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Research article

Existence of viscosity solutions to two-phase problems for fully nonlinear equations with distributed sources

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Abstract: In this paper we construct a viscosity solution of a two-phase free boundary problem for a class of fully nonlinear equation with distributed sources, via an adaptation of the Perron method. Our results extend those in [Caffarelli, 1988], [Wang, 2003] for the homogeneous case, and of [De Silva, Ferrari, Salsa, 2015] for divergence form operators with right hand side.

Keywords: Perron method; two-phase free boundary problems; fully nonlinear elliptic equations

1. Introduction

In the last years the regularity theory for two phase problems governed by uniformly elliptic equations with distributed sources has reached a considerable level of completeness (see for instance the survey paper [10]) extending the results in the seminal papers [2, 4] (for the Laplace operator) and in [17, 18] (for concave fully non linear operators) to the inhomogeneous case, through a different approach first introduced in [7].

In particular the papers [15] and [8] provides optimal Lipschitz regularity for viscosity solutions and their free boundary for a large class of fully nonlinear equations.

Existence of a continuous viscosity solution through a Perron method has been established for linear operators in divergence form in [3] (homogeneous case) and in [9] (inhomogeneous case), and for a class of concave operators in [19]. The main aim of this paper is to adapt the Perron method to extend the results of [19] to the inhomogeneous case. Although we are largely inspired by the papers [3] and [9], the presence of a right hand side and the nonlinearity of the governing equation presents several delicate points, significantly in Section 6, which require new arguments.

We now introduce our class of free boundary problems and their weak (or viscosity) solutions.

Let Sym_n denote the space of $n \times n$ symmetric matrices and let $F : \mathrm{Sym}_n \to \mathbb{R}$ denote a positively homogeneous map of degree one, smooth except at the origin, concave and uniformly elliptic, i.e. such that there exist constants $0 < \lambda \leq \Lambda$ with

$$\lambda\|N\| \leq F(M+N) - F(M) \leq \Lambda\|N\| \qquad \text{for every } M, N \in \mathrm{Sym}_n \text{ with } N \geq 0,$$

where $\|M\| = \max_{|x|=1} |Mx|$ denotes the (L^2, L^2)-norm of the matrix M.

Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and $f_1, f_2 \in C(\Omega) \cap L^\infty(\Omega)$. We consider the following two-phase inhomogeneous free boundary problem (f.b.p. in the sequel).

$$\begin{cases} F(D^2u^+) = f_1 & \text{in } \Omega^+(u) := \{u > 0\} \\ F(D^2u^-) = f_2\chi_{\{u<0\}} & \text{in } \Omega^-(u) = \{u \leq 0\}^o \\ u_\nu^+(x) = G(u_\nu^-, x, \nu) & \text{along } \mathcal{F}(u) := \partial\{u > 0\} \cap \Omega. \end{cases} \tag{1.1}$$

Here $\nu = \nu(x)$ denotes the unit normal to the free boundary $\mathcal{F} = \mathcal{F}(u)$ at the point x, pointing toward Ω^+, while the function $G(\beta, x, \nu)$ is Lipschitz continuous, strictly increasing in β, and

$$\inf_{x\in\Omega, |\nu|=1} G(0, x, \nu) > 0. \tag{1.2}$$

Moreover, u_ν^+ and u_ν^- denote the normal derivatives in the inward direction to $\Omega^+(u)$ and $\Omega^-(u)$ respectively.

As we said, the main aim of this paper is to adapt Perron's method in order to prove the existence of a weak (viscosity) solution of the above f.b.p., with assigned Dirichlet boundary conditions

For any u continuous in Ω we say that a point $x_0 \in \mathcal{F}(u)$ is *regular from the right* (resp. left) if there exists a ball $B \subset \Omega^+(u)$ (resp. $B \subset \Omega^-(u)$) such that $\overline{B} \cap \mathcal{F}(u) = x_0$. In both cases, we denote with $\nu = \nu(x_0)$ the unit normal to ∂B at x_0, pointing toward $\Omega^+(u)$.

Definition 1.1. A *weak* (or *viscosity*) *solution* of the free boundary problem (1.1) is a continuous function u which satisfies the first two equality of (1.1) in viscosity sense (see Appendix A), and such that the free boundary condition is satisfied in the following viscosity sense:

(i) (supersolution condition) if $x_0 \in \mathcal{F}$ is regular from the right with touching ball B, then, near x_0,

$$u^+(x) \geq \alpha \langle x - x_0, \nu\rangle^+ + o(|x - x_0|) \qquad \text{in } B, \text{ with } \alpha \geq 0$$

and

$$u^-(x) \leq \beta \langle x - x_0, \nu\rangle^- + o(|x - x_0|) \qquad \text{in } B^c, \text{ with } \beta \geq 0,$$

with equality along every non-tangential direction, and

$$\alpha \leq G(\beta, x_0, \nu(x_0));$$

(ii) (subsolution condition) if $x_0 \in \mathcal{F}$ is regular from the left with touching ball B, then, near x_0,

$$u^+(x) \leq \alpha \langle x - x_0, \nu\rangle^+ + o(|x - x_0|) \qquad \text{in } B^c, \text{ with } \alpha \geq 0$$

and

$$u^-(x) \geq \beta \langle x - x_0, \nu\rangle^- + o(|x - x_0|) \qquad \text{in } B, \text{ with } \beta \geq 0,$$

with equality along every non-tangential direction, and

$$\alpha \geq G(\beta, x_0, \nu(x_0));$$

We will construct our solution via Perron's method, by taking the infimum over the following class of *admissible supersolutions* $\mathcal{S}$.

Definition 1.2. A locally Lipschitz continuous function $w \in C(\overline{\Omega})$ is in the class $\mathcal{S}$ if

(a) w is a solution in viscosity sense to

$$\begin{cases} F(D^2 w^+) \le f_1 & \text{in } \Omega^+(w) \\ F(D^2 w^-) \ge f_2 \chi_{\{u<0\}} & \text{in } \Omega^-(w); \end{cases}$$

(b) if $x_0 \in \mathcal{F}(w)$ is regular from the left, with touching ball B, then

$$w^+(x) \le \alpha \langle x - x_0, \nu \rangle^+ + o(|x - x_0|) \qquad \text{in } B^c, \text{ with } \alpha \ge 0$$

and

$$w^-(x) \ge \beta \langle x - x_0, \nu \rangle^- + o(|x - x_0|) \qquad \text{in } B, \text{ with } \beta \ge 0,$$

with

$$\alpha \le G(\beta, x_0, \nu(x_0));$$

(c) if $x_0 \in \mathcal{F}(w)$ is not regular from the left then

$$w(x) = o(|x - x_0|).$$

The last ingredient we need is that of minorant subsolution.

Definition 1.3. A locally Lipschitz continuous function $\underline{u} \in C(\overline{\Omega})$ is a *strict minorant* if

(a) $\underline{u}$ is a solution in viscosity sense to

$$\begin{cases} F(D^2 \underline{u}^+) \ge f_1 & \text{in } \Omega^+(\underline{u}) \\ F(D^2 \underline{u}^-) \le f_2 \chi_{\{\underline{u}<0\}} & \text{in } \Omega^-(\underline{u}); \end{cases}$$

(b) every $x_0 \in \mathcal{F}(\underline{u})$ is regular from the right, with touching ball B, and near x_0

$$\underline{u}^+(x) \ge \alpha \langle x - x_0, \nu \rangle^+ + \omega(|x - x_0|)|x - x_0| \qquad \text{in } B, \text{ with } \alpha > 0,$$

where $\omega(r) \to 0$ as $r \to 0^+$, and

$$\underline{u}^-(x) \le \beta \langle x - x_0, \nu \rangle^- + o(|x - x_0|) \qquad \text{in } B^c, \text{ with } \beta \ge 0,$$

with

$$\alpha > G(\beta, x_0, \nu(x_0)).$$

Our main result is the following.

Theorem 1.4. *Let g be a continuous function on $\partial\Omega$. If*

(a) there exists a strict minorant $\underline{u}$ with $\underline{u} = g$ on $\partial\Omega$ and

(b) the set $\{w \in \mathcal{S} : w \geq \underline{u},\ w = g \text{ on } \partial\Omega\}$ *is not empty, then*

$$u = \inf\{w : w \in \mathcal{S}, w \geq \underline{u}\}$$

is a weak solution of (1.1) such that $u = g$ *on* $\partial\Omega$*.*

Once existence of a solution is established, we turn to the analysis of the regularity of the free boundary.

Theorem 1.5. *The free boundary* $\mathcal{F}(u)$ *has finite* $(n-1)$*-dimensional Hausdorff measure. More precisely, there exists a universal constant* $r_0 > 0$ *such that for every* $r < r_0$*, for every* $x_0 \in \mathcal{F}(u)$*,*

$$\mathcal{H}^{n-1}(\mathcal{F}(u) \cap B_r(x_0)) \leq r^{n-1}.$$

Moreover, the reduced boundary $\mathcal{F}^*(u)$ *of* $\Omega^+(u)$ *has positive density in* $\mathcal{H}^{n-1}$ *measure at any point of* $F(u)$*, i.e. for* $r < r_0$*,* r_0 *universal*

$$\mathcal{H}^{n-1}(\mathcal{F}^*(u) \cap B_r(x)) \geq cr^{n-1},$$

for every $x \in \mathcal{F}(u)$*. In particular*

$$\mathcal{H}^{n-1}(\mathcal{F}(u) \setminus \mathcal{F}^*(u)) = 0.$$

Using the results in [8] we deduce the following regularity result.

Corollary 1.6. $\mathcal{F}(u)$ *is a* $C^{1,\gamma}$ *surface in a neighborhood of* $\mathcal{H}^{n-1}$ *a.e. point* $x_0 \in \mathcal{F}(u)$*.*

Notation. Constants c, C and so on will be termed "universal" if they only depend on λ, Λ, n, Ω, $\|f_i\|_\infty$ and g.

2. Asymptotic developments

In this section we show that positive solutions of $F(D^2u) = f$ (with f continuous up to the boundary) have asymptotically linear behavior at any boundary point which admits a touching ball, either from inside or from outside the domain. We need the following preliminary result.

