Mathematical Biosciences and Engineering

https://www.aimspress.com/journal/mbe

MBE, 22(6): 1572–1597.
DOI: 10.3934/mbe.2025058
Received: 07 October 2024
Revised: 07 April 2025
Accepted: 18 April 2025
Published: 15 May 2025

Research article

Measures of cumulative residual Tsallis entropy for concomitants of generalized order statistics based on the Morgenstern family with application to medical data

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Abstract: Ever since Tsallis introduced Tsallis entropy theory, it has been applied to a wide variety of topics in physics and chemistry, with new applications being discovered annually. The amount of research suggests that the Tsallis entropy concept holds significant potential. This paper introduces weighted cumulative residual Tsallis entropy (WCRTE) and weighted cumulative past Tsallis entropy (WCPTE), as well as their dynamic counterparts for the concomitants of m-generalized order statistics (m-GOSs) derived from the Farlie-Gumbel-Morgenstern bivariate family. The characteristics of the proposed entropy measures were analyzed, demonstrating their ability to characterize the Pareto and exponential distributions. Applications of these findings were presented for order statistics (OSs) systems and record values with uniform, Weibull, and power marginal distributions. Furthermore, the empirical alternatives WCRTE and WCPTE were proposed for calculating new information measures. Two real-world data sets have been evaluated for illustrative purposes, demonstrating satisfactory performance.

Keywords: weighted cumulative residual Tsallis entropy; weighted cumulative past Tsallis entropy; Farlie-Gumbel-Morgenstern (FGM) family; concomitants; generalized order statistics

1. Introduction

The Farlie-Gumbel-Morgenstern (FGM) family of bivariate distributions has been widely utilized in practical applications. This family is defined by the given marginal distribution functions (DFs) $F_Y(y)$

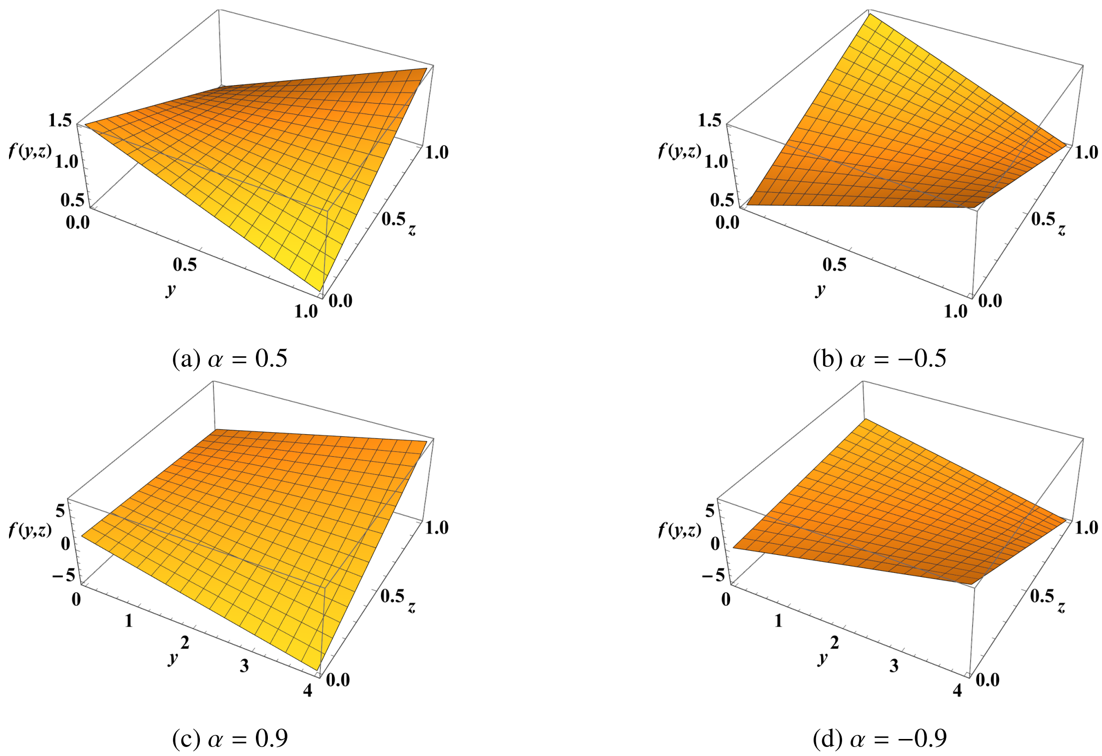
and $F_Z(z)$ for the random variables (RVs) Y and Z, respectively, along with a parameter α, resulting in the bivariate DF given by

$$F_{Y,Z}(y,z) = F_Y(y)F_Z(z)[1 + \alpha(1 - F_Y(y))(1 - F_Z(z))], \tag{1.1}$$

with the corresponding probability density function (PDF)

$$f_{Y,Z}(y,z) = f_Y(y)f_Z(z)[1 + \alpha(2F_Y(y) - 1)(2F_Z(z) - 1)]. \tag{1.2}$$

Here $f_Y(y)$ and $f_Z(z)$ are the marginals of $f_{Y,Z}(y,z)$, and α is referred to as the association parameter. The RVs Y and Z are treated as independent at $\alpha = 0$. The initial introduction of this model was done by [1] and further examined by [2] for exponential marginals. The overall structure described in Eq (1.1) is credited to [3, 4]. The parameter α must lie in the range $[-1, 1]$ to ensure that the function $F_{Y,Z}(y,z)$ remains a valid DF. The Pearson correlation coefficient ρ between Y and Z cannot exceed $\frac{1}{3}$. An aspect of the behavior of the bivariate PDF with the associated parameter α for the FGM association (i.e., the FGM family with uniform margins) is depicted in Figure 1.



(a) $\alpha = 0.5$ (b) $\alpha = -0.5$

(c) $\alpha = 0.9$ (d) $\alpha = -0.9$

Figure 1. The bivariate PDF for FGM copula.

Reference [5] proposed the concept of the GOSs as a comprehensive framework for RVs arranged in ascending order. The subclass $m-$GOSs of GOSs encompasses the most prominent models of ordered RVs. These models include OSs, lower and upper record values, $k-$records, sequential OSs, and progressive type II censoring with a constant scheme. We exclusively consider the $m-$GOSs model. Consider $F_Y(y) = P(Y \leq y)$ as an arbitrary continuous DF with the PDF $f_Y(y)$. Then the RVs

$Y(1,n,m,k) \leq Y(2,n,m,k) \leq ... \leq Y(n,n,m,k)$ $(k > 0,\ m \geq -1)$ are said to be $m-$GOSs if their joint PDF (JPDF) is given by

$$f^{(m,k)}_{1,2,...,n:n}(y_1, y_2, ..., y_n) = \left(\prod_{j=1}^{n} \gamma_j\right)\left(\prod_{j=1}^{n-1} \overline{F}^m_Y(y_j) f_Y(y_j)\right) \overline{F}^{k-1}_Y(y_n) f_Y(y_n),$$

where $F^{-1}_Y(1) \geq y_n \geq ... \geq y_1 \geq F^{-1}_Y(0)$ and $\gamma_j = k + (n-j)(m+1) > 0, j = 1,2,...,n$ (note that $\gamma_n = k$). The marginal PDF of rth $m-$GOS, $1 \leq r \leq n$, is given by (cf. [5])

$$f_{Y(r,n,m,k)}(y) = \frac{C_{r-1}}{(r-1)!}(\overline{F}_Y(y))^{\gamma_r - 1} f_Y(y) g^{r-1}_m(F_Y(y)),$$

where $C_{r-1} = \prod_{i=1}^{r} \gamma_i, r = 1,2,...,n,\ g_m(y) = h_m(y) - h_m(0),\ y \in [0,1)$, and $h_m(y) = \frac{-1}{(m+1)}(1-y)^{m+1}$, if $m \neq -1$, while $h_{-1}(y) = -\log(1-y)$.

The concomitant subject has recently gained traction again because of its applicability to prediction problems and selection processes. The concept of the concomitants of OSs may be traced back to [6], while [7] further developed the broad theory of concomitants of OSs. Reference [6] provided an exceptional examination of the concomitants of OSs. Compared to the concomitants of OSs, the concomitants of GOSs have not been as fully studied. Numerous researchers have studied this topic, including [8–10].

For the FGM family defined by (1.1) and (1.2), the PDF, DF, and survival function (SF) of the concomitant of the rth m-GOS are given by [9], respectively, as

$$f_{[r,n,m,k]}(z) = f_Z(z)\left[1 + \alpha C^*_{(r,n,m,k)}(1 - 2F_Z(z))\right],$$

$$F_{[r,n,m,k]}(z) = F_Z(z)\left[1 + \alpha C^*_{(r,n,m,k)}(1 - F_Z(z))\right], \tag{1.3}$$

and

$$\overline{F}_{[r,n,m,k]}(z) = \overline{F}_Z(z)\left[1 - \alpha C^*_{(r,n,m,k)} F_Z(z)\right], \tag{1.4}$$

where $C^*_{(r,n,m,k)} = \frac{2\prod_{j=1}^{r}\gamma_j}{\prod_{i=1}^{r}(\gamma_i+1)} - 1$ and $\overline{F}_Z(z) = 1 - F_Z(z)$.

Reference [9] conducted a study on the concomitants of m-GOSs in the FGM family. Reference [11] derived certain characteristics of the concomitants of m-GOSs from the bivariate Rayleigh distribution of FGM type. References [12, 13] examined various information measures in concomitants of m-GOSs under iterated FGM and Huang-Kotz FGM, respectively. References [14–16] examined certain characteristics of concomitants of m-GOSs from the bivariate Cambanis family.

Shannon entropy, established by [17], is one of the most extensively used measures of uncertainty. It is used to quantify the amount of uncertainty involved in an RV and has multiple uses in various fields. Let Y be a non-negative RV having PDF $f_Y(y)$. [17] defined the entropy of Y by

$$\mathcal{H}(Y) = -\int_0^\infty f_Y(y)\log f_Y(y)dy.$$

An entropic expression with an index θ was first presented by [18] and results in non-extensive statistics. The so-called non-extensive statistical mechanics, which generalize Boltzmann-Gibbs theory, is based on Tsallis entropy. Applications of Tsallis statistics can be found in a wide range of fields, including physics, chemistry, biology, medicine, economics, and geophysics. According to [18], the non-additive generalization of Shannon's entropy of order θ is called Tsallis entropy. This measure is crucial in the uncertainty assessments of an RV Y, defined as

$$\mathcal{T}_\theta(Y) = \frac{1}{\theta - 1}\left(1 - \int_0^\infty f_Y^\theta(y)dy\right),$$

where $\theta \neq 1, \theta > 0$. Tsallis entropy approaches Shannon entropy as $\theta \longrightarrow 1$.

Tsallis statistics, rooted in the generalized entropy framework, are particularly valuable for systems with long-range interactions, non-equilibrium dynamics, and fractal-like structures. These properties make them highly applicable to fields dealing with complex and noisy data, such as seismic inversion and magnetic resonance imaging evaluations. For example, seismic inversion involves extracting subsurface geological properties (e.g., porosity, fluid content, rock type) from seismic reflection data. These data sets are typically noisy, incomplete, and exhibit long-range spatial correlations, making them ideal candidates for Tsallis statistical approaches. In seismic inversion and MRI evaluations, Tsallis entropy and statistics provide enhanced tools for dealing with noisy, complex, and non-linear data. They enable better noise filtering, improved stability in inversion algorithms, and deeper insights into fractal-like patterns and tissue textures, ultimately leading to more accurate geological modeling and medical diagnoses. For additional information regarding Tsallis entropy, we suggest reading [19, 20]. Numerous generalizations of Shannon entropy have been formulated, rendering these entropies responsive to various types of probability distributions by the incorporation of several additional factors.

Reference [21] presented a novel measure of Shannon entropy known as cumulative residual entropy (CRE), which utilizes the SF rather than the PDF. The CRE is regarded as more stable and mathematically robust due to its more consistent SF compared to the PDF. Moreover, DFs exist even when PDFs do not exist (e.g., Govindarajulu, power-Pareto, and generalized lambda distributions). The CRE measure based on the SF $\overline{F}_Y(y)$ is defined as

$$J(Y) = -\int_0^\infty \overline{F}_Y(y)\log\overline{F}_Y(y)dy.$$

The cumulative residual Tsallis entropy (CRTE) of order θ is a helpful generalization of Shannon entropy that can be considered an alternate dispersion measure and has demonstrated promising results in several applications. Reference [22] devised the measure CRTE, which is defined by

$$\mathcal{J}_\theta(Y) = \frac{1}{\theta - 1}\left(1 - \int_0^\infty \overline{F}_Y^\theta(y)dy\right),\ \theta > 0,\ \theta \neq 1.$$

When $\theta \longrightarrow 1$, CRTE approaches CRE.

Reference [23] unveiled an alternative measure for CRTE, which is defined by

$$\mathcal{E}_\theta(Y) = \frac{1}{\theta - 1}\int_0^\infty \left(\overline{F}_Y(y) - \overline{F}_Y^\theta(y)\right)dy,\ \theta > 0,\ \theta \neq 1. \tag{1.5}$$

Unlike CRTE, the measure defined by (1.5) has some additional features and has simple relationships with other important information and reliability measures. Reference [24] established a parallel notion of CRE, the cumulative entropy, which is useful for measuring information when uncertainty is associated with the past. Motivated from (1.5), reference [25] introduced cumulative past Tsallis entropy (CPTE) which is defined as

$$\mathcal{P}_\theta(Y) = \frac{1}{\theta - 1} \int_0^\infty \left(F_Y(y) - F_Y^\theta(y)\right) dy, \ \theta > 0, \ \theta \neq 1.$$

Assigning appropriate weights is crucial for accurately reflecting the relative relevance or importance of different observations or events in a data set. Weighting methodologies ensure that observations or events in a data set are appropriately prioritized based on their relevance or importance. Key approaches include:

1) Statistical weighting: based on variance (lower variance = higher weight) or frequency (higher frequency = higher weight).
2) Entropy-based weighting: higher entropy events get higher weights due to their greater uncertainty.
3) Expert knowledge-based weighting: relies on domain experts' insights.
4) Information gain-based weighting: events providing more information gain are assigned higher weights.
5) Distance-based weighting: observations closer to a reference point have higher weights (e.g., inverse distance weighting).

