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Research article

$C^{1,1}$–smoothness of constrained solutions in the calculus of variations with application to mean field games

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Abstract: We derive necessary optimality conditions for minimizers of regular functionals in the calculus of variations under smooth state constraints. In the literature, this classical problem is widely investigated. The novelty of our result lies in the fact that the presence of state constraints enters the Euler-Lagrange equations as a local feedback, which allows to derive the $C^{1,1}$-smoothness of solutions. As an application, we discuss a constrained Mean Field Games problem, for which our optimality conditions allow to construct Lipschitz relaxed solutions, thus improving an existence result due to the first two authors.

Keywords: necessary conditions; state constraints; mean field games; constrained MFG equilibrium; mild solution

1. Introduction

The centrality of necessary conditions in optimal control is well-known and has originated an immense literature in the fields of optimization and nonsmooth analysis, see, e.g., [3, 16, 17, 29, 33, 35].

In control theory, the celebrated Pontryagin Maximum Principle plays the role of the classical Euler-Lagrange equations in the calculus of variations. In the case of unrestricted state space, such conditions provide Lagrange multipliers—the so-called co-states—in the form of solutions to a suitable adjoint system satisfying a certain transversality condition. Among various applications of necessary optimality conditions is the deduction of further regularity properties for minimizers which, a priori, would just be absolutely continuous.

When state constraints are present, a large body of results provide adaptations of the Pontryagin Principle by introducing appropriate corrections in the adjoint system. The price to pay for such extensions usually consists of reduced regularity for optimal trajectories which, due to constraint reactions, turn out to be just Lipschitz continuous while the associated co-states are of bounded variation, see [20].

The maximum principle under state constraints was first established by Dubovitskii and Milyutin [17] (see also the monograph [35] for different forms of such a result). It may happen that the maximum principle is degenerate and does not yield much information (abnormal maximum principle). As explained in [8, 10, 18, 19] in various contexts, the so-called "inward pointing condition" generally ensures the normality of the maximum principle under state constraints. In our setting (calculus of variation problem, with constraints on positions but not on velocities), this will never be an issue. The maximum principle under state constraints generally involves an adjoint state which is the sum of a $W^{1,1}$ map and a map of bounded variation. This latter mapping may be very irregular and have infinitely many jumps [32], which allows for discontinuities in optimal controls. However, under suitable assumptions (requiring regularity of the data and the affine dynamics with respect to controls), it has been shown that optimal controls and the corresponding adjoint states are continuous, and even Lipschitz continuous: see the seminal work by Hager [22] (in the convex setting) and the subsequent contributions by Malanowski [31] and Galbraith and Vinter [21] (in much more general frameworks). Generalization to less smooth frameworks can also be found in [9, 18].
Let $\Omega \subset \mathbb{R}^n$ be a bounded open domain with C^2 boundary. Let Γ be the metric subspace of $AC(0,T;\mathbb{R}^n)$ defined by

$$\Gamma = \Big\{\gamma \in AC(0,T;\mathbb{R}^n) : \ \gamma(t) \in \overline{\Omega}, \ \ \forall t \in [0,T]\Big\},$$

with the uniform metric. For any $x \in \overline{\Omega}$, we set

$$\Gamma[x] = \{\gamma \in \Gamma : \gamma(0) = x\}.$$

We consider the problem of minimizing the classical functional of the calculus of variations

$$J[\gamma] = \int_0^T f(t,\gamma(t),\dot\gamma(t))\,dt + g(\gamma(T)).$$

Let $U \subset \mathbb{R}^n$ be an open set such that $\overline{\Omega} \subset U$. Given $x \in \overline{\Omega}$, we consider the constrained minimization problem

$$\inf_{\gamma\in\Gamma[x]} J[\gamma], \quad \text{where} \quad J[\gamma] = \Big\{\int_0^T f(t,\gamma(t),\dot\gamma(t))\,dt + g(\gamma(T))\Big\}, \tag{1.1}$$

where $f : [0,T]\times U \times \mathbb{R}^n \to \mathbb{R}$ and $g : U \to \mathbb{R}$. In this paper, we obtain a certain formulation of the necessary optimality conditions for the above problem, which are particularly useful to study the regularity of minimizers. More precisely, given a minimizer $\gamma^\star \in \Gamma[x]$ of (1.1), we prove that there exists a Lipschitz continuous arc $p : [0,T] \to \mathbb{R}^n$ such that

$$\begin{cases} \dot\gamma^\star(t) = -D_pH(t,\gamma^\star(t),p(t)) & \text{for all } t \in [0,T] \\ \dot p(t) = D_xH(t,\gamma^\star(t),p(t)) - \Lambda(t,\gamma^\star,p)\mathbf{1}_{\partial\Omega}(\gamma^\star)Db_\Omega(\gamma^\star(t)) & \text{for a. e. } t \in [0,T] \end{cases} \tag{1.2}$$

where Λ is a bounded continuous function independent of $\gamma^\star$ and p (Theorem 3.1). By the above necessary conditions we derive a sort of maximal regularity, showing that any solutions $\gamma^\star$ is of class $C^{1,1}$. As is customary in this kind of problems, the proof relies on the analysis of suitable penalized functional which has the following form:

$$\inf_{\substack{\gamma \in AC(0,T;\mathbb{R}^n) \\ \gamma(0)=x}} \left\{ \int_0^T \left[f(t,\gamma(t),\dot\gamma(t)) + \frac{1}{\epsilon}\, d_\Omega(\gamma(t)) \right] dt + \frac{1}{\delta}\, d_\Omega(\gamma(T)) + g(\gamma(T)) \right\}.$$

Then, we show that all solutions of the penalized problem remain in $\overline{\Omega}$ (Lemma3.7).

A direct consequence of our necessary conditions is the Lipschitz regularity of the value function associated to (1.1) (Proposition 4.1).

Our interest is also motivated by application to mean field games, as we explain below. Mean field games (MFG) theory has been developed simultaneously by Lasry and Lions ([25, 26, 27]) and by Huang, Malhamé and Caines ([23, 24]) in order to study differential games with an infinite number of rational players in competition. The simplest MFG model leads to systems of partial differential equations involving two unknown functions: the value function u of an optimal control problem of a typical player and the density m of the population of players. In the presence of state constraints, the usual construction of solutions to the MFG system has to be completely revised because the minimizers of the problem lack many of the good properties of the unconstrained case. Such constructions are discussed in detail in [11], where a relaxed notion of solution to the constrained MFG problem was introduced following the so-called Lagrangian formulation (see [4, 5, 6, 7, 13, 14]. In this paper, applying our necessary conditions, we deduce the existence of more regular solutions than those constructed in [11], assuming data to be Lipschitz continuous.

This paper is organised as follows. In Section 2, we introduce the notation and recall preliminary results. In Section 3, we derive necessary conditions for the constrained problem. Moreover, we prove the $C^{1,1}$-smoothness of minimizers. In Section 4, we apply our necessary conditions to obtain the Lipschitz regularity of the value function for the constrained problem. Furthermore, we deduce the existence of more regular constrained MFG equilibria. Finally, in the Appendix, we prove a technical result on limiting subdifferentials.

2. Preliminaries

Throughout this paper we denote by $|\cdot|$ and $\langle\cdot\rangle$, respectively, the Euclidean norm and scalar product in $\mathbb{R}^n$. Let $A \in \mathbb{R}^{n\times n}$ be a matrix. We denote by $\|\cdot\|$ the norm of A defined as follows

$$\|A\| = \max_{x\in\mathbb{R}^n,\ |x|=1} \|Ax\| .$$

For any subset $S \subset \mathbb{R}^n$, $\overline{S}$ stands for its closure, ∂S for its boundary, and S^c for $\mathbb{R}^n \setminus S$. We denote by $\mathbf{1}_S : \mathbb{R}^n \to \{0,1\}$ the characteristic function of S, i.e.,

$$\mathbf{1}_S(x) = \begin{cases} 1 & x \in S, \\ 0 & x \in S^c. \end{cases}$$

We write $AC(0,T;\mathbb{R}^n)$ for the space of all absolutely continuous $\mathbb{R}^n$-valued functions on $[0,T]$, equipped with the uniform norm $\|\gamma\|_\infty = \sup_{[0,T]} |\gamma(t)|$. We observe that $AC(0,T;\mathbb{R}^n)$ is not a Banach

space.
Let U be an open subset of $\mathbb{R}^n$. $C(U)$ is the space of all continuous functions on U and $C_b(U)$ is the space of all bounded continuous functions on U. $C^k(U)$ is the space of all functions $\phi : U \to \mathbb{R}$ that are k-times continuously differentiable. Let $\phi \in C^1(U)$. The gradient vector of ϕ is denoted by $D\phi = (D_{x_1}\phi, \cdots, D_{x_n}\phi)$, where $D_{x_i}\phi = \frac{\partial\phi}{\partial x_i}$. Let $\phi \in C^k(U)$ and let $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n$ be a multiindex. We define $D^\alpha\phi = D^{\alpha_1}_{x_1} \cdots D^{\alpha_n}_{x_n}\phi$. $C^k_b(U)$ is the space of all function $\phi \in C^k(U)$ and such that

$$\|\phi\|_{k,\infty} := \sup_{\substack{x \in U \\ |\alpha| \leq k}} |D^\alpha\phi(x)| < \infty$$

Let Ω be a bounded open subset of $\mathbb{R}^n$ with C^2 boundary. $C^{1,1}(\overline{\Omega})$ is the space of all the functions C^1 in a neighborhood U of Ω and with locally Lipschitz continuous first order derivates in U.
The distance function from $\overline{\Omega}$ is the function $d_\Omega : \mathbb{R}^n \to [0, +\infty[$ defined by

$$d_\Omega(x) := \inf_{y\in\overline{\Omega}} |x - y| \quad (x \in \mathbb{R}^n).$$

We define the oriented boundary distance from $\partial\Omega$ by

$$b_\Omega(x) = d_\Omega(x) - d_{\Omega^c}(x) \quad (x \in \mathbb{R}^n).$$

We recall that, since the boundary of Ω is of class C^2, there exists $\rho_0 > 0$ such that

$$b_\Omega(\cdot) \in C^2_b \text{ on } \Sigma_{\rho_0} = \Big\{y \in B(x, \rho_0) : x \in \partial\Omega\Big\}. \tag{2.1}$$

Throughout the paper, we suppose that ρ_0 is fixed so that (2.1) holds.
Take a continuous function $f : \mathbb{R}^n \to \mathbb{R}$ and a point $x \in \mathbb{R}^n$. A vector $p \in \mathbb{R}^n$ is said to be a proximal subgradient of f at x if there exists $\epsilon > 0$ and $C \geq 0$ such that

$$p \cdot (y - x) \leq f(y) - f(x) + C|y - x|^2 \text{ for all } y \text{ that satisfy } |y - x| \leq \epsilon.$$

The set of all proximal subgradients of f at x is called the proximal subdifferential of f at x and is denoted by $\partial^p f(x)$. A vector $p \in \mathbb{R}^n$ is said to be a limiting subgradient of f at x if there exist sequences $x_i \in \mathbb{R}^n$, $p_i \in \partial^p f(x_i)$ such that $x_i \to x$ and $p_i \to p$ $(i \to \infty)$.
The set of all limiting subgradients of f at x is called the limiting subdifferential and is denoted by $\partial f(x)$. In particular, for the distance function we have the following result.

Lemma 2.1. *Let Ω be a bounded open subset of $\mathbb{R}^n$ with C^2 boundary. Then, for every $x \in \mathbb{R}^n$ it holds*

$$\partial^p d_\Omega(x) = \partial d_\Omega(x) = \begin{cases} Db_\Omega(x) & 0 < b_\Omega(x) < \rho_0, \\ Db_\Omega(x)[0, 1] & x \in \partial\Omega, \\ 0 & x \in \Omega, \end{cases}$$

where ρ_0 is as in (2.1) *and $Db_\Omega(x)[0, 1]$ denotes the set* $\{Db_\Omega(x)\alpha \ : \ \alpha \in [0, 1]\}$.

The proof is given in the Appendix.
Let X be a separable metric space. $C_b(X)$ is the space of all bounded continuous functions on X. We

denote by $\mathscr{B}(X)$ the family of the Borel subset of X and by $\mathcal{P}(X)$ the family of all Borel probability measures on X. The support of $\eta \in \mathcal{P}(X)$, $supp(\eta)$, is the closed set defined by

$$supp(\eta) := \Big\{x \in X : \eta(V) > 0 \text{ for each neighborhood V of } x\Big\}.$$

We say that a sequence $(\eta_i) \subset \mathcal{P}(X)$ is narrowly convergent to $\eta \in \mathcal{P}(X)$ if

$$\lim_{i\to\infty} \int_X f(x)\, d\eta_i(x) = \int_X f(x)\, d\eta \quad \forall f \in C_b(X).$$

We denote by d_1 the Kantorovich-Rubinstein distance on X, which—when X is compact—can be characterized as follows

$$d_1(m, m') = sup\Big\{ \int_X f(x)\, dm(x) - \int_X f(x)\, dm'(x) \,\Big|\, f : X \to \mathbb{R} \text{ is 1-Lipschitz}\Big\}, \tag{2.2}$$

for all $m, m' \in \mathcal{P}(X)$.
Let Ω be a bounded open subset of $\mathbb{R}^n$ with C^2 boundary. We write $\mathrm{Lip}(0, T; \mathcal{P}(\overline{\Omega}))$ for the space of all maps $m : [0, T] \to \mathcal{P}(\overline{\Omega})$ that are Lipschitz continuous with respect to d_1, i.e.,

$$d_1(m(t), m(s)) \le C|t - s|, \quad \forall t, s \in [0, T], \tag{2.3}$$

for some constant $C \ge 0$. We denote by $\mathrm{Lip}(m)$ the smallest constant that verifies (2.3).

