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Theory article

Cumulative STF coefficients evaluation and validation

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Abstract: Decomposition of geophysical functions in ranks on degrees of components of a single position vector with coefficients in the form of the indexes of tensors, symmetric and traceless on any couple (symmetric and trace free [STF] tensors or deviators), is applied along with decomposition on surface harmonics (scalar, vector, and tensor). The article considers the problem of deviator decomposition of a function having the special form of a series of degrees of components of a unit radius vector. The algorithm evaluation of STF coefficients using known values of series coefficients is under consideration. Taking into account that often only the first several of these coefficients are used, the author created and presented a table with several coefficient formulas for reference and validation. The STF-formalism is mainly used for the representation of radiative gravity fields and gravitational waves in general relativity; however, it can also be applied in mathematical physics to represent spherical harmonics, including fluid dynamics in Earth's outer core and seismic wave analysis.

Keywords: deviator; STF decomposition; algorithm; irreducible representation

1. Introduction

Decomposition of geophysical functions in ranks on degrees of components of a single position vector with coefficients in the form of the indexes of tensors, symmetric and traceless on any couple (symmetric and trace free [STF] tensors or deviators), is applied along with decomposition on surface harmonics (scalar, vector, tensor). The term deviator comes from the elasticity theory [1]. From the point of view of group theory, the deviator part of a tensor is the first member of decomposition of tensor on irreducible representations of rotation group *SO (dim)*.

In group theory, it has been proven that deviator decompositions are equivalent to decomposition on surface harmonics (scalar, vector, and tensor), which makes it possible to use such decompositions to solve mathematical physics equations. At the same time, in some cases (e.g., boundary conditions), there is a need to represent the sum of the inner products of symmetric tensors $\tilde{F}_M$ with multi components of a unit radial vector:

$$F = \sum_{m=1}^{N} \tilde{F}_M n_M \tag{1}$$

in the form of the sum of the inner products of deviators $\hat{F}_M$:

$$F = \sum_{m=1}^{N} \hat{F}_M n_M , \tag{2}$$

where $n_M \equiv n_{i_1} .. n_{i_M}$ is the multi component of a unit radial vector in Damour's notation [2]. The multi index in Thorn notation: $T_{I_l} \equiv T_{i_1 i_2 \ldots i_l}$ [3, formulae (1.6a)] will be used in section 2.

This task can be solved directly by evaluating the $\hat{F}_M$ coefficient with the integration of sum (1) multiplied by the STF-basis tensor of rank M by a unit sphere. In this paper, an easier route is considered.

Let us consider, for example, case *N=2*:

$$F = \sum_{m=0}^{2} \tilde{F}_M n_M = \tilde{F}_{ij} n_i n_j + \tilde{F}_i n_i + \tilde{F}_0 .$$

The Einstein summation notation is used whenever we have an expression with a repeated index (multi index). Thus, we implicitly know to sum over that index (indexes of multi index) from 1 to dim, where dim is notation of the dimension of the space. The dimension of space (dim) can be also written as $\delta_{kk} = \underbrace{1+1+\ldots+1}_{\dim} = \dim$.

One can obtain the following for STF part of tensor with rank 2: $\tilde{F}_{ij} = \hat{F}_{ij} + \frac{1}{\dim} \delta_{ij} F_{kk}$. Substituting this into an expression for *F* and introducing similar ones, one can get:

$$F = \sum_{m=0}^{2} \tilde{F}_M n_M = \left(\hat{F}_{ij} + \frac{1}{\dim} \delta_{ij} F_{kk} \right) n_i n_j + \tilde{F}_i n_i + \tilde{F}_0 = \hat{F}_{ij} n_i n_j + \hat{F}_i n_i + \hat{F}_0 ,$$

where deviator coefficients are equal:

$$\hat{F}_{ij} = \tilde{F}_{ij} - \frac{1}{\dim} \delta_{ij} F_{kk} , \quad \hat{F}_k = \tilde{F}_k , \quad \hat{F}_0 = \tilde{F}_0 + \frac{1}{\dim} F_{kk} .$$

In this simple case, formulas for finding deviator coefficients from symmetric coefficients are very easy, and it is not needed in some special evaluations. However, when the number of sum members *N* is growing, the difficulties increase because the fragments of the members of the senior ranks are "poured" into the members of the lower ranks and the more members there are in total, the more complex they are to enter there.