Lemma 2.1. *Let* $r > 0$*,* $\delta > 0$*,* $\sigma > 0$*,* $B_1^+ := B_1 \cap \{x_1 > 0\}$ *and let* $E \subset \partial B_1^+ \cap \{x_1 > 0\}$ *be any subset such that there exists* $\bar{x} \in E$ *with*

$$E \supset \partial B_1^+ \cap \{x_1 > 0\} \cap B_\sigma(\bar{x}).$$

Let u be the solution to

$$\begin{cases} F(D^2u) = r & \text{in } B_1^+ \\ u = \delta g_E & \text{on } \partial B_1^+, \end{cases} \tag{2.1}$$

where g_E *is a cut-off function,* $g_E = 1$ *on E. If r is sufficiently small then there exists a positive constant* $C = C(\delta, \sigma)$ *such that*

$$u(x) \geq Cx_1 \qquad \text{in } B_{1/2}^+.$$

Proof. We write

$$F(D^2u) = \sum_{i,j=1}^{n} a_{ij}(x)u_{x_ix_j} \equiv L_u u,$$

with ($F = F(M)$)

$$a_{ij} = \int_0^1 \frac{\partial F}{\partial M_{ij}}(tD^2u)dt.$$

We have

$$\lambda|\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2.$$

Denote $u = v + w$ with $L_u v = 0$, $v = \delta\chi_E$ on ∂B_1^+ and $L_u w = r$, $w = 0$ on ∂B_1^+. By [11] we have that $v(e_1/2) \geq C\delta$, for some constant $C = C(n, \lambda, \Lambda, \sigma)$, and by the Boundary Harnack principle applied to v and $u_1(x) = x_1$ we get that, in $B_{1/2}^+$, for some positive constants c_0 and c_1,

$$c_0\delta x_1 \leq v \leq c_1\delta x_1.$$

Put $z(x) = \frac{1}{2\min a_{11}}(x_1 - x_1^2)r$. The function z is positive in B_1^+ and

$$L_u z = -\frac{a_{11}}{\min a_{11}} r \leq -r.$$

Therefore $L_u(w + z) \leq 0$ in B_1^+ and $w + z \geq 0$ on ∂B_1^+. By the maximum principle $w \geq -c_2 r x_1$ in B_1^+, where $c_2 = \frac{a_{11}}{\min a_{11}} > 0$.

Summing up we get, in $B_{1/2}^+$,

$$u = v + w \geq (c_0\delta - c_2 r)x_1 \geq c_3 x_1$$

for r small enough, having $c_3 > 0$. □

Lemma 2.2. *Let Ω_1 be a bounded domain with $0 \in \partial\Omega_1$ and*

$$B_1^+ := B_1 \cap \{x_1 > 0\} \subset \Omega_1.$$

Let u be non-negative and Lipschitz in $\overline{\Omega_1} \cap B_2$, such that $F(D^2u) = f$ in $\Omega_1 \cap B_2$ and that $u = 0$ in $\partial\Omega_1 \cap B_2$. Then there exists $\alpha \geq 0$ such that

$$u(x) = \alpha x_1^+ + o(|x|) \qquad \text{as } x \to 0,\ x_1 > 0.$$

Proof. Let $\alpha_k = \sup\{\beta : u(x) \geq \beta x_1 \text{ in } B_{1/k}^+\}$ for $k \geq 1$. Then the sequence $\{\alpha_k\}_k$ is increasing and $\alpha_k \leq L$ for any k, where L is the Lipschitz constant of u. Let $\alpha = \lim_k \alpha_k$. By definition, $u(x) \geq \alpha x_1 + o(|x|)$ in B_1^+, where $x = (x_1, x_2, ..., x_n)$.

Suppose by contradiction that $u(x) \neq \alpha x_1 + o(|x|)$ in B_1^+. Then there exist a constant $\delta_0 > 0$ and a sequence $\{x_k\}_k = \{(x_{1,k}, x_{2,k}, ..., x_{n,k})\}_k \subset B_1^+$, with $|x_k| = r_k \to 0$, such that

$$u(x_k) \geq \alpha x_{1,k} + \delta_0 r_k.$$

Since u is Lipschitz, a simple computation implies that

$$u(x) \geq \alpha x_1 + \frac{\delta_0}{2} r_k \geq \alpha_k x_1 + \frac{\delta_0}{2} r_k \qquad \text{in } \left\{x : |x| = r_k, |x - x_k| \leq \frac{\delta_0 r_k}{4L}\right\}.$$

Let

$$u_k(x) = \frac{u(r_k x)}{r_k} - \alpha_k x_1.$$

The functions u_k are defined in B_1^+ and, by assumption of homogeneity on F, we have

$$F(D^2 u_k(x)) = F(r_k D^2 u(r_k x)) = r_k F(D^2 u(r_k x)) = r_k f(r_k x) \le r_k \|f\|_\infty.$$

Moreover $u_k(x) \ge 0$ on ∂B_1^+ and $u_k \ge \delta_0/2$ in $E_k = \{x : x \in \partial B_1^+, x_1 > 0, |x - x_k| \le \frac{\delta_0}{4L}\}$. We deduce that u_k is a supersolution of (2.1). By comparison and Lemma 2.1, there exists $C > 0$, not depending on k, such that

$$u_k(x) = \frac{1}{r_k} u(r_k x) - \alpha_k x_1 \ge C x_1 \qquad \text{in } B_{1/2}^+.$$

Writing $z = r_k x$ we obtain $u(z) \ge (\alpha_k + C) z_1$ in $B_{r_k/2}^+$. Choosing k, k' in such a way that $\alpha_k + C > \alpha$ and $k' > 2/r_k$ we obtain

$$\alpha_{k'} > \alpha,$$

a contradiction. □

Lemma 2.3. *Let Ω_1 be a bounded domain such that, writing $B_1^- := B_1 \cap \{x_1 < 0\}$,*

$$\overline{B_1^-} \cap \overline{\Omega}_1 = \{0\}.$$

Let u be non-negative and Lipschitz in $\overline{\Omega}_1 \cap B_2(0)$, such that $F(D^2 u) = f$ in $\Omega_1 \cap B_2(0)$ and that $u = 0$ in $\partial\Omega_1 \cap B_2(0)$. Then there exists $\alpha \ge 0$ such that

$$u(x) = \alpha x_1^+ + o(|x|) \qquad \text{as } x \to 0,\ x \in \Omega_1.$$

Proof. By assumption, we have that

$$\Omega_1 \cap B_1 \subset B_1^+.$$

Then we can extend u as the zero function on $B_1^+ \setminus \Omega_1$ so that it is a Lipschitz, non-negative solution to

$$F(D^2 u) \ge -\|f\|_\infty \qquad \text{in } B_1^+.$$

Reasoning in a similar way as in Lemma 2.2, we define $\alpha_k = \inf\{\beta : u(x) \le \beta x_1 \text{ in } B_{1/k}\}$, $k \ge 1$. Then $0 \le \alpha_k < +\infty$ (u is Lipschitz), and $\alpha_k \searrow \alpha \ge 0$, with $u(x) \le \alpha x_1 + o(|x|)$ in B_1^+. Again, let us suppose by contradiction that

$$u(x_k) \le \alpha x_{1,k} - \delta_0 r_k.$$

where $\delta_0 > 0$ and $\{x_k\}_k = \{(x_{1,k}, x_{2,k}, ..., x_{n,k})\}_k \subset B_1^+$, is such that $|x_k| = r_k \to 0$. As before, such inequality propagates by Lipschitz continuity:

$$u(x) \le \alpha x_1 - \frac{\delta_0}{2} r_k \le \alpha_k x_1 - \frac{\delta_0}{2} r_k \qquad \text{in } \left\{x : |x| = r_k, |x - x_k| \le \frac{\delta_0 r_k}{4L}\right\}.$$

Defining the elliptic, homogeneous operator $F^*(M) = -F(-M)$, we have that the functions

$$u_k(x) = \alpha_k x_1 - \frac{u(r_k x)}{r_k}$$

solve

$$F^*(D^2 u_k(x)) \le r_k \|f\|_\infty \qquad \text{in } B_1^+,$$

with $u_k(x) \ge 0$ on ∂B_1^+ and $u_k \ge \delta_0/2$ in $E_k = \{x : x \in \partial B_1^+, x_1 > 0, |x - x_k| \le \frac{\delta_0}{4L}\}$. As a consequence, a contradiction can be obtained by reasoning as in Lemma 2.2. □

Lemma 2.4. *Let Ω_1 be bounded domain with $0 \in \partial\Omega_1$ and*

$$\text{either } B_1(e_1) \subset \Omega_1 \qquad \text{or } \overline{B}_1(-e_1) \cap \overline{\Omega}_1 = \{0\}.$$

Let u be non-negative and Lipschitz in $\overline{\Omega}_1 \cap B_2(0)$, such that $F(D^2u) = f$ in $\Omega_1 \cap B_2(0)$ and that $u = 0$ in $\partial\Omega_1 \cap B_2(0)$. Then there exists $\alpha \geq 0$ such that

$$u(x) = \alpha x_1 + o(|x|)$$

as $x \to 0$ and either $x \in B_1(e_1)$ or $x \in \Omega$.

Proof. In both cases, we use the smooth change of variable

$$\begin{cases} y_1 = x_1 - \psi(x') \\ y' = x', \end{cases}$$

where $\psi(x')$ is smooth, with $\psi(x') = 1 - \sqrt{1 - |x'|^2}$ for $|x'|$ small. Then, by direct calculations, the function $\tilde{u}(y) = u(y_1 + \psi(y'), y')$ satisfies

$$\tilde{F}(D^2\tilde{u}, \nabla\tilde{u}, y') = F(D^2u),$$

where $\tilde{F}$ is still a uniformly elliptic operator. As a consequence the lemma follows by arguing as in the proofs of Lemmas 2.2, 2.3, with minor changes. □

We conclude this section by providing a uniform estimate from below of the development coefficient α, in case the touching ball is inside the domain.

Lemma 2.5. *Let $u \in C(\overline{B_r(re_1)})$, $r \leq 1$, be such that*

$$\begin{cases} F(D^2u) = f & \text{in } B_r(re_1), \\ u \geq 0, \\ u(0) = 0. \end{cases}$$

Moreover, assume that $u(re_1) \geq Cr$, for some $C > 0$. Then

$$u(x) \geq \alpha x_1 + o(|x|), \qquad \text{where } \alpha \geq c_1 \frac{u(re_1)}{r} - c_2 r \|f\|_\infty,$$

as $x \to 0$, for $r \leq \bar{r}$, where c_1, c_2 and $\bar{r}$ only depend on λ, Λ, n.

