $i = 1,2$. In view of (4.30) and (2.12)(iv), this yields $\partial^* E_{u_i} \cap \Omega', \partial^*(D \setminus E_{u_i}) \cap \Omega' \subset J_{u_i}$ up to $\mathcal{H}^{d-1}$-negligigble sets. Since $\hat{E} = E_{u_i} \setminus D$, this implies for $i = 1,2$ that

$$\partial^*(\hat{E} \cup D) \cap \Omega' \subset J_{u_i} \qquad \text{up to } \mathcal{H}^{d-1}\text{-negligigble sets.} \tag{4.32}$$

Recall that u_1, u_2 are both minimizers, that also $\hat{u}$ is a minimzer, and that there holds $\hat{u} = u_1 = u_2$ on $\Omega' \setminus (\hat{E} \cup D)$, see (4.31)(i). This along with (4.31)(ii) and (4.32) yields $e(u_i) = 0$ on $\hat{E} \cup D$ and $\mathcal{H}^{d-1}(J_{u_i} \cap (\hat{E} \cup D)^1) = 0$ for $i = 1,2$. Then (4.27) and (4.32) show that $e(u_1) = e(u_2)$ $\mathcal{L}^d$-a.e. on Ω', and $J_{u_1} = J_{u_2}$ up to an $\mathcal{H}^{d-1}$-negligible set. □

We finally provide an example that in case (a) the strains cannot be compared inside $E_{u_1} \cup E_{u_2}$.

Example 4.3. Similar to Example 2.5, we consider $\Omega' = (0,3) \times (0,1)$, $\Omega = (1,3) \times (0,1)$, $\Omega_1 = (0,2) \times (0,1)$, $\Omega_2 = (2,3) \times (0,1)$, and $h \equiv 0$. Let $z \in W^{2,\infty}(\Omega'; \mathbb{R}^d)$ with $\{z = 0\} = \emptyset$, and define

$$y_\varepsilon(x) = x\chi_{\Omega_1}(x) + \big(x + \varepsilon z(x)\big)\chi_{\Omega_2}(x) \quad \text{for } x \in \Omega'.$$

Note that $J_{y_\varepsilon} = \partial\Omega_1 \cap \Omega' = \partial\Omega_2 \cap \Omega'$. Then two possible alternatives are

$$\begin{aligned} &(1)\ \ P_1^\varepsilon = \Omega_1,\ \ P_2^\varepsilon = \Omega_2,\ \ R_1^\varepsilon = \mathbf{Id},\ \ R_2^\varepsilon = \bar{R}_\varepsilon,\\ &(2)\ \ \tilde{P}_1^\varepsilon = \Omega',\ \ \tilde{R}_1^\varepsilon = \mathbf{Id}, \end{aligned}$$

where $\bar{R}_\varepsilon \in SO(2)$ satisfies $\bar{R}_\varepsilon = \mathbf{Id} + \varepsilon^\gamma A + O(\varepsilon^{2\gamma})$ for some $A \in \mathbb{R}^{2\times 2}_{\rm skew}$, $\gamma \in (\frac{2}{3}, \beta)$. Let $u_\varepsilon = \varepsilon^{-1}(\sum_{j=1}^2 R_j^\varepsilon y_\varepsilon \chi_{P_j^\varepsilon} - \mathbf{id})$ and $\tilde{u}_\varepsilon = \varepsilon^{-1}(y_\varepsilon - \mathbf{id})$, We observe that $|u_\varepsilon| \to \infty$ on Ω_2. Possible limits identified in (2.12) are $u = \lambda\chi_{\Omega_2}$ for some $\lambda \in \mathbb{R}^d$, $\lambda \neq 0$, with $E_u = \Omega_2$, and $\tilde{u}(x) = z(x)\chi_{\Omega_2}(x)$ with $E_{\tilde{u}} = \emptyset$. This shows that in general there holds $e(u) \neq e(\tilde{u})$ in E_u.

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Conflict of interest

The author declares no conflict of interest.

References

1. Agostiniani V, Dal Maso G, DeSimone A (2012) Linear elasticity obtained from finite elasticity by Γ-convergence under weak coerciveness conditions. *Ann Inst H Poincaré Anal Non Linéaire* 29: 715–735.

2. Alicandro R, Dal Maso G, Lazzaroni G, et al. (2018) Derivation of a linearised elasticity model from singularly perturbed multiwell energy functionals. *Arch Ration Mech Anal* 230: 1–45.

