Mathematical Biosciences and Engineering

https://www.aimspress.com/journal/mbe

MBE, 22(4): 876–896.
DOI: 10.3934/mbe.2025031
Received: 26 December 2024
Revised: 25 February 2025
Accepted: 27 February 2025
Published: 07 March 2025

Research article

Event-triggered impulsive control for input-to-state stability of nonlinear time-delay system with delayed impulse

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Abstract: The problem of input-to-state stability (ISS) was studied in this paper for a class of nonlinear time-delay systems with delayed impulses under event-triggered impulsive control (ETIC), where delays were flexible and external inputs in continuous and impulse dynamics were different. To avoid Zeno behavior, an event-triggered mechanism (ETM) that used the information of the system state and external disturbances was proposed, and the feasibility was enhanced by introducing a forced impulse sequence. Furthermore, some sufficient conditions were formulated to enable ISS for the considered system by using Lyapunov–Razumikhin-like methods. For a class of nonlinear impulse control systems, an application was introduced that utilizes linear matrix inequalities (LMIs) to design an ETM and control gain. Finally, two numerical examples were given to validate the theoretical results.

Keywords: input-to-state stability; nonlinear time-delay systems; event-triggered impulsive control; delayed impulse; Zeno behavior

1. Introduction

Stability is the basic issue in the field of control, and input-to-state stability (ISS) constitutes a crucial aspect. The concept of ISS was first introduced by Sontag [1]. Distinct from the conventional concepts of stability, ISS not only demands that the state of the system exhibit excellent stability with respect to initial conditions and internal disturbances, but also requires that the system is able to make a controlled response to external inputs so that the system can maintain a certain degree of stability when confronted with external interferences. In recent years, the characteristics of ISS have received extensive attention, and a large number of significant results in theory and application fields have emerged in the relevant literature [2–7].

Impulse system is a hybrid system integrating continuous dynamics and discrete dynamics [8], which has extensive use in many domains, such as automatic control, signal processing, image encryption, and so on. In comparison to the traditional continuous control methods, impulse control breaks through the limitations of continuous control by using discrete impulse signals as the control input, and can effectively deal with the sudden change phenomena in the system and achieve good control results. The main advantages of impulse control lie in its low cost, low energy consumption, and high efficiency, so it has gradually become the focus of attention in many research fields [9, 10]. In order to enhance system stability and robustness, sliding mode control (SMC), as a widely used robust control method, can effectively handle system uncertainties [11–13]. SMC ensures system stability by designing an appropriate sliding surface and forcing the system state to slide along this surface. However, a common issue with SMC is the phenomenon of "chattering", which may lead to excessive control, thereby inducing high-frequency oscillations in the system. In contrast, impulse control effectively avoids the chattering phenomenon by applying discrete pulse signals when needed, reducing control power consumption and energy waste. The research of stability for impulsive systems mainly focuses on two aspects: impulse disturbance and impulse control. The former takes into account the system's robustness when unstable impulses are present, while the latter considers the stabilization of the systems incorporating stable impulses [14]. The traditional impulse control strategy is time-triggered impulse control, which is convenient for controllers design. However, since there is no clear criteria to select the appropriate trigger time, it often causes unnecessary waste of resources [15, 16]. Therefore, in order to address the drawbacks of the time-triggered control method, event-triggered control (ETC) is put forward, where data transmission only takes place when the control mechanism is activated, otherwise, the control signals are continuously updated [17–20]. Event-triggered impulse control (ETIC) merges the strengths of impulse control and event-triggered control by releasing control signals only when specific state-dependent criteria are met [21, 22]. Compared with conventional ETC using the zero-order-hold (ZOH) method [23], although ETIC also requires real-time monitoring, the absence of control transmission between two neighboring trigger signals leads to a reduction in communication load and resource consumption.

In practical engineering systems, time delay is almost everywhere, and it inevitably affects the performance of the controlled system, potentially even causing instability. The existence of time delay complicates the direct application of the event-triggering mechanism (ETM) designed for delay-free systems to systems that include delays, as the delay can result in infinitely fast triggering, a phenomenon known as Zeno behavior [24, 25]. This infinitely fast triggering behavior may not only cause serious degradation of system performance, but also may make the controller difficult to realize in practical applications. In addition, the existence of time delay makes the design of suitable ETM more complicated, and it is necessary to extract delayed state information while excluding Zeno phenomenon, which poses a higher challenge to the design of ETM. Therefore, designing an appropriate ETM to exclude Zeno behavior while achieving the desired performance of the delayed system, considering the effects of time delay, has become a hot and difficult issue of current research. Although there are some research results, the abovementioned literature on ETIC design of time-delay systems ignores the influence of external interference and has some limitations. Therefore, in recent years, a growing number of scholars have already started to study the ISS under the ETIC [26–31]. For example, based on ETIC, the ISS of nonlinear systems is studied in [26, 27], while excluding Zeno behavior. In [28] and [29], the external inputs in the impulse and the system are different, that is, ISS based on ETIC with different external inputs is explored. In particular, Yu et al. [29] proposed a continuous ETIC and a dynamic ETIC to analyze ISS for nonlinear systems. It is worth noting that although the aforementioned studies have made important progress in ISS for nonlinear systems based on ETIC, no directly related works have considered time delay. The time delay may significantly impact the stability and performance of the system. As a result, it is essential to consider the influence of time delay when studying ISS for nonlinear systems subject to ETIC. In fact, several literatures have begun to focus on this area, which extends the existing ISS theory by introducing time delay term and the exploration focuses on designing effective ETIC strategies to ensure the ISS of the system impacted by time delays. In [30], the time delay in the impulse is considered, Zeno behavior is avoided, and the ISS of the system is studied. Meanwhile, in [31], delays in the system are considered and a new ETM is designed to study ISS of the nonlinear system, but the external disturbances in impulse are the same as those in the system and delays in the impulses are not considered. In view of this, we should develop an effective mechanism to study ISS of nonlinear systems under the ETIC strategy, including the time delay of the system and the time delay of the impulse.