Proof. Let

$$u_r(x) = \frac{u(r(e_1 + x))}{r}, \qquad x \in B_1(0).$$

Then

$$\begin{cases} F(D^2u_r) = rf & \text{in } B_1, \\ u_r \geq 0, \\ u_r(-e_1) = 0. \end{cases}$$

By Harnack's inequality [5, Theorem 4.3] we have that

$$\inf_{\partial B_{1/2}} u_r \geq c(u_r(0) - r\|f\|_\infty) =: a,$$

where c only depends on λ, Λ, n. We are in a position to apply Lemma A.2, which provides

$$u_r(x) \geq \alpha(x_1 + 1) + o(|x + e_1|), \quad \text{with } \alpha \geq c_1 a - c_2 r\|f\|_\infty = c_1' u_r(0) - c_2' r\|f\|_\infty,$$

as $x \to -e_1$, and the lemma follows. □

Remark 2.6. Notice that the above results can be applied both to $F(D^2u^+) = f_1$ in $\Omega^+(u)$ and to $F(D^2u^-) = f_2\chi_{\{u<0\}}$ in $\Omega^-(w)$.

3. The function u^+ is Lipschitz continuous

In this section we adapt the strategy developed in [3], in order to show that u^+ is locally Lipschitz. To this aim we need to use the following almost-monotonicity formula, provided in [6, 14].

Proposition 3.1. *Let u_i, $i = 1, 2$ be continuous, non-negative functions in B_1, satisfying $\Delta u_i \geq -1$, $u_1 \cdot u_2 = 0$ in B_1. Then there exist universal constants C_0 and r_0, such that the functional*

$$\Phi(r) := \frac{1}{r^4} \int_{B_r} \frac{|\nabla u_1|^2}{r^{n-2}} \int_{B_r} \frac{|\nabla u_2|^2}{r^{n-2}}$$

satisfies, for $0 < r \leq r_0$,

$$\Phi(r) \leq C_0 \left(1 + \|u_1\|^2_{L^2(B_1)} + \|u_2\|^2_{L^2(B_1)}\right)^2.$$

Lemma 3.2. *Let $w \in \mathcal{S}$. There exists $\tilde{w} \in \mathcal{S}$ such that*

1. *$F(D^2\tilde{w}) = f_1$ in $\Omega^+(\tilde{w})$,*
2. *$\tilde{w} \leq w$, $\tilde{w}^- = w^-$, and*
3. *$\tilde{w} \geq \underline{u}$ in Ω.*

Proof. Let $w \in \mathcal{S}$ and $\Omega^+ = \Omega^+(w)$. We define

$$\mathcal{V} := \{v \in C(\overline{\Omega^+}) : F(D^2v) \geq f_1\chi_{\{v>0\}} \text{ in } \Omega^+,\ v \geq 0 \text{ in } \Omega^+,\ v = w \text{ on } \partial\Omega^+\}$$

and

$$\tilde{w}(x) := \begin{cases} \sup\{v(x) : v \in \mathcal{V}\} & x \in \Omega^+ \\ w(x) & \text{elsewhere.} \end{cases}$$

Since $\underline{u}^+ \in \mathcal{V}$ we obtain that $\mathcal{V}$ is not empty and that $\underline{u} \leq \tilde{w} \leq w$. Moreover $\tilde{w}$ is a solution of the obstacle problem (see [13])

$$\begin{cases} F(D^2\tilde{w}) = f_1 & \text{in } \{\tilde{w} > 0\} \\ \tilde{w} \geq 0 & \text{in } \Omega^+ \\ \tilde{w} = w & \text{on } \partial\Omega^+. \end{cases}$$

In particular, regularity results for the obstacle problem for fully nonlinear equations imply that $\tilde{w}$ is $C^{1,1}$ in Ω^+ (see [13]). To conclude that $\tilde{w} \in \mathcal{S}$, we need to show that the free boundary conditions in Definition 1.2 hold true. Let $x_0 \in \mathcal{F}(\tilde{w})$: if $x_0 \in \mathcal{F}(w)$ too, then the free boundary condition follows from the fact that $\tilde{w} \leq w$; otherwise, $x_0 \in \Omega^+$ is an interior zero of $\tilde{w}$, and the free boundary condition follows by the $C^{1,1}$ regularity of $\tilde{w}$. □

Lemma 3.3. *Let $w \in \mathcal{S}$ with $F(D^2 w) = f_1$ in $\Omega^+(w)$, and let $x_0 \in \mathcal{F}(w)$ be regular from the right. Then u admits developments*

$$w^+(x) = \alpha \langle x - x_0, \nu\rangle^+ + o(|x - x_0|),$$
$$w^-(x) \geq \beta \langle x - x_0, \nu\rangle^- + o(|x - x_0|),$$

with $0 \leq \alpha \leq G(\beta, x_0, \nu(x_0))$*, and*

$$\alpha\beta \leq C_0 \left(1 + \|u_1\|_2^2 + \|u_2\|_2^2\right).$$

Proof. If x_0 is not regular from the left, then by definition of $\mathcal{S}$ the asymptotic developments hold with $\alpha = \beta = 0$ and there is nothing to prove. On the other hand, if x_0 is also regular from the left, then the asymptotic developments and the free boundary condition hold true by definition of $\mathcal{S}$ and by Lemma 2.4. Also in this case, if $\alpha = 0$ then there is nothing else to prove, thus we are left to deal with the case $\alpha > 0$.

Reasoning as in [3, Lemma 3], see also [19, Lemma 4.3], one can show that

$$\Phi(r) \geq C(n)(\alpha + o(1))^2(\beta + o(1))^2 \tag{3.1}$$

(recall that $\Phi(r)$ is defined in Proposition 3.1). On the other hand, since F is concave,

$$\Delta w^\pm \geq -c\|f\|_\infty.$$

The conclusion follows by combining Proposition 3.1 with (3.1). □

Proposition 3.4. *For every $D \subset\subset \Omega$ there exists a constant L_D, depending only on D, G, $\underline{u}$ and $\mathcal{S}$, such that*

$$\frac{|w^+(x) - w^+(y)|}{|x - y|} \leq L_D$$

for every $x, y \in D$, $x \neq y$, and for every $w \in \mathcal{S}$ with $F(D^2 w) = f_1$ in $\Omega^+(w)$.

Proof. Let $x_0 \in \Omega^+(w) \cap D$ such that

$$r := \operatorname{dist}(x_0, \mathcal{F}(w)) < \frac{1}{2} \operatorname{dist}(\overline{D}, \partial\Omega).$$

We will show that there exists $M > 0$, not depending on w, such that

$$\frac{w(x_0)}{r} \leq M,$$

and the lemma will follow by Schauder estimates and Harnack inequality. By contradiction, let M large to be fixed and let as assume that

$$\frac{w(x_0)}{r} > M.$$

Then Lemma 2.5 applies and we obtain

$$w(x) \geq \alpha_M \langle x - x_0, \nu\rangle^+ + o(|x - x_0|),$$

where $\alpha_M = c_1 M - c_2 r\|f\|_\infty > 0$ for M sufficiently large. Then x_0 is regular from the right and Lemma 3.3 applies, with $\alpha_M \leq \alpha \leq G(\beta, x_0, \nu(x_0))$, providing

$$\alpha_M G^{-1}(\alpha_M) \leq C_0 \left(1 + \|u_1\|_2^2 + \|u_2\|_2^2\right),$$

where $G^{-1}(\alpha) := \inf_{x,\nu} G^{-1}(\alpha, x, \nu)$. This provides a contradiction for M sufficiently large. □

Corollary 3.5. *u^+ is locally Lipschitz and satisfies $F(D^2 u) = f_1$ in $\Omega^+(u)$.*

4. The function u is Lipschitz continuous

Now we turn to the Lipschitz continuity of u.

Lemma 4.1. *If* $w_1, w_2 \in \mathcal{S}$ *then* $\min\{w_1, w_2\} \in \mathcal{S}$.

Proof. This follows by standard arguments, see e.g. [9, Lemma 4.1]. □

To prove that u is Lipschitz continuous we use the double replacement technique introduced in [9]. Let $w \in \mathcal{S}$ with $w(x_0) < 0$ and

$$B := B_R(x_0), \qquad \Omega_1 := \Omega^+(w) \setminus \overline{B}.$$

Working on Ω_1, we define

$$\mathcal{V}_1 := \{v : F(D^2 v) \geq f_1 \chi_{\{v>0\}} \text{ in } \Omega_1,\ v \geq 0 \text{ in } \Omega_1,\ v = w \text{ on } \partial\Omega_1 \setminus \partial B,\ v = 0 \text{ on } \partial B\}$$

(which is non empty, for R sufficiently small, as $\underline{u}^+ \in \mathcal{V}_1$). Then

$$w_1 := \sup \mathcal{V}_1$$

solves the obstacle problem (see [13])

$$F(D^2 w_1) \geq f_1 \text{ in } \{w_1 > 0\}, \quad w_1 \geq 0 \text{ in } \Omega_1. \tag{4.1}$$

On the other hand, working on B, let

$$\mathcal{V}_2 := \{v : F(D^2 v) \geq f_2 \chi_{\{v>0\}} \text{ in } B,\ v \geq 0 \text{ in } B,\ v = w^- \text{ on } \partial B\}$$

(which is non empty, as $w^- \in \mathcal{V}_2$). Again,

$$w_2 := \sup \mathcal{V}_2$$

solves the obstacle problem

$$F(D^2 w_2) \geq f_1 \text{ in } \{w_2 > 0\}, \quad w_2 \geq 0 \text{ in } B. \tag{4.2}$$

Under the above notation, the double replacement $\tilde{w}$ of w, relative to B, is defined as

$$\tilde{w} := \begin{cases} w_1 & \text{in } \Omega_1 \\ -w_2 & \text{in } B \\ w & \text{otherwise.} \end{cases}$$

Lemma 4.2. *Let* $w \in \mathcal{S}$ *with* $w(x_0) = -h < 0$*. There exists* $>_\varepsilon 0$ *(depending on* $\operatorname{dist}(x_0, \partial\Omega)$ *and* $\underline{u}$*) such that:*

1. *the double replacement* $\tilde{w}$ *of* w*, relative to* $B_{h_\varepsilon}(x_0)$*, satisfies* $\underline{u} \leq \tilde{w} \leq w$ *in* Ω*;*

2. $\tilde{w} < 0$ *and* $F(D^2\tilde{w}) = f_2$ *in* $B_{h_\varepsilon}(x_0)$, *with*

$$|\nabla\tilde{w}| \leq \frac{C}{\varepsilon} + C_\varepsilon \|f_2\|_\infty \quad in\ B_{h_\varepsilon/2}(x_0);$$

3. $\tilde{w} \in \mathcal{S}$.