3. Ambrosio L (1990) Existence theory for a new class of variational problems. *Arch Ration Mech Anal* 111: 291–322.

4. Ambrosio L (1994) On the lower semicontinuity of quasi-convex integrals in $SBV(\Omega;\mathbb{R}^k)$. *Nonlinear Anal Theor* 23: 405–425.

5. Ambrosio L, Coscia A, Dal Maso G (1997) Fine properties of functions with bounded deformation. *Arch Ration Mech Anal* 139: 201–238.

6. Ambrosio L, Fusco N, Pallara D (2000) *Functions of Bounded Variation and Free Discontinuity Problems*, Oxford: Oxford University Press.

7. Babadjian JF, Giacomini A (2014) Existence of strong solutions for quasi-static evolution in brittle fracture. *Ann Sc Norm Super Pisa Cl Sci* 13: 925–974.

8. Ball JM, Currie JC, Olver PL (1981) Null Lagrangians, weak continuity, and variational problems of arbitrary order. *J Funct Anal* 41: 135–174.

9. Batra RC (1976) Thermodynamics of non-simple elastic materials. *J Elasticity* 6: 451–456.

10. Blake A, Zisserman A (1987) *Visual Reconstruction*, Cambridge: The MIT Press.

11. Bourdin B, Francfort GA, Marigo JJ (2008) The variational approach to fracture. *J Elasticity* 91: 5–148.

12. Braides A (2002) *Γ-Convergence for Beginners*, Oxford: Oxford University Press.

13. Braides A, Solci M, Vitali E (2007) A derivation of linear elastic energies from pair-interaction atomistic systems. *Netw Heterog Media* 2: 551–567.

14. Capriz G (1985) Continua with latent microstructure. *Arch Ration Mech Anal* 90: 43–56.

15. Carriero M, Leaci A, Tomarelli F (1996) A second order model in image segmentation: Blake & Zisserman functional, In: *Variational Methods for Discontinuous Structures*, Basel: Birkhäuser, 57–72.

16. Carriero M, Leaci A, Tomarelli F (2004) Second order variational problems with free discontinuity and free gradient discontinuity, In: *Calculus of Variations: Topics from the Mathematical Heritage of Ennio De Giorgi*, Quad Mat, 14: 135–186.

17. Carriero M, Leaci A, Tomarelli F (1992) Plastic free discontinuities and special bounded hessian. *C R Acad Sci* 314: 595–600.

18. Chambolle A (2003) A density result in two-dimensional linearized elasticity, and applications. *Arch Ration Mech Anal* 167: 167–211.

19. Chambolle A, Conti S, Francfort G (2016) Korn-Poincaré inequalities for functions with a small jump set. *Indiana Univ Math J* 65: 1373–1399.

20. Chambolle A, Conti S, Francfort G (2018) Approximation of a brittle fracture energy with a constraint of non-interpenetration. *Arch Ration Mech Anal* 228: 867–889.

21. Chambolle A, Conti S, Francfort G (2019) Approximation of functions with small jump sets and existence of strong minimizers of Griffith's energy. *J Math Pures Appl* 128: 119–139.

22. Chambolle A, Giacomini A, Ponsiglione M (2007) Piecewise rigidity. *J Funct Anal* 244: 134–153.

23. Conti S, Focardi M, Iurlano F (2018) Which special functions of bounded deformation have bounded variation? *P Roy Soc Edinb A* 148: 33–50.

24. Conti S, Focardi M, Iurlano F (2017) Integral representation for functionals defined on SBD^p in dimension two. *Arch Rat Mech Anal* 223: 1337–1374.

25. Chambolle A, Crismale V (2019) A density result in $GSBD^p$ with applications to the approximation of brittle fracture energies. *Arch Ration Mech Anal* 232: 1329–1378.

26. Chambolle A, Crismale V (2018) *Compactness and lower semicontinuity in GSBD*. *J Eur Math Soc* http://arxiv.org/abs/1802.03302.

27. Chambolle A, Crismale V (2018) Existence of strong solutions to the Dirichlet problem for Griffith energy. *Calc Var Partial Dif* 58: 136.

28. Crismale V (2018) On the approximation of SBD functions and some applications. http://arxiv.org/abs/1806.03076.

29. Cortesani G, Toader R (1999) A density result in SBV with respect to non-isotropic energies. *Nonlinear Anal* 38: 585–604.