Each methodology suits different scenarios and data set characteristics, and the choice depends on the objective and data context. Reference [26] defined weighted CPTE (WCPTE) of order θ as

$$\mathcal{P}_\theta^w(Y) = \frac{1}{\theta - 1} \int_0^\infty y\left(F_Y(y) - F_Y^\theta(y)\right) dy, \ \theta > 0, \ \theta \neq 1. \tag{1.6}$$

It was shown that the WCPTE can be used as a risk measure. Moreover, [27] proposed weighted CRTE (WCRTE) defined as

$$\zeta_\theta^w(Y) = \frac{1}{\theta - 1}\left(1 - \int_0^\infty y\overline{F}_Y^\theta(y) dy\right), \ \theta > 0, \ \theta \neq 1. \tag{1.7}$$

For a non-negative continuous RV Y, [26] defined a new measure called "alternative weighted cumulative residual Tsallis entropy" (denoted as AWCRTE) by

$$\xi_\theta^w(Y) = \frac{1}{\theta - 1} \int_0^\infty y\left(\overline{F}_Y(y) - \overline{F}_Y^\theta(y)\right) dy, \ \theta > 0, \ \theta \neq 1. \tag{1.8}$$

Despite the extensive use of Tsallis entropy in fields such as medical imaging (e.g., see [28,29]), there remains a need for further exploration of how Tsallis entropy can be applied to more complex, dynamic systems−especially when considering weighted measures that reflect cumulative effects over time.

This paper introduces the concepts of WCRTE and WCPTE, along with their dynamic versions, derived from m-GOSs in the context of the FGM family. The motivation for this work stems from the desire to enhance the versatility and applicability of Tsallis entropy in systems where dynamic and weighted measures are essential, such as in medical imaging, environmental studies, and artificial intelligence. The novel entropy measures introduced here offer new approaches for analyzing non-extensive and complex systems that cannot be adequately described by traditional entropy measures, such as Shannon entropy. The integration of these new tools with existing models opens up new possibilities for understanding and predicting behaviors in dynamic systems.

This is how the remainder of the paper is structured. Section 2 discusses some cases and provides the WCRTE for the concomitant $Z_{[r,n,m,k]}$ based on the FGM family of bivariate distributions. In the same section, we suggest a dynamic version of AWCRTE and its characteristics for $Z_{[r,n,m,k]}$ based on this family. In Section 3, the WCPTE and its dynamic form are examined. In Section 4, we offer the empirical AWCRTE and WCPTE. In Section 5, two real data sets are analyzed for illustration. In Section 6, the paper outlines the study's essential findings and their ramifications.

2. Weighted cumulative residual Tsallis entropy of order θ

Theorem 2.1. *The WCRTE of concomitant $Z_{[r,n,m,k]}$ of the rth $m-$GOS based on the FGM family is given by*

$$\zeta^w_{\theta,\alpha}(Z_{[r,n,m,k]}) = \frac{1}{\theta-1}\left(1-\sum_{i=0}^{\mathcal{N}(\theta)}\binom{\theta}{i}(-1)^i(\alpha C^*_{(r,n,m,k)})^i E[U^i(1-U)^\theta Q(U)q(U)]\right),$$

where $\mathcal{N}(z)=\infty$, if z is a non-integer, $\mathcal{N}(z)=z$, if z is an integer, U is the uniform RV on $(0,1)$, $Q(u)$ is the quantile function of the RV Z, and $q(u)$ is the quantile density function, i.e., $q(u)=\frac{d}{du}Q(u)$.

Proof. Using (1.4) and (1.7), then WCRTE is provided by

$$\begin{aligned}
\zeta^w_{\theta,\alpha}(Z_{[r,n,m,k]}) &= \frac{1}{\theta-1}\left(1-\int_0^\infty z\overline{F}^\theta_{[r,n,m,k]}(z)dz\right)\\
&= \frac{1}{\theta-1}\left(1-\int_0^\infty z\overline{F}^\theta_Z(z)\left(1-\alpha C^*_{(r,n,m,k)}F_Z(z)\right)^\theta dz\right)\\
&= \frac{1}{\theta-1}\left(1-\sum_{i=0}^{\mathcal{N}(\theta)}\binom{\theta}{i}(-1)^i(\alpha C^*_{(r,n,m,k)})^i E\left[\frac{Z(1-F_Z(Z))^\theta(F_Z(Z))^i}{f_Z(Z)}\right]\right). \qquad (2.1)
\end{aligned}$$

Let $Q(u)=F_Z^{-1}(z)$ be the quantile function. By using the well-known relation $q(u)f_Z(Q(u))=1$ in (2.1), we get

$$\zeta^w_{\theta,\alpha}(Z_{[r,n,m,k]}) = \frac{1}{\theta-1}\left(1-\sum_{i=0}^{\mathcal{N}(\theta)}\binom{\theta}{i}(-1)^i(\alpha C^*_{(r,n,m,k)})^i E\left[Q(U)(1-U)^\theta U^i q(U)\right]\right). \qquad (2.2)$$

□

Remark 2.1. *If $m=0$ and $k=1$, the WCRTE of the concomitant of the rth OS, $Z_{[r,n,0,1]} := Z_{[r:n]}$, based on the FGM family is given by*

$$\zeta^w_{\theta,\alpha}(Z_{[r:n]}) = \frac{1}{\theta-1}\left(1-\sum_{i=0}^{\mathcal{N}(\theta)}\binom{\theta}{i}(-1)^i(\alpha\phi_{(r:n)})^i E\left[U^i(1-U)^\theta Q(U)q(U)\right]\right),$$

where $\phi_{(r:n)}=\frac{n-2r+1}{n+1}$ (cf. [30]).

Remark 2.2. *The model of record values is a special case of the m-GOSs by putting $m = -1$ and $k = 1$. Therefore, the WCRTE of the concomitant $Z_{[n]}$ of the nth upper record value based on the FGM family is given by*

$$\zeta^w_{\theta,\alpha}(Z_{[n]}) = \frac{1}{\theta - 1}\left(1 - \sum_{i=0}^{\mathcal{N}(\theta)} \binom{\theta}{i} (-1)^i (\alpha\delta_{(n)})^i E\left[U^i (1-U)^\theta Q(U) q(U)\right]\right),$$

where $\delta_{(n)} = 2^{-(n-1)} - 1$.

Example 2.1. *Consider the two variables, Y and Z, that possess exponential distribution (ED) from the FGM family (represented by FGM-ED) (i.e., $F_Z(z) = 1 - e^{-\lambda z}, z, \lambda > 0$). Then*

$$\int_0^\infty z(1 - e^{-\lambda z})^i (e^{-\lambda z})^\theta dz = \sum_{p=0}^{i} \binom{i}{p} (-1)^p \frac{1}{\lambda^2 (p+\theta)^2}.$$

Based on (2.2), we get the WCRTE in $Z_{[r,n,m,k]}$ as follows:

$$\zeta^w_{\theta,\alpha}(Z_{[r,n,m,k]}) = \frac{1}{(\theta - 1)}\left(1 - \sum_{i=0}^{\mathcal{N}(\theta)} \sum_{p=0}^{i} \binom{\theta}{i}\binom{i}{p} (-1)^{i+p} \frac{(\alpha C^*_{(r,n,m,k)})^i}{\lambda^2 (p+\theta)^2}\right).$$

Example 2.2. *Consider Y and Z to be Weibull distributions (WD) derived from the FGM family (i.e., $F_Z(z) = 1 - e^{-(\frac{z}{\beta})^\eta}, z > 0,\ \eta, \beta > 0$). Then*

$$\int_0^\infty z(1 - e^{-(\frac{z}{\beta})^\eta})^i (e^{-(\frac{z}{\beta})^\eta})^\theta dz = \sum_{p=0}^{i} \binom{i}{p} (-1)^p \frac{(p+\theta)^{\frac{-2}{\eta}} \Gamma(\frac{2}{\eta})}{\eta \beta^\eta}.$$

Thus, $\zeta^w_{\theta,\alpha}(Z_{[r,n,m,k]})$ is given by

$$\zeta^w_{\theta,\alpha}(Z_{[r,n,m,k]}) = \frac{1}{(\theta - 1)}\left(1 - \sum_{i=0}^{\mathcal{N}(\theta)} \sum_{p=0}^{i} \binom{\theta}{i}\binom{i}{p} (-1)^{i+p} (\alpha C^*_{(r,n,m,k)})^i \frac{(p+\theta)^{\frac{-2}{\eta}} \Gamma(\frac{2}{\eta})}{\eta \beta^\eta}\right).$$

Tables 1 and 2 show aspects of the behavior of the WCRTE for $Z_{[r:n]}$ and $Z_{[n]}$, respectively. In Tables 1 and 2, the following properties can be extracted:

- Generally, $\zeta^w_{\theta,\alpha}(Z_{[r:n]}) = \zeta^w_{\theta,-\alpha}(Z_{[n-r+1:n]})$. Also, the value of $\zeta^w_{\theta,\alpha}(Z_{[r:n]})$ slowly increases as the value of n increases (cf. Table 1).
- We see that the value of $\zeta^w_{\theta,-\alpha}(Z_{[n]})$ goes up as n goes up, the value of $\zeta^w_{\theta,\alpha}(Z_{[n]})$ goes down as n goes up, and it almost stays the same when $n = 15$ (cf. Table 2).

Remark 2.3. *It is worth noting that the relation $\zeta^w_{\theta,\alpha}(Z_{[r:n]}) = \zeta^w_{\theta,-\alpha}(Z_{[n-r+1:n]})$ may be theoretically proved for every $\theta > 0$, $-1 \le \alpha \le 1$, and any non-negative continuous RV Z by applying Theorem 2.1 and remarking that $C^*_{(n-r+1,n,0,1)} = -C^*_{(r,n,0,1)}$.*

Table 1. $\zeta^{w}_{\theta,\alpha}(Z_{[r:n]})$ based on FGM-WD.

		$\theta = 4, \beta = 1.2, \eta = 0.5$						$\theta = 9, \beta = 1.5, \eta = 0.2$					
n	**r**	$\alpha = -0.1$	$\alpha = 0.1$	$\alpha = -0.5$	$\alpha = 0.5$	$\alpha = -0.9$	$\alpha = 0.9$	$\alpha = -0.1$	$\alpha = 0.1$	$\alpha = -0.5$	$\alpha = 0.5$	$\alpha = -0.9$	$\alpha = 0.9$
4	1	0.31693	0.32098	0.30563	0.32664	0.28891	0.32991	0.12492	0.12496	0.12469	0.12499	0.12405	0.12501
4	2	0.31838	0.31973	0.31536	0.32213	0.31187	0.32417	0.12493	0.12495	0.12489	0.12497	0.12484	0.12498
4	3	0.31973	0.31838	0.32213	0.31536	0.32417	0.31187	0.12495	0.12493	0.12497	0.12489	0.12498	0.12484
4	4	0.32098	0.31693	0.32664	0.30563	0.32991	0.28891	0.12496	0.12492	0.12499	0.12469	0.12501	0.12405
6	1	0.31649	0.32132	0.3022	0.3276	0.27969	0.3308	0.12491	0.12496	0.12459	0.12499	0.12352	0.12503
6	2	0.31757	0.32046	0.31021	0.32494	0.30036	0.32803	0.12492	0.12495	0.1248	0.12498	0.12453	0.125
6	3	0.31858	0.31954	0.31649	0.32132	0.31417	0.3229	0.12493	0.12494	0.12491	0.12496	0.12488	0.12497
6	4	0.31954	0.31858	0.32132	0.31649	0.3229	0.31417	0.12494	0.12493	0.12496	0.12491	0.12497	0.12488
6	5	0.32046	0.31757	0.32494	0.31021	0.32803	0.30036	0.12495	0.12492	0.12498	0.1248	0.125	0.12453
6	6	0.32132	0.31649	0.3276	0.3022	0.3308	0.27969	0.12496	0.12491	0.12499	0.12459	0.12503	0.12352
8	1	0.31625	0.3215	0.30015	0.32808	0.27397	0.33119	0.12491	0.12496	0.12452	0.125	0.12313	0.12505
8	2	0.3171	0.32084	0.30688	0.32623	0.29214	0.32949	0.12492	0.12496	0.12473	0.12499	0.1242	0.12501
8	3	0.31791	0.32016	0.31249	0.32385	0.30563	0.32664	0.12493	0.12495	0.12485	0.12498	0.12469	0.12499
8	4	0.31869	0.31944	0.3171	0.32084	0.31536	0.32213	0.12494	0.12494	0.12492	0.12496	0.12489	0.12497
8	5	0.31944	0.31869	0.32084	0.3171	0.32213	0.31536	0.12494	0.12494	0.12496	0.12492	0.12497	0.12489
8	6	0.32016	0.31791	0.32385	0.31249	0.32664	0.30563	0.12495	0.12493	0.12498	0.12485	0.12499	0.12469
8	7	0.32084	0.3171	0.32623	0.30688	0.32949	0.29214	0.12496	0.12492	0.12499	0.12473	0.12501	0.1242
8	8	0.3215	0.31625	0.32808	0.30015	0.33119	0.27397	0.12496	0.12491	0.125	0.12452	0.12505	0.12313
10	1	0.31609	0.32162	0.29879	0.32836	0.27008	0.33141	0.1249	0.12496	0.12448	0.125	0.12284	0.12507
10	2	0.31679	0.32109	0.30457	0.32696	0.28612	0.33022	0.12491	0.12496	0.12466	0.12499	0.1239	0.12502
10	3	0.31747	0.32054	0.30956	0.32522	0.29879	0.32836	0.12492	0.12495	0.12479	0.12499	0.12448	0.125
10	4	0.31813	0.31996	0.31384	0.3231	0.30862	0.32559	0.12493	0.12495	0.12487	0.12497	0.12477	0.12499
10	5	0.31876	0.31937	0.31747	0.32054	0.31609	0.32162	0.12494	0.12494	0.12492	0.12495	0.1249	0.12496
10	6	0.31937	0.31876	0.32054	0.31747	0.32162	0.31609	0.12494	0.12494	0.12495	0.12492	0.12496	0.1249
10	7	0.31996	0.31813	0.3231	0.31384	0.32559	0.30862	0.12495	0.12493	0.12497	0.12487	0.12499	0.12477
10	8	0.32054	0.31747	0.32522	0.30956	0.32836	0.29879	0.12495	0.12492	0.12499	0.12479	0.125	0.12448
10	9	0.32109	0.31679	0.32696	0.30457	0.33022	0.28612	0.12496	0.12491	0.12499	0.12466	0.12502	0.1239
10	10	0.32162	0.31609	0.32836	0.29879	0.33141	0.27008	0.12496	0.1249	0.125	0.12448	0.12507	0.12284

Table 2. $\zeta^{w}_{\theta,\alpha}(Z_{[n]})$ based on FGM-WD.