Proof. The inequality $w_1 \leq w$ in Ω_1 follows by the maximum principle, while $w_2 \geq -w$ in B because $w^- \in \mathcal{V}_2$. On the other hand, provided ε is sufficiently small (depending on the Lipschitz constant of $\underline{u}$), we have that $\underline{u} < 0$ in $B := B_{h_\varepsilon}(x_0)$ and $u^* \in \mathcal{V}_1$, so that $w_1 \geq \underline{u}$; finally, by the maximum principle in $\{w_2 > 0\}$, also $-w_2 \geq \underline{u}$, and part 1. follows.

Turning to part 2., assume by contradiction that $\partial\{w_2 > 0\} \cap B_{h_\varepsilon}(x_0) \neq \emptyset$. Then, by the regularity properties of the obstacle problem (4.2) (see [13]), we obtain that

$$w_2(x_0) \leq C(h_\varepsilon)^2.$$

Since $-w_2(x_0) \leq w(x_0) = -h$, we obtain a contradiction for ε sufficiently small. Then $w_2 > 0$ in $B_{h_\varepsilon}(x_0)$, w_2 solves the equation by (4.2), and the remaining part of 2. follows by standard Schauder estimates and Harnack inequality.

Coming to part 3., the fact that $\tilde{w}$ satisfies (a) in Definition 1.2 follows by equations (4.1), (4.2) and by part 2. above, and we are left to check the free boundary conditions. For $\bar{x} \in \mathcal{F}(\tilde{w})$, three possibilities may occur. If $\bar{x} \in \mathcal{F}(w)$ then, since $\tilde{w} \leq w$, then $\tilde{w}$ has the correct asymptotic behavior both when $\bar{x}$ is regular and when it is not (recall that $G(0,\cdot,\cdot) > 0$. If $\bar{x} \in \partial\{w_1 > 0\} \cap \Omega_1$, then we can use again the regularity of the obstacle problem (4.1) to obtain the correct asymptotic behavior. We are left to the final case, when $\bar{x} \in \partial B \cap \Omega^+(w)$. By Proposition 3.4, let us denote with L the Lipschitz constant of w in $B_{\mathrm{dist}(x_0,\partial\Omega)/2}(x_0)$. Then

$$\tilde{w} \leq w^+ \leq Lh_\varepsilon \qquad \text{in } B_{2h_\varepsilon}(x_0).$$

Defining

$$\tilde{w}_\varepsilon(x) := \frac{\tilde{w}(x_0 + \varepsilon h x)}{\varepsilon h},$$

we have that

$$\begin{cases} F(D^2\tilde{w}_\varepsilon^+) = \varepsilon h f_1 & \text{in } (B_2 \setminus \overline{B}_1) \cap \Omega^+(w_\varepsilon) \\ \tilde{w}_\varepsilon^+ \leq L & \text{on } \partial B_2 \\ \tilde{w}_\varepsilon^+ \leq 0 & \text{on } \partial B_1 \cap \partial\Omega^+(w_\varepsilon). \end{cases}$$

Then Lemma A.1 applies, yielding

$$\tilde{w}_\varepsilon^+ \leq \alpha\langle x - \bar{x}_\varepsilon, \nu(\bar{x}_\varepsilon)\rangle^+ + o(|x - \bar{x}_\varepsilon|),$$

where

$$\alpha \leq c_1 L + c_2 \varepsilon h \|f_1\|_\infty, \qquad \bar{x}_\varepsilon := \frac{\bar{x} - x_0}{\varepsilon h},$$

for universal c_1, c_2. Going back to $\tilde{w}$ we obtain

$$\tilde{w}^+ \leq \alpha\langle x - \bar{x}, \nu\rangle^+ + o(|x - \bar{x}|), \qquad \alpha \leq \bar{L} \tag{4.3}$$

where $\nu = (\bar{x} - x_0)/|\bar{x} - x_0|$.

On the other hand, we can apply Lemma 2.5 to $(-w_2)_\varepsilon$, obtaining

$$\tilde{w}_\varepsilon^- = -(w_2)_\varepsilon \geq \beta\langle x - \bar{x}_\varepsilon, \nu(\bar{x}_\varepsilon)\rangle^+ + o(|x - \bar{x}_\varepsilon|),$$

where

$$\beta \geq \frac{c_1'}{\varepsilon} - c_2'\varepsilon h\|f_1\|_\infty,$$

for universal c_1', c_2', and thus

$$\tilde{w}^- \geq \beta\langle x - \bar{x}, \nu\rangle^- + o(|x - \bar{x}|), \qquad \beta \geq \frac{\bar{c}}{\varepsilon}. \tag{4.4}$$

Comparing (4.3) and (4.4) we have that, choosing ε small so that

$$\bar{L} < \inf_{x,\nu} G(\bar{c}/\varepsilon, x, \nu),$$

the free boundary condition holds true. □

Corollary 4.3. *Let $u(x_0) = -h < 0$. There exist an non-increasing sequence $\{\tilde{w}_k\} \subset \mathcal{S}$, $\tilde{w}_k \geq \underline{u}$, and $\varepsilon > 0$, depending on $\mathrm{dist}(x_0, \partial\Omega)$ and $\underline{u}$, such that the following hold:*

1. *$\tilde{w}_k(x_0) \searrow u(x_0)$;*
2. *$\tilde{w}_k < 0$ and $F(D^2\tilde{w}_k^-) = f_2$ in $B_{\varepsilon h}(x_0)$;*
3. *the sequence $\{\tilde{w}_k\}$ is uniformly Lipschitz in $B_{\varepsilon h/2}(x_0)$, with Lipschitz constant L_0 depending on $\mathrm{dist}(x_0, \partial\Omega)$.*
4. *$\tilde{w}_k \searrow u$ uniformly on $B_{h\epsilon/4}$*

Proof. Let $u(x_0) = -h < 0$, $\{w_k\} \subset \mathcal{S}$ be such that $w_k \searrow u$ in some neighborhood of x_0 and $\{\tilde{w}_k\} \subset \mathcal{S}$ be the corresponding double replacements, as in Lemma 4.2. Then first three points are direct consequence of the lemma above, and we are left to prove that $\tilde{w}_k \searrow u$ uniformly on $B_{h\epsilon/4}$. By equicontinuity, $\tilde{w}_k \to \tilde{w}$ in $B_{\epsilon h/2}(x_0)$, and suppose by contradiction that $\tilde{w}(x_1) > u(x_1)$ for some $x_1 \in B_{\epsilon h/4}(x_0)$. Then consider a new sequence $\{v_k\}_k$ converging to u at x_1 and define $\{\tilde{u}_k\}_k$ as the double replacement of $\{\min\{\tilde{v}_k, \tilde{w}_k\}\}_k$ in $B_{\epsilon h/2}(x_0)$. Then $\tilde{u}_k \to \tilde{u}$, $\tilde{u} \leq \tilde{w}$ in $B_{\epsilon h/2}(x_0)$, $\tilde{u}(x_0) = \tilde{w}(x_0)$ and $\tilde{u}(x_1) < \tilde{w}(x_1)$. Since $F(D^2\tilde{w}) = F(D^2\tilde{u}) = f_2$ in $B_{\epsilon h/2}(x_0)$, this contradicts the strong maximum principle. □

Corollary 4.4. *For any $\overline{D} \subset \Omega$ there exists $\{w_k\}_k \subset \mathcal{S}$ such that $w_k \searrow u$ uniformly in $\overline{D}$. Furthermore, if $\overline{D} \subset \Omega^-(u)$, then each w_k may be taken non-positive in $\overline{D}$.*

Proof. The first part follows from the previous corollary. By compactness, it is enough to prove the second part for balls $\overline{B}_\varepsilon(x_0) \subset \Omega^-(u)$, with ε small. Let $w_k \searrow u$ uniformly in $\overline{B}_{2\varepsilon}(x_0) \subset \Omega^-(u)$, and let

$$w_k^\varepsilon(x) = \frac{w_k(x_0 + \varepsilon x)}{\varepsilon} \searrow u_\varepsilon \qquad \text{in } B_2.$$

Let ϕ be such that

$$\begin{cases} \Delta\phi = -c\|_\varepsilon f\|_\infty & \text{in } B_2 \setminus \overline{B}_1 \\ \phi = a & \text{on } \partial B_2 \\ \phi = 0 & \text{on } \partial B_1, \end{cases}$$

with a and ε positive and sufficiently small so that

$$\nabla\phi(e_1)\cdot e_1 < \inf_{x,\nu} G(0,x,\nu)$$

(this is possible by explicit calculations, see for instance Lemma A.1); notice that this condition insure that ϕ, extended to zero in B_1, is a supersolution in B_2 (when c universal is suitably chosen). Since $u_\varepsilon \le 0$ in $\overline{B}_2$, for k sufficiently large $w_k \le a/2$ in $\overline{B}_2$. Let us define

$$\bar{w}_k^\varepsilon = \begin{cases} \min\{w_k^\varepsilon, \phi\} & \text{in} \quad \overline{B}_2, \\ w^\varepsilon & \text{otherwise.} \end{cases}$$

Then, by Lemma 4.1, the function

$$\bar{w}_k(x) = \varepsilon \bar{w}_k^\varepsilon\left(\frac{x-x_0}{\varepsilon}\right)$$

satisfies $\bar{w}_k \in \mathcal{S}$, $\bar{w}_k \le 0$ in $\overline{B}_\varepsilon(x_0)$ and $\bar{w}_k \searrow u$ in $\overline{B}_\varepsilon(x_0)$, as required. □

Corollary 4.5. *u is locally Lipschitz in Ω, continuous in $\overline{\Omega}$, $u = g$ on $\partial\Omega$. Moreover u solves*

$$\mathcal{L}u = f_2\chi_{\{u<0\}}, \quad \text{in } \Omega^-(u).$$

5. The function u^+ is non-degenerate

In this section we will show that u^+ is non-degenerate, in the sense of the following result.