The algorithm of estimation of coefficients $\hat{F}_K$ of sum (2) using knowing values of symmetric coefficients $\tilde{F}_J$ of sum (1) is under consideration. Taking into account that often only the first several of these coefficients are used, the author created and presented a table with several coefficient formulas for reference and validation.

2. The algebraic algorithm for cumulative STF coefficients

The algebraic algorithm for cumulative STF coefficients estimation was proposed and proved by author in [4] and is represented here in Supplementary in the English translation. It can be formulated as the following theorem about cumulative STF coefficients.

Theorem about cumulative STF coefficients. Sum (1) can be represented as exactly equal to its sum (2) with the deviator coefficients estimated according to formulas:

$$\hat{F}_{2J} = \sum_{s=j}^{\left[\frac{N}{2}\right]} \tilde{a}(2s, s-j)\tilde{F}_{<2J>K_{S-J}K_{S-J}}, \qquad \hat{F}_{2J+1} = \sum_{s=j}^{\left[\frac{N-1}{2}\right]} \tilde{a}(2s+1, s-j)\tilde{F}_{<2J+1>K_{S-J}K_{S-J}}, \tag{3}$$

where $\tilde{a}(n,l)$ are cumulative STF coefficients and calculated by values coefficients $a(n,l)$ of STF part of tensor by following the recurrent algorithm.

1) for *l=0,1,2* cumulative coefficients are evaluated by formulas:

$$\tilde{a}(n,0) = 1, \quad \tilde{a}(n,1) = -a(n,1), \quad \tilde{a}(n,2) = a(n,1)a(n-2,1) - a(n,2). \tag{4}$$

2) for *l >2*:

– evaluated $a(n, l-2, 1)$ by formula:

$$a(n, l-2, 1) = a(n,1)a(n-2, l-1) - a(n,l) \tag{5}$$

– for *k=1,…,l-2* evaluated:

$$a(n, l-2-k, k+1) = a(n, l-1-k, k) - a(n-2(k+1), l-1-k)\tilde{a}(n, k+1) \tag{6}$$

Last coefficient $a(n, 0, l-1)$ is equal to $\tilde{a}(n,l)$.

Remark. At the same time, the coefficients $\tilde{a}(n, k+1)$ for *k=1,…,l-2* have to be evaluated on previous steps.

The formulae for the coefficients $a(n,l)$ of STF part of tensor for 3-dimentional space one can be found in [1, formulae (2.2c)]. For multidimensional space, it can be found in [4, formulae (4)].

3. Results

Taking into account that often only the first several of these coefficients are used, the algebraic algorithm for cumulative STF coefficients estimation was used by author for the evaluation the first several cumulative STF coefficients (see Supplementary). For reference, author prepared the following tables.

4. Validation of Table 1 formulas

The following procedure was used for the validation of formulas for cumulative *STF* coefficients presented at Table 1. At first, the *STF* coefficients of sum (2) must be expressed in terms of sum (1) for given sum limit *N*, according to formulae (3) with cumulative STF coefficient from Table 1. After that, these expressions must be inserted in sum (2) and similar ones have to be introduced. If the sum (1) will be obtained, then the validation is assumed to be successful. Validation for *N=2* (see example in Introduction) obviously is successful.