	$\theta = 4, \beta = 1.2, \eta = 0.5$					$\theta = 9, \beta = 1.5, \eta = 0.2$			
n	$\alpha = -0.5$	$\alpha = 0.5$	$\alpha = -0.9$	$\alpha = 0.9$	**n**	$\alpha = -0.5$	$\alpha = 0.5$	$\alpha = -0.9$	$\alpha = 0.9$
2	0.32569	0.30839	0.3289	0.29591	2	0.12499	0.12476	0.125	0.12437
3	0.32787	0.30106	0.33103	0.27653	3	0.125	0.12456	0.12504	0.12331
4	0.32874	0.29681	0.33168	0.26429	4	0.125	0.1244	0.1251	0.12237
5	0.32913	0.29453	0.33193	0.25746	5	0.125	0.12431	0.12516	0.12176
6	0.32931	0.29335	0.33204	0.25386	6	0.125	0.12426	0.12519	0.12141
7	0.3294	0.29274	0.3321	0.252	7	0.125	0.12423	0.12521	0.12122
8	0.32945	0.29244	0.33212	0.25107	8	0.125	0.12422	0.12522	0.12112
9	0.32947	0.29229	0.33213	0.25059	9	0.12501	0.12421	0.12522	0.12107
10	0.32948	0.29221	0.33214	0.25036	10	0.12501	0.12421	0.12522	0.12105
11	0.32948	0.29217	0.33214	0.25024	11	0.12501	0.1242	0.12522	0.12104
12	0.32949	0.29215	0.33214	0.25018	12	0.12501	0.1242	0.12523	0.12103
13	0.32949	0.29215	0.33214	0.25015	13	0.12501	0.1242	0.12523	0.12103
14	0.32949	0.29214	0.33215	0.25013	14	0.12501	0.1242	0.12523	0.12103
15	0.32949	0.29214	0.33215	0.25013	15	0.12501	0.1242	0.12523	0.12102
16	0.32949	0.29214	0.33215	0.25012	16	0.12501	0.1242	0.12523	0.12102
17	0.32949	0.29214	0.33215	0.25012	17	0.12501	0.1242	0.12523	0.12102

2.1. AWCRTE for the concomitant of the m−GOS of order θ

Theorem 2.2. *The AWCRTE for the concomitant $Z_{[r,n,m,k]}$ of the m−GOS based on the FGM family is given by*

$$\xi^{w}_{\theta,\alpha}(Z_{[r,n,m,k]}) = \frac{1}{\theta-1}\left(\frac{1}{2}E\left(Z^{2}_{[r,n,m,k]}\right) - \sum_{i=0}^{\mathcal{N}(\theta)}\binom{\theta}{i}(-1)^{i}(\alpha C^{*}_{(r,n,m,k)})^{i}\mathrm{E}\left[U^{i}(1-U)^{\theta}Q(U)q(U)\right]\right).$$

Proof. Using (1.4) and (1.8), we get

$$\begin{aligned}\xi^{w}_{\theta,\alpha}(Z_{[r,n,m,k]}) &= \frac{1}{\theta-1}\int_{0}^{\infty} z\left(\overline{F}_{[r,n,m,k]}(z) - \overline{F}^{\theta}_{[r,n,m,k]}(z)\right)dz \\ &= \frac{1}{\theta-1}\left(\int_{0}^{\infty} z\overline{F}_{[r,n,m,k]}(z)dz - \sum_{i=0}^{\mathcal{N}(\theta)}\binom{\theta}{i}(-1)^{i}(\alpha C^{*}_{(r,n,m,k)})^{i}E\left[\frac{Z\overline{F}^{\theta}_{Z}(Z)F^{i}_{Z}(Z)}{f_{Z}(Z)}\right]\right)\end{aligned}$$

$$= \frac{1}{\theta-1}\left(\frac{1}{2}E\left(Z^2_{[r,n,m,k]}\right) - \sum_{i=0}^{\mathcal{N}(\theta)}\binom{\theta}{i}(-1)^i(\alpha C^*_{(r,n,m,k)})^i E\left[U^i(1-U)^\theta Q(U)q(U)\right]\right). \qquad (2.3)$$

□

Remark 2.4. *If $m = 0$ and $k = 1$, the AWCRTE of the concomitant $Z_{[r:n]}$ of the rth OS based on the FGM family is given by*

$$\xi^w_{\theta,\alpha}(Z_{[r:n]}) = \frac{1}{\theta-1}\left(\frac{1}{2}E\left(Z^2_{[r:n]}\right) - \sum_{i=0}^{\mathcal{N}(\theta)}\binom{\theta}{i}(-1)^i(\alpha\phi_{(r:n)})^i E\left[U^i(1-U)^\theta Q(U)q(U)\right]\right).$$

Remark 2.5. *If $m = -1$ and $k = 1$, the AWCRTE of concomitant of the nth upper record value based on the FGM family is given by*

$$\xi^w_{\theta,\alpha}(Z_{[n]}) = \frac{1}{\theta-1}\left(\frac{1}{2}E\left(Z^2_{[n]}\right) - \sum_{i=0}^{\mathcal{N}(\theta)}\binom{\theta}{i}(-1)^i(\alpha\delta_{(n)})^i E\left[U^i(1-U)^\theta Q(U)q(U)\right]\right).$$

In survival analysis, if an RV T^* given $\Theta = \theta$ has a DF $F(t|\theta)$, the weighted mean residual lifetime (WMRL) of T^* given $\Theta = \theta$ is then obtained as

$$m^*_F(t) = \int_t^\infty z\frac{\overline{F}(z|\theta)}{\overline{F}(t|\theta)}dz,$$

which is valid for all θ that $\overline{F}(t|\theta) > 0$ and also plays a crucial role in survival analysis. Now, we provide a relationship between the WMRL, $m^*_{F_{[r,n,m,k]}}(t) = \int_t^\infty z\frac{\overline{F}_{[r,n,m,k]}(z)}{\overline{F}_{[r,n,m,k]}(t)}dz$, $\overline{F}_{[r,n,m,k]}(t) > 0$, and the AWCRTE of an RV Z. This relationship illustrates AWCRTE's crucial role in survival analysis.

Lemma 2.1. *For a non-negative continuous RV Z with SF $\overline{F}_Z(z)$,*

$$\xi^w_{\theta,\alpha}(Z_{[r,n,m,k]}) = E\left[m^*_{F_{[r,n,m,k]}}(Z_{[r,n,m,k]})\overline{F}^{\theta-1}_{[r,n,m,k]}(Z_{[r,n,m,k]})\right].$$

Proof. First, we note $\frac{d}{dz}(m^*_{F_{[r,n,m,k]}}(z)\overline{F}_{[r,n,m,k]}(z)) = -z\overline{F}_{[r,n,m,k]}(z)$. Thus, by using the fact in (1.8), we get

$$\xi^w_{\theta,\alpha}(Z_{[r,n,m,k]}) = \frac{1}{\theta-1}\left[-\int_0^\infty \frac{d}{dz}(m^*_{F_{[r,n,m,k]}}(z)\overline{F}_{[r,n,m,k]}(z))(1-\overline{F}^{\theta-1}_{[r,n,m,k]}(z))dz\right].$$

Now the result follows by using integration by parts. □

Example 2.3. *Consider the RVs, Y and Z, that follow FGM-ED. Based on (2.3), we get the AWCRTE in $Z_{[r,n,m,k]}$ as follows:*

$$\xi^w_{\theta,\alpha}(Z_{[r,n,m,k]}) = \frac{1}{\lambda^2(\theta-1)}\left(\left(1-\frac{3}{4}\alpha C^*_{(r,n,m,k)}\right) - \sum_{i=0}^{\mathcal{N}(\theta)}\sum_{p=0}^{i}\binom{\theta}{i}\binom{i}{p}(-1)^{i+p}\frac{(\alpha C^*_{(r,n,m,k)})^i}{(p+\theta)^2}\right).$$

Moreover, it is easy to check that

$$m^*_{F_{[r,n,m,k]}}(t) = \frac{e^{-t\lambda}(1+t\lambda)}{\lambda^2\overline{F}_{[r,n,m,k]}(t)}\left[1-\alpha C^*_{(r,n,m,k)}(1-\frac{e^{-2t\lambda}(1+2t\lambda)}{4\lambda^2})\right],$$

which yields

$$E\left[m^*_{F_{[r,n,m,k]}}(Z_{[r,n,m,k]})\overline{F}^{\theta-1}_{[r,n,m,k]}(Z_{[r,n,m,k]})\right] = \frac{\left(\left(1-\frac{3}{4}\alpha C^*_{(r,n,m,k)}\right)-\sum\limits_{i=0}^{\mathcal{N}(\theta)}\sum\limits_{p=0}^{i}\binom{\theta}{i}\binom{i}{p}(-1)^{i+p}\frac{(\alpha C^*_{(r,n,m,k)})^i}{(p+\theta)^2}\right)}{\lambda^2(\theta-1)}$$

$$= \xi^w_{\theta,\alpha}(Z_{[r,n,m,k]}).$$

Example 2.4. *Consider Y and Z to be WD derived from the FGM family. Then* $\int_0^\infty z(e^{-(\frac{z}{\beta})^\eta})dz = \frac{\Gamma(\frac{2}{\eta})\beta^2}{\eta}$,

$$\int_0^\infty z(e^{-(\frac{z}{\beta})^\eta})(1-e^{-(\frac{z}{\beta})^\eta})dz = \beta^2\Gamma(\frac{2}{\eta}+1)(4^{-\eta}-1)2^{-(\frac{2}{\eta}+1)},$$

and

$$\int_0^\infty z(1-e^{-(\frac{z}{\beta})^\eta})^i(e^{-(\frac{z}{\beta})^\eta})^\theta dz = \sum_{j=0}^{i}\binom{i}{j}(-1)^j\frac{(j+\theta)^{\frac{-2}{\eta}}\Gamma(\frac{2}{\eta})}{\eta\beta^\eta}.$$

Therefore, the measure $\xi^w_{\theta,\alpha}(Z_{[r,n,m,k]})$ *is given by*

$$\xi^w_{\theta,\alpha}(Z_{[r,n,m,k]}) = \frac{1}{\theta-1}\left(\frac{\beta^2}{\eta}\Gamma(\frac{2}{\eta})\left(1-\alpha C^*_{(r,n,m,k)}(4^{-\eta}-1)2^{\frac{-2}{\eta}}\right)-\sum_{i=0}^{\mathcal{N}(\theta)}\sum_{j=0}^{i}\binom{\theta}{i}\binom{i}{j}(-1)^{i+j}\right.$$
$$\left.\times\quad(\alpha C^*_{(r,n,m,k)})^i\left(\frac{(j+\theta)^{\frac{-2}{\eta}}\Gamma(\frac{2}{\eta})}{\eta\beta^\eta}\right)\right).$$

Example 2.5. *Suppose Y and Z are continuous RVs with respective PDF* $f_Y(y) = \frac{1}{a},\ 0 < y < a$, *and* $g_Z(z) = \frac{1}{a},\ h < z < a+h,\ h > 0$. *From (1.5), we have*

$$\mathcal{E}_{\theta,\alpha}(Y_{[r,n,m,k]}) = \mathcal{E}_{\theta,\alpha}(Z_{[r,n,m,k]}) = \frac{1}{\theta-1}\left(\frac{a}{2}(1-\frac{\alpha C^*_{(r,n,m,k)}}{3})-\sum_{i=0}^{\mathcal{N}(\theta)}\binom{\theta}{i}(-1)^i(\alpha C^*_{(r,n,m,k)})^i\,(a\beta(1+i,1+\theta))\right).$$

Now, from (1.8), we get

$$\xi^w_{\theta,\alpha}(Y_{[r,n,m,k]}) = \frac{1}{\theta-1}\left(\frac{a^2}{12}(2-\alpha C^*_{(r,n,m,k)})-\sum_{i=0}^{\mathcal{N}(\theta)}\binom{\theta}{i}(-1)^i(\alpha C^*_{(r,n,m,k)})^i\left(a^2\beta(2+i,1+\theta)\right)\right)$$

and

$$\xi^w_{\theta,\alpha}(Z_{[r,n,m,k]}) = \frac{1}{\theta-1}\left(\left(\frac{a^2}{6}+\frac{ah}{2}\right)\left(1-\alpha C^*_{(r,n,m,k)}\frac{a^2+2ah}{2(a^2+3ah)}\right)-\sum_{i=0}^{\mathcal{N}(\theta)}\binom{\theta}{i}(-1)^i(\alpha C^*_{(r,n,m,k)})^i\right.$$
$$\left.\left(\frac{a\,(a(i+1)+h(2+i+\theta))}{(i+\theta+2)}\beta(1+i,1+\theta)\right)\right).$$

Therefore, although $\mathcal{E}_{\theta,\alpha}(Y_{[r,n,m,k]}) = \mathcal{E}_{\theta,\alpha}(Z_{[r,n,m,k]})$, $\xi^w_{\theta,\alpha}(Y_{[r,n,m,k]}) \neq \xi^w_{\theta,\alpha}(Z_{[r,n,m,k]})$.