3. Necessary conditions and smoothness of minimizers

3.1. Assumptions and main result

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with C^2 boundary. Let Γ be the metric subspace of $AC(0, T; \mathbb{R}^n)$ defined by

$$\Gamma = \Big\{\gamma \in AC(0, T; \mathbb{R}^n) : \ \gamma(t) \in \overline{\Omega}, \ \ \forall t \in [0, T]\Big\}.$$

For any $x \in \overline{\Omega}$, we set

$$\Gamma[x] = \{\gamma \in \Gamma : \gamma(0) = x\}.$$

Let $U \subset \mathbb{R}^n$ be an open set such that $\overline{\Omega} \subset U$. Given $x \in \overline{\Omega}$, we consider the constrained minimization problem

$$\inf_{\gamma\in\Gamma[x]} J[\gamma], \quad \text{where} \quad J[\gamma] = \Big\{ \int_0^T f(t, \gamma(t), \dot\gamma(t))\, dt + g(\gamma(T))\Big\}. \tag{3.1}$$

We denote by $\mathcal{X}[x]$ the set of solutions of (3.1), that is

$$\mathcal{X}[x] = \Big\{\gamma^\star \in \Gamma[x] : J[\gamma^\star] = \inf_{\Gamma[x]} J[\gamma]\Big\}.$$

We assume that $f : [0, T] \times U \times \mathbb{R}^n \to \mathbb{R}$ and $g : U \to \mathbb{R}$ satisfy the following conditions.

(g1) $g \in C^1_b(U)$

(f0) $f \in C([0,T] \times U \times \mathbb{R}^n)$ and for all $t \in [0,T]$ the function $(x,v) \longmapsto f(t,x,v)$ is differentiable. Moreover, $D_x f$, $D_v f$ are continuous on $[0,T] \times U \times \mathbb{R}^n$ and there exists a constant $M \geq 0$ such that

$$|f(t,x,0)| + |D_x f(t,x,0)| + |D_v f(t,x,0)| \leq M \quad \forall\ (t,x) \in [0,T] \times U. \tag{3.2}$$

(f1) For all $t \in [0,T]$ the map $(x,v) \longmapsto D_v f(t,x,v)$ is continuously differentiable and there exists a constant $\mu \geq 1$ such that

$$\frac{I}{\mu} \leq D^2_{vv} f(t,x,v) \leq I\mu, \tag{3.3}$$

$$\|D^2_{vx} f(t,x,v)\| \leq \mu(1+|v|), \tag{3.4}$$

for all $(t,x,v) \in [0,T] \times U \times \mathbb{R}^n$, where I denotes the identity matrix.

(f2) For all $(x,v) \in U \times \mathbb{R}^n$ the function $t \longmapsto f(t,x,v)$ and the map $t \longmapsto D_v f(t,x,v)$ are Lipschitz continuous. Moreover, there exists a constant $\kappa \geq 0$ such that

$$|f(t,x,v) - f(s,x,v)| \leq \kappa(1+|v|^2)|t-s| \tag{3.5}$$

$$|D_v f(t,x,v) - D_v f(s,x,v)| \leq \kappa(1+|v|)|t-s| \tag{3.6}$$

for all t, $s \in [0,T]$, $x \in U$, $v \in \mathbb{R}^n$.

Remark 3.1. By classical results in the calculus of variation (see, e.g., [15, Theorem 11.1i]), there exists at least one minimizer of (3.1) in Γ for any fixed point $x \in \overline{\Omega}$.

In the next lemma we show that (f0)-(f2) imply the useful growth conditions for f and for its derivatives.

Lemma 3.1. *Suppose that (f0)-(f2) hold. Then, there exists a positive constant $C(\mu, M)$ depending only on μ and M such that*

$$|D_v f(t,x,v)| \leq C(\mu,M)(1+|v|), \tag{3.7}$$

$$|D_x f(t,x,v)| \leq C(\mu,M)(1+|v|^2), \tag{3.8}$$

$$\frac{1}{4\mu}|v|^2 - C(\mu,M) \leq f(t,x,v) \leq 4\mu|v|^2 + C(\mu,M), \tag{3.9}$$

for all $(t,x,v) \in [0,T] \times U \times \mathbb{R}^n$.

Proof. By (3.2), and by (3.3) one has that

$$\begin{aligned} |D_v f(t,x,v)| &\leq |D_v f(t,x,v) - D_v f(t,x,0)| + |D_v f(t,x,0)| \\ &\leq \int_0^1 \left|D^2_{vv} f(t,x,\tau v)\right||v|\,d\tau + |D_v f(t,x,0)| \leq \mu|v| + M \leq C(\mu,M)(1+|v|) \end{aligned}$$

and so (3.7) holds. Furthermore, by (3.2), and by (3.4) we have that

$$|D_x f(t,x,v)| \leq |D_x f(t,x,v) - D_x f(t,x,0)| + |D_x f(t,x,0)| \leq \int_0^1 \left|D^2_{xv} f(t,x,\tau v)\right||v|\,d\tau + M$$

$$\leq \mu(1+|v|)|v| + M \leq C(\mu, M)(1+|v|^2).$$

Therefore, (3.8) holds. Moreover, fixed $v \in \mathbb{R}^n$ there exists a point ξ of the segment with endpoints 0, v such that

$$f(t,x,v) = f(t,x,0) + \langle D_v f(t,x,0), v\rangle + \frac{1}{2}\langle D^2_{vv} f(t,x,\xi)v, v\rangle.$$

By (3.2), (3.3), and by (3.7) we have that

$$\begin{aligned}
&-C(\mu, M) + \frac{1}{4\mu}|v|^2 \leq -M - C(\mu, M)|v| + \frac{1}{2\mu}|v|^2 \leq f(t,x,v) \leq M + C(\mu, M)|v| + \frac{\mu}{2}|v|^2 \\
&\leq C(\mu, M) + 4\mu|v|^2,
\end{aligned}$$

and so (3.9) holds. This completes the proof. □

In the next result we show a special property of the minimizers of (3.1).

Lemma 3.2. *For any $x \in \overline{\Omega}$ and for any $\gamma^\star \in \mathcal{X}[x]$ we have that*

$$\int_0^T \frac{1}{4\mu}|\dot\gamma^\star(t)|^2\,dt \leq K,$$

where

$$K := T\Big(C(\mu, M) + M\Big) + 2\max_U |g(x)|. \tag{3.10}$$

Proof. Let $x \in \overline{\Omega}$ and let $\gamma^\star \in \mathcal{X}[x]$. By comparing the cost of $\gamma^\star$ with the cost of the constant trajectory $\gamma^\star(t) \equiv x$, one has that

$$\begin{aligned}
&\int_0^T f(t, \gamma^\star(t), \dot\gamma^\star(t))\,dt + g(\gamma^\star(T)) \leq \int_0^T f(t,x,0)\,dt + g(x) \\
&\leq T \max_{[0,T]\times U} |f(t,x,0)| + \max_U |g(x)|.
\end{aligned} \tag{3.11}$$

Using (3.2) and (3.9) in (3.11), one has that

$$\int_0^T \frac{1}{4\mu}|\dot\gamma^\star(t)|^2\,dt \leq K,$$

where

$$K := T\Big(C(\mu, M) + M\Big) + 2\max_U |g(x)|.$$

□

We denote by $H : [0,T] \times U \times \mathbb{R}^n \to \mathbb{R}$ the Hamiltonian

$$H(t,x,p) = \sup_{v\in\mathbb{R}^n}\Big\{-\langle p, v\rangle - f(t,x,v)\Big\}, \qquad \forall\,(t,x,p) \in [0,T]\times U \times \mathbb{R}^n.$$

Our assumptions on f imply that H satisfies the following conditions.

(H0) $H \in C([0,T] \times U \times \mathbb{R}^n)$ and for all $t \in [0,T]$ the function $(x,p) \longmapsto H(t,x,p)$ is differentiable. Moreover, D_xH, D_pH are continuous on $[0,T] \times U \times \mathbb{R}^n$ and there exists a constant $M' \geq 0$ such that

$$|H(t,x,0)| + |D_xH(t,x,0)| + |D_pH(t,x,0)| \leq M' \quad \forall\, (t,x) \in [0,T] \times U. \tag{3.12}$$

(H1) For all $t \in [0,T]$ the map $(x,p) \longmapsto D_pH(t,x,p)$ is continuously differentiable and

$$\frac{I}{\mu} \leq D^2_{pp}H(t,x,p) \leq I\mu, \tag{3.13}$$

$$\|D^2_{px}H(t,x,p)\| \leq C(\mu,M')(1+|p|), \tag{3.14}$$

for all $(t,x,p) \in [0,T] \times U \times \mathbb{R}^n$, where μ is the constant given in (f1) and $C(\mu,M')$ depends only on μ and M'.

(H2) For all $(x,p) \in U \times \mathbb{R}^n$ the function $t \longmapsto H(t,x,p)$ and the map $t \longmapsto D_pH(t,x,p)$ are Lipschitz continuous. Moreover

$$|H(t,x,p) - H(s,x,p)| \leq \kappa C(\mu,M')(1+|p|^2)|t-s|, \tag{3.15}$$

$$|D_pH(t,x,p) - D_pH(s,x,p)| \leq \kappa C(\mu,M')(1+|p|)|t-s|, \tag{3.16}$$

for all $t,\ s \in [0,T]$, $x \in U$, $p \in \mathbb{R}^n$, where κ is the constant given in (f2) and $C(\mu,M')$ depends only on μ and M'.

Remark 3.2. Arguing as in Lemma 3.1 we deduce that

$$|D_pH(t,x,p)| \leq C(\mu,M')(1+|p|), \tag{3.17}$$

$$|D_xH(t,x,p)| \leq C(\mu,M')(1+|p|^2), \tag{3.18}$$

$$\frac{1}{4\mu}|p|^2 - C(\mu,M') \leq H(t,x,p) \leq 4\mu|p|^2 + C(\mu,M'), \tag{3.19}$$

for all $(t,x,p) \in [0,T] \times U \times \mathbb{R}^n$ and $C(\mu,M')$ depends only on μ and M'.

Under the above assumptions on Ω, f and g our necessary conditions can be stated as follows.

Theorem 3.1. *For any $x \in \overline{\Omega}$ and any $\gamma^\star \in \mathcal{X}[x]$ the following holds true.*

(i) $\gamma^\star$ is of class $C^{1,1}([0,T];\overline{\Omega})$.

(ii) There exist:

(a) a Lipschitz continuous arc $p : [0,T] \to \mathbb{R}^n$,

(b) a constant $\nu \in \mathbb{R}$ such that

$$0 \leq \nu \leq \max\left\{1, 2\mu \sup_{x\in U}\left|D_pH(T,x,Dg(x))\right|\right\},$$

which satisfy the adjoint system

$$\begin{cases} \dot{\gamma}^\star = -D_pH(t,\gamma^\star,p) & \text{for all } t \in [0,T], \\ \dot{p} = D_xH(t,\gamma^\star,p) - \Lambda(t,\gamma^\star,p)\mathbf{1}_{\partial\Omega}(\gamma^\star)Db_\Omega(\gamma^\star) & \text{for a.e. } t \in [0,T], \end{cases} \tag{3.20}$$

and the transversality condition

$$p(T) = Dg(\gamma^\star(T)) + \nu Db_\Omega(\gamma^\star(T))\mathbf{1}_{\partial\Omega}(\gamma^\star(T)),$$

where $\Lambda : [0,T] \times \Sigma_{\rho_0} \times \mathbb{R}^n \to \mathbb{R}$ *is a bounded continuous function independent of* $\gamma^\star$ *and* p.

Moreover,

(iii) the following estimate holds

$$\|\dot\gamma^\star\|_\infty \leq L^\star, \quad \forall \gamma^\star \in \mathcal{X}[x], \tag{3.21}$$

where $L^\star = L^\star(\mu, M', M, \kappa, T, \|Dg\|_\infty, \|g\|_\infty)$.

The (feedback) function Λ in (3.20) can be computed explicitly, see Remark 3.4 below.

3.2. *Proof of Theorem 3.1 for* $U = \mathbb{R}^n$

In this section, we prove Theorem 3.1 in the special case of $U = \mathbb{R}^n$. The proof for a general open set U will be given in the next section.

The proof is based on [12, Theorem 2.1] where the Maximum Principle under state constraints is obtained for a Mayer problem. The reasoning requires several intermediate steps.

Fix $x \in \overline{\Omega}$. The key point is to approximate the constrained problem by penalized problems as follows

$$\inf_{\substack{\gamma \in AC(0,T;\mathbb{R}^n) \\ \gamma(0)=x}} \left\{ \int_0^T \Big[f(t,\gamma(t),\dot\gamma(t)) + \frac{1}{\epsilon}\, d_\Omega(\gamma(t)) \Big] dt + \frac{1}{\delta}\, d_\Omega(\gamma(T)) + g(\gamma(T)) \right\}. \tag{3.22}$$

Then, we will show that, for $\epsilon > 0$ and $\delta \in (0,1]$ small enough, the solutions of the penalized problem remain in $\overline{\Omega}$.

Observe that the Hamiltonian associated with the penalized problem is given by

$$H_\epsilon(t,x,p) = \sup_{v\in\mathbb{R}^n} \Big\{ -\langle p, v\rangle - f(t,x,v) - \frac{1}{\epsilon}\, d_\Omega(x) \Big\} = H(t,x,p) - \frac{1}{\epsilon}\, d_\Omega(x), \tag{3.23}$$

for all $(t,x,p) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^n$.

By classical results in the calculus of variation (see, e.g., [15, Section 11.2]), there exists at least one mimimizer of (3.22) in $AC(0,T;\mathbb{R}^n)$ for any fixed initial point $x \in \overline{\Omega}$. We denote by $\mathcal{X}_{\epsilon,\delta}[x]$ the set of solutions of (3.22).