Table 1. The cumulative STF coefficients $\tilde{a}(n,l)$ for n from 1 to 7.

n	*L* 0	1	2	3
0	1	–	–	–
1	1	–	–	–
2	1	$\frac{1}{\dim}$	–	–
3	1	$\frac{3}{(\dim+2)}$	–	–
4	1	$\frac{6}{(4+\dim)}$	$\frac{3}{(2+\dim)\dim}$	–
5	1	$\frac{10}{(6+\dim)}$	$\frac{15}{(4+\dim)(2+\dim)}$	–
6	1	$\frac{15}{(8+\dim)}$	$\frac{45}{(6+\dim)(4+\dim)}$	$\frac{15}{(4+\dim)(2+\dim)\dim}$
7	1	$\frac{21}{(10+\dim)}$	$\frac{105}{(8+\dim)(6+\dim)}$	$\frac{105}{(6+\dim)(4+\dim)(2+\dim)}$

4.1. Validation formulas for N=3

Formulas (3) for N=3 can be written as:

$$\hat{F}_0 = \tilde{F}_0 + \tilde{a}(2,1)\tilde{F}_{ss} = \tilde{F}_0 + \frac{1}{\dim}\tilde{F}_{ss}, \quad \hat{F}_i = \tilde{F}_i + \tilde{a}(3,1)\tilde{F}_{iss} = \tilde{F}_i + \frac{3}{(2+\dim)}\tilde{F}_{iss},$$

$$\hat{F}_{ij} = \tilde{F}_{\langle ij\rangle} = \tilde{F}_{ij} - \frac{1}{\dim}\delta_{ij}\tilde{F}_{ss}, \quad \hat{F}_{ijk} = \tilde{F}_{\langle ijk\rangle} = \tilde{F}_{ijk} - \frac{3}{(2+\dim)}\frac{1}{3}\left(\delta_{ij}\tilde{F}_{ssk} + \delta_{ik}\tilde{F}_{ssj} + \delta_{jk}\tilde{F}_{ssi}\right).$$

Now, inserting these in (2) and introducing similar ones, one can obtain:

$$F = \tilde{F}_0 + \left\{\frac{1}{\dim}\tilde{F}_{ss}\right\} + \tilde{F}_i n_i + \left[\frac{3}{(2+\dim)}\tilde{F}_{iss}n_i\right] + \tilde{F}_{ij}n_i n_j + \left\{-\frac{1}{\dim}n_i n_j \delta_{ij}\tilde{F}_{ss}\right\} + \tilde{F}_{ijk}n_i n_j n_k +$$

$$+\left[-\frac{3}{(2+\dim)}\frac{1}{3}\left(\delta_{ij}\tilde{F}_{kss} + \delta_{ik}\tilde{F}_{jss} + \delta_{jk}\tilde{F}_{iss}\right)n_i n_j n_k\right] = \tilde{F}_0 + \tilde{F}_i n_i + \tilde{F}_{ij}n_i n_j + \tilde{F}_{ijk}n_i n_j n_k.$$

Thus, validation is successful. The shrinking members are enclosed in big curly and square brackets.

4.2. Validation formulas for N=4

Formulas (3) for N=4 can be written as:

$$\hat{F}_0 = \tilde{F}_0 + \tilde{a}(2,1)\tilde{F}_{ss} + \tilde{a}(4,2)\tilde{F}_{ssmm} = \tilde{F}_0 + \frac{1}{\delta_{kk}}\tilde{F}_{ss} + \left\{\frac{3}{(2+\dim)\dim}\tilde{F}_{ssmm}\right\},$$

$$\hat{F}_i = \tilde{F}_i + \tilde{a}(3,1)\tilde{F}_{iss} = \tilde{F}_i + \frac{3}{(2+\dim)}\tilde{F}_{iss},$$