Example 2.6. *Suppose Z has a Pareto distribution with DF $F_Z(z) = 1 - (\frac{b}{z})^a, z \geq b, b > 0, a > 0$. Then*

$$\xi^w_{\theta,\alpha}(Z_{[r,n,m,k]}) = \frac{1}{\theta-1}\left(\frac{b^2}{(a-2)}\left(1 - \frac{a\alpha C^*_{(r,n,m,k)}}{2(a-1)}\right) - \sum_{i=0}^{\mathcal{N}(\theta)}\sum_{s=0}^{i}\binom{\theta}{i}\binom{i}{s}(-1)^{i+s}\frac{b^2(\alpha C^*_{(r,n,m,k)})^i}{(a(s+\theta)-2)}\right).$$

Also, we can show that

$$m^*_{F_{[r,n,m,k]}}(t) = \frac{b^a t^{2-a}}{(a-2)\overline{F}_{[r,n,m,k]}(t)}\left[1 - \alpha C^*_{(r,n,m,k)}\left(1 - \frac{b^a t^{-a}(a-2)}{2(a-1)}\right)\right]$$

and

$$\begin{aligned} E\left[m^*_{F_{[r,n,m,k]}}(Z_{[r,n,m,k]})\overline{F}^{\theta-1}_{[r,n,m,k]}(Z_{[r,n,m,k]})\right] &= \frac{\frac{b^2}{a-2}\left(1-\frac{a\alpha C^*_{(r,n,m,k)}}{2(a-1)}\right) - \sum_{i=0}^{\mathcal{N}(\theta)}\sum_{s=0}^{i}\binom{\theta}{i}\binom{i}{s}(-1)^{i+s}\frac{b^2(\alpha C^*_{(r,n,m,k)})^i}{(a(s+\theta)-2)}}{\theta-1} \\ &= \xi^w_{\theta,\alpha}(Z_{[r,n,m,k]}). \end{aligned}$$

The following lemma shows that $\xi^w_{\theta,\alpha}(Z_{[r,n,m,k]})$ is a shift-dependent measure.

Lemma 2.2. *Let $Z = aX + b$ with $a > 0$ and $b \geq 0$. Then*

$$\xi^w_{\theta,\alpha}(Z_{[r,n,m,k]}) = a^2\xi^w_{\theta,\alpha}(X_{[r,n,m,k]}) + ab\mathcal{E}_{\theta,\alpha}(X_{[r,n,m,k]}).$$

Proof. First, since the linear transformation, with $b \geq 0$, preserves the order relation, then for every $1 \leq r \leq n$, we get $Z_{r,n,m,k} = aX_{r,n,m,k} + b$ and $Z_{[r,n,m,k]} = aX_{[r,n,m,k]} + b$. The proof follows using the fact that $\overline{F}_{Z_{[r,n,m,k]}}(u) = \overline{F}_{X_{[r,n,m,k]}}(\frac{u-b}{a})$. □

As shown in Tables 3–6 of the FGM-ED and FGM-WD, respectively, the AWCRTE for $Z_{[r:n]}$ and $Z_{[n]}$ are presented. After running the numbers through MATHEMATICA version 12, we can deduce the following properties from Tables 3–6.

- Generally, $\xi^w_{\theta,\alpha}(Z_{[r:n]}) = \xi^w_{\theta,-\alpha}(Z_{[n-r+1:n]})$ (similar to Remark 2.3, this symmetry relation can easily be proved theoretically). Also, the value of $\xi^w_{\theta,\alpha}(Z_{[r:n]})$ slowly increases as the value of n increases (see Tables 3 and 5).
- We can see that the value of $\xi^w_{\theta,-\alpha}(Z_{[n]})$ increases as n increases, the value of $\xi^w_{\theta,\alpha}(Z_{[n]})$ decreases as n decreases, and it almost stays the same when $n = 15$ (see Table 4).
- At most, the value of $\xi^w_{\theta,-\alpha}(Z_{[n]})$ increases as n increases, but the value of $\xi^w_{\theta,\alpha}(Z_{[n]})$ decreases as n increases, and it almost stays the same when $n = 17$ (see Table 6).

Table 3. $\xi^{w}_{\theta,\alpha}(Z_{[r:n]})$ based on FGM-ED.

		$\theta = 4, \lambda = 0.5$						$\theta = 9, \lambda = 0.9$					
n	**r**	$\alpha = -0.1$	$\alpha = 0.1$	$\alpha = -0.5$	$\alpha = 0.5$	$\alpha = -0.9$	$\alpha = 0.9$	$\alpha = -0.1$	$\alpha = 0.1$	$\alpha = -0.5$	$\alpha = 0.5$	$\alpha = -0.9$	$\alpha = 0.9$
4	1	0.2425	0.2569	0.2058	0.2793	0.1563	0.295	0.1229	0.1233	0.1217	0.1238	0.1198	0.1242
4	2	0.2476	0.2524	0.2371	0.2612	0.2256	0.2691	0.123	0.1232	0.1227	0.1234	0.1224	0.1236
4	3	0.2524	0.2476	0.2612	0.2371	0.2691	0.2256	0.1232	0.123	0.1234	0.1227	0.1236	0.1224
4	4	0.2569	0.2425	0.2793	0.2058	0.295	0.1563	0.1233	0.1229	0.1238	0.1217	0.1242	0.1198
6	1	0.241	0.2582	0.1953	0.2836	0.1305	0.3001	0.1228	0.1233	0.1214	0.1239	0.1186	0.1243
6	2	0.2447	0.255	0.2202	0.2722	0.1897	0.2856	0.1229	0.1232	0.1222	0.1237	0.1212	0.124
6	3	0.2483	0.2517	0.241	0.2582	0.2331	0.2641	0.123	0.1231	0.1228	0.1233	0.1226	0.1235
6	4	0.2517	0.2483	0.2582	0.241	0.2641	0.2331	0.1231	0.123	0.1233	0.1228	0.1235	0.1226
6	5	0.255	0.2447	0.2722	0.2202	0.2856	0.1897	0.1232	0.1229	0.1237	0.1222	0.124	0.1212
6	6	0.2582	0.241	0.2836	0.1953	0.3001	0.1305	0.1233	0.1228	0.1239	0.1214	0.1243	0.1186
8	1	0.2402	0.2588	0.1891	0.2858	0.1148	0.3026	0.1228	0.1233	0.1211	0.124	0.1179	0.1243
8	2	0.2431	0.2564	0.2097	0.2776	0.1656	0.2928	0.1229	0.1233	0.1219	0.1238	0.1202	0.1241
8	3	0.2459	0.2539	0.2276	0.2678	0.2058	0.2793	0.123	0.1232	0.1224	0.1236	0.1217	0.1238
8	4	0.2487	0.2513	0.2431	0.2564	0.2371	0.2612	0.1231	0.1231	0.1229	0.1233	0.1227	0.1234
8	5	0.2513	0.2487	0.2564	0.2431	0.2612	0.2371	0.1231	0.1231	0.1233	0.1229	0.1234	0.1227
8	6	0.2539	0.2459	0.2678	0.2276	0.2793	0.2058	0.1232	0.123	0.1236	0.1224	0.1238	0.1217
8	7	0.2564	0.2431	0.2776	0.2097	0.2928	0.1656	0.1233	0.1229	0.1238	0.1219	0.1241	0.1202
8	8	0.2588	0.2402	0.2858	0.1891	0.3026	0.1148	0.1233	0.1228	0.124	0.1211	0.1243	0.1179
10	1	0.2396	0.2593	0.185	0.2872	0.1043	0.3041	0.1228	0.1233	0.121	0.124	0.1173	0.1244
10	2	0.242	0.2573	0.2025	0.2807	0.1484	0.2967	0.1229	0.1233	0.1216	0.1239	0.1195	0.1242
10	3	0.2444	0.2553	0.2181	0.2734	0.185	0.2872	0.1229	0.1232	0.1221	0.1237	0.121	0.124
10	4	0.2467	0.2532	0.232	0.2649	0.2151	0.2749	0.123	0.1232	0.1226	0.1235	0.122	0.1237
10	5	0.2489	0.2511	0.2444	0.2553	0.2396	0.2593	0.1231	0.1231	0.1229	0.1232	0.1228	0.1233
10	6	0.2511	0.2489	0.2553	0.2444	0.2593	0.2396	0.1231	0.1231	0.1232	0.1229	0.1233	0.1228
10	7	0.2532	0.2467	0.2649	0.232	0.2749	0.2151	0.1232	0.123	0.1235	0.1226	0.1237	0.122
10	8	0.2553	0.2444	0.2734	0.2181	0.2872	0.185	0.1232	0.1229	0.1237	0.1221	0.124	0.121
10	9	0.2573	0.242	0.2807	0.2025	0.2967	0.1484	0.1233	0.1229	0.1239	0.1216	0.1242	0.1195
10	10	0.2593	0.2396	0.2872	0.185	0.3041	0.1043	0.1233	0.1228	0.124	0.121	0.1244	0.1173

Table 4. $\xi^{w}_{\theta,\alpha}(Z_{[n]})$ based on FGM-ED.

	$\theta = 4, \lambda = 0.5$					$\theta = 9, \lambda = 0.9$			
n	$\alpha = -0.5$	$\alpha = 0.5$	$\alpha = -0.9$	$\alpha = 0.9$	**n**	$\alpha = -0.5$	$\alpha = 0.5$	$\alpha = -0.9$	$\alpha = 0.9$
2	0.2753	0.21439	0.28981	0.17654	2	0.12373	0.12202	0.12406	0.12064
3	0.28488	0.19183	0.30156	0.12176	3	0.12395	0.12123	0.12432	0.1182
4	0.28903	0.17918	0.30602	0.08876	4	0.12404	0.12075	0.12443	0.11648
5	0.29096	0.1725	0.30796	0.0707	5	0.12408	0.12048	0.12448	0.11546
6	0.29189	0.16906	0.30887	0.06126	6	0.1241	0.12034	0.12451	0.11491
7	0.29235	0.16732	0.3093	0.05644	7	0.12411	0.12027	0.12453	0.11462
8	0.29257	0.16645	0.30952	0.054	8	0.12412	0.12023	0.12454	0.11447
9	0.29269	0.16601	0.30962	0.05277	9	0.12412	0.12022	0.12454	0.1144
10	0.29274	0.16579	0.30968	0.05216	10	0.12412	0.12021	0.12454	0.11436
11	0.29277	0.16567	0.3097	0.05185	11	0.12412	0.1202	0.12454	0.11434
12	0.29278	0.16562	0.30972	0.0517	12	0.12412	0.1202	0.12454	0.11433
13	0.29279	0.16559	0.30972	0.05162	13	0.12412	0.1202	0.12454	0.11433
14	0.29279	0.16558	0.30973	0.05158	14	0.12412	0.1202	0.12454	0.11432
15	0.2928	0.16557	0.30973	0.05156	15	0.12412	0.1202	0.12454	0.11432
16	0.2928	0.16557	0.30973	0.05155	16	0.12412	0.1202	0.12454	0.11432
17	0.2928	0.16557	0.30973	0.05155	17	0.12412	0.1202	0.12454	0.11432
18	0.2928	0.16557	0.30973	0.05155	18	0.12412	0.1202	0.12454	0.11432
19	0.2928	0.16556	0.30973	0.05154	19	0.12412	0.1202	0.12454	0.11432

2.2. *Dynamic AWCRTE of order θ*

When doing a reliability analysis, one crucial quantity to consider is the component or system's residual lifetime. Assuming a component Z has survived for a certain amount of time t, its residual lifetime is $Z_t = (Z - t)|Z > t$. This is the AWCRTE with residual lifetime Z_t, which is dynamic and has order θ. Then, the dynamic AWCRTE in the rth concomitant based on the FGM family is given by

$$\begin{aligned}
\xi^{w}_{\theta,\alpha}(Z_{[r,n,m,k]};t) &= \frac{1}{\theta-1}\int_t^{\infty} z\left(\overline{F}_{Z_t[r,n,m,k]}(z) - \overline{F}^{\theta}_{Z_t[r,n,m,k]}(z)\right)dz \\
&= \frac{1}{\theta-1}\int_t^{\infty} z\left(\frac{\overline{F}_{[r,n,m,k]}(z)}{\overline{F}_{[r,n,m,k]}(t)} - \left(\frac{\overline{F}_{[r,n,m,k]}(z)}{\overline{F}_{[r,n,m,k]}(t)}\right)^{\theta}\right)dz
\end{aligned}$$

$$= \frac{1}{\theta-1}\left(m^{*}_{F_{[r,n,m,k]}}(t) - \int_t^{\infty} z\left(\frac{\overline{F}_{[r,n,m,k]}(z)}{\overline{F}_{[r,n,m,k]}(t)}\right)^{\theta} dz\right). \tag{2.4}$$

In the following theorem, we establish the connection between the dynamic AWCRTE and WMRL.

Table 5. $\xi^{w}_{\theta,\alpha}(Z_{[r:n]})$ based on FGM-WD.