Remark 3.3. Arguing as in Lemma 3.2 we have that, for any $x \in \overline{\Omega}$, all $\gamma \in \mathcal{X}_{\epsilon,\delta}[x]$ satisfy

$$\int_0^T \Big[\frac{1}{4\mu} |\dot\gamma(t)|^2 + \frac{1}{\epsilon}\, d_\Omega(\gamma(t)) \Big] dt \leq K, \tag{3.24}$$

where K is the constant given in (3.10).

The first step of the proof consists in showing that the solutions of the penalized problem remain in a neighborhood of $\overline{\Omega}$.

Lemma 3.3. *Let* ρ_0 *be such that* (2.1) *holds. For any* $\rho \in (0, \rho_0]$, *there exists* $\epsilon(\rho) > 0$ *such that for all* $\epsilon \in (0, \epsilon(\rho)]$ *and all* $\delta \in (0,1]$ *we have that*

$$\forall\, x \in \overline{\Omega},\ \gamma \in \mathcal{X}_{\epsilon,\delta}[x] \implies \sup_{t\in[0,T]} d_\Omega(\gamma(t)) \leq \rho. \tag{3.25}$$

Proof. We argue by contradiction. Assume that, for some $\rho > 0$, there exist sequences $\{\epsilon_k\}$, $\{\delta_k\}$, $\{t_k\}$, $\{x_k\}$ and $\{\gamma_k\}$ such that

$$\epsilon_k \downarrow 0,\ \delta_k > 0,\ t_k \in [0,T],\ x_k \in \overline{\Omega},\ \gamma_k \in \mathcal{X}_{\epsilon_k,\delta_k}[x_k] \text{ and } d_\Omega(\gamma_k(t_k)) > \rho, \quad \text{for all } k \geq 1.$$

By Remark 3.3, one has that for all $k \geq 1$

$$\int_0^T \Big[\frac{1}{4\mu}|\dot\gamma_k(t)|^2 + \frac{1}{\epsilon_k}\, d_\Omega(\gamma_k(t))\Big]\, dt \leq K,$$

where K is the constant given in (3.10). The above inequality implies that γ_k is $1/2-$Hölder continuous with Hölder constant $(4\mu K)^{1/2}$. Then, by the Lipschitz continuity of d_Ω and the regularity of γ_k, we have that

$$d_\Omega(\gamma_k(t_k)) - d_\Omega(\gamma_k(s)) \leq (4\mu K)^{1/2}|t_k - s|^{1/2}, \quad s \in [0,T].$$

Since $d_\Omega(\gamma_k(t_k)) > \rho$, one has that

$$d_\Omega(\gamma_k(s)) > \rho - (4\mu K)^{1/2}|t_k - s|^{1/2}.$$

Hence, $d_\Omega(\gamma_k(s)) \geq \rho/2$ for all $s \in J := [t_k - \frac{\rho^2}{16\mu K}, t_k + \frac{\rho^2}{16\mu K}] \cap [0,T]$ and all $k \geq 1$. So,

$$K \geq \frac{1}{\epsilon_k}\int_0^T d_\Omega(\gamma_k(t))\, dt \geq \frac{1}{\epsilon_k}\int_J d_\Omega(\gamma_k(t))\, dt \geq \frac{1}{\epsilon_k}\frac{\rho^3}{32\mu K}.$$

But the above inequality contradicts the fact that $\epsilon_k \downarrow 0$. So, (3.25) holds true. □

In the next lemma, we show the necessary conditions for the minimizers of the penalized problem.

Lemma 3.4. *Let $\rho \in (0,\rho_0]$ and let $\epsilon \in (0,\epsilon(\rho)]$, where $\epsilon(\rho)$ is given by Lemma 3.3. Fix $\delta \in (0,1]$, let $x_0 \in \overline{\Omega}$, and let $\gamma \in \mathcal{X}_{\epsilon,\delta}[x_0]$. Then,*

(i) γ is of class $C^{1,1}([0,T];\mathbb{R}^n)$;

(ii) there exists an arc $p \in \mathrm{Lip}(0,T;\mathbb{R}^n)$, a measurable map $\lambda : [0,T] \to [0,1]$, and a constant $\beta \in [0,1]$ such that

$$\begin{cases} \dot\gamma(t) = -D_pH(t,\gamma(t),p(t)), & \textit{for all } t \in [0,T],\\ \dot p(t) = D_xH(t,\gamma(t),p(t)) - \frac{\lambda(t)}{\epsilon}\, Db_\Omega(\gamma(t)), & \textit{for a.e. } t \in [0,T],\\ p(T) = Dg(\gamma(T)) + \frac{\beta}{\delta}\, Db_\Omega(\gamma(T)), \end{cases} \tag{3.26}$$

where

$$\lambda(t) \in \begin{cases} \{0\} & \text{if } \gamma(t) \in \Omega,\\ \{1\} & \text{if } 0 < d_\Omega(\gamma(t)) < \rho,\\ [0,1] & \text{if } \gamma(t) \in \partial\Omega, \end{cases} \tag{3.27}$$

and

$$\beta \in \begin{cases} \{0\} & \text{if } \gamma(T) \in \Omega,\\ \{1\} & \text{if } 0 < d_\Omega(\gamma(T)) < \rho,\\ [0,1] & \text{if } \gamma(T) \in \partial\Omega. \end{cases} \tag{3.28}$$

Moreover,

(iii) the function

$$r(t) := H(t, \gamma(t), p(t)) - \frac{1}{\epsilon}\, d_\Omega(\gamma(t)), \quad \forall t \in [0, T]$$

belongs to $AC(0, T; \mathbb{R})$ *and satisfies*

$$\int_0^T |\dot{r}(t)|\, dt \le \kappa(T + 4\mu K),$$

where K *is the constant given in* (3.10) *and* μ, κ *are the constants in* (3.5) *and* (3.9)*, respectively;*

(iv) the following estimate holds

$$|p(t)|^2 \le 4\mu \left[\frac{1}{\epsilon} d_\Omega(\gamma(t)) + \frac{C_1}{\delta^2}\right], \qquad \forall t \in [0, T], \tag{3.29}$$

where $C_1 = 8\mu + 8\mu\|Dg\|_\infty^2 + 2C(\mu, M') + \kappa(T + 4\mu K)$.

Proof. In order to use the Maximum Principle in the version of [35, Theorem 8.7.1], we rewrite (3.22) as a Mayer problem in a higher dimensional state space. Define $X(t) \in \mathbb{R}^n \times \mathbb{R}$ as

$$X(t) = \begin{pmatrix} \gamma(t) \\ z(t) \end{pmatrix},$$

where $z(t) = \int_0^t \left[f(s, \gamma(s), \dot\gamma(s)) + \frac{1}{\epsilon}\, d_\Omega(\gamma(s))\right] ds$. Then the state equation becomes

$$\begin{cases} \dot{X}(t) = \begin{pmatrix} \dot\gamma(t) \\ \dot z(t) \end{pmatrix} = \mathcal{F}_\epsilon(t, X(t), u(t)), \\ \\ X(0) = \begin{pmatrix} x_0 \\ 0 \end{pmatrix}. \end{cases}$$

where

$$\mathcal{F}_\epsilon(t, X, u) = \begin{pmatrix} u \\ \mathcal{L}_\epsilon(t, x, u) \end{pmatrix}$$

and $\mathcal{L}_\epsilon(t, x, u) = f(t, x, u) + \frac{1}{\epsilon}\, d_\Omega(x)$ for $X = (x, z)$ and $(t, x, z, u) \in [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$. Thus, (3.22) can be written as

$$\min\left\{\Phi(X^u(T)) : u \in L^1\right\}, \tag{3.30}$$

where $\Phi(X) = g(x) + \frac{1}{\delta}\, d_\Omega(x) + z$ for any $X = (x, z) \in \mathbb{R}^n \times \mathbb{R}$. The associated unmaximized Hamiltonian is given by

$$\mathcal{H}_\epsilon(t, X, P, u) = -\langle P, \mathcal{F}_\epsilon(t, X, u)\rangle, \qquad \forall (t, X, P, u) \in [0, T] \times \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \times \mathbb{R}^n.$$

We observe that, as $\gamma(\cdot)$ is minimizer for (3.22), X is minimizer for (3.30). Hence, the hypotheses of [35, Theorem 8.7.1] are satisfied. It follows that there exist $P(\cdot) = (p(\cdot), b(\cdot)) \in AC(0, T; \mathbb{R}^{n+1})$, $r(\cdot) \in AC(0, T; \mathbb{R})$, and $\lambda_0 \ge 0$ such that

(i) $(P, \lambda_0) \not\equiv (0,0)$,

(ii) $(\dot{r}(t), \dot{P}(t)) \in co\ \partial_{t,X}\mathcal{H}_\epsilon(t, X(t), P(t), \dot{\gamma}(t))$, a.e $t \in [0,T]$,

(iii) $P(T) \in \lambda_0 \partial\Phi(X^u(T))$,

(iv) $\mathcal{H}_\epsilon(t, X(t), P(t), \dot{\gamma}(t)) = \max_{u\in\mathbb{R}^n} \mathcal{H}_\epsilon(t, X(t), P(t), u)$, a.e. $t \in [0,T]$,

(v) $\mathcal{H}_\epsilon(t, X(t), P(t), \dot{\gamma}(t)) = r(t)$, a.e. $t \in [0,T]$,

where $\partial_{t,X}\mathcal{H}_\epsilon$ and $\partial\Phi$ denote the limiting subdifferential of $\mathcal{H}_\epsilon$ and Φ with respect to (t, X) and X respectively, while co stands for the closed convex hull. Using the definition of $\mathcal{H}_\epsilon$ we have that

$$(p, b, \lambda_0) \not\equiv (0,0,0), \tag{3.31}$$
$$(\dot{r}(t), \dot{p}(t)) \in -b(t)\ co\ \partial_{t,x}\mathcal{L}_\epsilon(t, \gamma(t), \dot{\gamma}(t)), \tag{3.32}$$
$$\dot{b}(t) = 0, \tag{3.33}$$
$$p(T) \in \lambda_0\ \partial(g + \frac{1}{\delta}\ d_\Omega)(\gamma(T)), \tag{3.34}$$
$$b(T) = \lambda_0, \tag{3.35}$$
$$r(t) = H_\epsilon(t, \gamma(t), p(t)), \tag{3.36}$$

where $\partial_{t,x}\mathcal{L}_\epsilon$ and $\partial(g + \frac{1}{\delta}\ d_\Omega)$ stands for the limiting subdifferential of $\mathcal{L}_\epsilon(\cdot,\cdot,u)$ and $g(\cdot) + \frac{1}{\delta}d_\Omega(\cdot)$. We claim that $\lambda_0 > 0$. Indeed, suppose that $\lambda_0 = 0$. Then $b \equiv 0$ by (3.33) and (3.35). Moreover, $p(T) = 0$ by (3.34). It follows from (3.32) that $p \equiv 0$, which is in contradiction with (3.31). So, $\lambda_0 > 0$ and we may rescale p and b so that $b(t) = \lambda_0 = 1$ for any $t \in [0,T]$.
Note that the Weierstrass Condition (iv) becomes

$$-\langle p(t), \dot{\gamma}(t)\rangle - f(t, \gamma(t), \dot{\gamma}(t)) = \sup_{u\in\mathbb{R}^n}\Big\{-\langle p(t), u\rangle - f(t, \gamma(t), u)\Big\}. \tag{3.37}$$

Therefore

$$\dot{\gamma}(t) = -D_pH(t, \gamma(t), p(t)), \qquad \text{a.e. } t \in [0,T]. \tag{3.38}$$

By Lemma 2.1, by the definition of ρ, and by (3.5) we have that

$$\partial_{t,x}\mathcal{L}_\epsilon(t,x,u) \subset \begin{cases} [-\kappa(1+|u|^2), \kappa(1+|u|^2)] \times D_xf(t,x,u) & \text{if } x \in \Omega, \\ [-\kappa(1+|u|^2), \kappa(1+|u|^2)] \times (D_xf(t,x,u) + \frac{1}{\epsilon}\ Db_\Omega(x)) & \text{if } 0 < b_\Omega(x) < \rho, \\ [-\kappa(1+|u|^2), \kappa(1+|u|^2)] \times (D_xf(t,x,u) + \frac{1}{\epsilon}[0,1]\ Db_\Omega(x)) & \text{if } x \in \partial\Omega. \end{cases}$$

Thus (3.32) implies that there exists $\lambda(t) \in [0,1]$ as in (3.27) such that

$$|\dot{r}(t)| \le \kappa(1 + |\dot{\gamma}(t)|^2), \quad \forall t \in [0,T], \tag{3.39}$$
$$\dot{p}(t) = -D_xf(t, \gamma(t), \dot{\gamma}(t)) - \frac{\lambda(t)}{\epsilon}\ Db_\Omega(\gamma(t)), \quad \text{a.e. } t \in [0,T]. \tag{3.40}$$

Hence, by (3.39), and by Remark 3.3 we conclude that

$$\int_0^T |\dot{r}(t)|\, dt \le \kappa \int_0^T (1 + |\dot{\gamma}(t)|^2)\, dt \le \kappa(T + 4\mu K). \tag{3.41}$$

Moreover, by Lemma 2.1, and by assumption on g, one has that

$$\partial\Big(g + \frac{1}{\delta}\, d_\Omega\Big)(x) \subset \begin{cases} Dg(x) & \text{if } x \in \Omega, \\ Dg(x) + \frac{1}{\delta}\, Db_\Omega(x) & \text{if } 0 < b_\Omega(x) < \rho, \\ Dg(x) + \frac{1}{\delta}[0,1]\, Db_\Omega(x) & \text{if } x \in \partial\Omega. \end{cases}$$