Lemma 5.1. *Let $x_0 \in \mathcal{F}(u)$ and let A be a connected component of $\Omega^+(u)\cap(B_r(x_0)\setminus\overline{B}_{r/2}(x_0))$ satisfying*

$$\overline{A}\cap\partial B_{r/2}(x_0) \ne \emptyset, \quad \overline{A}\cap\partial B_r(x_0)\ne\emptyset,$$

for $r \le r_0$ universal. Then

$$\sup_A u \ge Cr.$$

Moreover

$$\frac{|A\cap B_r(x_0)|}{|B_r(x_0)|} \ge C > 0,$$

where all the constants C depend on $d(x,\partial\Omega)$ and on $\underline{u}$.

Corollary 5.2. *$\mathcal{F}(w_k) \to \mathcal{F}(u)$ locally in Hausdorff distance and $\chi_{\{w_k>0\}} \to \chi_{\{u>0\}}$ in L^1_{loc}.*

The proof of the above result will follow by the two following lemmas.

Lemma 5.3. *Let u be a Lipschitz function in $\overline{\Omega}\cap B_1(0)$, with $0\in\partial\Omega$, satisfying*

$$\begin{cases} F(D^2u) = f & \text{in } \Omega\cap B_1 \\ u = 0 & \text{on } \partial\Omega\cap B_1. \end{cases}$$

If there exists $c > 0$ such that

$$u(x) \ge c\,\text{dist}(x,\partial\Omega) \qquad \text{for every } x\in\Omega\cap B_{1/2} \tag{5.1}$$

then there exists a constant $C > 0$ *such that*

$$\sup_{B_r(0)} u \geq Cr,$$

for all $r \leq r_0$ *universal.*

Proof. Let $x_0 \in \Omega \cap B_1$, $\varepsilon = \mathrm{dist}(x_0, \partial\Omega)$, and let L denote the Lipschitz constant of u. Then

$$c\varepsilon \leq u(x_0) \leq L\varepsilon.$$

We will show that, for $\delta > 0$ to be fixed, there exists $x_1 \in B_\varepsilon(x_0)$ such that

$$u(x_1) \geq (1+\delta)u(x_0). \tag{5.2}$$

Then, iterating the procedure, one can conclude as in [9, Lemma 5.1].

Assume by contradiction that (5.2) does not hold. Then, defining the elliptic, homogeneous operator $F^*(M) = -F(-M)$, we infer that

$$v(x) := (1+\delta)u(x_0) - u(x) > 0 \quad \text{in } B_\varepsilon(x_0) \qquad \text{satisfies } F^*(D^2 v) = -f.$$

Let $r(L) = 1 - c/(4L)$; using the Harnack inequality we have that there exists $C(L)$ such that

$$v \leq C(L)(\delta u(x_0) + \varepsilon^2 \|f\|_\infty) \leq \frac{1}{2}u(x_0) \qquad \text{in } \overline{B}_{r(L)\varepsilon}(x_0),$$

provided both δ and ε are sufficiently small (depending on c, L and $\|f\|_\infty$). In terms of u, the previous inequality writes as

$$u \geq \frac{c\varepsilon}{2} \quad \text{in } \overline{B}_{r(L)\varepsilon}(x_0).$$

On the other hand, there exists $y_0 \in \partial B_{r(L)\varepsilon}(x_0)$ such that $\mathrm{dist}(y_0, \partial\Omega) = (1 - r(L))\varepsilon$ and hence

$$\min_{\overline{B}_{r(L)\varepsilon}(x_0)} u \leq u(y_0) \leq L\,\mathrm{dist}(y_0, \partial\Omega) = \frac{c\varepsilon}{4}.$$

This is a contradiction, therefore (5.2) holds true. □

Lemma 5.4. *There exist universal constants* $\bar{r}$, $\bar{C}$ *such that*

$$u(x_0) \geq \bar{C}\,\mathrm{dist}(x_0, \mathcal{F}(u)) \qquad \textit{for every } x_0 \in \{x \in \Omega^+(u) : \mathrm{dist}(x, \mathcal{F}(u)) \leq \bar{r}\}.$$

Proof. Let $x_0 \in \{x \in \Omega^+(u) : \mathrm{dist}(x, \mathcal{F}(u)) \leq \bar{r}\}$, with $\bar{r}$ universal to be specified later, and let $r := \mathrm{dist}(x_0, \mathcal{F}(u))$. We distinguish two cases.

First let us assume that

$$\mathrm{dist}(x_0, \Omega^+(\underline{u})) \leq \frac{r}{2}.$$

In this case, for any $x \in \mathcal{F}(\underline{u})$ we define

$$\rho(x) := \max\{r > 0 : \text{for some } z,\ x \in \partial B_r(z) \text{ and } B_r(z) \subset \Omega^+(\underline{u})\}.$$

Notice that $\rho(x) > 0$ for every x, since any point in $\mathcal{F}(\underline{u})$ is regular from the right by assumption. Thus, recalling that $\underline{u}^+$ has linear growth bounded below by $\inf_{x,\nu} G(0, x, \nu)$, and noticing that $B_{3r/4}(x_0) \cap \Omega^+(\underline{u})$ contains a ball of radius comparable with r (at least for a suitable choice of $\bar{r}$):

$$\sup_{B_{3r/4}(x_0)} u^+ \geq \sup_{B_{3r/4}(x_0)} \underline{u}^+ \geq \bar{C}r,$$

where $\bar{C}$ only depends on $\underline{u}$.

On the other hand, in case

$$\mathrm{dist}(x_0, \Omega^+(\underline{u})) \geq \frac{r}{2},$$

we have $\underline{u} \leq 0$ in $B_{r/2}(x_0)$. By Corollary 4.4 we can find $\{w_k\}_k \subset \mathcal{S}$ converging uniformly to u on some $D \supset B_r(x_0)$. By scaling

$$u_r(x) = \frac{u(x_0 + rx)}{r}, \qquad w_k^r(x) = \frac{w_k(x_0 + rx)}{r},$$

we need to find $\bar{C}$ universal such that $u_r(0) \geq \bar{C}$. Let us assume by contradiction that

$$u_r(0) < \bar{C}.$$

Then by Harnack inequality

$$u_r \leq C(\bar{C} + r\|f_1\|_\infty) \qquad \text{in } B_{1/2}$$

and, for k sufficiently large,

$$w_k^r \leq C'(\bar{C} + r\|f_1\|_\infty) \qquad \text{in } B_{1/2}.$$

Now, reasoning as in the proof of Corollary 4.4, let ϕ be such that

$$\begin{cases} \Delta\phi = -cr\|f\|_\infty & \text{in } B_{1/2} \setminus \overline{B}_{1/4} \\ \phi = a & \text{on } \partial B_{1/2} \\ \phi = 0 & \text{on } \partial B_{1/4}, \end{cases}$$

with a and r positive and sufficiently small so that $\nabla\phi(e_1/4) \cdot e_1 < \inf_{x,\nu} G(0, x, \nu)$, in such a way that ϕ, extended to zero in $B_{1/4}$, is a supersolution in $B_{1/2}$. Then, choosing $\bar{C} < (a - r\|f_1\|_\infty)/C'$ we obtain that $w_k^r < \phi$ on $\partial B_{1/2}$ and then the functions

$$\bar{w}_k^r = \begin{cases} 0 & \text{in } \overline{B}_{1/4}, \\ \min\{w_k^r, \phi\} & \text{in } \overline{B}_{1/2} \setminus B_{1/4}, \\ w_k^r & \text{otherwise} \end{cases}$$

are continuous, while

$$\bar{w}_k(x) = r\bar{w}_k^r\left(\frac{x - x_0}{r}\right)$$

satisfy $\bar{w}_k \in \mathcal{S}$, $\bar{w}_k \equiv 0$ in $\overline{B}_{r/4}(x_0)$. This is in contradiction with the fact that $u(x_0) > 0$, and the lemma follows. □

6. The function u is a supersolution

This section is devoted to the proof that u satisfies the supersolution condition (i) in Definition 1.1. Thanks to Lemma 2.4 we only need to prove that, whenever u admits asymptotic developments at $x_0 \in \mathcal{F}(u)$, with coefficients α and β, then $\alpha \leq G(\beta, x_0, \nu_{x_0})$. To do that, we need to distinguish the two cases $\beta > 0$ and $\beta = 0$.

Lemma 6.1. *Let $x_0 \in \mathcal{F}(u)$, and*

$$u^+(x) = \alpha \langle x - x_0, \nu\rangle^+ + o(|x - x_0|),$$
$$u^-(x) = \beta \langle x - x_0, \nu\rangle^- + o(|x - x_0|),$$

with

$$\beta > 0.$$

Then $\alpha \leq G(\beta, x_0, \nu_{x_0})$.

Proof. Since $\beta > 0$, then $\mathcal{F}(u)$ is tangent at x_0 to the hyperplane

$$\pi : \langle x - x_0, \nu\rangle = 0$$

in the following sense: for any point $x \in \mathcal{F}(u)$, $\operatorname{dist}(x, \mathcal{F}(u)) = o(|x - x_0|)$. Otherwise we get a contradiction to the asymptotic development of u.

Let $\{w_k\}_k \subset \mathcal{S}$ be uniformly decreasing to u, as in Corollary 4.4. By the non-degeneracy of u^+ we have that, for k large, w_k can not remain strictly positive near x_0. Let $d_k = d_H(\mathcal{F}(w_k), \mathcal{F}(u))$ be the Hausdorff distance between the two free boundaries. In the ball $B_{2\sqrt{d_k}}(x_0)$, $\mathcal{F}(u)$ is contained in a strip parallel to π of width $o\left(\sqrt{d_k}\right)$ and, since $d_k \to 0$, $\mathcal{F}(w_k)$ is contained in a strip S_k of width $d_k + o\left(\sqrt{d_k}\right) = o\left(\sqrt{d_k}\right)$.