		$\theta=4,\beta=0.2,\eta=0.5$						$\theta=9,\beta=2,\eta=1.5$					
n	**r**	$\alpha=-0.1$	$\alpha=0.1$	$\alpha=-0.5$	$\alpha=0.5$	$\alpha=-0.9$	$\alpha=0.9$	$\alpha=-0.1$	$\alpha=0.1$	$\alpha=-0.5$	$\alpha=0.5$	$\alpha=-0.9$	$\alpha=0.9$
4	1	0.1195	0.13	0.0906	0.1451	0.0485	0.1543	0.2899	0.3026	0.2646	0.3277	0.239	0.3527
4	2	0.1233	0.1268	0.1155	0.1331	0.1065	0.1384	0.2942	0.2984	0.2857	0.3067	0.2773	0.3151
4	3	0.1268	0.1233	0.1331	0.1155	0.1384	0.1065	0.2984	0.2942	0.3067	0.2857	0.3151	0.2773
4	4	0.13	0.1195	0.1451	0.0906	0.1543	0.0485	0.3026	0.2899	0.3277	0.2646	0.3527	0.239
6	1	0.1184	0.1309	0.082	0.1477	0.0254	0.157	0.2887	0.3037	0.2585	0.3336	0.2278	0.3634
6	2	0.1212	0.1287	0.1023	0.1405	0.0773	0.1489	0.2918	0.3008	0.2737	0.3187	0.2555	0.3366
6	3	0.1238	0.1263	0.1184	0.1309	0.1124	0.1351	0.2948	0.2978	0.2887	0.3037	0.2827	0.3097
6	4	0.1263	0.1238	0.1309	0.1184	0.1351	0.1124	0.2978	0.2948	0.3037	0.2887	0.3097	0.2827
6	5	0.1287	0.1212	0.1405	0.1023	0.1489	0.0773	0.3008	0.2918	0.3187	0.2737	0.3366	0.2555
6	6	0.1309	0.1184	0.1477	0.082	0.157	0.0254	0.3037	0.2887	0.3336	0.2585	0.3634	0.2278
8	1	0.1178	0.1314	0.0768	0.1491	0.0111	0.1583	0.2881	0.3044	0.2551	0.3369	0.2216	0.3693
8	2	0.1199	0.1297	0.0938	0.144	0.0566	0.1531	0.2904	0.3021	0.267	0.3253	0.2433	0.3485
8	3	0.1221	0.1279	0.1081	0.1376	0.0906	0.1451	0.2928	0.2998	0.2787	0.3137	0.2646	0.3277
8	4	0.1241	0.126	0.1199	0.1297	0.1155	0.1331	0.2951	0.2974	0.2904	0.3021	0.2857	0.3067
8	5	0.126	0.1241	0.1297	0.1199	0.1331	0.1155	0.2974	0.2951	0.3021	0.2904	0.3067	0.2857
8	6	0.1279	0.1221	0.1376	0.1081	0.1451	0.0906	0.2998	0.2928	0.3137	0.2787	0.3277	0.2646
8	7	0.1297	0.1199	0.144	0.0938	0.1531	0.0566	0.3021	0.2904	0.3253	0.267	0.3485	0.2433
8	8	0.1314	0.1178	0.1491	0.0768	0.1583	0.0111	0.3044	0.2881	0.3369	0.2551	0.3693	0.2216
10	1	0.1174	0.1317	0.0733	0.1499	0.0014	0.159	0.2877	0.3048	0.253	0.339	0.2177	0.3731
10	2	0.1192	0.1303	0.088	0.146	0.0415	0.1552	0.2896	0.3029	0.2627	0.3296	0.2354	0.3561
10	3	0.1209	0.1289	0.1006	0.1413	0.0733	0.1499	0.2915	0.301	0.2723	0.3201	0.253	0.339
10	4	0.1226	0.1274	0.1116	0.1356	0.0982	0.1423	0.2934	0.2991	0.2819	0.3106	0.2704	0.322
10	5	0.1243	0.1259	0.1209	0.1289	0.1174	0.1317	0.2953	0.2972	0.2915	0.301	0.2877	0.3048
10	6	0.1259	0.1243	0.1289	0.1209	0.1317	0.1174	0.2972	0.2953	0.301	0.2915	0.3048	0.2877
10	7	0.1274	0.1226	0.1356	0.1116	0.1423	0.0982	0.2991	0.2934	0.3106	0.2819	0.322	0.2704
10	8	0.1289	0.1209	0.1413	0.1006	0.1499	0.0733	0.301	0.2915	0.3201	0.2723	0.339	0.253
10	9	0.1303	0.1192	0.146	0.088	0.1552	0.0415	0.3029	0.2896	0.3296	0.2627	0.3561	0.2354
10	10	0.1317	0.1174	0.1499	0.0733	0.159	0.0014	0.3048	0.2877	0.339	0.253	0.3731	0.2177

Table 6. $\xi^{w}_{\theta,\alpha}(Z_{[n]})$ based on FGM-WD.

	$\theta=4,\beta=0.2,\eta=0.5$					$\theta=9,\beta=2,\eta=1.5$			
n	$\alpha=-0.5$	$\alpha=0.5$	$\alpha=-0.9$	$\alpha=0.9$	**n**	$\alpha=-0.5$	$\alpha=0.5$	$\alpha=-0.9$	$\alpha=0.9$
2	0.14252	0.09764	0.15139	0.06608	2	0.32243	0.2699	0.3433	0.24861
3	0.14849	0.07907	0.15772	0.01748	3	0.33548	0.25662	0.36672	0.22435
4	0.15094	0.06835	0.15989	−0.01305	4	0.342	0.24995	0.37841	0.21204
5	0.15205	0.0626	0.16079	−0.03006	5	0.34525	0.24661	0.38426	0.20584
6	0.15257	0.05963	0.1612	−0.03904	6	0.34688	0.24493	0.38718	0.20272
7	0.15283	0.05812	0.1614	−0.04364	7	0.34769	0.24409	0.38864	0.20116
8	0.15296	0.05735	0.1615	−0.04598	8	0.3481	0.24368	0.38937	0.20037
9	0.15302	0.05697	0.16154	−0.04715	9	0.3483	0.24347	0.38974	0.19998
10	0.15305	0.05678	0.16157	−0.04774	10	0.3484	0.24336	0.38992	0.19979
11	0.15307	0.05668	0.16158	−0.04804	11	0.34846	0.24331	0.39001	0.19969
12	0.15307	0.05663	0.16158	−0.04818	12	0.34848	0.24328	0.39006	0.19964
13	0.15308	0.05661	0.16159	−0.04826	13	0.34849	0.24327	0.39008	0.19961
14	0.15308	0.0566	0.16159	−0.04829	14	0.3485	0.24326	0.39009	0.1996
15	0.15308	0.05659	0.16159	−0.04831	15	0.3485	0.24326	0.3901	0.1996
16	0.15308	0.05659	0.16159	−0.04832	16	0.3485	0.24326	0.3901	0.19959
17	0.15308	0.05659	0.16159	−0.04833	17	0.34851	0.24326	0.3901	0.19959

Theorem 2.3. *Let Z be an absolutely continuous non-negative RV with WMRL function $m^{*}_{F_{[r,n,m,k]}}(t)$, and then,*

$$\xi^{w}_{\theta,\alpha}(Z_{[r,n,m,k]};t) = \frac{E\left[m^{*}_{F[r,n,m,k]}(Z_{[r,n,m,k]})\overline{F}^{\theta-1}_{[r,n,m,k]}(Z_{[r,n,m,k]})|Z>t\right]}{\overline{F}^{\theta-1}_{[r,n,m,k]}(t)}.$$

Proof. From (2.4), we have

$$\xi^{w}_{\theta,\alpha}(Z_{[r,n,m,k]};t) = \frac{1}{\theta-1}\left[m^{*}_{F_{[r,n,m,k]}}(t) + \frac{1}{\overline{F}^{\theta}_{[r,n,m,k]}(t)}\int_t^{\infty}\left[\frac{d}{dz}\left(m^{*}_{F_{[r,n,m,k]}}(z)\overline{F}_{[r,n,m,k]}(z)\right)\overline{F}^{\theta-1}_{[r,n,m,k]}(z)\right]dz\right]$$

$$= \frac{1}{\theta-1}\left[m^*_{F_{[r,n,m,k]}}(t)+\frac{1}{\overline{F}^{\theta}_{[r,n,m,k]}(t)}\left(-m^*_{F_{[r,n,m,k]}}(t)\overline{F}^{\theta}_{[r,n,m,k]}(t)+(\theta-1)\int_t^{\infty} m^*_{F_{[r,n,m,k]}}(z)\right.\right.$$

$$\left.\left.\times\ \overline{F}^{\theta-1}_{[r,n,m,k]}(z)f_{[r,n,m,k]}(z)dz\right)\right] = \frac{1}{\overline{F}^{\theta-1}_{[r,n,m,k]}(t)}\int_t^{\infty} m^*_{F_{[r,n,m,k]}}(z)\overline{F}^{\theta-1}_{[r,n,m,k]}(z)\frac{f_{[r,n,m,k]}(z)}{\overline{F}_{[r,n,m,k]}(t)}dz. \quad (2.5)$$

The proof is complete. □

3. Weighted cumulative past Tsallis entropy of order θ

Here, we study the WCPTE and its dynamic version in concomitant $Z_{[r,n,m,k]}$, from the FGM family, with numerical illustrations according to the sub-model OSs and record values.

Theorem 3.1. *The WCPTE for the rth concomitant of m−GOSs based on the FGM family is given by*

$$\mathcal{P}^w_{\theta,\alpha}(Z_{[r,n,m,k]}) = \frac{1}{\theta-1}\left(\int_0^{\infty} zF_{[r,n,m,k]}(z)dz - \sum_{j=0}^{\mathcal{N}(\theta)}\binom{\theta}{j}(\alpha C^*_{(r,n,m,k)})^j \mathrm{E}\left[U^{\theta}(1-U)^j Q(U)q(U)\right]\right).$$

Proof. Using (1.3) and (1.6), then the WCPTE is provided by

$$\begin{aligned}
\mathcal{P}^w_{\theta,\alpha}(Z_{[r,n,m,k]}) &= \frac{1}{\theta-1}\int_0^{\infty} z\left(F_{[r,n,m,k]}(z) - F^{\theta}_{[r,n,m,k]}(z)\right)dz \\
&= \frac{1}{\theta-1}\left(\int_0^{\infty} zF_{[r,n,m,k]}(z)dz - \sum_{j=0}^{\mathcal{N}(\theta)}\binom{\theta}{j}(\alpha C^*_{(r,n,m,k)})^j\right. \\
&\times\ \left. E\left[\frac{Z(1-F_Z(Z))^j(F_Z(Z))^{\theta}}{f_Z(Z)}\right]\right) \\
&= \frac{1}{\theta-1}\left(\int_0^{\infty} zF_{[r,n,m,k]}(z)dz - \sum_{j=0}^{\mathcal{N}(\theta)}\binom{\theta}{j}(\alpha C^*_{(r,n,m,k)})^j\right. \\
&\times\ \left. E\left[U^{\theta}(1-U)^j Q(U)q(U)\right]\right). \qquad (3.1)
\end{aligned}$$

□

Remark 3.1. *Let $F_{[r:n]}(z)$ be the DF of the rth concomitant $Z_{[r:n]}$. By putting $m = 0$ and $k = 1$, then the WCPTE in the rth concomitant of the OSs is given by*

$$\mathcal{P}^w_{\theta,\alpha}(Z_{[r:n]}) = \frac{1}{\theta-1}\left(\int_0^{\infty} zF_{[r:n]}(z)dz - \sum_{j=0}^{\mathcal{N}(\theta)}\binom{\theta}{j}(\alpha\phi_{(r:n)})^j E\left[U^{\theta}(1-U)^j Q(U)q(U)\right]\right).$$

Remark 3.2. *Let $F_{[n]}(z)$ be the DF of the nth upper record. By putting $k = 1$ and $m = -1$, then the WCPTE in the concomitant of the nth upper record value is given by*

$$\mathcal{P}^w_{\theta,\alpha}(Z_{[n]}) = \frac{1}{\theta-1}\left(\int_0^{\infty} zF_{[n]}(z)dz - \sum_{j=0}^{\mathcal{N}(\theta)}\binom{\theta}{j}(\alpha\delta_{(n)})^j E\left[U^{\theta}(1-U)^j Q(U)q(U)\right]\right).$$

The link between the weighted mean past lifetime (WMPL), $\mu^*_{F_{[r,n,m,k]}}(t) = \int_0^t z\frac{F_{[r,n,m,k]}(z)}{F_{[r,n,m,k]}(t)}dz$, $F_{[r,n,m,k]}(t) > 0$, and the WCPTE is then determined.

Lemma 3.1. *Let Z be a non-negative continuous RV with DF $F_Z(z)$, and then*

$$\mathcal{P}^w_{\theta,\alpha}(Z_{[r,n,m,k]}) = E\left[\mu^*_{F_{[r,n,m,k]}}(Z_{[r,n,m,k]})F^{\theta-1}_{[r,n,m,k]}(Z_{[r,n,m,k]})\right].$$

Proof. We have $\frac{d}{dz}(\mu^*_{F[r,n,m,k]}(z)F_{[r,n,m,k]}(z)) = zF_{[r,n,m,k]}(z)$. Using (3.1), we get

$$\mathcal{P}^w_{\theta,\alpha}(Z_{[r,n,m,k]}) = \frac{1}{\theta-1}\left[\int_0^\infty \frac{d}{dz}(\mu^*_{F_{[r,n,m,k]}}(z)F_{[r,n,m,k]}(z))(1 - F^{\theta-1}_{[r,n,m,k]}(z))dz\right].$$

□

Example 3.1. *Assume that the uniform distribution (UD) of Y and Z results from the FGM family (i.e., $F_Z(z) = z, 0 \leq z \leq 1$). After simple algebra, we get*

$$\int_0^1 zF_{[r,n,m,k]}(z)dz = \frac{1}{3}\left[1 + \frac{(\alpha C^*_{(r,n,m,k)})}{4}\right],$$

and

$$E\left[U^\theta(1-U)^jQ(U)q(U)\right] = \int_0^1 z^{1+\theta}(1-z)^jdz = \beta(1+j, 2+\theta).$$

Then, based on (3.1), this leads to the following WCPTE in $Z_{[r,n,m,k]}$:

$$\mathcal{P}^w_{\theta,\alpha}(Z_{[r,n,m,k]}) = \frac{1}{\theta-1}\left(\frac{1}{3}\left[1 + \frac{(\alpha C^*_{(r,n,m,k)})}{4}\right] - \sum_{j=0}^{\mathcal{N}(\theta)}\binom{\theta}{j}(\alpha C^*_{(r,n,m,k)})^j\beta(1+j, 2+\theta)\right),$$

and

$$\mu^*_{F_{[r,n,m,k]}}(t) = \frac{1}{F_{[r,n,m,k]}(t)}\int_0^t zF_{[r,n,m,k]}(z)dz = \frac{t^3}{3F_{[r,n,m,k]}(t)}\left[1 + \alpha C^*_{(r,n,m,k)}\left(\frac{4-3t}{4}\right)\right].$$