So, by (3.34), there exists $\beta \in [0,1]$ as in (3.28) such that

$$p(T) = Dg(x) + \frac{\beta}{\delta}\, Db_\Omega(x). \tag{3.42}$$

Finally, by well-known properties of the Legendre transform one has that

$$D_x H(t,x,p) = -D_x f(t,x,-D_p H(t,x,p)).$$

So, recalling (3.38), (3.40) can be rewritten as

$$\dot p(t) = D_x H(t,\gamma(t),p(t)) - \frac{\lambda(t)}{\epsilon}\, Db_\Omega(\gamma(t)), \text{ a.e. } t \in [0,T].$$

We have to prove estimate (3.29). Recalling (3.23) and (3.19), we have that

$$H_\epsilon(t,\gamma(t),p(t)) = H(t,\gamma(t),p(t)) - \frac{1}{\epsilon}\, d_\Omega(\gamma(t)) \geq \frac{1}{4\mu}|p(t)|^2 - C(\mu,M') - \frac{1}{\epsilon}\, d_\Omega(\gamma(t)).$$

So, using (3.41) one has that

$$|H_\epsilon(T,\gamma(T),p(T)) - H_\epsilon(t,\gamma(t),p(t))| = |r(T) - r(t)| \leq \int_t^T |\dot r(s)|\, ds \leq \kappa(T + 4\mu K).$$

Moreover, (3.42) implies that $|p(T)| \leq \frac{1}{\delta} + \|Dg\|_\infty$. Therefore, using again (3.19), we obtain

$$\begin{aligned} &\frac{1}{4\mu}|p(t)|^2 - C(\mu,M') - \frac{1}{\epsilon}\, d_\Omega(\gamma(t)) \leq H_\epsilon(t,\gamma(t),p(t)) \leq H_\epsilon(T,\gamma(T),p(T)) + \kappa(T + 4\mu K) \\ &\leq 4\mu|p(T)|^2 + C(\mu,M') + \kappa(T+4\mu K) \leq 8\mu\left[\frac{1}{\delta^2} + \|Dg\|_\infty^2\right] + C(\mu,M') + \kappa(T + 4\mu K). \end{aligned}$$

Hence,

$$|p(t)|^2 \leq 4\mu\left[\frac{1}{\epsilon} d_\Omega(\gamma(t)) + \frac{C_1}{\delta^2}\right],$$

where $C_1 = 8\mu + 8\mu\|Dg\|_\infty^2 + 2C(\mu,M') + \kappa(T + 4\mu K)$. This completes the proof of (3.29).
Finally, by the regularity of H, we have that $p \in \mathrm{Lip}(0,T;\mathbb{R}^n)$. So, $\gamma \in C^{1,1}([0,T];\mathbb{R}^n)$. Observing that the right-hand side of the equality $\dot\gamma(t) = -D_p H(t,\gamma(t),p(t))$ is continuous we conclude that this equality holds for all t in $[0,T]$. □

Lemma 3.5. *Let $\rho \in (0,\rho_0]$ and let $\epsilon \in (0,\epsilon(\rho)]$, where $\epsilon(\rho)$ is given by Lemma 3.3. Fix $\delta \in (0,1]$, let $x \in \overline{\Omega}$, and let $\gamma \in \mathcal{X}_{\epsilon,\delta}[x]$. If $\gamma(\bar t) \notin \partial\Omega$ for some $\bar t \in [0,T]$, then there exists $\tau > 0$ such that $\gamma \in C^2\left(\left(\bar t - \tau, \bar t + \tau\right) \cap [0,T];\mathbb{R}^n\right)$.*

Proof. Let $\gamma \in \mathcal{X}_{\epsilon,\delta}[x]$ and let $\bar{t} \in [0, T]$ be such that $\gamma(\bar{t}) \in \Omega \cup (\mathbb{R}^n \setminus \overline{\Omega})$. If $\gamma(\bar{t}) \in \mathbb{R}^n \setminus \overline{\Omega}$, then there exists $\tau > 0$ such that $\gamma(t) \in \mathbb{R}^n \setminus \overline{\Omega}$ for all $t \in I := (\bar{t} - \tau, \bar{t} + \tau) \cap [0, T]$. By Lemma 3.4, we have that there exists $p \in \mathrm{Lip}(0, T; \mathbb{R}^n)$ such that

$$\begin{aligned} \dot{\gamma}(t) &= -D_p H(t, \gamma(t), p(t)), \\ \dot{p}(t) &= D_x H(t, \gamma(t), p(t)) - \frac{1}{\epsilon} Db_\Omega(\gamma(t)), \end{aligned}$$

for $t \in I$. Since $p(t)$ is Lipschitz continuous for $t \in I$, and $\dot{\gamma}(t) = -D_p H(t, \gamma(t), p(t))$, then γ belongs to $C^1(I; \mathbb{R}^n)$. Moreover, by the regularity of H, b_Ω, p, and γ one has that $\dot{p}(t)$ is continuous for $t \in I$. Then $p \in C^1(I; \mathbb{R}^n)$. Hence, $\dot{\gamma} \in C^1(I; \mathbb{R}^n)$. So, $\gamma \in C^2(I; \mathbb{R}^n)$. Finally, if $\gamma(\bar{t}) \in \Omega$, the conclusion follows by a similar argument. □

In the next two lemmas, we show that, for $\epsilon > 0$ and $\delta \in (0, 1]$ small enough, any solution γ of problem (3.22) belongs to $\overline{\Omega}$ for all $t \in [0, T]$. For this we first establish that, if $\delta \in (0, 1]$ is small enough and $\gamma(T) \notin \overline{\Omega}$, then the function $t \longmapsto b_\Omega(\gamma(t))$ has nonpositive slope at $t = T$. Then we prove that the entire trajectory γ remains in $\overline{\Omega}$ provided ϵ is small enough. Hereafter, we set

$$\epsilon_0 = \epsilon(\rho_0), \quad \text{where } \rho_0 \text{ is such that (2.1) holds and } \epsilon(\cdot) \text{ is given by Lemma 3.3.}$$

Lemma 3.6. *Let*

$$\delta = \frac{1}{2\mu N} \wedge 1, \tag{3.43}$$

where

$$N = \sup_{x \in \mathbb{R}^n} |D_p H(T, x, Dg(x))|.$$

Fix any $\delta_1 \in (0, \delta]$ *and let* $x \in \overline{\Omega}$. *Let* $\epsilon \in (0, \epsilon_0]$. *If* $\gamma \in \mathcal{X}_{\delta_1,\epsilon}[x]$ *is such that* $\gamma(T) \notin \overline{\Omega}$, *then*

$$\langle \dot{\gamma}(T), Db_\Omega(\gamma(T)) \rangle \le 0.$$

Proof. As $\gamma(T) \notin \overline{\Omega}$, by Lemma 3.4 we have that $p(T) = Dg(\gamma(T)) + \frac{1}{\delta} Db_\Omega(\gamma(T))$. Hence,

$$\begin{aligned} &\Big\langle D_p H(T, \gamma(T), p(T)), Db_\Omega(\gamma(T)) \Big\rangle = \Big\langle D_p H(T, \gamma(T), Dg(\gamma(T))), Db_\Omega(\gamma(T)) \Big\rangle \\ &\quad + \Big\langle D_p H(T, \gamma(T), Dg(\gamma(T)) + \frac{1}{\delta} Db_\Omega(\gamma(T))) - D_p H(T, \gamma(T), Dg(\gamma(T))), Db_\Omega(\gamma(T)) \Big\rangle. \end{aligned}$$

Recalling that $D^2_{pp} H(t, x, p) \ge \frac{I}{\mu}$, one has that

$$\begin{aligned} &\Big\langle D_p H(T, \gamma(T), Dg(\gamma(T)) + \frac{1}{\delta} Db_\Omega(\gamma(T))) - D_p H(T, \gamma(T), Dg(\gamma(T))), \frac{1}{\delta} Db_\Omega(\gamma(T)) \Big\rangle \\ &\ge \frac{1}{2\mu} \frac{1}{\delta^2} |Db_\Omega(\gamma(T))|^2 = \frac{1}{2\delta^2 \mu}. \end{aligned}$$

So,

$$\Big\langle D_p H(T, \gamma(T), p(T)), Db_\Omega(\gamma(T)) \Big\rangle \ge \frac{1}{2\delta\mu} - |D_p H(T, \gamma(T), Dg(\gamma(T)))|.$$

Therefore, we obtain

$$\langle \dot\gamma(T), Db_\Omega(\gamma(T))\rangle = -\Big\langle D_pH(T,\gamma(T),p(T)), Db_\Omega(\gamma(T))\Big\rangle$$
$$\leq -\frac{1}{2\delta\mu} + |D_pH(T,\gamma(T),Dg(\gamma(T)))|.$$

Thus, choosing δ as in (3.43) gives the result. □

Lemma 3.7. *Fix δ as in* (3.43). *Then there exists $\epsilon_1 \in (0,\epsilon_0]$, such that for any $\epsilon \in (0,\epsilon_1]$*

$$\forall x \in \overline{\Omega},\ \gamma \in \mathcal{X}_{\epsilon,\delta}[x] \implies \gamma(t) \in \overline{\Omega} \quad \forall t \in [0,T].$$

Proof. We argue by contradiction. Assume that there exist sequences $\{\epsilon_k\}$, $\{t_k\}$, $\{x_k\}$, $\{\gamma_k\}$ such that

$$\epsilon_k \downarrow 0,\ t_k \in [0,T],\ x_k \in \overline{\Omega},\ \gamma_k \in \mathcal{X}_{\epsilon_k,\delta}[x_k] \text{ and } \gamma_k(t_k) \notin \overline{\Omega}, \quad \text{for all } k \geq 1. \tag{3.44}$$

Then, for each $k \geq 1$ one could find an interval with end-points $0 \leq a_k < b_k \leq T$ such that

$$\begin{cases} d_\Omega(\gamma_k(a_k)) = 0, \\ d_\Omega(\gamma_k(t)) > 0 \quad t \in (a_k,b_k), \\ d_\Omega(\gamma_k(b_k)) = 0 \ \text{ or else } \ b_k = T. \end{cases}$$

Let $\bar t_k \in (a_k,b_k]$ be such that

$$d_\Omega(\gamma_k(\bar t_k)) = \max_{t\in[a_k,b_k]} d_\Omega(\gamma_k(t)).$$

We note that, by Lemma 3.5, γ_k is of class C^2 in a neighborhood of $\bar t_k$.

<u>Step 1</u>

We claim that

$$\frac{d^2}{dt^2} d_\Omega(\gamma_k(t))\Big|_{t=\bar t_k} \leq 0. \tag{3.45}$$

Indeed, (3.45) is trivial if $\bar t_k \in (a_k,b_k)$. Suppose $\bar t_k = b_k$. Since $\bar t_k$ is a maximum point of the map $t \longmapsto d_\Omega(\gamma_k(t))$ and $\gamma_k(\bar t_k) \notin \overline{\Omega}$, we have that $d_\Omega(\gamma_k(\bar t_k)) \neq 0$. So, $b_k = T = \bar t_k$ and we get

$$\frac{d}{dt} d_\Omega(\gamma_k(t))\Big|_{t=\bar t_k} \geq 0.$$

Moreover, Lemma 3.6 yields

$$\frac{d}{dt} d_\Omega(\gamma_k(t))\Big|_{t=\bar t_k} \leq 0.$$

So,

$$\frac{d}{dt} d_\Omega(\gamma_k(t))\Big|_{t=\bar t_k} = 0,$$

and we have that (3.45) holds true at $\bar t_k = T$.

<u>Step 2</u>

Now, we prove that

$$\frac{1}{\mu\epsilon_k} \leq C(\mu, M', \kappa)\left[1 + 4\mu\frac{C_1}{\delta^2} + \frac{4\mu}{\epsilon_k}\, d_\Omega(\gamma_k(\bar t_k))\right], \quad \forall k \geq 1, \tag{3.46}$$

where $C_1 = 8\mu + 8\mu\|Dg\|_\infty^2 + 2C(\mu, M') + \kappa(T + 4\mu K)$ and the constant $C(\mu, M', \kappa)$ depends only on μ, M' and κ. Indeed, since γ is of class C^2 in a neighborhood of $\bar{t}_k$ one has that

$$\begin{aligned}\ddot{\gamma}(\bar{t}_k) = &- D_{pt}^2 H(\bar{t}_k, \gamma(\bar{t}_k), p(\bar{t}_k)) - \left\langle D_{px}^2 H(\bar{t}_k, \gamma(\bar{t}_k), p(\bar{t}_k)), \dot{\gamma}(\bar{t}_k)\right\rangle \\ &- \left\langle D_{pp}^2 H(\bar{t}_k, \gamma(\bar{t}_k), p(\bar{t}_k)), \dot{p}(\bar{t}_k)\right\rangle.\end{aligned} \tag{3.47}$$

Developing the second order derivative of $d_\Omega \circ \gamma$, by (3.47) and the expression of the derivatives of γ and p in Lemma 3.4 one has that

$$\begin{aligned} 0 \;\geq\; & \left\langle D^2 d_\Omega(\gamma(\bar{t}_k))\dot{\gamma}(\bar{t}_k), \dot{\gamma}(\bar{t}_k)\right\rangle + \left\langle Dd_\Omega(\gamma(\bar{t}_k)), \ddot{\gamma}(\bar{t}_k)\right\rangle \\ =\; & \left\langle D^2 d_\Omega(\gamma(\bar{t}_k)) D_p H(\bar{t}_k, \gamma(\bar{t}_k), p(\bar{t}_k)), D_p H(\bar{t}_k, \gamma(\bar{t}_k), p(\bar{t}_k))\right\rangle \\ & - \left\langle Dd_\Omega(\gamma(\bar{t}_k)), D_{pt}^2 H(\bar{t}_k, \gamma(\bar{t}_k), p(\bar{t}_k))\right\rangle \\ & + \left\langle Dd_\Omega(\gamma(\bar{t}_k)), D_{px}^2 H(\bar{t}_k, \gamma(\bar{t}_k), p(\bar{t}_k)) D_p H(\bar{t}_k, \gamma(\bar{t}_k), p(\bar{t}_k))\right\rangle \\ & - \left\langle Dd_\Omega(\gamma(\bar{t}_k)), D_{pp}^2 H(\bar{t}_k, \gamma(\bar{t}_k), p(\bar{t}_k)) D_x H(\bar{t}_k, \gamma(\bar{t}_k), p(\bar{t}_k))\right\rangle \\ & + \frac{1}{\epsilon}\left\langle Dd_\Omega(\gamma(\bar{t}_k)), D_{pp}^2 H(\bar{t}_k, \gamma(\bar{t}_k), p(\bar{t}_k)) Dd_\Omega(\gamma(\bar{t}_k))\right\rangle. \end{aligned}$$