$$\hat{F}_{ij} = \tilde{F}_{\langle ij \rangle} + \tilde{a}(4,1)\tilde{F}_{\langle ij \rangle ss} = \tilde{F}_{ij} - \frac{1}{\dim}\delta_{ij}\tilde{F}_{ss} + \left[\frac{6}{(4+\dim)}\tilde{F}_{ijss}\right] + \left\{-\frac{6}{(4+\dim)}\frac{1}{\dim}\delta_{ij}\tilde{F}_{mmss}\right\},$$

$$\hat{F}_{ijk} = \tilde{F}_{\langle ijk \rangle} = \tilde{F}_{ijk} - \frac{3}{(2+\dim)}\frac{1}{3}\left(\delta_{ij}\tilde{F}_{ssk} + \delta_{ik}\tilde{F}_{ssj} + \delta_{jk}\tilde{F}_{ssi}\right),$$

$$\hat{F}_{ijkl} = \tilde{F}_{\langle ijkl \rangle} = \tilde{F}_{ijkl} + \left[-\frac{6}{(4+\dim)}\frac{1}{6}\left(\delta_{ij}\tilde{F}_{sskl} + \delta_{ik}\tilde{F}_{ssjl} + \delta_{il}\tilde{F}_{ssjk} + \delta_{jk}\tilde{F}_{ssil} + \delta_{jl}\tilde{F}_{ssik} + \delta_{lk}\tilde{F}_{ssij}\right)\right] +$$
$$+\left\{\frac{3}{(4+\dim)(2+\dim)}\frac{1}{3}\left(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)\tilde{F}_{ssmm}\right\}.$$

As shown above, the members which not enclosed in big brackets, give us the first 4 members of sum (1), excluding one with coefficient $\tilde{F}_{ijkl}$. Thus, only the new members with coefficients enclosed in big brackets and $\tilde{F}_{ijkl}$ have to be checked. The members of sum (2) of these must give the result $\tilde{F}_{ijkl} n_i n_j n_k n_k$. After evaluating these members and introducing similar ones, one can obtain $\tilde{F}_{ijkl} n_i n_j n_k n_k$. So, validation is successful.

Remark. In paper [4], the example for *N*=4 has some mistakes because of a big rush before publication; however, all formulas here are correct.

4.3. Validation formulas for N=5

Similar to the previous case, it is enough to check only for new odd-ranked members:

$$\hat{F}_{i} = \tilde{F}_{i} + \tilde{a}(3,1)\tilde{F}_{iss} + \tilde{a}(5,2)\tilde{F}_{isskk} = \tilde{F}_{i} + \frac{3}{(2+\dim)}\tilde{F}_{iss} + \left\{\frac{15}{(4+\dim)(2+\dim)}\tilde{F}_{isskk}\right\},$$

$$\hat{F}_{ijk} = \tilde{F}_{\langle ijk \rangle} + \tilde{a}(5,1)\tilde{F}_{\langle ijk \rangle ss} = \tilde{F}_{ijk} - \frac{3}{(2+\dim)}\frac{1}{3}\left(\delta_{ij}\tilde{F}_{ssk} + \delta_{ik}\tilde{F}_{ssj} + \delta_{jk}\tilde{F}_{ssi}\right) +$$
$$+\left[\frac{10}{(6+\dim)}\tilde{F}_{ijkss}\right] + \left\{-\frac{10}{(6+\dim)}\frac{3}{(2+\dim)}\frac{1}{3}\left(\delta_{ij}\tilde{F}_{sswwk} + \delta_{ik}\tilde{F}_{sswwj} + \delta_{jk}\tilde{F}_{wwssi}\right)\right\},$$