30. Dal Maso G (1993) *An Introduction to* Γ*-Convergence*, Basel: Birkhäuser.

31. Dal Maso G (2013) Generalized functions of bounded deformation. *J Eur Math Soc* 15: 1943–1997.

32. Dal Maso G, Francfort GA, Toader R (2005) Quasistatic crack growth in nonlinear elasticity. *Arch Ration Mech Anal* 176: 165–225.

33. Dal Maso G, Negri M, Percivale D (2002) Linearized elasticity as Γ-limit of finite elasticity. *Set-valued Anal* 10: 165–183.

34. Dal Maso G, Lazzaroni G (2010) Quasistatic crack growth in finite elasticity with non-interpenetration. *Ann Inst H Poincaré Anal Non Linéaire* 27: 257–290.

35. De Giorgi E, Ambrosio L (1988) Un nuovo tipo di funzionale del calcolo delle variazioni. *Acc Naz Lincei, Rend Cl Sci Fis Mat Natur* 82: 199–210.

36. Dunn JE, Serrin J (1985) On the thermomechanics of interstitial working. *Arch Ration Mech Anal* 88: 95–133.

37. Francfort GA, Marigo JJ (1998) Revisiting brittle fracture as an energy minimization problem. *J Mech Phys Solids* 46: 1319–1342.

38. Francfort GA, Larsen CJ (2003) Existence and convergence for quasi-static evolution in brittle fracture. *Comm Pure Appl Math* 56: 1465–1500.

39. Friedrich M (2017) A derivation of linearized Griffith energies from nonlinear models. *Arch Ration Mech Anal* 225: 425–467.

40. Friedrich M (2017) A Korn-type inequality in SBD for functions with small jump sets. *Math Mod Meth Appl Sci* 27: 2461–2484.

41. Friedrich M (2018) A piecewise Korn inequality in SBD and applications to embedding and density results. *SIAM J Math Anal* 50: 3842–3918.

42. Friedrich M (2019) A compactness result in $GSBV^p$ and applications to Γ-convergence for free discontinuity problems. *Calc Var Partial Dif* 58: 86.

43. Friedrich M, Kružík M (2018) On the passage from nonlinear to linearized viscoelasticity. *SIAM J Math Anal* 50: 4426–4456.

44. Friedrich M, Schmidt B (2015) On a discrete-to-continuum convergence result for a two dimensional brittle material in the small displacement regime. *Netw Heterog Media* 10: 321–342.

45. Friedrich M, Solombrino F (2018) Quasistatic crack growth in 2d-linearized elasticity. *Ann Inst H Poincaré Anal Non Linéaire* 35: 27–64.

46. Friesecke G, James RD, Müller S (2002) A theorem on geometric rigidity and the derivation of nonlinear plate theory from three-dimensional elasticity. *Comm Pure Appl Math* 55: 1461–1506.

47. Giacomini A (2005) Ambrosio-Tortorelli approximation of quasi-static evolution of brittle fractures. *Calc Var Partial Dif* 22: 129–172.

48. Giacomini A, Ponsiglione M (2006) A Γ-convergence approach to stability of unilateral minimality properties. *Arch Ration Mech Anal* 180: 399–447.

49. Iurlano F (2014) A density result for GSBD and its application to the approximation of brittle fracture energies. *Calc Var Partial Dif* 51: 315–342.

50. Mielke A, Roubíček T (2015) *Rate-Independent Systems: Theory and Application*, New York: Springer.

51. Mielke A, Stefanelli U (2013) Linearized plasticity is the evolutionary Γ-limit of finite plasticity. *J Eur Math Soc* 15: 923–948.

52. Negri M, Toader R (2015) Scaling in fracture mechanics by Bažant's law: from finite to linearized elasticity. *Math Mod Meth Appl Sci* 25: 1389–1420.

53. Negri M, Zanini C (2014) From finite to linear elastic fracture mechanics by scaling. *Calc Var Partial Dif* 50: 525–548.

54. Schmidt B (2008) Linear Γ-limits of multiwell energies in nonlinear elasticity theory. *Continuum Mech Thermodyn* 20: 375–396.

55. Schmidt B (2009) On the derivation of linear elasticity from atomistic models. *Netw Heterog Media* 4: 789–812.

56. Schmidt B, Fraternali F, Ortiz M (2009) Eigenfracture: An eigendeformation approach to variational fracture. *SIAM Mult Model Simul* 7: 1237–1266.

57. Toupin RA (1962) Elastic materials with couple stresses. *Arch Ration Mech Anal* 11: 385–414.

58. Toupin RA (1964) Theory of elasticity with couple stress. *Arch Ration Mech Anal* 17: 85–112.

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