We now use the growth properties of H in (3.14), and (3.16)-(3.18), the lower bound for $D_{pp}^2 H$ in (3.13), and the regularity of the boundary of Ω to obtain:

$$\frac{1}{\mu\epsilon_k} \leq C(\mu, M')(1 + |p(\bar{t}_k)|)^2 + \kappa C(\mu, M')(1 + |p(\bar{t}_k)|) \leq C(\mu, M', \kappa)(1 + |p(\bar{t}_k)|^2),$$

where the constant $C(\mu, M', \kappa)$ depends only on μ, M' and κ. By our estimate for p in (3.29) we get:

$$\frac{1}{\mu\epsilon_k} \leq C(\mu, M', \kappa)\left[1 + 4\mu\frac{C_1}{\delta^2} + \frac{4\mu}{\epsilon_k} d_\Omega(\gamma(\bar{t}_k))\right], \quad \forall\, k \geq 1,$$

where $C_1 = 8\mu + 8\mu\|Dg\|_\infty^2 + 2C(\mu, M') + \kappa(T + 4\mu K)$.
Conclusion
Let $\rho = \min\left\{\rho_0, \frac{1}{32C(\mu, M', \kappa)\mu^2}\right\}$. Owing to Lemma 3.3, for all $\epsilon \in (0, \epsilon(\rho)]$ we have that

$$\sup_{t\in[0,T]} d_\Omega(\gamma(t)) \leq \rho, \quad \forall \gamma \in \mathcal{X}_{\epsilon,\delta}[x].$$

Hence, using (3.46), we deduce that

$$\frac{1}{2\mu\epsilon_k} \leq 4C(\mu, M', \kappa)\left[1 + 4\mu\frac{C_1}{\delta^2}\right].$$

Since the above inequality fails for k large enough, we conclude that (3.44) cannot hold true. So, $\gamma(t)$ belongs to $\overline{\Omega}$ for all $t \in [0, T]$. □

An obvious consequence of Lemma 3.7 is the following:

Corollary 3.1. *Fix δ as in* (3.43) *and take $\epsilon = \epsilon_1$, where ϵ_1 is defined as in Lemma 3.7. Then an arc $\gamma(\cdot)$ is a solution of problem (3.22) if and only if it is also a solution of (3.1).*

We are now ready to complete the proof of Theorem 3.1.

Proof of Theorem 3.1. Let $x \in \overline{\Omega}$ and $\gamma^\star \in \mathcal{X}[x]$. By Corollary 3.1 we have that $\gamma^\star$ is a solution of problem (3.22) with δ as in (3.43) and $\epsilon = \epsilon_1$ as in Lemma 3.7. Let $p(\cdot)$ be the associated adjoint map such that $(\gamma^\star(\cdot), p(\cdot))$ satisfies (3.26). Moreover, let $\lambda(\cdot)$ and β be defined as in Lemma 3.4. Define $\nu = \frac{\beta}{\delta}$. Then we have $0 \leq \nu \leq \frac{1}{\delta}$ and, by (3.26),

$$p(T) = Dg(\gamma^\star(T)) + \nu\, Db_\Omega(\gamma^\star(T)). \tag{3.48}$$

By Lemma 3.4 $\gamma^\star \in C^{1,1}([0,T];\overline{\Omega})$ and

$$\dot{\gamma}^\star(t) = -D_pH(t, \gamma^\star(t), p(t)), \quad \forall\, t \in [0,T]. \tag{3.49}$$

Moreover, $p(\cdot) \in \mathrm{Lip}(0,T;\mathbb{R}^n)$ and by (3.29) one has that

$$|p(t)| \leq 2\frac{\sqrt{\mu C_1}}{\delta}, \quad \forall t \in [0,T],$$

where $C_1 = 8\mu + 8\mu\|Dg\|_\infty^2 + 2C(\mu, M') + \kappa(T + 4\mu K)$. Hence, p is bounded. By (3.49), and by (3.17) one has that

$$\|\dot{\gamma}^\star\|_\infty = \sup_{t\in[0,T]} |D_pH(t,\gamma^\star(t),p(t))| \leq C(\mu, M')\Big(\sup_{t\in[0,T]} |p(t)| + 1\Big) \leq C(\mu, M')\Big(2\frac{\sqrt{\mu C_1}}{\delta} + 1\Big) = L^\star,$$

where $L^\star = L^\star(\mu, M', M, \kappa, T, \|Dg\|_\infty, \|g\|_\infty)$. Thus, (3.21) holds
Finally, we want to find an explicit expression for $\lambda(t)$. For this, we set

$$D = \Big\{t \in [0,T] : \gamma^\star(t) \in \partial\Omega\Big\} \text{ and } D_{\rho_0} = \Big\{t \in [0,T] : |b_\Omega(\gamma^\star(t))| < \rho_0\Big\},$$

where ρ_0 is as in assumption (2.1). Note that $\psi(t) := b_\Omega \circ \gamma^\star$ is of class $C^{1,1}$ on the open set D_{ρ_0}, with

$$\dot{\psi}(t) = \Big\langle Db_\Omega(\gamma^\star(t)), \dot{\gamma}^\star(t)\Big\rangle = \Big\langle Db_\Omega(\gamma^\star(t)), -D_pH(t,\gamma^\star(t),p(t))\Big\rangle.$$

Since $p \in \mathrm{Lip}(0,T;\mathbb{R}^n)$, $\dot{\psi}$ is absolutely continuous on D_{ρ_0} with

$$\begin{aligned}
\ddot{\psi}(t) &= -\Big\langle D^2b_\Omega(\gamma^\star(t))\dot{\gamma}^\star(t), D_pH(t,\gamma^\star(t),p(t))\Big\rangle - \Big\langle Db_\Omega(\gamma^\star(t)), D^2_{pt}H(t,\gamma^\star(t),p(t))\Big\rangle \\
&- \Big\langle Db_\Omega(\gamma^\star(t)), D^2_{px}H(t,\gamma^\star(t),p(t))\dot{\gamma}^\star(t)\Big\rangle - \Big\langle Db_\Omega(\gamma^\star(t)), D^2_{pp}H(t,\gamma^\star(t),p(t))\dot{p}(t)\Big\rangle \\
&= \Big\langle D^2b_\Omega(\gamma^\star(t))D_pH(t,\gamma^\star(t),p(t)), D_pH(t,\gamma^\star(t),p(t))\Big\rangle \\
&- \Big\langle Db_\Omega(\gamma^\star(t)), D^2_{pt}H(t,\gamma^\star(t),p(t))\Big\rangle \\
&+ \Big\langle Db_\Omega(\gamma^\star(t)), D^2_{px}H(t,\gamma^\star(t),p(t))D_pH(t,\gamma^\star(t),p(t))\Big\rangle \\
&- \Big\langle Db_\Omega(\gamma^\star(t)), D^2_{pp}H(t,\gamma^\star(t),p(t))D_xH(t,\gamma^\star(t),p(t))\rangle \\
&+ \frac{\lambda(t)}{\epsilon}\Big\langle Db_\Omega(\gamma^\star(t)), D^2_{pp}H(t,\gamma^\star(t),p(t))Db_\Omega(\gamma^\star(t))\Big\rangle.
\end{aligned}$$

Let $N_{\gamma^\star} = \{t \in D \cap (0,T) |\ \dot{\psi}(t) \neq 0\}$. Let $t \in N_{\gamma^\star}$, then there exists $\sigma > 0$ such that $\gamma^\star(s) \notin \partial\Omega$ for any $s \in ((t-\sigma, t+\sigma) \setminus \{t\}) \cap (0,T)$. Therefore, $N_{\gamma^\star}$ is composed of isolated points and so it is a discrete set. Hence, $\dot{\psi}(t) = 0$ a.e. $t \in D \cap (0, T)$. So, $\ddot{\psi}(t) = 0$ a.e. in D, because $\dot{\psi}$ is absolutely continuous. Moreover, since $D^2_{pp}H(t, x, p) > 0$ and $|Db_\Omega(\gamma^\star(t))| = 1$, we have that

$$\left\langle Db_\Omega(\gamma^\star(t)), D^2_{pp}H(t, \gamma^\star(t), p(t))Db_\Omega(\gamma^\star(t))\right\rangle > 0, \qquad \text{a.e. } t \in D_{\rho_0}.$$

So, for a.e. $t \in D$, $\lambda(t)$ is given by

$$\begin{aligned}\frac{\lambda(t)}{\epsilon} =& \frac{1}{\langle Db_\Omega(\gamma^\star(t)), D^2_{pp}H(t, \gamma^\star(t), p(t))Db_\Omega(\gamma^\star(t))\rangle} \Big[\left\langle Db_\Omega(\gamma^\star(t)), D^2_{pt}H(t, \gamma^\star(t), p(t))\right\rangle \\ &- \left\langle D^2 b_\Omega(\gamma^\star(t))D_pH(t, \gamma^\star(t), p(t)), D_pH(t, \gamma^\star(t), p(t))\right\rangle \\ &- \left\langle Db_\Omega(\gamma^\star(t)), D^2_{px}H(t, \gamma^\star(t), p(t))D_pH(t, \gamma^\star(t), p(t))\right\rangle \\ &+ \left\langle Db_\Omega(\gamma^\star(t)), D^2_{pp}H(t, \gamma^\star(t), p(t))D_xH(t, \gamma^\star(t), p(t))\right\rangle\Big].\end{aligned}$$

Since $\lambda(t) = 0$ for all $t \in [0, T]\backslash D$ by (3.27), taking $\Lambda(t, \gamma^\star(t), p(t)) = \frac{\lambda(t)}{\epsilon}$, we obtain the conclusion. □

Remark 3.4. The above proof gives a representation of Λ, i.e., for all $(t, x, p) \in [0, T] \times \Sigma_{\rho_0} \times \mathbb{R}^n$ one has that

$$\begin{aligned}\Lambda(t, x, p) =& \frac{1}{\theta(t, x, p)} \Big[- \left\langle D^2 b_\Omega(x)D_pH(t, x, p), D_pH(t, x, p)\right\rangle - \left\langle Db_\Omega(x), D^2_{pt}H(t, x, p)\right\rangle - \\ &\left\langle Db_\Omega(x), D^2_{px}H(t, x, p)D_pH(t, x, p)\right\rangle + \left\langle Db_\Omega(x), D^2_{pp}H(t, x, p)D_xH(t, x, p)\right\rangle\Big],\end{aligned}$$

where $\theta(t, x, p) := \langle Db_\Omega(x), D^2_{pp}H(t, x, p)Db_\Omega(x)\rangle$. Observe that (3.13) ensures that $\theta(t, x, p) > 0$ for all $t \in [0, T]$, for all $x \in \Sigma_{\rho_0}$ and for all $p \in \mathbb{R}^n$.

3.3. Proof of Theorem 3.1 for general U

We now want to remove the extra assumption $U = \mathbb{R}^n$. For this purpose, it suffices to show that the data f and g—a priori defined just on U—can be extended to $\mathbb{R}^n$ preserving the conditions in (f0)-(f2) and (g1). So, we proceed to construct such an extension by taking a cut-off function $\xi \in C^\infty(\mathbb{R})$ such that

$$\begin{cases} \xi(x) = 0 & \text{if } x \in (-\infty, \frac{1}{3}], \\ 0 < \xi(x) < 1 & \text{if } x \in (\frac{1}{3}, \frac{2}{3}), \\ \xi = 1 & \text{if } x \in [\frac{2}{3}, +\infty). \end{cases} \tag{3.50}$$

Lemma 3.8. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with C^2 boundary. Let U be a open subset of $\mathbb{R}^n$ such that $\overline{\Omega} \subset U$ and set*

$$\sigma_0 = dist(\overline{\Omega}, \mathbb{R}^n \setminus U) > 0.$$

Suppose that $f : [0, T] \times U \times \mathbb{R}^n \to \mathbb{R}$ and $g : U \to \mathbb{R}$ satisfy (f0)-(f2) and (g1), respectively. Set $\sigma = \sigma_0 \wedge \rho_0$. Then, the function f admits the extension

$$\widetilde{f}(t, x, v) = \xi\left(\frac{b_\Omega(x)}{\sigma}\right)\frac{|v|^2}{2} + \left(1 - \xi\left(\frac{b_\Omega(x)}{\sigma}\right)\right)f(t, x, v), \quad \forall\ (t, x, v) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n,$$

that satisfies conditions (f0)-(f2) with $U = \mathbb{R}^n$. Moreover, g admits the extension

$$\widetilde{g}(x) = \left(1 - \xi\left(\frac{b_\Omega(x)}{\sigma}\right)\right)g(x), \quad \forall x \in \mathbb{R}^n,$$

that satisfies condition (g1) with $U = \mathbb{R}^n$.

Note that, since Ω is bounded and U is open, the distance between $\overline{\Omega}$ and $\mathbb{R}^n \setminus U$ is positive.