$$\hat{F}_{ijklm} = \tilde{F}_{\langle ijklm \rangle} = \tilde{F}_{ijklm} -$$
$$-\left[\frac{10}{(6+\dim)}\frac{1}{10}\left(\delta_{ij}\tilde{F}_{ssklm} + \delta_{ik}\tilde{F}_{ssjlm} + \delta_{il}\tilde{F}_{ssjkm} + \delta_{jk}\tilde{F}_{ssilm} + \delta_{jl}\tilde{F}_{ssikm} + \delta_{lk}\tilde{F}_{ssijm} + \delta_{im}\tilde{F}_{ssklj} + \delta_{km}\tilde{F}_{ssijl} + \delta_{lm}\tilde{F}_{ssijk} + \delta_{jm}\tilde{F}_{ssilk}\right)\right] +$$
$$+\left\{\frac{15}{(6+\dim)(4+\dim)}\frac{1}{15}\left(\delta_{ij}\delta_{kl}\tilde{F}_{sswwm} + \delta_{ij}\delta_{km}\tilde{F}_{sswwl} + \delta_{ij}\delta_{lm}\tilde{F}_{sswwk} + \delta_{ik}\delta_{jl}\tilde{F}_{sswwm} + \delta_{ik}\delta_{jm}\tilde{F}_{sswwl} + \delta_{ik}\delta_{lm}\tilde{F}_{sswwj} + \delta_{il}\delta_{jk}\tilde{F}_{sswwm} +\right.\right.$$
$$\left.\left. + \delta_{il}\delta_{jm}\tilde{F}_{sswwk} + \delta_{il}\delta_{km}\tilde{F}_{sswwj} + \delta_{im}\delta_{jk}\tilde{F}_{sswwl} + \delta_{im}\delta_{jl}\tilde{F}_{sswwk} + \delta_{im}\delta_{kl}\tilde{F}_{sswwj} + \delta_{jk}\delta_{lm}\tilde{F}_{sswwi} + \delta_{jl}\delta_{km}\tilde{F}_{sswwi} + \delta_{jm}\delta_{kl}\tilde{F}_{sswwi}\right)\right\}$$

Other members are the same as in previous case. After inserting the coefficients in (2) and introducing similar ones, one can obtain $\tilde{F}_{ijklm} n_i n_j n_k n_k n_m$ for the sum members with coefficients enclosed in big brackets and $\tilde{F}_{ijklm}$. So, validation is successful.

4.4. Validation formulas for N=6

Similar to the previous case, it is enough to check only for new even-ranked members:

$$\hat{F}_0 = \tilde{F}_0 + \tilde{a}(2,1)\tilde{F}_{ss} + \tilde{a}(4,2)\tilde{F}_{ssmm} + \tilde{a}(6,3)\tilde{F}_{ssmm} = \tilde{F}_0 + \frac{1}{\dim}\tilde{F}_{ss} + \frac{3}{(2+\dim)\dim}\tilde{F}_{ssmm} +$$

$$+\left\{\frac{15}{(4+\dim)(2+\dim)\dim}\tilde{F}_{ssmmrr}\right\},$$

$$\hat{F}_{ij} = \tilde{F}_{\langle ij\rangle} + \tilde{a}(4,1)\tilde{F}_{\langle ij\rangle ss} + \tilde{a}(6,2)\tilde{F}_{\langle ij\rangle ssrr} = \tilde{F}_{ij} - \frac{1}{\dim}\delta_{ij}\tilde{F}_{ss} + \frac{6}{(4+\dim)}\left(\tilde{F}_{ijss} - \frac{1}{\dim}\delta_{ij}\tilde{F}_{mmss}\right) +$$

$$+\left[\frac{45}{(6+\dim)(4+\dim)}\tilde{F}_{ijssrr}\right] + \left\{-\frac{45}{(6+\dim)(4+\dim)}\frac{1}{\dim}\delta_{ij}\tilde{F}_{mmssrr}\right\},$$

$$\hat{F}_{ijkl} = \tilde{F}_{\langle ijkl\rangle} + \tilde{a}(6,1)\tilde{F}_{\langle ijkl\rangle ss} = \tilde{F}_{ijkl} - \frac{6}{(4+\dim)}\frac{1}{6}\left(\delta_{ij}\tilde{F}_{sskl} + \delta_{ik}\tilde{F}_{ssjl} + \delta_{il}\tilde{F}_{ssjk} + \delta_{jk}\tilde{F}_{ssil} + \delta_{jl}\tilde{F}_{ssik} + \delta_{lk}\tilde{F}_{ssij}\right) +$$