We can easily show that

$$E\left[\mu^*_{F_{[r,n,m,k]}}(Z_{[r,n,m,k]})F^{\theta-1}_{[r,n,m,k]}(Z_{[r,n,m,k]})\right]$$
$$= \frac{1}{\theta-1}\left(\frac{1}{3}\left[1 + \frac{\alpha C^*_{(r,n,m,k)}}{4}\right] - \sum_{j=0}^{\mathcal{N}(\theta)}\binom{\theta}{j}(\alpha C^*_{(r,n,m,k)})^j\beta(1+j, 2+\theta)\right).$$

Example 3.2. *Let Y and Z be two variables that represent power distributions obtained from the FGM family (i.e., $F_Z(z) = z^c,\ 0 \leq z \leq 1, c > 0$). Then*

$$\int_0^1 zF_{[r,n,m,k]}(z)dz = \frac{1}{2+c}\left[1 + \frac{c(\alpha C^*_{(r,n,m,k)})}{2(c+1)}\right],$$

and

$$E\left[U^\theta(1-U)^jQ(U)q(U)\right] = \int_0^1 z^{1+c\theta}(1-z^c)^jdz = \frac{\beta(1+j, \frac{2}{c}+\theta)}{c}.$$

Thus, based on (3.1), this leads to the following WCPTE in $Z_{[r,n,m,k]}$:

$$\mathcal{P}^w_{\theta,\alpha}(Z_{[r,n,m,k]}) = \frac{1}{\theta-1}\left(\frac{1}{2+c}\left[1 + \frac{c(\alpha C^*_{(r,n,m,k)})}{2(c+1)}\right] - \sum_{j=0}^{\mathcal{N}(\theta)}\binom{\theta}{j}(\alpha C^*_{(r,n,m,k)})^j\frac{\beta(1+j, \frac{2}{c}+\theta)}{c}\right).$$

Lemma 3.2. *If $Z = aX + b$ with $a > 0$ and $b \geq 0$, then*

$$\mathcal{P}^{w}_{\theta,\alpha}(Z_{[r,n,m,k]}) = a^2\mathcal{P}^{w}_{\theta,\alpha}(X_{[r,n,m,k]}) + ab\mathcal{P}^{w}_{\theta,\alpha}(X_{[r,n,m,k]}).$$

Proof. The proof follows using the fact that $F_{Z_{[r,n,m,k]}}(u) = F_{X_{[r,n,m,k]}}(\frac{u-b}{a})$. □

Tables 7 and 8 show some aspects of the behavior of the WCPTE for $Z_{[n]}$ and $Z_{[r:n]}$ based on the FGM-UD. From Tables 7 and 8, the ensuing characteristics are extractable:

- We see that the value of $\mathcal{P}^{w}_{\theta,\alpha}(Z_{[n]})$ increases as n increases, the value of $\mathcal{P}^{w}_{\theta,-\alpha}(Z_{[n]})$ decreases as n increases, and it almost stays the same when $n = 15$ (see Table 7).
- Generally, $\mathcal{P}^{w}_{\theta,\alpha}(Z_{[r:n]}) = \mathcal{P}^{w}_{\theta,-\alpha}(Z_{[n-r+1:n]})$. Also, the value of $\mathcal{P}^{w}_{\theta,\alpha}(Z_{[r:n]})$ slowly increases at $r \leq (\frac{n}{2} + 1)$, and decreases at $r \geq (\frac{n}{2} + 1)$ (see Table 8).

Table 7. $\mathcal{P}^{w}_{\theta,\alpha}(Z_{[n]})$ based on FGM-UD.

	$\theta = 4$					$\theta = 9$			
n	$\alpha = -0.5$	$\alpha = 0.5$	$\alpha = -0.9$	$\alpha = 0.9$	**n**	$\alpha = -0.5$	$\alpha = 0.5$	$\alpha = -0.9$	$\alpha = 0.9$
2	0.0538	0.0558	0.0511	0.0552	2	0.0304	0.0295	0.0298	0.0286
3	0.0522	0.0555	0.0466	0.0536	3	0.0301	0.029	0.0281	0.0273
4	0.0513	0.0552	0.0437	0.0525	4	0.0299	0.0287	0.0268	0.0265
5	0.0508	0.0551	0.042	0.0519	5	0.0297	0.0285	0.0259	0.0262
6	0.0505	0.055	0.0411	0.0516	6	0.0296	0.0284	0.0255	0.026
7	0.0504	0.0549	0.0407	0.0514	7	0.0296	0.0284	0.0252	0.0259
8	0.0503	0.0549	0.0405	0.0513	8	0.0296	0.0283	0.0251	0.0258
9	0.0503	0.0549	0.0404	0.0513	9	0.0295	0.0283	0.025	0.0258
10	0.0503	0.0549	0.0403	0.0513	10	0.0295	0.0283	0.025	0.0258
11	0.0503	0.0549	0.0403	0.0512	11	0.0295	0.0283	0.025	0.0258
12	0.0503	0.0549	0.0403	0.0512	12	0.0295	0.0283	0.025	0.0258
13	0.0503	0.0549	0.0402	0.0512	13	0.0295	0.0283	0.025	0.0258
14	0.0502	0.0549	0.0402	0.0512	14	0.0295	0.0283	0.025	0.0258
15	0.0502	0.0549	0.0402	0.0512	15	0.0295	0.0283	0.025	0.0258
16	0.0502	0.0549	0.0402	0.0512	16	0.0295	0.0283	0.025	0.0258

Table 8. $\mathcal{P}^{w}_{\theta,\alpha}(Z_{[r:n]})$ based on FGM-UD.

		$\theta = 4$						$\theta = 9$					
n	**r**	$\alpha = -0.1$	$\alpha = 0.1$	$\alpha = -0.5$	$\alpha = 0.5$	$\alpha = -0.9$	$\alpha = 0.9$	$\alpha = -0.1$	$\alpha = 0.1$	$\alpha = -0.5$	$\alpha = 0.5$	$\alpha = -0.9$	$\alpha = 0.9$
4	1	0.0558	0.0553	0.0557	0.0532	0.0546	0.0495	0.0302	0.0304	0.0293	0.0303	0.0281	0.0293
4	2	0.0556	0.0555	0.0558	0.055	0.0559	0.0544	0.0303	0.0303	0.0301	0.0304	0.0298	0.0305
4	3	0.0555	0.0556	0.055	0.0558	0.0544	0.0559	0.0303	0.0303	0.0304	0.0301	0.0305	0.0298
4	4	0.0553	0.0558	0.0532	0.0557	0.0495	0.0546	0.0304	0.0302	0.0303	0.0293	0.0293	0.0281
6	1	0.0558	0.0552	0.0556	0.0525	0.0538	0.0473	0.0301	0.0304	0.0291	0.0302	0.0275	0.0284
6	2	0.0557	0.0554	0.0559	0.0541	0.0555	0.0521	0.0302	0.0304	0.0297	0.0304	0.0289	0.0301
6	3	0.0556	0.0555	0.0558	0.0552	0.0559	0.0548	0.0303	0.0303	0.0301	0.0304	0.03	0.0305
6	4	0.0555	0.0556	0.0552	0.0558	0.0548	0.0559	0.0303	0.0303	0.0304	0.0301	0.0305	0.03
6	5	0.0554	0.0557	0.0541	0.0559	0.0521	0.0555	0.0304	0.0302	0.0304	0.0297	0.0301	0.0289
6	6	0.0552	0.0558	0.0525	0.0556	0.0473	0.0538	0.0304	0.0301	0.0302	0.0291	0.0284	0.0275
8	1	0.0558	0.0552	0.0554	0.0521	0.0533	0.046	0.0301	0.0304	0.0289	0.0301	0.0271	0.0279
8	2	0.0557	0.0553	0.0558	0.0535	0.0549	0.0502	0.0302	0.0304	0.0294	0.0304	0.0283	0.0295
8	3	0.0557	0.0554	0.0559	0.0546	0.0557	0.0532	0.0302	0.0304	0.0299	0.0305	0.0293	0.0303
8	4	0.0556	0.0555	0.0557	0.0553	0.0558	0.055	0.0303	0.0303	0.0302	0.0304	0.0301	0.0304
8	5	0.0555	0.0556	0.0553	0.0557	0.055	0.0558	0.0303	0.0303	0.0304	0.0302	0.0304	0.0301
8	6	0.0554	0.0557	0.0546	0.0559	0.0532	0.0557	0.0304	0.0302	0.0305	0.0299	0.0303	0.0293
8	7	0.0553	0.0557	0.0535	0.0558	0.0502	0.0549	0.0304	0.0302	0.0304	0.0294	0.0295	0.0283
8	8	0.0552	0.0558	0.0521	0.0554	0.046	0.0533	0.0304	0.0301	0.0301	0.0289	0.0279	0.0271
10	1	0.0558	0.0551	0.0554	0.0518	0.053	0.045	0.0301	0.0304	0.0288	0.03	0.0269	0.0274
10	2	0.0558	0.0553	0.0557	0.053	0.0544	0.0489	0.0302	0.0304	0.0292	0.0303	0.0279	0.029
10	3	0.0557	0.0554	0.0559	0.054	0.0554	0.0518	0.0302	0.0304	0.0296	0.0304	0.0288	0.03
10	4	0.0557	0.0554	0.0559	0.0548	0.0559	0.0538	0.0302	0.0304	0.03	0.0305	0.0296	0.0304
10	5	0.0556	0.0555	0.0557	0.0554	0.0558	0.0551	0.0303	0.0303	0.0302	0.0304	0.0301	0.0304
10	6	0.0555	0.0556	0.0554	0.0557	0.0551	0.0558	0.0303	0.0303	0.0304	0.0302	0.0304	0.0301
10	7	0.0554	0.0557	0.0548	0.0559	0.0538	0.0559	0.0304	0.0302	0.0305	0.03	0.0304	0.0296
10	8	0.0554	0.0557	0.054	0.0559	0.0518	0.0554	0.0304	0.0302	0.0304	0.0296	0.03	0.0288
10	9	0.0553	0.0558	0.053	0.0557	0.0489	0.0544	0.0304	0.0302	0.0303	0.0292	0.029	0.0279
10	10	0.0551	0.0558	0.0518	0.0554	0.045	0.053	0.0304	0.0301	0.03	0.0288	0.0274	0.0269

Dynamic WCPTE measure

Dynamic WCPTE (DWCPTE) of an RV Z is the WCPTE of the RV $[t - Z|Z < t], t > 0$. The DWCPTE variant from the FGM family in $Z_{[r,n,m,k]}$ is provided as

$$\begin{aligned}
\mathcal{P}^{w}_{\theta,\alpha}(Z_{[r,n,m,k]};t) &= \frac{1}{\theta-1}\int_0^t z\left(F_{Z_t[r,n,m,k]}(z) - F^{\theta}_{Z_t[r,n,m,k]}(z)\right)dz \\
&= \frac{1}{\theta-1}\int_0^t z\left(\frac{F_{[r,n,m,k]}(z)}{F_{[r,n,m,k]}(t)} - \left(\frac{F_{[r,n,m,k]}(z)}{F_{[r,n,m,k]}(t)}\right)^{\theta}\right)dz \\
&= \frac{1}{\theta-1}\left(\mu^{*}_{F_{[r,n,m,k]}}(t) - \int_0^t z\left(\frac{F_{[r,n,m,k]}(z)}{F_{[r,n,m,k]}(t)}\right)^{\theta} dz\right).
\end{aligned}$$

Theorem 3.2. *Let Z be an absolutely continuous non-negative RV with WMPL function $\mu^{*}_{F[r,n,m,k]}(t)$, and then,*

$$\mathcal{P}^{w}_{\theta,\alpha}(Z_{[r,n,m,k]},t) = \frac{E\left[\mu^{*}_{F_{[r,n,m,k]}}(Z_{[r,n,m,k]})F^{\theta-1}_{[r,n,m,k]}(Z_{[r,n,m,k]})|Z < t\right]}{F^{\theta-1}_{[r,n,m,k]}(t)}.$$

Proof. The Proof is similar to (2.5). □

4. Empirical AWCRTE and WCPTE

The issue of estimating the AWCRTE and WCPTE for concomitant $Z_{[r,n,m,k]}$ utilizing the empirical AWCRTE will be examined next. For every $i = 1, 2, ..., n$, consider the FGM sequence (Y_i, Z_i). In accordance with (2.3), the empirical AWCRTE of the set $Z_{[r,n,m,k]}$ can be computed as follows:

$$\begin{aligned}
\hat{\xi}^{w}_{\theta,\alpha}(Z_{[r,n,m,k]}) &= \frac{1}{\theta-1}\int_0^{\infty} z\left(\hat{\bar{F}}_{[r,n,m,k]}(z) - \hat{\bar{F}}^{\theta}_{[r,n,m,k]}(z)\right)dz \\
&= \frac{1}{\theta-1}\sum_{j=1}^{n-1}\int_{z_{(j)}}^{z_{(j+1)}} z\left((1-\hat{F}_Z(z))\left(1-\alpha C^{*}_{(r,n,m,k)}\hat{F}_Z(z)\right) - (1-\hat{F}_Z(z))^{\theta}\right. \\
&\times \left.\left(1-\alpha C^{*}_{(r,n,m,k)}\hat{F}_Z(z)\right)^{\theta}\right)dz \\
&= \frac{1}{\theta-1}\sum_{j=1}^{n-1}\Delta_j\left(\left(1-\frac{j}{n}\right)\left(1-\alpha C^{*}_{(r,n,m,k)}\frac{j}{n}\right) - \left(1-\frac{j}{n}\right)^{\theta}\left(1-\alpha C^{*}_{(r,n,m,k)}\frac{j}{n}\right)^{\theta}\right),
\end{aligned}$$

where for any DF $F(.)$, the symbol $\hat{F}(.)$ stands for the empirical DF of $F(.)$ and $\Delta_j = \frac{Z^2_{(j+1)} - Z^2_{(j)}}{2}$, $j = 1, 2..., n-1$, are the sample spacings based on ordered random samples of Z_j. Similarly, based on (3.1), the empirical WCPTE of the set $Z_{[r,n,m,k]}$ can be expressed as

$$\hat{\mathcal{P}}^{w}_{\theta,\alpha}(Z_{[r,n,m,k]}) = \frac{1}{\theta-1}\sum_{j=1}^{n-1}\Delta_j\left(\frac{j}{n}\left(1+\alpha C^{*}_{(r,n,m,k)}\left(1-\frac{j}{n}\right)\right) - \left(\frac{j}{n}\right)^{\theta}\left(1+\alpha C^{*}_{(r,n,m,k)}\left(1-\frac{j}{n}\right)\right)^{\theta}\right).$$