Proof. By construction we note that $\widetilde{f} \in C([0,T] \times \mathbb{R}^n \times \mathbb{R}^n)$. Moreover, for all $t \in [0,T]$ the function $(x, v) \longmapsto \widetilde{f}(t, x, v)$ is differentiable and the map $(x, v) \longmapsto D_v\widetilde{f}(t, x, v)$ is continuously differentiable by construction. Furthermore, $D_x\widetilde{f}$, $D_v\widetilde{f}$ are continuous on $[0,T] \times \mathbb{R}^n \times \mathbb{R}^n$ and $\widetilde{f}$ satisfies (3.2). In order to prove (3.3) for $\widetilde{f}$, we observe that

$$D_v\widetilde{f}(t, x, v) = \xi\left(\frac{b_\Omega(x)}{\sigma}\right)v + \left(1 - \xi\left(\frac{b_\Omega(x)}{\sigma}\right)\right)D_v f(t, x, v),$$

and

$$D^2_{vv}\widetilde{f}(t, x, v) = \xi\left(\frac{b_\Omega(x)}{\sigma}\right)I + \left(1 - \xi\left(\frac{b_\Omega(x)}{\sigma}\right)\right)D^2_{vv} f(t, x, v).$$

Hence, by the definition of ξ and (3.3) we obtain that

$$\left(1 \wedge \frac{1}{\mu}\right)I \le D^2_{vv}\widetilde{f}(t, x, v) \le (1 \vee \mu)I, \quad \forall\, (t, x, v) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^n.$$

Since $\mu \ge 1$, we have that $\widetilde{f}$ verifies the estimate in (3.3).
Moreover, since

$$\begin{aligned} D_x(D_v\widetilde{f}(t, x, v)) &= \dot{\xi}\left(\frac{b_\Omega(x)}{\sigma}\right)v \otimes \frac{Db_\Omega(x)}{\sigma} + \left(1 - \xi\left(\frac{b_\Omega(x)}{\sigma}\right)\right)D^2_{vx} f(t, x, v) \\ &- \dot{\xi}\left(\frac{b_\Omega(x)}{\sigma}\right)D_v f(t, x, v) \otimes \frac{Db_\Omega(x)}{\sigma}, \end{aligned}$$

and by (3.4) we obtain that

$$\|D^2_{vx}\widetilde{f}(t, x, v)\| \le C(\mu, M)(1 + |v|) \quad \forall (t, x, v) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^n.$$

For all $(x, v) \in \mathbb{R}^n \times \mathbb{R}^n$ the function $t \longmapsto \widetilde{f}(t, x, v)$ and the map $t \longmapsto D_v\widetilde{f}(t, x, v)$ are Lipschitz continuous by construction. Moreover, by (3.5) and the definition of ξ one has that

$$\left|\widetilde{f}(t, x, v) - \widetilde{f}(s, x, v)\right| = \left|\left(1 - \xi\left(\frac{b_\Omega(x)}{\sigma}\right)\right)[f(t, x, v) - f(s, x, v)]\right| \le \kappa(1 + |v|^2)|t - s|$$

for all $t,\ s \in [0,T]$, $x \in \mathbb{R}^n$, $v \in \mathbb{R}^n$. Now, we have to prove that (3.6) holds for $\widetilde{f}$. Indeed, using (3.6) we deduce that

$$\begin{aligned} &\left|D_v\widetilde{f}(t, x, v)) - D_v\widetilde{f}(s, x, v))\right| \le \left|\left(1 - \xi\left(\frac{b_\Omega(x)}{\sigma}\right)\right)[D_v f(t, x, v) - D_v f(s, x, v))]\right| \\ &\le \kappa(1 + |v|)|t - s|, \end{aligned}$$

for all $t,\ s \in [0,T]$, $x \in \mathbb{R}^n$, $v \in \mathbb{R}^n$. Therefore, $\widetilde{f}$ verifies the assumptions (f0)-(f2).
Finally, by the regularity of b_Ω, ξ, and g we have that $\widetilde{g}$ is of class $C^1_b(\mathbb{R}^n)$. This completes the proof. □

4. Applications of Theorem 3.1

4.1. Lipschitz regularity for constrained minimization problems

Suppose that $f : [0,T] \times U \times \mathbb{R}^n \to \mathbb{R}$ and $g : U \to \mathbb{R}$ satisfy the assumptions (f0)-(f2) and (g1), respectively. Let $(t,x) \in [0,T]\times\overline{\Omega}$. Define $u : [0,T]\times\overline{\Omega} \to \mathbb{R}$ as the value function of the minimization problem (3.1), i.e.,

$$u(t,x) = \inf_{\substack{\gamma \in \Gamma \\ \gamma(t) = x}} \int_t^T f(s,\gamma(s),\dot\gamma(s))\,ds + g(\gamma(T)). \tag{4.1}$$

Proposition 4.1. *Let Ω be a bounded open subset of $\mathbb{R}^n$ with C^2 boundary. Suppose that f and g satisfy (f0)-(f2) and (g1), respectively. Then, u is Lipschitz continuous in $[0,T]\times\overline{\Omega}$.*

Proof. First, we shall prove that $u(t,\cdot)$ is Lipschitz continuous on Ω, uniformly for $t \in [0,T]$. Since $u(T,\cdot) = g$, it suffices to consider the case of $t \in [0,T)$. Let $x_0 \in \Omega$ and choose $0 < r < 1$ such that $B_r(x_0) \subset B_{2r}(x_0) \subset B_{4r}(x_0) \subset \Omega$. To prove that $u(t,\cdot)$ is Lipschitz continuous in $B_r(x_0)$, take $x \neq y$ in $B_r(x_0)$. Let γ be an optimal trajectory for u at (t,x) and let $\overline{\gamma}$ be the trajectory defined by

$$\begin{cases} \overline{\gamma}(t) = y, \\ \dot{\overline{\gamma}}(s) = \dot\gamma(s) + \frac{x-y}{\tau} & \text{if } s \in [t,t+\tau] \text{ a.e.,} \\ \dot{\overline{\gamma}}(s) = \dot\gamma(s) & \text{otherwise,} \end{cases}$$

where $\tau = \frac{|x-y|}{2L^\star} < T - t$. We claim that

(a) $\overline{\gamma}(t+\tau) = \gamma(t+\tau)$;

(b) $\overline{\gamma}(s) = \gamma(s)$ for any $s \in [t+\tau, T]$;

(c) $|\overline{\gamma}(s) - \gamma(s)| \leq |y - x|$ for any $s \in [t, t+\tau]$;

(d) $\overline{\gamma}(s) \in \overline{\Omega}$ for any $s \in [t,T]$.

Indeed, by the definition of $\overline{\gamma}$ we have that

$$\overline{\gamma}(t+\tau) - \overline{\gamma}(t) = \overline{\gamma}(t+\tau) - y = \int_t^{t+\tau} \Big(\dot\gamma(s) + \frac{x-y}{\tau}\Big)\,ds = \gamma(t+\tau) - y,$$

and this gives (a). Moreover, by (a), and by the definition of $\overline{\gamma}$ one has that $\overline{\gamma}(s) = \gamma(s)$ for any $s \in [t+\tau, T]$. Hence, $\overline{\gamma}$ verifies (b). By the definition of $\overline{\gamma}$, for any $s \in [t, t+\tau]$ we obtain that

$$\Big|\overline{\gamma}(s) - \gamma(s)\Big| \leq \Big|y - x + \int_t^s (\dot{\overline{\gamma}}(\sigma) - \dot\gamma(\sigma))\,d\sigma\Big| = \Big|y - x + \int_t^s \frac{x-y}{\tau}\,d\sigma\Big| \leq |y - x|$$

and so (c) holds. Since γ is an optimal trajectory for u and by $\overline{\gamma}(s) = \gamma(s)$ for all $s \in [t+\tau, T]$, we only have to prove that $\overline{\gamma}(s)$ belongs to $\overline{\Omega}$ for all $s \in [t, t+\tau]$. Let $s \in [t, t+\tau]$, by Theorem 3.1 one has that

$$|\overline{\gamma}(s) - x_0| \leq |\overline{\gamma}(s) - y| + |y - x_0| \leq \Big|\int_t^s \dot{\overline{\gamma}}(\sigma)\,d\sigma\Big| + r \leq \int_t^s \Big|\dot\gamma(\sigma) + \frac{x-y}{\tau}\Big|\,d\sigma + r$$

$$\leq \int_t^s \left[|\dot{\gamma}(\sigma)| + \frac{|x-y|}{\tau}\right] d\sigma + r \leq L^\star(s-t) + \frac{|x-y|}{\tau}(s-t) + r \leq L^\star\tau + |x-y| + r.$$

Recalling that $\tau = \frac{|x-y|}{2L^\star}$ one has that

$$|\overline{\gamma}(s) - x_0| \leq \frac{|x-y|}{2} + |x-y| + r \leq 4r.$$

Therefore, $\overline{\gamma}(s) \in B_{4r}(x_0) \subset \overline{\Omega}$ for all $s \in [t, t+\tau]$.

Now, owing to the dynamic programming principle, by (a) one has that

$$u(t,y) \leq \int_t^{t+\tau} f(s, \overline{\gamma}(s), \dot{\overline{\gamma}}(s))\, ds + u(t+\tau, \gamma(t+\tau)). \tag{4.2}$$

Since γ is an optimal trajectory for u at (t,x), we obtain that

$$u(t,y) \leq u(t,x) + \int_t^{t+\tau} \left[f(s, \overline{\gamma}(s), \dot{\overline{\gamma}}(s)) - f(s, \gamma(s), \dot{\gamma}(s))\right] ds.$$

By (3.7), (3.8), and the definition of $\overline{\gamma}$, for $s \in [t, t+\tau]$ we have that

$$\begin{aligned}
&|f(s, \overline{\gamma}(s), \dot{\overline{\gamma}}(s)) - f(s, \gamma(s), \dot{\gamma}(s))| \\
&\leq |f(s, \overline{\gamma}(s), \dot{\overline{\gamma}}(s)) - f(s, \overline{\gamma}(s), \dot{\gamma}(s))| + |f(s, \overline{\gamma}(s), \dot{\gamma}(s)) - f(s, \gamma(s), \dot{\gamma}(s))| \\
&\leq \int_0^1 |\langle D_v f(s, \overline{\gamma}(s), \lambda\dot{\overline{\gamma}}(s) + (1-\lambda)\dot{\gamma}(s)), \dot{\overline{\gamma}}(s) - \dot{\gamma}(s)\rangle|\, d\lambda \\
&+ \int_0^1 |D_x f(s, \lambda\overline{\gamma}(s) + (1-\lambda)\gamma(s), \dot{\gamma}(s)), \overline{\gamma}(s) - \gamma(s)\rangle|\, d\lambda \\
&\leq C(\mu, M)|\dot{\overline{\gamma}}(s) - \dot{\gamma}(s)| \int_0^1 (1 + |\lambda\dot{\overline{\gamma}}(s) + (1-\lambda)\dot{\gamma}(s)|)\, d\lambda \\
&+ C(\mu, M)|\overline{\gamma}(s) - \gamma(s)| \int_0^1 (1 + |\dot{\gamma}(s)|^2)\, d\lambda.
\end{aligned}$$

By Theorem 3.1 one has that

$$\int_0^1 (1 + |\lambda\dot{\overline{\gamma}}(s) + (1-\lambda)\dot{\gamma}(s)|)\, d\lambda \leq 1 + 4L^\star, \tag{4.3}$$

$$\int_0^1 (1 + |\dot{\gamma}(s)|^2)\, d\lambda \leq 1 + (L^\star)^2. \tag{4.4}$$

Using (4.3), (4.4), and (c), by the definition of $\overline{\gamma}$ one has that

$$|f(s, \overline{\gamma}(s), \dot{\overline{\gamma}}(s)) - f(s, \gamma(s), \dot{\gamma}(s))| \leq C(\mu, M)(1 + 4L^\star)\frac{|x-y|}{\tau} + C(\mu, M)(1 + (L^\star)^2)|x-y|, \tag{4.5}$$

for a.e. $s \in [t, t+\tau]$. By (4.5), and the choice of τ we deduce that

$$u(t,y) \leq u(t,x) + C(\mu, M)(1 + 4L^\star) \int_t^{t+\tau} \frac{|x-y|}{\tau}\, ds + C(\mu, M)(1 + (L^\star)^2) \int_t^{t+\tau} |x-y|\, ds$$

$$\leq u(t,x) + C(\mu, M)(1+4L^\star)\big|x-y\big| + \tau C(\mu, M)(1+(L^\star)^2)\big|x-y\big| \leq u(t,x) + C_{L^\star}|x-y|$$

where $C_{L^\star} = C(\mu, M)(1+4L^\star) + \frac{1}{2L^\star}C(\mu, M)(1+(L^\star)^2)$. Thus, u is locally Lipschitz continuous in space and one has that $\|Du\|_\infty \leq \vartheta$, where ϑ is a constant not depending on Ω. Owing to the smoothness of Ω, u is globally Lipschitz continuous in space, uniformly for $t \in [0,T]$.