Consider now the points $x_k = x_0 - \sqrt{d_k}\nu$ and let $B_k = B_{r_k}(x_k)$ be the largest ball contained in $\Omega^-(w_k)$ with touching point $z_k \in \mathcal{F}(w_k)$. Then $z_k \in S_k$ and, since $w_k \geq u$, from the asymptotic developments of w_k and u we have

$$\beta\sqrt{d_k} + o\left(\sqrt{d_k}\right) = u^-(x_k) \geq w^-(x_k) = \beta_k r_k + o(r_k),$$

since

$$\sqrt{d_k} + o\left(\sqrt{d_k}\right) \leq r_k \leq \sqrt{d_k}.$$

Passing to the limit we infer

$$\limsup \beta_k \leq \beta.$$

Reasoning in the same way on the other side th the points $y_k = x_0 + \sqrt{d_k}$ (and the same z_k, which are regular from the left), we get

$$\alpha \leq \liminf \alpha_k.$$

From $\alpha_k \leq G(\beta_k, z_k, \nu_k)$, where $\nu_k = (x_k - z_k)/|x_k - z_k|$, we get $\alpha \leq G(\beta, x_0, \nu_{x_0})$. □

To treat the case $\beta = 0$ we need the following preliminary lemma.

Lemma 6.2. *Let $v \geq 0$ continous in $B_1(x_0)$ be such that $\Delta v \geq -M$. Let*

$$\Psi_r(x_0, v) = \frac{1}{r^2} \int_{B_r(x_0)} \frac{|\nabla v|^2}{|x - x_0|^{n-2}} dx.$$

Then, for r small,

$$\Psi_r(x_0, v) \leq c(n) \left\{ \sup_{B_{2r}(x_0)} \left(\frac{v}{r} \right)^2 + M \sup_{B_{2r}(x_0)} v \right\}. \tag{6.1}$$

Proof. We may assume $x_0 = 0$ and write $\Psi_r(0, v) = \Psi_r(v)$. Rescale setting $v_r(x) = v(rx)/r$; we have $\Delta v_r \geq -rM$ and

$$\Psi_r(v) = \Psi_1(v_r).$$

Let $\eta \in C_0^\infty(B_2)$, $\eta = 1$ in B_1. Since $2|\nabla v_r|^2 \leq 2rMv_r + \Delta v_r^2$, we have:

$$\begin{aligned} \Psi_1(v_r) &\leq C \int_{B_2} \eta \frac{|\nabla v_r|^2}{|x|^{n-2}} \leq C \int_{B_2} \eta \frac{2Mv_r + \Delta v_r^2}{|x|^{n-2}} \\ &= C \int_{B_2} \left[\frac{2Mv_r}{|x|^{n-2}} + v_r^2 \Delta \left(\frac{\eta}{|x|^{n-2}} \right) \right], \end{aligned}$$

so that

$$\begin{aligned} \Psi_r(v) &= \Psi_1(v_r) \leq c(n) \left(|v_r|^2_{L^\infty(B_2)} + rM |v_r|_{L^\infty(B_2)} \right) \\ &= c(n) \left\{ \sup_{B_{2r}} \left(\frac{v}{r} \right)^2 + M \sup_{B_{2r}} v \right\}, \end{aligned}$$

which is (6.1). □

Lemma 6.3. *Let $x_0 \in \mathcal{F}(u)$, and*

$$\begin{aligned} u^+(x) &= \alpha \langle x - x_0, \nu \rangle^+ + o(|x - x_0|), \\ u^-(x) &= o(|x - x_0|). \end{aligned}$$

Then $\alpha \leq G(\beta, x_0, \nu_{x_0})$.

Proof. As before, let $\{w_k\}_k \subset \mathcal{S}$ be uniformly decreasing to u, with w_k that is not strictly positive near x_0, for k large. The first part of the proof is exactly as in Lemma 6.3 of [9], until equation (6.2) below. For the reader's convenience, we recall such argument here.

For each k we denote with

$$B_{m,k} = B_{\lambda_{m,k}} \left(x_0 + \frac{1}{m} \nu \right)$$

the largest ball centered at $x_0 + \nu/m$ contained in $\Omega^+(w_k)$, touching $\mathcal{F}(w_k)$ at $x_{m,k}$ where $\nu_{m,k}$ is the unit inward normal of $\mathcal{F}(w_k)$ at $x_{m,k}$. Then up to proper subsequences we deduce that

$$\lambda_{m,k} \to \lambda_m, \quad x_{m,k} \to x_m, \quad \nu_{m,k} \to \nu_m$$

and $B_{\lambda_m}(x_0 + \nu/m)$ touches $\mathcal{F}(u)$ at x_m, with unit inward normal ν_m. From the behavior of u^+, we get that

$$|x_m - x_0| = o\left(\frac{1}{m} \right),$$

$$\frac{1}{m} + o\left(\frac{1}{m}\right) \leq \lambda_m \leq \frac{1}{m}$$

and

$$|\nu_m - \nu| = o(1).$$

Now since $w_k \in \mathcal{F}$, near $x_{m,k}$ in $B_{m,k}$:

$$w_k^+ \leq \alpha_{m,k}\langle x - x_{m,k}, \nu_{m,k}\rangle^+ + o(|x - x_{m,k}|)$$

and in $\Omega \setminus B_{m,k}$

$$w_k^- \geq \beta_{m,k}\langle x - x_{m,k}, \nu_{m,k}\rangle^- + o(|x - x_{m,k}|)$$

with

$$0 \leq \alpha_{m,k} \leq G(\beta_{m,k}, x_{m,k}, \nu_{m,k}),$$

(by Lemma 2.5 the touching occurs at a regular point, for m, k large.) We know that

$$w_k^+ \geq u^+ \geq \alpha\langle x - x_0, \nu\rangle^+ + o(|x - x_0|),$$

hence

$$\underline{\alpha}_m = \liminf_{k\to\infty} \alpha_{m,k} \geq \alpha - \epsilon_m$$

and $\epsilon_m \to 0$, as $m \to \infty$. We have to show that

$$\underline{\beta} = \liminf_{m,k\to+\infty} \beta_{m,k} = 0.$$

We assume by contradiction that $\bar{\beta} > 0$. Acting as in [9, Lemma 6.3] we obtain, for r small,

$$(1 + \omega(r))\Phi_r(x_{m,k}, w_k) + C\omega(r) \geq c_n\ \alpha_{m,k}^2\beta_{m,k}^2, \tag{6.2}$$

where

$$\Phi_r\left(x_{m,k}, w_k\right) = \Psi_r\left(x_{m,k}, w_k^+\right)\Psi_r\left(x_{m,k}, w_k^-\right).$$

By concavity we have that $\Delta w_k^{\pm} \geq -M$ where $M = c\min\left(\|f_1\|_\infty, \|f_2\|_\infty\right)$. Lemma 6.2 implies

$$\begin{aligned} c_n\alpha_{m,k}^2\beta_{m,k}^2 &\leq (1 + \omega\left(r\right))\Psi_r\left(x_{m,k}, w_k^+\right)\Psi_r\left(x_{m,k}, w_k^-\right) + C\omega\left(r\right) \\ &\leq c^2\left(n\right)\left(1 + \omega\left(r\right)\right)\left\{\sup_{B_{2r}\left(x_{m,k}\right)}\left(\frac{w_k^+}{r}\right)^2 + M\sup_{B_{2r}\left(x_{m,k}\right)} w_k^+\right\}\times \\ &\quad\times\left\{\sup_{B_{2r}\left(x_{m,k}\right)}\left(\frac{w_k^-}{r}\right)^2 + M\sup_{B_{2r}\left(x_{m,k}\right)} w_k^-\right\} + C\omega\left(r\right) \\ &\leq C_1\left(n, M, L\right)\left\{\sup_{B_{2r}\left(x_{m,k}\right)}\left(\frac{w_k^-}{r}\right)^2 + M\sup_{B_{2r}\left(x_{m,k}\right)} w_k^-\right\} + C\omega\left(r\right), \end{aligned}$$

where L is the uniform Lipschitz constant of $\{w_k^+\}_k$ (recall Lemma 3.4). Taking the $\liminf$ as $m, k \to \infty$ and using the uniform convergence of w_k to u we infer

$$0 < c_n\alpha^2\bar{\beta}^2 \leq C_1\left(n, M, L\right)\left\{\sup_{B_{2r}(x_0)}\left(\frac{u^-}{r}\right)^2 + M\sup_{B_{2r}(x_0)} u^-\right\} + C\omega\left(r\right).$$

Recalling that, by assumption, $u^-(x) = o(|x - x_0|)$ as $x \to x_0$, we have

$$\sup_{B_{2r}(0)} \left(\frac{u^-}{r}\right)^2 = o(1) \qquad \text{as } r \to 0,$$

and we get a contradiction. □

7. The function u is a subsolution

In this section we want to show that u is a subsolution according to Definition 1.1. Note that, if $x_0 \in \mathcal{F}(u)$ is a regular point from the left with touching ball $B \subset \Omega^-(u)$, then near to x_0

$$u^-(x) = \beta\langle x - x_0, \nu\rangle^- + o(|x - x_0|), \quad \beta \geq 0,$$

in B, and

$$u^+(x) = \alpha\langle x - x_0, \nu\rangle^+ + o(|x - x_0|), \quad \alpha \geq 0$$

in $\Omega \backslash B$. Indeed, even if $\beta = 0$, then $\Omega^+(u)$ and $\Omega^-(u)$ are tangent to $\{\langle x - x_0, \nu\rangle = 0\}$ at x_0 since u^+ is non-degenerate. Thus u has a full asymptotic development as in the next lemma. We want to show that $\alpha \geq G(\beta, x_0, \nu)$. We follow closely [3] and [9].