$$+\frac{3}{(4+\dim)(2+\dim)}\frac{1}{3}\left(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)\tilde{F}_{ssmm} + \left(\frac{15}{(8+\dim)}\tilde{F}_{ijklss}\right) + \left[-\frac{15}{(8+\dim)}\frac{6}{(4+\dim)}\frac{1}{6}\left(\delta_{ij}\tilde{F}_{ssklrr} +\right.\right.$$

$$\left.\left.+\delta_{ik}\tilde{F}_{ssjlrr} + \delta_{il}\tilde{F}_{ssrrjk} + \delta_{jk}\tilde{F}_{rrssil} + \delta_{jl}\tilde{F}_{rrssik} + \delta_{lk}\tilde{F}_{rrssij}\right)\right] + \left\{\frac{15}{(8+\dim)}\frac{3}{(4+\dim)(2+\dim)}\frac{1}{3}\left(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)\tilde{F}_{ssmmrr}\right\},$$

$$\hat{F}_{ijklmp} = \tilde{F}_{\langle ijklmp\rangle} = \tilde{F}_{ijklmp} + \left(-\frac{15}{(8+\dim)}\frac{1}{15}\left(\delta_{ij}\tilde{F}_{ssklmp} + \ldots + \delta_{jm}\tilde{F}_{ssilk}\right)\right) + \left[\frac{45}{(8+\dim)(6+\dim)}\frac{1}{45}\left(\delta_{ij}\delta_{kl}\tilde{F}_{sswwmp} + \ldots\ \right)\right] +$$

$$+\left\{-\frac{15}{(8+\dim)(6+\dim)(4+\dim)}\frac{1}{15}\left(\delta_{ij}\delta_{kl}\delta_{mp} + \ldots\ \right)\tilde{F}_{ssmmrr}\right\}.$$

Other members are the same as in previous case. The ellipsis denotes similar terms, the form of which is understandable from context. After inserting the coefficients in (2) and introducing the similar ones, one can obtain $\tilde{F}_{ijklmp}n_i n_j n_k n_k n_m n_p$ for sum of members with coefficients enclosed in big brackets and $\tilde{F}_{ijklmp}$. So, validation is successful.

4.5. Validation formulas for N=7

Similar to the previous case, it is enough to check only for new odd-ranked members:

$$\hat{F}_i = \tilde{F}_i + \tilde{a}(3,1)\tilde{F}_{iss} + \tilde{a}(5,2)\tilde{F}_{isskk} + \tilde{a}(7,3)\tilde{F}_{isskkrr} = \tilde{F}_i + \frac{3}{(2+\dim)}\tilde{F}_{iss} + \frac{15}{(4+\dim)(2+\dim)}\tilde{F}_{isskk} +$$

$$+\left\{\frac{105}{(6+\dim)(4+\dim)(2+\dim)}\tilde{F}_{isskkrr}\right\},$$

$$\hat{F}_{ijk} = \tilde{F}_{\langle ijk \rangle} + \tilde{a}(5,1)\tilde{F}_{\langle ijk \rangle ss} + \tilde{a}(7,2)\tilde{F}_{\langle ijk \rangle ssrr} = \tilde{F}_{ijk} - \frac{3}{(2+\dim)}\frac{1}{3}\left(\delta_{ij}\tilde{F}_{ssk} + \delta_{ik}\tilde{F}_{ssj} + \delta_{jk}\tilde{F}_{ssi}\right) +$$