In order to prove Lipschitz continuity in time, let $x \in \overline{\Omega}$ and fix $t_1, t_2 \in [0,T]$ with $t_2 \geq t_1$. Let γ be an optimal trajectory for u at (t_1, x). Then,

$$|u(t_2,x) - u(t_1,x)| \leq |u(t_2,x) - u(t_2,\gamma(t_2))| + |u(t_2,\gamma(t_2)) - u(t_1,x)|. \tag{4.6}$$

The first term on the right-side of (4.6) can be estimated using the Lipschitz continuity in space of u and Theorem 3.1. Thus, we get

$$|u(t_2,x) - u(t_2,\gamma(t_2))| \leq C_{L^\star}|x-\gamma(t_2)| \leq C_{L^\star}\int_{t_1}^{t_2}|\dot\gamma(s)|\,ds \leq L^\star C_{L^\star}(t_2-t_1). \tag{4.7}$$

We only have to estimate the second term on the right-side of (4.6). By the dynamic programming principle, (3.9), and the assumptions on F we deduce that

$$\begin{aligned}|u(t_2,\gamma(t_2)) - u(t_1,x)| &= \Big|\int_{t_1}^{t_2} f(s,\gamma(s),\dot\gamma(s))\,ds\Big| \leq \int_{t_1}^{t_2}|f(s,\gamma(s),\dot\gamma(s))|\,ds \\ &\leq \int_{t_1}^{t_2}\Big[C(\mu,M) + 4\mu|\dot\gamma(s)|^2\Big]\,ds \leq \Big[C(\mu,M) + 4\mu L^\star\Big](t_2-t_1)\end{aligned} \tag{4.8}$$

Using (4.7) and (4.8) to bound the right-hand side of (4.6), we obtain that u is Lipschitz continuous in time. This completes the proof. □

4.2. Lipschitz regularity for constrained MFG equilibria

In this section we want to apply Theorem 3.1 to a mean field game (MFG) problem with state constraints. Such a problem was studied in [11], where the existence and uniqueness of constrained equilibria was obtained under fairly general assumptions on the data. Here, we will apply our necessary conditions to deduce the existence of more regular equilibria than those constructed in [11], assuming the data F and G to be Lipschitz continuous.

Assumptions

Let Ω be a bounded open subset of $\mathbb{R}^n$ with C^2 boundary. Let $\mathcal{P}(\overline{\Omega})$ be the set of all Borel probability measures on $\overline{\Omega}$ endowed with the Kantorovich-Rubinstein distance d_1 defined in (2.2). Let U be an open subset of $\mathbb{R}^n$ and such that $\overline{\Omega} \subset U$. Assume that $F : U \times \mathcal{P}(\overline{\Omega}) \to \mathbb{R}$ and $G : U \times \mathcal{P}(\overline{\Omega}) \to \mathbb{R}$ satisfy the following hypotheses.

(D1) For all $x \in U$, the functions $m \longmapsto F(x,m)$ and $m \longmapsto G(x,m)$ are Lipschitz continuous, i.e., there exists a constant $\kappa \geq 0$ such that

$$|F(x,m_1) - F(x,m_2)| + |G(x,m_1) - G(x,m_2)| \leq \kappa d_1(m_1,m_2), \tag{4.9}$$

for any $m_1, m_2 \in \mathcal{P}(\overline{\Omega})$.

(D2) For all $m \in \mathcal{P}(\overline{\Omega})$, the functions $x \longmapsto G(x,m)$ and $x \longmapsto F(x,m)$ belong to $C^1_b(U)$. Moreover

$$|D_x F(x,m)| + |D_x G(x,m)| \le \kappa, \quad \forall\, x \in U,\ \forall\, m \in \mathcal{P}(\overline{\Omega}).$$

Let $L : U \times \mathbb{R}^n \to \mathbb{R}$ be a function that satisfies the following assumptions.

(L0) $L \in C^1(U \times \mathbb{R}^n)$ and there exists a constant $M \ge 0$ such that

$$|L(x,0)| + |D_x L(x,0)| + |D_v L(x,0)| \le M, \quad \forall\, x \in U. \tag{4.10}$$

(L1) $D_v L$ is differentiable on $U \times \mathbb{R}^n$ and there exists a constant $\mu \ge 1$ such that

$$\frac{I}{\mu} \le D^2_{vv} L(x,v) \le I\mu, \tag{4.11}$$

$$\|D^2_{vx} L(x,v)\| \le \mu(1+|v|), \tag{4.12}$$

for all $(x,v) \in U \times \mathbb{R}^n$.

Remark 4.1. (i) F, G and L are assumed to be defined on $U \times \mathcal{P}(\overline{\Omega})$ and on $U \times \mathbb{R}^n$, respectively, just for simplicity. All the results of this section hold true if we replace U by $\overline{\Omega}$. This fact can be easily checked by using well-known extension techniques (see, e.g. [1, Theorem 4.26]).
(ii) Arguing as Lemma 3.1 we deduce that there exists a positive constant $C(\mu, M)$ that dependes only on M, μ such that

$$|D_x L(x,v)| \le C(\mu,M)(1+|v|^2), \tag{4.13}$$

$$|D_v L(x,v)| \le C(\mu,M)(1+|v|), \tag{4.14}$$

$$\frac{|v|^2}{4\mu} - C(\mu,M) \le L(x,v) \le 4\mu|v|^2 + C(\mu,M), \tag{4.15}$$

for all $(x,v) \in U \times \mathbb{R}^n$.
Let $m \in \mathrm{Lip}(0,T;\mathcal{P}(\overline{\Omega}))$. If we set $f(t,x,v) = L(x,v) + F(x,m(t))$, then the associated Hamiltonian H takes the form

$$H(t,x,p) = H_L(x,p) - F(x,m(t)), \quad \forall\, (t,x,p) \in [0,T] \times U \times \mathbb{R}^n,$$

where

$$H_L(x,p) = \sup_{v \in \mathbb{R}^n} \Big\{ -\langle p, v\rangle - L(x,v) \Big\}, \qquad \forall\, (x,p) \in U \times \mathbb{R}^n.$$

The assumptions on L imply that H_L satisfies the following conditions.

1. $H_L \in C^1(U \times \mathbb{R}^n)$ and there exists a constant $M' \ge 0$ such that

$$|H_L(x,0)| + |D_x H_L(x,0)| + |D_p H_L(x,0)| \le M', \quad \forall x \in U. \tag{4.16}$$

2. $D_p H_L$ is differentiable on $U \times \mathbb{R}^n$ and satisfies

$$\frac{I}{\mu} \le D_{pp} H_L(x,p) \le I\mu, \qquad \forall\, (x,p) \in U \times \mathbb{R}^n, \tag{4.17}$$

$$\|D^2_{px} H_L(x,p)\| \le C(\mu,M')(1+|p|), \quad \forall\, (x,p) \in U \times \mathbb{R}^n, \tag{4.18}$$

where μ is the constant in (L1) and $C(\mu, M')$ depends only on μ and M'.

For any $t \in [0, T]$, we denote by $e_t : \Gamma \to \overline{\Omega}$ the evaluation map defined by

$$e_t(\gamma) = \gamma(t), \quad \forall \gamma \in \Gamma.$$

For any $\eta \in \mathcal{P}(\Gamma)$, we define

$$m^\eta(t) = e_t \sharp \eta \quad \forall t \in [0, T].$$

Remark 4.2. We observe that for any $\eta \in \mathcal{P}(\Gamma)$, the following holds true (see [11] for a proof).

(i) $m^\eta \in C([0, T]; \mathcal{P}(\overline{\Omega}))$.

(ii) Let η_i, $\eta \in \mathcal{P}(\Gamma)$, $i \geq 1$, be such that η_i is narrowly convergent to η. Then $m^{\eta_i}(t)$ is narrowly convergent to $m^\eta(t)$ for all $t \in [0, T]$.

For any fixed $m_0 \in \mathcal{P}(\overline{\Omega})$, we denote by $\mathcal{P}_{m_0}(\Gamma)$ the set of all Borel probability measures η on Γ such that $e_0 \sharp \eta = m_0$. For all $\eta \in \mathcal{P}_{m_0}(\Gamma)$, we set

$$J_\eta[\gamma] = \int_0^T \Big[L(\gamma(t), \dot{\gamma}(t)) + F(\gamma(t), m^\eta(t)) \Big] \, dt + G(\gamma(T), m^\eta(T)), \qquad \forall \gamma \in \Gamma.$$

For all $x \in \overline{\Omega}$ and $\eta \in \mathcal{P}_{m_0}(\Gamma)$, we define

$$\Gamma^\eta[x] = \Big\{ \gamma \in \Gamma[x] : J_\eta[\gamma] = \min_{\Gamma[x]} J_\eta \Big\}.$$

It is shown in [11] that, for every $\eta \in \mathcal{P}_{m_0}(\Gamma)$, the set $\Gamma^\eta[x]$ is nonempty and $\Gamma^\eta[\cdot]$ has closed graph. We recall the definition of constrained MFG equilibria for m_0, given in [11].

Definition 4.1. *Let $m_0 \in \mathcal{P}(\overline{\Omega})$. We say that $\eta \in \mathcal{P}_{m_0}(\Gamma)$ is a contrained MFG equilibrium for m_0 if*

$$supp(\eta) \subseteq \bigcup_{x \in \overline{\Omega}} \Gamma^\eta[x].$$

Let Γ' be a nonempty subset of Γ. We denote by $\mathcal{P}_{m_0}(\Gamma')$ the set of all Borel probability measures η on Γ' such that $e_0 \sharp \eta = m_0$. We now introduce special subfamilies of $\mathcal{P}_{m_0}(\Gamma)$ that play a key role in what follows.

Definition 4.2. *Let Γ' be a nonempty subset of Γ. We define by $\mathcal{P}^{\mathrm{Lip}}_{m_0}(\Gamma')$ the set of $\eta \in \mathcal{P}_{m_0}(\Gamma')$ such that $m^\eta(t) = e_t \sharp \eta$ is Lipschitz continuous, i.e.,*

$$\mathcal{P}^{\mathrm{Lip}}_{m_0}(\Gamma') = \{ \eta \in \mathcal{P}_{m_0}(\Gamma') : m^\eta \in \mathrm{Lip}(0, T; \mathcal{P}(\overline{\Omega})) \}.$$

Remark 4.3. We note that $\mathcal{P}^{\mathrm{Lip}}_{m_0}(\Gamma)$ is a nonempty convex set. Indeed, let $j : \overline{\Omega} \to \Gamma$ be the continuous map defined by

$$j(x)(t) = x \quad \forall t \in [0, T].$$

Then,

$$\eta := j \sharp m_0$$

is a Borel probability measure on Γ and $\eta \in \mathcal{P}^{\mathrm{Lip}}_{m_0}(\Gamma)$.
In order to show that $\mathcal{P}^{\mathrm{Lip}}_{m_0}(\Gamma)$ is convex, let $\{\eta_i\}_{i=1,2} \subset \mathcal{P}^{\mathrm{Lip}}_{m_0}(\Gamma)$ and let $\lambda_1, \lambda_2 \geq 0$ be such that $\lambda_1 + \lambda_2 = 1$.

Since η_i are Borel probability measures, $\eta := \lambda\eta_1 + (1-\lambda)\eta_2$ is a Borel probability measure as well. Moreover, for any Borel set $B \in \mathscr{B}(\overline{\Omega})$ we have that

$$e_0\sharp\eta(B) = \eta(e_0^{-1}(B)) = \sum_{i=1}^{2} \lambda_i\eta_i(e_0^{-1}(B)) = \sum_{i=1}^{2} \lambda_i e_0\sharp\eta_i(B) = \sum_{i=1}^{2} \lambda_i m_0(B) = m_0(B).$$

So, $\eta \in \mathcal{P}_{m_0}(\Gamma)$. Since $m^{\eta_1}, m^{\eta_2} \in \mathrm{Lip}(0,T;\mathcal{P}(\overline{\Omega}))$, we have that $m^\eta(t) = \lambda_1 m^{\eta_1}(t) + \lambda_2 m^{\eta_2}(t)$ belongs to $\mathrm{Lip}(0,T;\mathcal{P}(\overline{\Omega}))$.

In the next result, we apply Theorem 3.1 to prove a useful property of minimizers of J_η.

Proposition 4.2. *Let Ω be a bounded open subset of $\mathbb{R}^n$ with C^2 boundary and let $m_0 \in \mathcal{P}(\overline{\Omega})$. Suppose that (L0), (L1), (D1), and (D2) hold true. Let $\eta \in \mathcal{P}^{\mathrm{Lip}}_{m_0}(\Gamma)$ and fix $x \in \overline{\Omega}$. Then $\Gamma^\eta[x] \subset C^{1,1}([0,T];\mathbb{R}^n)$ and*

$$\|\dot\gamma\|_\infty \leq L_0, \quad \forall \gamma \in \Gamma^\eta[x], \tag{4.19}$$

where $L_0 = L_0(\mu, M', M, \kappa, T, \|G\|_\infty, \|DG\|_\infty)$.

Proof. Let $\eta \in \mathcal{P}^{\mathrm{Lip}}_{m_0}(\Gamma)$, $x \in \overline{\Omega}$ and $\gamma \in \Gamma^\eta[x]$. Since $m \in \mathrm{Lip}(0,T;\mathcal{P}(\overline{\Omega}))$, taking $f(t,x,v) = L(x,v) + F(x,m(t))$, one can easly check that all the assumptions of Theorem 3.1 are satisfied by f and G. Therefore, we have that $\Gamma^\eta[x] \subset C^{1,1}([0,T];\mathbb{R}^n)$ and, in this case, (3.21) becomes

$$\|\dot\gamma\|_\infty \leq L_0, \quad \forall \gamma \in \Gamma^\eta[x],$$

where $L_0 = L_0(\mu, M', M, \kappa, T, \|G\|_\infty, \|DG\|_\infty)$. □

We denote by Γ_{L_0} the set of $\gamma \in \Gamma$ such that (4.19) holds, i.e.,

$$\Gamma_{L_0} = \{\gamma \in \Gamma : \|\dot\gamma\|_\infty \leq L_0\}. \tag{4.20}$$

Lemma 4.1. *Let $m_0 \in \mathcal{P}(\overline{\Omega})$. Then, $\mathcal{P}^{\mathrm{Lip}}_{m_0}(\Gamma_{L_0})$ is a nonempty convex compact subset of $\mathcal{P}_{m_0}(\Gamma)$. Moreover, for every $\eta \in \mathcal{P}_{m_0}(\Gamma_{L_0})$, $m^\eta(t) := e_t\sharp\eta$ is Lipschitz continuous of constant L_0, where L_0 is as in Proposition 4.2.*

Proof. Arguing as in Remark 4.3, we obtain that $\mathcal{P}^{\mathrm{Lip}}_{m_0}(\Gamma_{L_0})$ is a nonempty convex set. Moreover, since Γ_{L_0} is compactly embedded in Γ, one has that $\mathcal{P}^{\mathrm{Lip}}_{m_0}(\Gamma_{L_0})$ is compact.