Inspired by the above discussion, the goal of this paper is to propose an ETM based on the Lyapunov method and to design an appropriate triggering time sequence to ensure TSS of the system under external disturbances, while avoiding Zeno behavior. The challenges faced in this process include the potential degradation or instability of the control system due to time delays, as well as the fact that existing ETMs cannot be directly applied to systems with time delays, especially in cases where Zeno phenomena need to be avoided. Moreover, the presence of external disturbances adds complexity to the analysis and control of system stability. Therefore, this paper explores the ISS of nonlinear systems under ETIC, with particular attention to time delay and exogenous disturbances. The contributions of this paper are summarized as follows. First, unlike most existing methods [27, 32, 33], the proposed ETM introduces a forced impulse sequence with greater flexibility, allowing the system to adapt to varying external inputs. In addition, the proposed ETM in this paper includes information from both the system state and external inputs, and the threshold signal can be adjusted based on the robustness margin. Second, a Lyapunov–Razumikhin-like method is used to design an ETM to guarantee the ISS of the system, while excluding the possibility of Zeno behavior. Third, in contrast to existing works that focus solely on system delay [31] or impulse delay [30], this paper addresses both system dynamics delay and impulse control delay simultaneously. Additionally, compared to the work in [31], we explicitly differentiate between the external inputs in the continuous and impulsive dynamics, which broadens the applicability of the existing results. Fourth, this paper imposes no limitations on delay, meaning that the time delay can exceed the impulsive interval, thus overcoming the constraints of many studies that have a strict upper bound on time delay [34]. Finally, by using the linear matrix inequality (LMI) technique, the impulsive control gain and ETM are derived by resolving LMIs.

The remaining part of this paper is structured as follows. Several common notations and some relevant preliminaries are introduced in Section 2. Furthermore, we describe the impulsive systems under consideration. In Section 3, an ETM and main results are proposed. Section 4 presents an application. Illustrative examples accompanied by simulation figures are presented in Section 5. Finally, Section 6 presents the conclusions of this paper.

2. Preliminaries

Notation: The set of real numbers (nonnegative real numbers) is denoted by $\mathbb{R}$ ($\mathbb{R}_+$). $\mathbb{Z}_+$ denotes the set of positive integer numbers. The space $\mathbb{R}^n$ refers to the real n-dimensional space, where the

Euclidean norm is denoted by $|\cdot|$, and $\mathbb{R}^{n\times m}$ denotes the set of $n \times m$ real matrices. v_0 is a class of locally Lipschitz functions. A positive (negative) definite matrix $\mathcal{A} \in \mathbb{R}^{n\times n}$ is defined by $\mathcal{A} > 0$ ($\mathcal{A} < 0$). $\mathcal{A}^{-1}$ and $\mathcal{A}^T$ represent the inverse and transpose of matrix $\mathcal{A} \in \mathbb{R}^{n\times n}$, respectively. The maximum eigenvalue of the symmetric matrix $\mathcal{B} \in \mathbb{R}^{n\times n}$ is denoted by $\lambda_{\max}(\mathcal{B})$. $A \vee B = \max\{A, B\}$, $A \wedge B = \min\{A, B\}$. A function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is classified as $\mathcal{K}$ if it is continuous, strictly increasing and $\alpha(0) = 0$. It is classified as $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and radially infinite. A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is of class $\mathcal{KL}$ if for each fixed time $t \geq 0$, $\beta(\cdot, t)$ is of class $\mathcal{K}$, and $\beta(\cdot, t)$ tends to zero as $t \to +\infty$. Symbol $\star$ represents a symmetric block in a symmetric matrix.