Example 4.1. *Suppose that Z has a Rayleigh distribution with PDF $f_Z(z) = 2\lambda z e^{-\lambda z^2}, z > 0, \lambda > 0$. Then, Z^2 has ED with mean $\frac{1}{\lambda}$ and $\Delta_j = \frac{Z^2_{(j+1)} - Z^2_{(j)}}{2}$ has ED with mean $\frac{1}{2\lambda(n-j)}$, $j = 1, 2..., n-1$. The* *expected value and variance of the empirical AWCRTE in* $\hat{\xi}^{w}_{\theta,\alpha}(Z_{[r,n,m,k]})$ *are given by*

$$E\left[\hat{\xi}^{w}_{\theta,\alpha}(Z_{[r,n,m,k]})\right] = \frac{1}{2\lambda(\theta-1)}\sum_{j=1}^{n-1}\frac{1}{(n-j)}\left(\left(1-\frac{j}{n}\right)\left(1-\alpha C^{*}_{(r,n,m,k)}\frac{j}{n}\right)-\left(1-\frac{j}{n}\right)^{\theta}\left(1-\alpha C^{*}_{(r,n,m,k)}\frac{j}{n}\right)^{\theta}\right),$$

and

$$Var\left[\hat{\xi}^{w}_{\theta,\alpha}(Z_{[r,n,m,k]})\right] = \frac{1}{4\lambda^{2}(\theta-1)^{2}}\sum_{j=1}^{n-1}\frac{1}{(n-j)^{2}}\left(\left(1-\frac{j}{n}\right)\left(1-\alpha C^{*}_{(r,n,m,k)}\frac{j}{n}\right)-\left(1-\frac{j}{n}\right)^{\theta}\left(1-\alpha C^{*}_{(r,n,m,k)}\frac{j}{n}\right)^{\theta}\right)^{2}.$$

Example 4.2. *Let* $(Y_i, Z_i), i = 1, 2, ..., n$, *be a random sample from the FGM family with PDF* $f_Z(z) = 2z,\ 0 < z < 1$. *Then,* Z^2 *has a standard uniform distribution. Furthermore,* $\Delta_j = \frac{Z^2_{(j+1)}-Z^2_{(j)}}{2}$ *follows the beta distribution with mean* $\frac{1}{2(n+1)}$ *and variance* $\frac{n}{4(n+1)^2(n+2)}$. *The mean and variance of the empirical WCPTE in* $\hat{\mathcal{P}}^{w}_{\theta,\alpha}(Z_{[r,n,m,k]})$ *are given by*

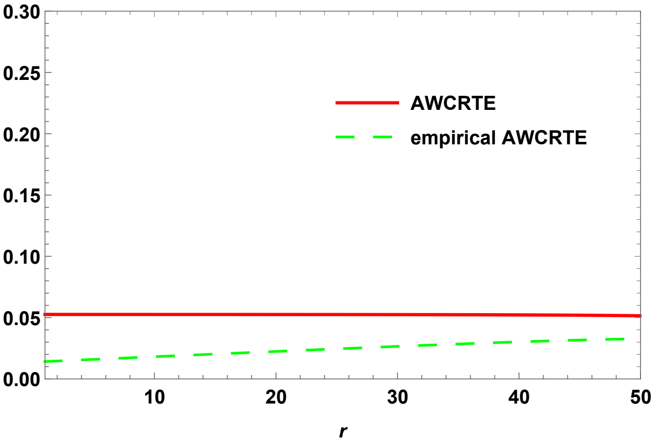
$$E\left[\hat{\mathcal{P}}^{w}_{\theta,\alpha}(Z_{[r,n,m,k]})\right] = \frac{1}{2(n+1)(\theta-1)}\sum_{j=1}^{n-1}\left(\frac{j}{n}\left(1+\alpha C^{*}_{(r,n,m,k)}\left(1-\frac{j}{n}\right)\right)-\left(\frac{j}{n}\right)^{\theta}\left(1+\alpha C^{*}_{(r,n,m,k)}\left(1-\frac{j}{n}\right)\right)^{\theta}\right),$$

and

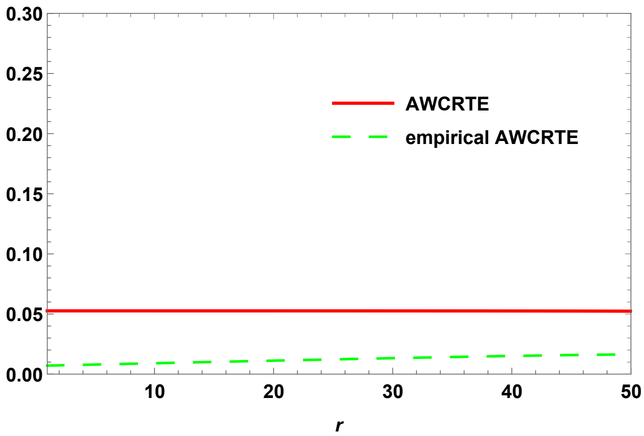
$$\begin{aligned} Var\left[\hat{\mathcal{P}}^{w}_{\theta,\alpha}(Z_{[r,n,m,k]})\right] = & \frac{n}{4(n+1)^{2}(n+2)(\theta-1)^{2}}\sum_{j=1}^{n-1}\left(\frac{j}{n}\left(1+\alpha C^{*}_{(r,n,m,k)}\left(1-\frac{j}{n}\right)\right)-\left(\frac{j}{n}\right)^{\theta}\right. \\ & \times \left.\left(1+\alpha C^{*}_{(r,n,m,k)}\left(1-\frac{j}{n}\right)\right)^{\theta}\right)^{2}. \end{aligned}$$

Both the AWCRTE and empirical AWCRTE in $Z_{[r:n]}$ from FGM-ED at $n = 50$, as well as the WCPTE and empirical WCPTE in $Z_{[r:n]}$ from FGM-UD, are shown in Figures 2 and 3. Figures 2 and 3 can be utilized to ascertain the subsequent properties:

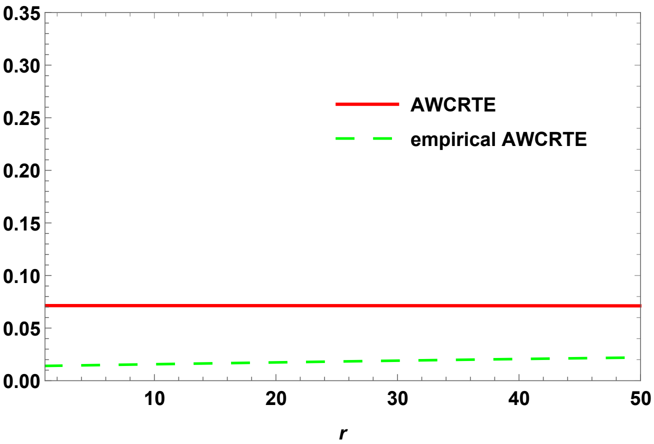
1) Mostly, the AWCRTE and the empirical AWCRTE are close together.
2) Generally, the WCPTE and the empirical WCPTE are very close.



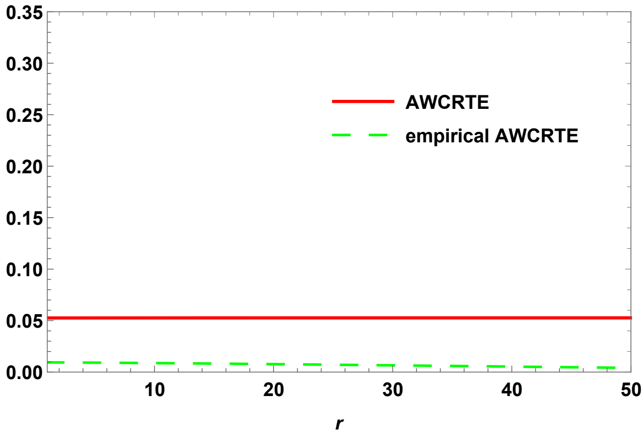
(a) $\alpha = 0.9, \lambda = 1$, and $\theta = 20$



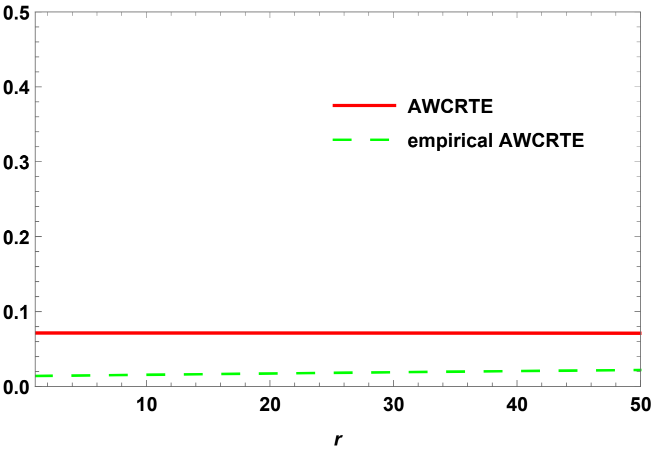
(b) $\alpha = 0.9, \lambda = 2$, and $\theta = 20$



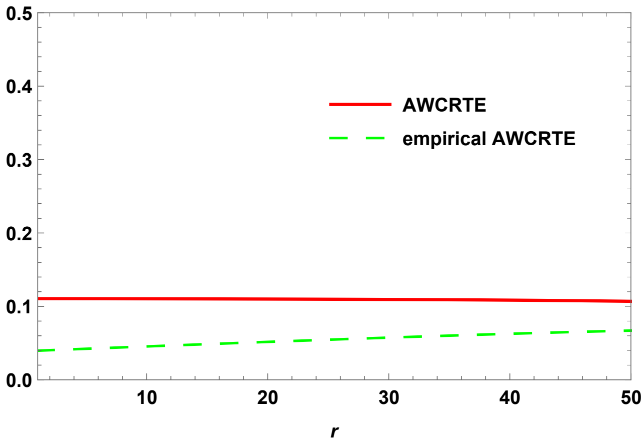
(c) $\alpha = 0.9, \lambda = 2$, and $\theta = 15$



(d) $\alpha = -0.9, \lambda = 3.5$, and $\theta = 20$

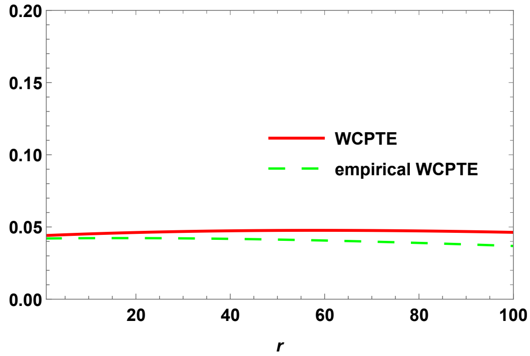


(e) $\alpha = 0.5, \lambda = 1.8$, and $\theta = 15$

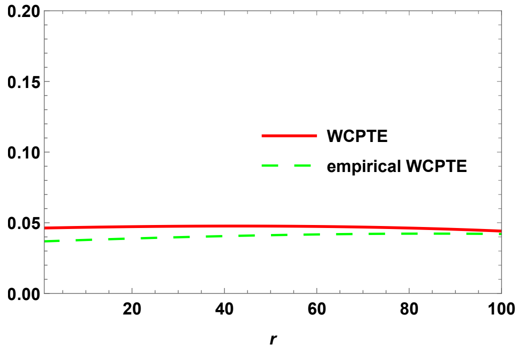


(f) $\alpha = 0.6, \lambda = 0.9$, and $\theta = 10$

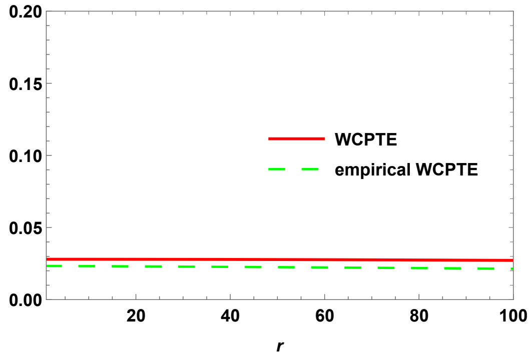
Figure 2. Representation of AWCRTE and empirical AWCRTE in $Z_{[r:n]}$ based on FGM-ED for $n = 50$.



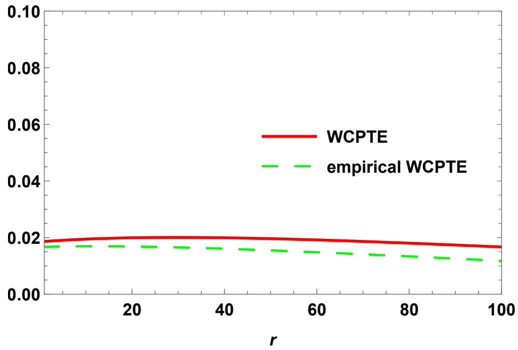
(a) $\alpha = 0.5$ and $\theta = 5$



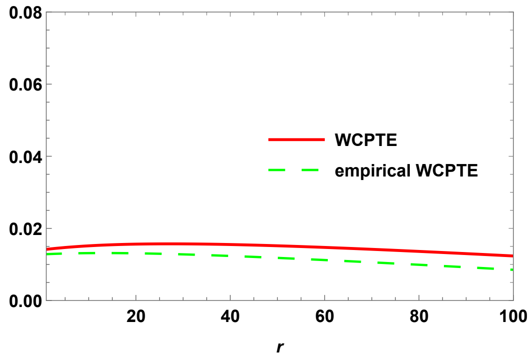
(b) $\alpha = -0.5$ and $\theta = 5$



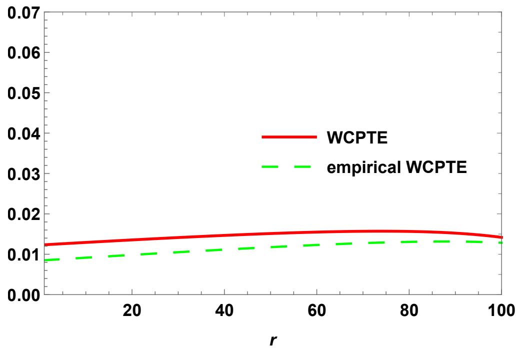
(c) $\alpha = 0.2$ and $\theta = 10$



(d) $\alpha = 0.2$ and $\theta = 15$



(e) $\alpha = 0.9$ and $\theta = 20$



(f) $\alpha = -0.9$ and $\theta = 20$

Figure 3. Representation of WCPTE and empirical WCPTE in $Z_{[r:n]}$ based on FGM-UD for $n = 100$.