Let $\eta \in \mathcal{P}_{m_0}(\Gamma_{L_0})$ and $m^\eta(t) = e_t\sharp\eta$. For any $t_1, t_2 \in [0,T]$, we recall that

$$d_1(m^\eta(t_2), m^\eta(t_1)) = \sup\Big\{ \int_{\overline{\Omega}} \phi(x)(m^\eta(t_2,\,dx) - m^\eta(t_1,\,dx)) \,\Big|\, \phi : \overline{\Omega} \to \mathbb{R} \text{ is 1-Lipschitz}\Big\}.$$

Since ϕ is 1-Lipschitz continuous, one has that

$$\int_{\overline{\Omega}} \phi(x)\,(m^\eta(t_2,dx) - m^\eta(t_1,dx)) = \int_\Gamma \Big[\phi(e_{t_2}(\gamma)) - \phi(e_{t_1}(\gamma))\Big]\,d\eta(\gamma)$$
$$= \int_\Gamma \Big[\phi(\gamma(t_2)) - \phi(\gamma(t_1))\Big]\,d\eta(\gamma) \leq \int_\Gamma |\gamma(t_2) - \gamma(t_1)|\,d\eta(\gamma).$$

Since $\eta \in \mathcal{P}_{m_0}(\Gamma_{L_0})$, we deduce that

$$\int_\Gamma |\gamma(t_2) - \gamma(t_1)|\,d\eta(\gamma) \leq L_0 \int_\Gamma |t_2 - t_1|\,d\eta(\gamma) = L_0|t_2 - t_1|$$

and so $m^\eta(t)$ is Lipschitz continuous of constant L_0. □

In the next result, we deduce the existence of more regular equilibria than those constructed in [11].

Theorem 4.1. *Let Ω be a bounded open subset of $\mathbb{R}^n$ with C^2 boundary and $m_0 \in \mathcal{P}(\overline{\Omega})$. Suppose that (L0), (L1), (D1), and (D2) hold true. Then, there exists at least one constrained MFG equilibrium $\eta \in \mathcal{P}^{\mathrm{Lip}}_{m_0}(\Gamma)$.*

Proof. First of all, we recall that for any $\eta \in \mathcal{P}^{\mathrm{Lip}}_{m_0}(\Gamma)$, there exists a unique Borel measurable family * of probabilities $\{\eta_x\}_{x\in\overline{\Omega}}$ on Γ which disintegrates η in the sense that

$$\begin{cases} \eta(d\gamma) = \int_{\overline{\Omega}} \eta_x(d\gamma) m_0(\,dx), \\ supp(\eta_x) \subset \Gamma[x] \;\; m_0 - \text{a.e. } x \in \overline{\Omega} \end{cases} \tag{4.21}$$

(see, e.g., [2, Theorem 5.3.1]). Proceeding as in [11], we introduce the set-valued map

$$E : \mathcal{P}_{m_0}(\Gamma) \rightrightarrows \mathcal{P}_{m_0}(\Gamma),$$

by defining, for any $\eta \in \mathcal{P}_{m_0}(\Gamma)$,

$$E(\eta) = \left\{ \widehat{\eta} \in \mathcal{P}_{m_0}(\Gamma) : supp(\widehat{\eta}_x) \subseteq \Gamma^{\eta}[x] \;\; m_0 - \text{a.e. } x \in \overline{\Omega} \right\}. \tag{4.22}$$

We recall that, by [11, Lemma 3.6], the map E has closed graph.
Now, we consider the restriction E_0 of E to $\mathcal{P}^{\mathrm{Lip}}_{m_0}(\Gamma)$, i.e.,

$$E_0 : \mathcal{P}^{\mathrm{Lip}}_{m_0}(\Gamma_{L_0}) \rightrightarrows \mathcal{P}_{m_0}(\Gamma), \quad E_0(\eta) = E(\eta) \;\; \forall \eta \in \mathcal{P}^{\mathrm{Lip}}_{m_0}(\Gamma_{L_0}).$$

We will show that the set-valued map E_0 has a fixed point, i.e., there exists $\eta \in \mathcal{P}^{\mathrm{Lip}}_{m_0}(\Gamma_{L_0})$ such that $\eta \in E_0(\eta)$. By [11, Lemma 3.5] we have that for any $\eta \in \mathcal{P}^{\mathrm{Lip}}_{m_0}(\Gamma_{L_0})$, $E_0(\eta)$ is a nonempty convex set. Moreover, we have that

$$E_0(\mathcal{P}^{\mathrm{Lip}}_{m_0}(\Gamma_{L_0})) \subseteq \mathcal{P}^{\mathrm{Lip}}_{m_0}(\Gamma_{L_0}). \tag{4.23}$$

Indeed, let $\eta \in \mathcal{P}^{\mathrm{Lip}}_{m_0}(\Gamma_{L_0})$ and $\hat{\eta} \in E_0(\eta)$. Since, by Proposition 4.2 one has that

$$\Gamma^{\eta}[x] \subset \Gamma_{L_0} \quad \forall x \in \overline{\Omega},$$

and by definition of E_0 we deduce that

$$supp(\widehat{\eta}) \subset \Gamma_{L_0}.$$

So, $\widehat{\eta} \in \mathcal{P}_{m_0}(\Gamma_{L_0})$. By Lemma 4.1, $\widehat{\eta} \in \mathcal{P}^{\mathrm{Lip}}_{m_0}(\Gamma_{L_0})$.
Since E has closed graph, by Lemma 4.1 and (4.23) we have that E_0 has closed graph as well. Then, the assumptions of Kakutani's Theorem [30] are satisfied and so, there exists $\overline{\eta} \in \mathcal{P}^{\mathrm{Lip}}_{m_0}(\Gamma_{L_0})$ such that $\overline{\eta} \in E_0(\overline{\eta})$. □

We recall the definition of a mild solution of the constrained MFG problem, given in [11].

Definition 4.3. *We say that $(u, m) \in C([0, T]\times\overline{\Omega})\times C([0, T]; \mathcal{P}(\overline{\Omega}))$ is a mild solution of the constrained MFG problem in $\overline{\Omega}$ if there exists a constrained MFG equilibrium $\eta \in \mathcal{P}_{m_0}(\Gamma)$ such that*

(i) $m(t) = e_t \sharp \eta$ for all $t \in [0, T]$;

*We say that $\{\eta_x\}_{x\in\overline{\Omega}}$ is a Borel family (of probability measures) if $x \in \overline{\Omega} \longmapsto \eta_x(B) \in \mathbb{R}$ is Borel for any Borel set $B \subset \Gamma$.

(ii) u is given by

$$u(t,x) = \inf_{\substack{\gamma\in\Gamma \\ \gamma(t)=x}} \left\{ \int_t^T [L(\gamma(s),\dot\gamma(s)) + F(\gamma(s),m(s))]\ ds + G(\gamma(T),m(T)) \right\}, \tag{4.24}$$

for $(t,x) \in [0,T]\times\overline{\Omega}$.

Theorem 4.2. *Let* Ω *be a bounded open subset of* $\mathbb{R}^n$ *with* C^2 *boundary. Suppose that (L0),(L1), (D1) and (D2) hold true. There exists at least one mild solution* (u,m) *of the constrained MFG problem in* $\overline{\Omega}$*. Moreover,*

(i) u is Lipschitz continuous in $[0,T]\times\overline{\Omega}$*;*

(ii) $m \in \mathrm{Lip}(0,T;\mathcal{P}(\overline{\Omega}))$ *and* $\mathrm{Lip}(m) = L_0$*, where* L_0 *is the constant in* (4.19).

The question of the Lipschitz continuity up to the boundary of the value function under state constraints was addressed in [28] and [34], for stationary problems, and in a very large literature that has been published since. We refer to the survey paper [20] for references.

Proof. Let $m_0 \in \mathcal{P}(\overline{\Omega})$ and let $\eta \in \mathcal{P}^{\mathrm{Lip}}_{m_0}(\Gamma)$ be a constrained MFG equilibrium for m_0. Then, by Theorem 4.1 there exists at least one mild solution (u,m) of the constrained MFG problem in $\overline{\Omega}$. Moreover, by Theorem 4.1 one has that $m \in \mathrm{Lip}(0,T;\mathcal{P}(\overline{\Omega}))$ and $\mathrm{Lip}(m) = L_0$, where L_0 is the constant in (4.19). Finally, by Proposition 4.1 we conclude that u is Lipschitz continuous in $(0,T)\times\overline{\Omega}$. □

Remark 4.4. Recall that $F : U\times\mathcal{P}(\overline{\Omega}) \to \mathbb{R}$ is strictly monotone if

$$\int_{\overline{\Omega}} (F(x,m_1) - F(x,m_2))d(m_1-m_2)(x) \ \geq\ 0, \tag{4.25}$$

for any $m_1, m_2 \in \mathcal{P}(\overline{\Omega})$, and $\int_{\overline{\Omega}}(F(x,m_1)-F(x,m_2))d(m_1-m_2)(x) = 0$ if and only if $F(x,m_1) = F(x,m_2)$ for all $x \in \overline{\Omega}$.
Suppose that F and G satisfy (4.25). Let $\eta_1,\ \eta_2 \in \mathcal{P}^{\mathrm{Lip}}_{m_0}(\Gamma)$ be constrained MFG equilibria and let J_{η_1} and J_{η_2} be the associated functionals, respectively. Then J_{η_1} is equal to J_{η_2}. Consequently, if (u_1,m_1), (u_2,m_2) are mild solutions of the constrained MFG problem in $\overline{\Omega}$, then $u_1 = u_2$ (see [11] for a proof).

5. Appendix

In this Appendix we prove Lemma 2.1. The only case which needs to be analyzed is when $x \in \partial\Omega$. We recall that $p \in \partial^p d_\Omega(x)$ if and only if there exists $\epsilon > 0$ such that

$$d_\Omega(y) - d_\Omega(x) - \langle p, y-x\rangle \geq C|y-x|^2, \quad \text{for any } y \text{ such that } |y-x| \leq \epsilon, \tag{5.1}$$

for some constant $C \geq 0$. Let us show that $\partial^p d_\Omega(x) = Db_\Omega(x)[0,1]$. By the regularity of b_Ω, one has that

$$d_\Omega(y) - d_\Omega(x) - \langle Db_\Omega(x), y-x\rangle \geq b_\Omega(y) - b_\Omega(x) - \langle Db_\Omega(x), y-x\rangle \geq C|y-x|^2.$$

This shows that $Db_\Omega(x) \in \partial^p d_\Omega(x)$. Moreover, since

$$d_\Omega(y) - d_\Omega(x) - \langle \lambda Db_\Omega(x), y-x\rangle \geq \lambda\left(d_\Omega(y) - d_\Omega(x) - \langle Db_\Omega(x), y-x\rangle\right) \quad \forall\lambda\in[0,1],$$

we further obtain the inclusion

$$Db_\Omega(x)[0,1] \subset \partial d_\Omega(x).$$

Next, in order to show the reverse inclusion, let $p \in \partial^p d_\Omega(x) \setminus \{0\}$ and let $y \in \Omega^c$. Then, we can rewrite (5.1) as

$$b_\Omega(y) - b_\Omega(x) - \langle p, y - x\rangle \geq C|y - x|^2, \quad |y - x| \leq \epsilon. \tag{5.2}$$

Since $y \in \Omega^c$, by the regularity of b_Ω one has that

$$b_\Omega(y) - b_\Omega(x) \leq \langle Db_\Omega(x), y - x\rangle + C|y - x|^2 \tag{5.3}$$

for some constant $C \in \mathbb{R}$. By (5.2) and (5.3) one has that

$$\left\langle Db_\Omega(x) - p, \frac{y-x}{|y-x|}\right\rangle \geq C|y - x|.$$

Hence, passing to the limit for $y \to x$, we have that

$$\langle Db_\Omega(x) - p, v\rangle \geq 0, \quad \forall v \in T_{\Omega^c}(x),$$

where $T_{\Omega^c}(x)$ is the contingent cone to Ω^c at x (see e.g. [35] for a definition). Therefore, by the regularity of $\partial\Omega$,

$$Db_\Omega(x) - p = \lambda \nu(x),$$

where $\lambda \geq 0$ and $\nu(x)$ is the exterior unit normal vector to $\partial\Omega$ in x. Since $\nu(x) = Db_\Omega(x)$, we have that

$$p = (1 - \lambda)Db_\Omega(x).$$

Now, we prove that $\lambda \leq 1$. Suppose that $y \in \Omega$, then, by (5.1) one has that

$$0 = d_\Omega(y) \geq (1 - \lambda)\langle Db_\Omega(x), y - x\rangle + C|y - x|^2.$$

Hence,

$$(1 - \lambda)\left\langle Db_\Omega(x), \frac{y-x}{|y-x|}\right\rangle \leq -C|y - x|.$$

Passing to the limit for $y \to x$, we obtain

$$(1 - \lambda)\langle Db_\Omega(x), w\rangle \leq 0, \quad \forall w \in T_{\overline{\Omega}}(x),$$

where $T_{\overline{\Omega}}(x)$ is the contingent cone to Ω at x. We now claim that $\lambda \leq 1$. If $\lambda > 1$, then $\langle Db_\Omega(x), w\rangle \geq 0$ for all $w \in T_{\overline{\Omega}}(x)$ but this is impossible since $Db_\Omega(x)$ is the exterior unit normal vector to $\partial\Omega$ in x.
Using the regularity of b_Ω, simple limit-taking procedures permit us to prove that $\partial d_\Omega(x) = Db_\Omega(x)[0,1]$ when $x \in \partial\Omega$. This completes the proof of Lemma 2.1.

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Conflict of interest

The authors declare no conflict of interest.

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