$$+\frac{10}{(6+\dim)}\left(\tilde{F}_{ijkss} - \frac{3}{(2+\dim)}\frac{1}{3}\left(\delta_{ij}\tilde{F}_{sswwk} + \delta_{ik}\tilde{F}_{sswwj} + \delta_{jk}\tilde{F}_{wwssi}\right)\right) +$$

$$+\left[\frac{105}{(8+\dim)(6+\dim)}\tilde{F}_{ijkssrr}\right] + \left\{-\frac{105}{(8+\dim)(6+\dim)}\frac{3}{(2+\dim)}\frac{1}{3}\left(\delta_{ij}\tilde{F}_{sswwrrk} + \delta_{ik}\tilde{F}_{sswwrrj} + \delta_{jk}\tilde{F}_{wwssrri}\right)\right\},$$

$$\hat{F}_{ijklm} = \tilde{F}_{\langle ijklm \rangle} + \tilde{a}(7,1)\tilde{F}_{\langle ijklm \rangle ss} = \tilde{F}_{ijklm} - \frac{10}{(6+\dim)}\frac{1}{10}\left(\delta_{ij}\tilde{F}_{ssklm} + ... + \delta_{jm}\tilde{F}_{ssilk}\right) + \frac{15}{(6+\dim)(4+\dim)}\frac{1}{15}\left(\delta_{ij}\delta_{kl}\tilde{F}_{sswwm} +\right.$$

$$\left. + ... + \delta_{jm}\delta_{kl}\tilde{F}_{sswwi}\right) + \left(\frac{21}{(10+\dim)}\tilde{F}_{ijklmrr}\right) + \left[-\frac{21}{(10+\dim)}\frac{10}{(6+\dim)}\frac{1}{10}\left(\delta_{ij}\tilde{F}_{sswwklm} + \ldots + \delta_{jm}\tilde{F}_{sswwilk}\right)\right]$$

$$+\left\{\frac{15}{(6+\dim)(4+\dim)}\frac{1}{15}\left(\delta_{ij}\delta_{kl}\tilde{F}_{sswwrrm} + \ldots + \delta_{jm}\delta_{kl}\tilde{F}_{sswwi}\right)\right\},$$

$$\hat{F}_{ijklmps} = \tilde{F}_{\langle ijklmps \rangle} = \tilde{F}_{ijklmps} + \left(-\frac{21}{(10+\dim)}\frac{1}{21}\left(\delta_{ij}\tilde{F}_{wwklmps} + \ldots + \delta_{jm}\tilde{F}_{wwilks}\right)\right) +$$

$$+\left[\frac{105}{(10+\dim)(8+\dim)}\frac{1}{105}\left(\delta_{ij}\delta_{kl}\tilde{F}_{rrwwmps} + \ldots\right)\right] + \left\{-\frac{105}{(10+\dim)(8+\dim)(6+\dim)}\frac{1}{105}\left(\delta_{ij}\delta_{kl}\delta_{mp}\tilde{F}_{wwmmrrs} + \ldots\right)\right\},$$

Other members are the same as in previous case. The ellipsis denotes similar terms, the form of which is understandable from context. After inserting coefficients in (2) and introducing similar ones, one can obtain $\tilde{F}_{ijklmps} n_i n_j n_k n_k n_m n_p n_s$ for sum of members with coefficients enclosed in big brackets and $\tilde{F}_{ijklmps}$. So, validation is successful.

5. Conclusions

The cumulative STF coefficients formulas were obtained based on algebraic algorithm [4] for space with finite dimension. The formulas were checked analytically, and validation was successful. The formulas of Table 1 can be used for easier transition from sum in form (1) to sum in form (2) compared with directly using the theorem about cumulative STF coefficients algorithm. In conclusion, it should be noted that STF-formalism can be used for the solution of all mathematical physics tasks that include spherical harmonics. For example, it was used by author for calculating the added masses tensor in the solution of the Earth's solid inner core motion [5].

Use of AI tools declaration

The author declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of Interest

The authors declare no conflict of interest.

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