5. Real data application

As we have previously highlighted, Tsallis entropy is a generalization of Shannon entropy that works better with non-extensive systems. This makes it possible to analyze complex data, like those found in medical settings, such as electroencephalogram (EEG) signals used to diagnose epilepsy, in a more flexible and reliable manner. Tsallis entropy offers a more nuanced view of the system's non-linear and non-extensive dynamics, providing insights that may not be captured by Shannon entropy. EEG signals are an example of a complicated medical data set used in epilepsy diagnosis.

Using Tsallis entropy and a few of its associated measures that are covered in this article, we examine two medical data sets below. Tsallis entropy provides information that Shannon entropy might miss, despite the fact that it is challenging to practically verify the level of complexity and non-linearity of the handled systems.

Example 5.1. *We use the data for 30 patients from [31]. Let Y refer to the first recurrence time and Z to the second recurrence time, as follows:* Y *is (8, 23, 22, 447, 30, 24, 7, 511, 53, 15, 7, 141, 96, 149, 536, 17, 185, 292, 22, 15, 152, 402, 13, 39, 12, 113, 132, 34, 2, 130) and Z is (16, 13, 28, 318, 12, 245, 9, 30, 196, 154, 333, 8, 38, 70, 25, 4, 117, 114, 159, 108, 362, 24, 66, 46, 40, 201, 156, 30, 25, 26). Reference [32] introduced FGM bivariate WD, and they discussed the estimation of the parameters of this model and found the maximum likelihood estimates (MLEs) of the shape and scale parameters* $(\eta_i, \beta_i), i = 1, 2$, *as* $(0.75106, 100.11993)$ *and* $(0.92435, 98.24665)$, *respectively, and* $\alpha = 0.34801$. *To find a trust region or confidence intervals for the parameters of the FGM bivariate WD, we can further use, among many other methods, the fisher information matrix to derive asymptotic confidence intervals (which provides the standard method). However, the primary objective is to determine whether the FGM-WD model, after estimating its unknown parameters, adequately fits the given data and then examines the AWCRTE and WCRTE.*

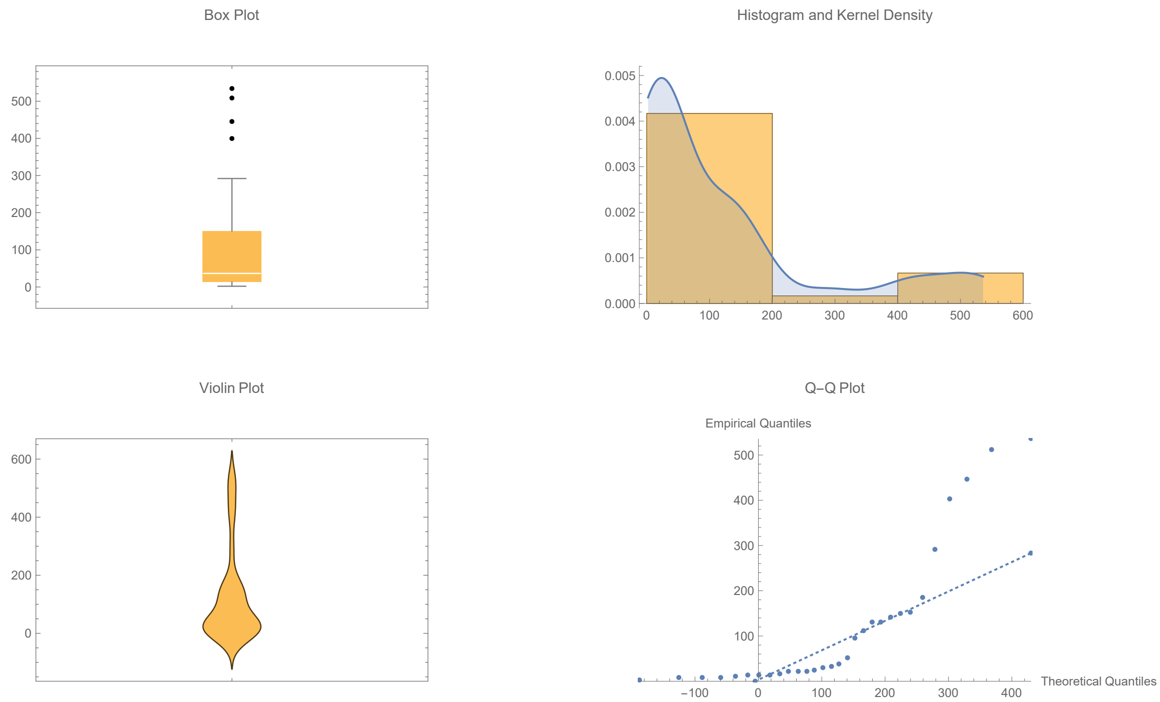


Figure 4. Plots of the first recurrence time data set.

Table 9 examines the AWCRTE and WCRTE for FGM-WD (0.75106,100.11993,0.92435, 98.24665). For the concomitants $Z_{[r:30]}, r = 1, 2, 14, 15, 29, 30$, *i.e., the lower and upper extremes concomitants, and the central values concomitants. We observe that the* $\xi^{w}_{\theta,\alpha}(Z_{[r:30]})$ *and* $\zeta^{w}_{\theta,\alpha}(Z_{[r:30]})$ *have maximum values at extremes. Figures 4 and 5 provide a fundamental statistical analysis illustrating the data.*

Table 9. The AWCRTE and WCRTE of FGM-WD.

θ	r	1	2	14	15	29	30
4	$\xi^w_{\theta,\alpha=0.348}(Z_{[r:30]})$	3958.94	3945.32	3781.93	3768.32	3577.69	3564.08
	$\zeta^w_{\theta,\alpha=0.348}(Z_{[r:30]})$	0.333164	0.333158	0.333068	0.333059	0.332893	0.332878
9	$\xi^w_{\theta,\alpha=0.348}(Z_{[r:30]})$	1484.6	1479.49	1418.22	1413.12	1341.64	1336.53
	$\zeta^w_{\theta,\alpha=0.348}(Z_{[r:30]})$	0.12499	0.124989	0.124983	0.124982	0.124968	0.124966

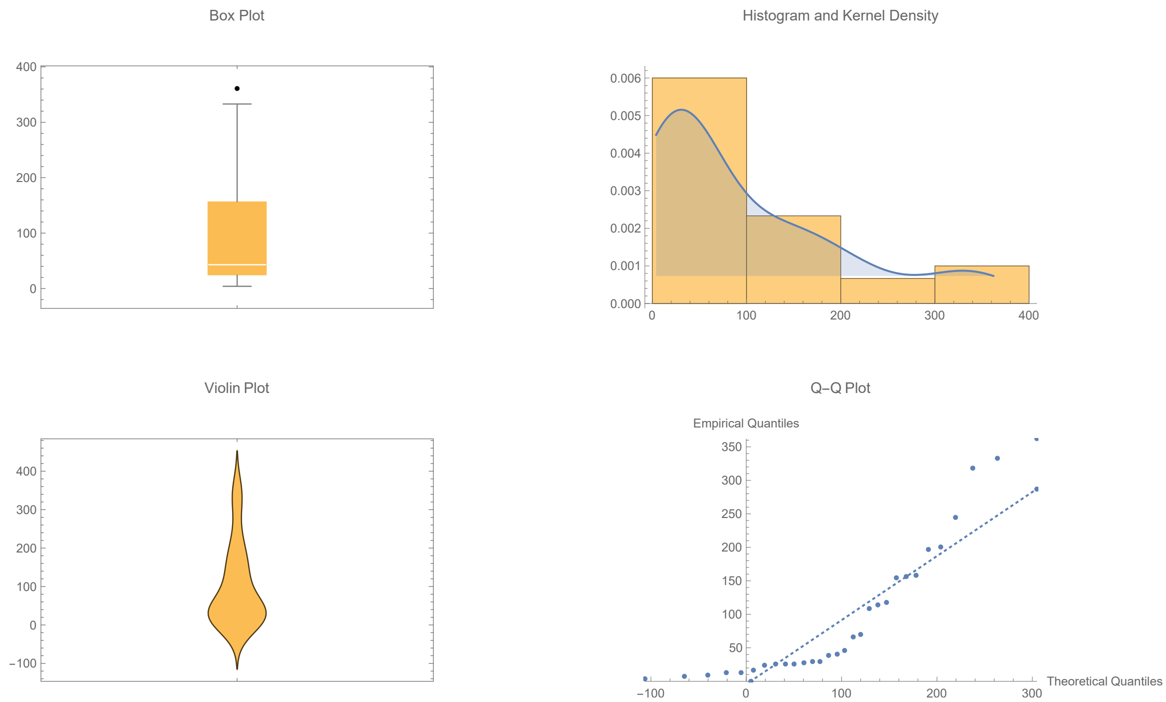


Figure 5. Plots of the second recurrence time data set.

Example 5.2 (Cholesterol data set)**.** *This data set includes cholesterol levels measured at 5 and 25 weeks after treatment in 30 patients (see [33]). We fit the data based on FGM-WD* $(\eta_1,\beta_1;\ \eta_2,\beta_2)$*. The MLEs of parameters are* $\widehat{\eta}_1 = 2.93893, \widehat{\beta}_1 = 1.21085, \widehat{\eta}_2 = 2.589$, $\widehat{\beta}_2 = 1.10099$, *and* $\widehat{\alpha} = 1$*. Table 10 examines the AWCRTE and WCRTE for FGM-WD (2.93893,1.21085,2.589,1.10099). For the concomitants* $Z_{[r:30]}, r = 1, 2, 14, 15, 29, 30$, *i.e., the lower and upper extremes concomitants, and the central values concomitants. We observe that the* $\xi^w_{\theta[r:30]}(Z)$ *and* $\zeta^w_{\theta[r:30]}(Z)$ *have maximum values at extremes.*

Table 10. The AWCRTE and WCRTE of FGM-WD.

θ	r	1	2	14	15	29	30
4	$\xi^w_{\theta[r:30]}(Z)$	0.26141	0.253816	0.158299	0.149854	0.0190433	0.00855513
	$\zeta^w_{\theta[r:30]}(Z)$	0.308546	0.30781	0.294593	0.293007	0.258213	0.254584
15	$\xi^w_{\theta[r:30]}(Z)$	0.0594999	0.0579511	0.0392552	0.0376493	0.0123585	0.0101515
	$\zeta^w_{\theta[r:30]}(Z)$	0.0696003	0.0695212	0.0684611	0.0683249	0.0636092	0.0628719
30	$\xi^w_{\theta[r:30]}(Z)$	0.0291314	0.0283863	0.019546	0.0187973	0.00726299	0.00626894
	$\zeta^w_{\theta[r:30]}(Z)$	0.0340075	0.0339718	0.0336455	0.0336062	0.0320047	0.0317202

6. Conclusions

Among bivariate distributions, the FGM model is one of the most well-known and useful in recent years. The FGM bivariate distribution family has been widely accepted for many practical applications. Furthermore, the recently introduced redundant concomitants of OSs have regained popularity due to their usefulness in prediction and selection contexts. Another important concept that has gained attention is Tsallis entropy, which has been applied in various fields, including physics and chemistry. Every year, new applications of these measures are discovered. Some of the most significant related measures recently introduced are WCRTE and WCPTE. AWCRTE, WCPTE, and their dynamic counterparts for the concomitants of m-GOSs were derived from the FGM bivariate family. The characteristics of the proposed entropy measures were analyzed. These entropy measures were used to characterize the exponential and Pareto distributions. Applications of these findings were presented for OS and record values with uniform, Weibull, and exponential marginal distributions. Additionally, non-parametric estimators of AWCRTE and WCPTE were proposed for calculating the new information measures.

Two real-world data sets were evaluated for illustration, yielding satisfactory results. The parameter θ in Tsallis entropy controls the sensitivity of the entropy to rare events and affects the system's "distributional" properties. Specifically, when $\theta = 1$, it corresponds to the standard Shannon entropy, while $\theta > 1$ and $\theta < 1$ modify the sensitivity to the tail and central parts of the distribution, respectively.

To estimate θ empirically, the Tsallis model is typically fitted to data using methods such as maximum likelihood estimation, least squares fitting, or numerical optimization. For more details, see [34]. However, in this study, we did not address the estimation of θ. Instead, we selected different values of θ to demonstrate how the results change as the value of θ varies.

Validating the quality of estimates for unknown parameters in real data cases can be quite challenging and often requires specialized techniques like bootstrapping. However, this aspect is beyond the scope of our current study. It is important to note that parameter estimation is just one part of our statistical analysis. The primary objective is to determine whether the FGM-WD model, after estimating its unknown parameters, adequately fits the given data, which was confirmed in the examples we explored. In future work, where we will conduct an in-depth study of a specific practical situation, we plan to address this issue more thoroughly.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The authors are grateful to the editor and anonymous referees for their insightful comments and suggestions, which helped to improve the paper's presentation. This research was conducted under a project titled "Ongoing Research Funding Program", funded by King Saudi University, Riyadh, Saudi Arabia, under grant number (ORF-2025-969).

Conflict of interest

The authors declare no conflict of interest.

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