Lemma 7.1. *Assume that near $x_0 \in \mathcal{F}(u)$,*

$$u(x) = \alpha\langle x - x_0, \nu\rangle^+ - \beta\langle x - x_0, \nu\rangle^- + o(|x - x_0|),$$

with $\alpha > 0$, $\beta \geq 0$. Then

$$\alpha \geq G(\beta, x_0, \nu).$$

Proof. Assume by contradiction that $\alpha < G(\beta, x_0, \nu)$. We construct a supersolution $w \in \mathcal{S}$ which is strictly smaller than u at some point, contradicting the minimality of u. Let u_0 be the two-plane solution, i.e.

$$u_0(x) := \lim_{r \to 0} \frac{u(x_0 + rx)}{r} = \alpha\langle x, \nu\rangle^+ - \beta\langle x, \nu\rangle^-.$$

Suppose that $\alpha \leq G(\beta, x_0, \nu) - \delta_0$ with $\delta_0 > 0$. Fix $\zeta = \zeta(\delta_0)$, to be chosen later. By Corollary 4.4, we can find $w_k \in F \searrow u$ locally uniformly and, for r small, k large, the rescaling $w_{k,r}$ satisfies the following conditions:

if $\beta > 0$, then

$$w_{k,r}(x) \leq u_0 + \zeta \min\{\alpha, \beta\} \text{ on } \partial B_1;$$

if $\beta = 0$, then

$$w_{k,r}(x) \leq u_0 + \alpha\zeta \ \text{ on } \partial B_1$$

and

$$w_{k,r}(x) \leq 0, \quad \text{in } \{\langle x, \nu\rangle < -\zeta\} \cap \overline{B}_1.$$

In particular,

$$w_{k,r}(x) \leq u_0(x + \zeta\nu) \quad \text{on } \partial B_1.$$

If $\beta > 0$, let v satisfy

$$\begin{cases} F(D^2 v) = r f_1^r, & \text{in } \{\langle x, \nu\rangle > -\zeta + \epsilon\phi(x)\} \\ F(D^2 v^-) = r f_2^r, & \text{in } \{\langle x, \nu\rangle < -\zeta + \epsilon\phi(x)\} \\ v(x) = 0, & \text{on } \{\langle x, \nu\rangle = -\zeta + \epsilon\phi(x)\} \\ v(x) = u_0(x + \zeta\nu), & \text{on } \partial B_1, \end{cases} \tag{7.1}$$

where $\phi \geq 0$ is a cut-off function, $\phi \equiv 0$ outside $B_{1/2}$, $\phi \equiv 1$ inside $B_{1/4}$.

For $\beta = 0$, replace the second equation with $v = 0$.

Along the new free boundary, $\mathcal{F}(v) = \{\langle x, \nu\rangle = -\zeta + \epsilon\phi(x)\}$ we have the following estimates:

$$|v_\nu^+ - \alpha| \leq c(\varepsilon + \zeta) + Cr, \quad |v_\nu^- - \beta| \leq c(\varepsilon + \zeta) + Cr,$$

with c, C universal.

Indeed,

$$v^+ - \alpha\langle x, \nu\rangle^+$$

is a solution of

$$F(D^2(v - \alpha\langle x, \nu\rangle^+)) = r f_1^r.$$

Thus, by standard $C^{1,\gamma}$ regularity estimates (see [16, Theorem 1.1])

$$|v_\nu^+ - \alpha| \leq C\left(\|v - \alpha\langle x, \nu\rangle^+\|_\infty + [-\zeta + \epsilon\phi]_{1,\gamma} + r\|f_1\|_\infty\right),$$

which gives the desired bound. Similarly, one gets the bound for v_ν^-.

Hence, since $\alpha \leq G(\beta, x_0, \nu(x_0)) - \delta_0$, say for $\varepsilon = 2\zeta$ and ζ, r small depending on δ_0

$$v_\nu^+ < G(v_\nu^-, x_0, \nu),$$

and the function,

$$\bar{w}_k = \begin{cases} \min\{w_k, \lambda v(\frac{x - x_0}{\lambda})\} & \text{in } B_\lambda(x_0), \\ w_k \quad \text{in } \Omega \setminus B_\lambda(x_0), \end{cases}$$

is still in $\mathcal{S}$. However, the set

$$\{\langle x, \nu\rangle \leq -\zeta + \epsilon\phi\}$$

contains a neighborhood of the origin, hence rescaling back $x_0 \in \Omega^-(\bar{w}_k)$. We get a contradiction since $x_0 \in F(u)$ and $\Omega^+(u) \subseteq \Omega^+(\bar{w}_k)$. □

8. Properties of the free boundary

In this section we prove the weak regularity properties of the free boundary. Both statements and proofs are by now rather standard and follows the papers [3] and [9] for problems governed by homogeneous and inhomogeneous divergence equations, respectively. Thus we limit ourselves to the few points in which differences from the previous cases emerge. Denote by $\mathcal{N}_\varepsilon(A)$ an ε-neighborhood of the set A. The following lemma provides a control of the $\mathcal{H}^{n-1}$ measure of $\mathcal{F}(u)$ and implies that $\Omega^+(u)$ is a set of finite perimeter.

Lemma 8.1. *Let u be our Perron solution. Let $x_0 \in \mathcal{F}(u) \cap B_1$. There exists a positive universal $\delta_0 < 1$ such that, for every $0 < \varepsilon < \delta \le \delta_0$, the following quantities are comparable:*

1. $\frac{1}{\varepsilon} |\{0 < u < \varepsilon\} \cap B_\delta(x_0)|$,
2. $\frac{1}{\varepsilon} |\mathcal{N}_\varepsilon(\mathcal{F}(u)) \cap B_\delta(x_0)|$,
3. $N\varepsilon^{n-1}$, *where N is the number of any family of balls of radius ε, with finite overlapping, covering $\mathcal{F}(u) \cap B_\delta(x_0)$,*
4. $\mathcal{H}^{n-1}(\mathcal{F}(u) \cap B_\delta(x_0))$.

Proof. From [3], it is sufficient to prove the following two equivalences:

$$c_1\varepsilon^n \le \int_{B_\varepsilon(x_0)} |\nabla u|^2 \le C_1\varepsilon^n \tag{8.1}$$

and

$$c_3\varepsilon\delta^{n-1} \le \int_{\{0<u<\varepsilon\}\cap B_\delta(x_0)} |\nabla u|^2 \le C_2\varepsilon\delta^{n-1} \tag{8.2}$$

with universal constants c_1, c_2, C_1, C_2.

Since $F\left(D^2u\right) = \inf_\alpha L_\alpha u$ where L_α is a uniformly elliptic operator with constant coefficients and ellipticity constant λ, Λ, we have $L_\alpha u^+ \ge f_1$ in $\Omega^+(u)$. Fix $\alpha = \alpha_0$ and set

$$L_{\alpha_0} = L = \sum_{i,j=1}^{n} a_{ij}\partial_{ij}, \quad A = \left(a_{ij}\right).$$

The upper bound in (8.1) follows by the Lipschitz continuity of u. The lower bound follows from $\sup_{B_\varepsilon(x_0)} u^+ \ge c\varepsilon$, c universal, $\inf_{B_\varepsilon(x_0)} u^+ = 0$, the Lipschitz continuity of u, and the Poincaré inequality (see [1, Lemma 1.15]).

To prove (8.2), rescale by setting

$$u_\delta(x) = \frac{u(x_0 + \delta x)}{\delta}, \; f_1^\delta(x) = f_1(x_0 + \delta x) \quad x \in B_1 = B_1(0).$$

Then $Lu_\delta \ge \delta\, f_1^\delta$ in $\Omega^+\left(u^\delta\right) \cap B_1$. For $0 < \varepsilon < \delta$, let

$$u_{\delta,s,\varepsilon} = u_{s,\varepsilon} := \max\{s/\delta, \min\{u_\delta, \varepsilon/\delta\}\}.$$

We have:

$$\begin{aligned}
-\delta \int_{B_1} f_1^\delta u_{\varepsilon,s} &= -\int_{B_1} u_{\varepsilon,s} L u_\delta^+ \\
&= \int_{B_1} \langle A\nabla u_\delta^+, \nabla u_{\varepsilon,s}^+\rangle dx - \int_{\partial B_1} \langle A\nabla u_\delta^+, \nu\rangle u_{\varepsilon,s} d\mathcal{H}^{n-1} \\
&= \int_{B_1\cap\{0<s/\delta<u_\delta<\varepsilon/\delta\}} \langle A\nabla u_\delta, \nabla u_\delta\rangle dx - \int_{\partial B_1} \langle A\nabla u_\delta^+, \nu\rangle u_{\varepsilon,s} d\mathcal{H}^{n-1}
\end{aligned}$$

since $\nabla u_{\varepsilon,s} = \nabla u_\delta \cdot \chi_{\{s/\delta<u_\delta<\varepsilon/\delta\}}$.

By uniform ellipticity, since u^+ is Lipschitz and f_1 is bounded, we get ($\delta < 1$)

$$\int_{B_1\cap\{0<s/\delta<u_\delta<\varepsilon/\delta\}} |\nabla u_\delta|^2 dx \leq C\frac{\varepsilon}{\delta},$$

with C universal. Letting $s \to 0$ and rescaling back, we obtain the upper bound in (8.2).

For the lower bound, let V be the solution to

$$\begin{cases} LV = -\frac{\chi_{B_\sigma}}{|B_\sigma|}, & \text{in} \quad B_1 \\ V = 0, & \text{on} \quad \partial B_1 \end{cases} \tag{8.3}$$

with σ to be chosen later. By standard estimates, see for example [12], $V \leq C\sigma^{2-n}$ and $-\langle A\nabla V, \nu\rangle \sim C^*$ on ∂B_1, with C^* independent of σ. By Green's formula

$$\int_{B_1} (LV)\frac{u_\delta^+ u_{\varepsilon,0}}{\varepsilon} - \left(L\frac{u_\delta^+ u_{\varepsilon,0}}{\varepsilon}\right)V = \int_{\partial B_1} \frac{u_\delta^+ u_{\varepsilon,0}}{\varepsilon}\langle A\nabla V, \nu\rangle d\mathcal{H}^{n-1} \tag{8.4}$$

since $V = 0$ on ∂B_1. We estimate

$$\delta\left|\int_{B_1} (LV)\frac{u_\delta^+ u_\varepsilon}{\varepsilon} dx\right| = \frac{\delta}{|B_\sigma|}\left|\int_{B_\sigma} \frac{u_\delta^+ u_\varepsilon}{\varepsilon} dx\right| \leq \bar{C}\sigma, \tag{8.5}$$

since u is Lipschitz and $0 \leq u_{\varepsilon,0} \leq \varepsilon/\delta$. From (8.4) and (8.5) and the fact that $\langle A_\delta \nabla V, \nu\rangle \sim -C^*$ on ∂B_1 we deduce that

$$\begin{aligned} \delta \int_{B_1} \left(L\frac{u_\delta^+ u_{\varepsilon,0}}{\varepsilon}\right) V dx &\geq -\bar{C}\sigma - \delta \int_{\partial B_1} \frac{u_\delta^+ u_{\varepsilon,0}}{\varepsilon}\langle A\nabla V, \nu\rangle d\mathcal{H}^{n-1} \\ &\geq -\bar{C}\sigma + C^*\delta \int_{\partial B_1} \frac{u_\delta^+ u_{\varepsilon,0}}{\varepsilon} d\mathcal{H}^{n-1}. \end{aligned}$$

Thus using that u^+ is non-degenerate and choosing σ small enough (universal) we get that ($\delta > \varepsilon$)

$$\delta \int_{B_1} \left(L\frac{u_\delta^+ u_{\varepsilon,0}}{\varepsilon}\right) V dx \geq \tilde{C}. \tag{8.6}$$

On the other hand in $\{0 < u_\delta^+ < \varepsilon/\delta\} \cap B_1$,

$$Lu_\delta^+ u_{\varepsilon,0} = 2\delta u_\varepsilon f_1^\delta + \langle A\nabla u_\delta, \nabla u_\delta\rangle. \tag{8.7}$$

Combining (8.6), (8.7) and using the ellipticity of A we get that

$$\frac{2\delta^2}{\varepsilon}\int_{B_1} u_\varepsilon f_1^\delta V + \frac{\delta\Lambda}{\varepsilon}\int_{B_1} |\nabla u_\delta|^2 V \geq \bar{C}.$$

From the estimate on V we obtain that for δ small enough

$$\frac{\delta}{\varepsilon}\int_{B_1} |\nabla u_\delta|^2 \geq C$$

for some C universal. Rescaling, we obtain the desired lower bound. □

Lemma 8.1 implies that $\Omega^+(u) \cap B_r(x)$, $x \in \mathcal{F}(u)$, is a set of finite perimeter. Next we show that in fact this perimeter is of order r^{n-1}.

Theorem 8.2. *Let u be our Perron solution. Then, the reduced boundary of $\Omega^+(u)$ has positive density in $\mathcal{H}^{n-1}$-measure at any point of $\mathcal{F}(u)$, i.e. for $r < r_0$, r_0 universal,*

$$\mathcal{H}^{n-1}(\mathcal{F}^*(u) \cap B_r(x)) \geq cr^{n-1}$$

for every $x \in \mathcal{F}(u)$.

Proof. The proof follows the lines of Corollary 4 in [3] and Theorem 8.2 in [9]. Let $w_k \in \mathcal{S}$, $w_k \searrow u$ in $\overline{B}_1$ and L as in Lemma 8.1. Then $\Omega^+(u) \subset\subset \Omega^+(w_k)$ and $Lw_k \geq F\left(D^2 w_k\right) = f_1$ in $\Omega^+(u)$. Let $x_0 \in \mathcal{F}(u)$. We rescale by setting

$$u_r(x) = \frac{u(x_0 + rx)}{r}, \quad w_{k,r} = \frac{w_k(x_0 + rx)}{r} \quad x \in B_1.$$

Let V be the solution to (8.3). Since $\nabla w_{k,r}$ is a continuous vector field in $\overline{\Omega_r^+(u_r) \cap B_1}$, we can use it to test for perimeter. We get

$$\begin{aligned} &\int_{B_1 \cap \Omega_r^+(u_r)} (Vrf_1^r - w_{k,r} LV) \leq \int_{B_1 \cap \Omega_r^+(u_r)} (VLw_{kr} - w_{k,r} LV) \\ &= \int_{\mathcal{F}^*(u_r) \cap B_1} (V\langle A\nabla w_{k,r}, \nu\rangle - w_{kr}\langle A\nabla V, \nu\rangle)\, d\mathcal{H}^{n-1} - \int_{\partial B_1 \cap \Omega_r^+(u_r)} w_{kr}\langle A\nabla V, \nu\rangle d\mathcal{H}^{n-1}. \end{aligned} \tag{8.8}$$

Using the estimates for V and the fact that the w_k are uniformly Lipschitz, we get that

$$\left| \int_{\mathcal{F}^*(u_r) \cap B_1} V\langle A\nabla w_{k,r}, \nu\rangle d\mathcal{H}^{n-1} \right| \leq C(\sigma)\mathcal{H}^{n-1}(\mathcal{F}^*(u_r) \cap B_1). \tag{8.9}$$

As in [3] we have, as $k \to \infty$,

$$\int_{\mathcal{F}^*(u_r) \cap B_1} w_{k,r}\langle A\nabla V, \nu\rangle d\mathcal{H}^{n-1} \to 0,$$

$$\int_{\partial B_1 \cap \Omega_r^+(u_r)} w_{kr}\langle A\nabla V, \nu\rangle d\mathcal{H}^{n-1} \to \int_{\partial B_1} u_r^+\langle A\nabla V, \nu\rangle d\mathcal{H}^{n-1}$$

and

$$-\int_{B_1 \cap \Omega_r^+(u_r)} w_{k,r} LV \to \fint_{B_\sigma} u_r^+.$$

Passing to the limit in (8.8) and using all of the above we get

$$\begin{aligned} &\left| r\int_{B_1 \cap \Omega^+(u_r)} Vf_1^r + \fint_{B_\sigma} u_r^+ + \int_{\partial B_1} u_r^+\langle A\nabla V, \nu\rangle d\mathcal{H}^{n-1} \right| \\ &\leq C(\sigma)\mathcal{H}^{n-1}(\mathcal{F}^*(u_r) \cap B_1). \end{aligned} \tag{8.10}$$

Since u is Lipschitz and non-degenerate, for σ small

$$\frac{1}{|B_\sigma|}\int_{B_\sigma} u_r^+ \leq \bar{C}\sigma,$$

and using the estimate for $\langle A\nabla V, \nu\rangle$

$$-\int_{\partial B_1} u_r^+ \langle A\nabla V, \nu\rangle d\mathcal{H}^{n-1} \geq \bar{c} > 0.$$

Also, since f_1^r is bounded,

$$\int_{B_1\cap\Omega_r^+(u_r)} V f_1^r \leq \bar{C}(\sigma).$$

Hence choosing first σ and then r sufficiently small we get that

$$\mathcal{H}^{n-1}(\mathcal{F}^*(u_r)\cap B_1) \geq \tilde{C},$$

$\tilde{C}$ universal. □

A. Some explicit barrier functions

For the reader's convenience we collect here some explicit barrier functions which arise frequently in our arguments. Their proof is based on comparison arguments, together with the well known chain of inequalities

$$\mathcal{P}^-_{\lambda/n,\Lambda} u \leq F(D^2u) \leq c\Delta u, \tag{A.1}$$

where $\mathcal{P}^-_{\lambda/n,\Lambda}$ denotes the lower Pucci operator, and $c = c(\lambda,\Lambda,n) > 0$ since F is concave (see [5] for further details).

Lemma A.1 (Barrier for subsolutions)**.** *Let u satisfy*

$$\begin{cases} F(D^2u) \geq f & \text{in } B_2(0)\setminus \overline{B}_1(0) \\ u \leq a & \text{on } \partial B_2(0) \\ u \leq 0 & \text{on } \partial B_1(0). \end{cases}$$

Then

$$u(x) \leq \alpha(x_1 - 1) + o(|x - e_1|) \qquad \text{where } \alpha \leq c_1 a + c_2\|f\|_\infty,$$

as $x \to e_1$, where the positive constants c_1, c_2 only depend on λ, Λ, n.

Proof. By comparison and (A.1) we infer that $u \geq \phi$ in $B_2 \setminus \overline{B}_1$, where ϕ solves

$$\begin{cases} \Delta\phi = -c\|f\|_\infty & \text{in } B_2\setminus \overline{B}_1 \\ \phi = a & \text{on } \partial B_2 \\ \phi = 0 & \text{on } \partial B_1, \end{cases}$$

for a universal c. Then direct calculations show that, for $n \geq 3$,

$$\phi(x) = A(|x|^2 - 1) + B(|x|^{-n+2} - 1),$$

where

$$A = -\frac{c}{2n}\|f\|_\infty, \qquad B = \frac{3}{1-2^{-n+2}}A - \frac{1}{1-2^{-n+2}}a.$$

Then the Lemma follows by choosing

$$\alpha := \nabla\phi(e_1)\cdot e_1 = 2A - (n-2)B.$$

The proof in dimension $n = 2$ is analogous. □

Lemma A.2 (Barrier for supersolutions)**.** *Let u satisfy*

$$\begin{cases} F(D^2u) \le rf & \text{in } B_2(0)\setminus B_1(0) \\ u \ge 0 & \text{on } \partial B_2(0) \\ u \ge a > 0 & \text{on } \partial B_1(0). \end{cases}$$

Then

$$u(x) \ge \alpha(x_1+2) + o(|x+2e_1|) \qquad \text{where } \alpha \ge c_1 a - c_2 r\|f\|_\infty,$$

as $x \to -2e_1$, whenever $r \le \bar{r}$, where the positive constants c_1, c_2 and $\bar{r}$ only depend on λ, Λ, n.

Proof. By comparison and (A.1) we infer that $u \ge \phi$ in $B_2 \setminus \overline{B}_1$, where ϕ solves

$$\begin{cases} \mathcal{P}^-_{\lambda/n,\Lambda}\phi = r\|f\|_\infty & \text{in } B_2\setminus\overline{B}_1 \\ \phi = 0 & \text{on } \partial B_2 \\ \phi = a & \text{on } \partial B_1. \end{cases}$$

Then direct calculations show that

$$\phi(x) = A(|x|^2 - 4) + B(|x|^{-\gamma} - 2^{-\gamma}), \qquad \text{where } \gamma = \frac{\Lambda n(n-1)}{\lambda} - 1 \ge 1$$

and

$$A = \frac{n}{2(\gamma+2)\lambda}r\|f\|_\infty > 0, \qquad B = \frac{1}{1-2^{-\gamma}}a + \frac{3}{1-2^{-\gamma}}A > 0.$$

To check this, one needs to choose $r \le \bar{r} = \bar{r}(\gamma)$, in such a way that $D^2\phi(x)$ has exactly one positive eigenvalue, for $1 \le |x| \le 2$. Then the Lemma follows by choosing

$$\alpha := \nabla\phi(-2e_1)\cdot e_1 = -4A + \gamma 2^{-\gamma-1}B.$$

□

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Conflict of interest

The authors declare no conflict of interest.

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