Consider the following nonlinear impulsive systems with delays:

$$\begin{aligned} \dot{z}(t) &= f(z_t, u_c(t)),\ t \neq t_k,\ t \geq t_0 \\ z(t) &= g_k(z(t^- - \tau), u_d(t^-)),\ t = t_k,\ k \in \mathbb{Z}_+ \\ z_{t_0} &= \varrho, \end{aligned} \tag{1}$$

where z is the state; the derivative of z from the right is represented by $\dot{z}$; $\tau > 0$ is a constant delay; and $u_c(t), u_d(t) \in \mathbb{R}^n$ are the locally bounded exogenous perturbation and impulsive perturbation input. The initial condition is given by $\varrho \in \mathbb{PC}_h$; $f,\ g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ satisfies $f(0, 0) = g(0, 0) = 0$ and some appropriate conditions such that existence and uniqueness of the solution of system (1) are guaranteed in relevant time intervals [35]; and $\{t_k\}_{k\in\mathbb{Z}_+}$ is the impulse time sequence, satisfying $0 = t_0 < t_1 < \cdots < t_k < \cdots$ and $\lim_{k\to+\infty} t_k = +\infty$. For any time t that meets $t \geq t_0$, $z_t \in \mathbb{PC}_h$ with $z_t(s) = z(t + s)$, $s \in [-h, 0], h > 0$. Let $z(t^+) = \lim_{s\to t^+} z(s)$ and $z(t^-) = \lim_{s\to t^-} z(s)$. Assume the solution of system (1) is right continuous, that is, $z(t^+) = z(t)$.

Definition 1 ([36])**.** *System (1) is ISS if there exist functions $\xi \in \mathcal{KL}$ and $\varsigma_c, \varsigma_d \in \mathcal{K}_\infty$ such that for every initial condition (ϱ, t_0) and every pair of inputs (u_c, u_d), the corresponding solution $z(t)$ of (1) satisfies*

$$|z(t)| \leq \xi(\|\varrho\|_h, t - t_0) + \varsigma_c(\|u_c\|_{[t_0,t]}) + \varsigma_d(\max_{t_0\leq t_k\leq t}\{|u_d(t_k^- - \tau)|\}),\ t \geq t_0,$$

where $\|\cdot\|_I$ is the supremum norm on the interval I.

Definition 2. *For a given locally Lipschitz function $V : \mathbb{R}^n \to \mathbb{R}_+$, the upper righthand Dini derivative D^+V in relation to system (1) is defined by*

$$D^+V(t, z) = \limsup_{h\to 0^+} \frac{1}{h} \left(V\left(t + h, z + hf\left(t, z, u_c\right)\right) - V(t, z)\right).$$

Lemma 1 ([29])**.** *For any real matrices $\Lambda > 0$, x, y, and constant $\theta > 0$, the following inequality is valid:*

$$x^T y + y^T x \leq \theta x^T \Lambda x + \theta^{-1} y^T \Lambda^{-1} y.$$

Remark 1. *The system (1) in this paper is related to the system (7) in [36]. Both systems describe hybrid dynamical systems that include discrete events t_k. The difference is that system (1) introduces time delay in the impulse dynamics and considers different external disturbances acting on the continuous and discrete parts, allowing the system to model more complex dynamic processes in reality. In contrast, system (7) does not fully account for time delays and the differences in disturbances between the continuous and discrete parts, making the model relatively simpler.*

3. Main results

In this section, in the framework of the ETIC, considering the effect of delayed impulses, some conditions are given to realize that the nonlinear delayed systems' ISS and Zeno behavior is eliminated. Let $V(t, z(t))$ denote the Lyapunov function depending on solution $z(t)$ of system (1) at time t. For the sake of simplicity, $V(t) := V(t, z(t))$, then consider the following ETM:

$$\begin{aligned} t_k &= \min\{t_k^\diamond, \tau_k\} \\ t_k^\diamond &= \inf\{t \geq t_{k-1} : V(t, z(t)) - \Omega_{k-1}(t) \geq 0\} \end{aligned} \tag{2}$$

in which

$$\tau_k = t_{k-1}^\diamond + \sigma,\ \Omega_{k-1}(t) = \mu V(t_{k-1}) + \nu e^{-\iota(t-t_0)} V_0 + r\phi_1(\|u_c\|_{[t_0,t]}),$$

with $k \in \mathbb{Z}_+$, $\phi_1 \in \mathcal{K}_\infty$, $V_0 = \sup_{s\in[-h,0]} V(t_0 + s)$, and event-triggering parameters $\mu, \nu, r > 1, \iota > 0$. $\sigma \in \mathbb{R}_+$ is the forced impulse interval to satisfy

$$\inf_{k\in N^+} \{\sigma\} > 0. \tag{3}$$

Figure 1 illustrates the basic idea of ETIM. The block diagram consists of a controlled plant, an external disturbance, an ETM, and an impulsive generator. The controlled plant is affected by the external disturbance and transmits its state to the ETM, which decides whether to trigger an event by comparing the system state $V(t, z(t))$ with a preset threshold $\Omega_{k-1}(t)$. If $V(t, z(t)) > \Omega_{k-1}(t)$, then the event is triggered, otherwise it is not. When the event is triggered, the impulse generator generates an impulse signal to act on the controlled plant to adjust its state, which improves the efficiency and performance of the system and reduces unnecessary communication and control actions.

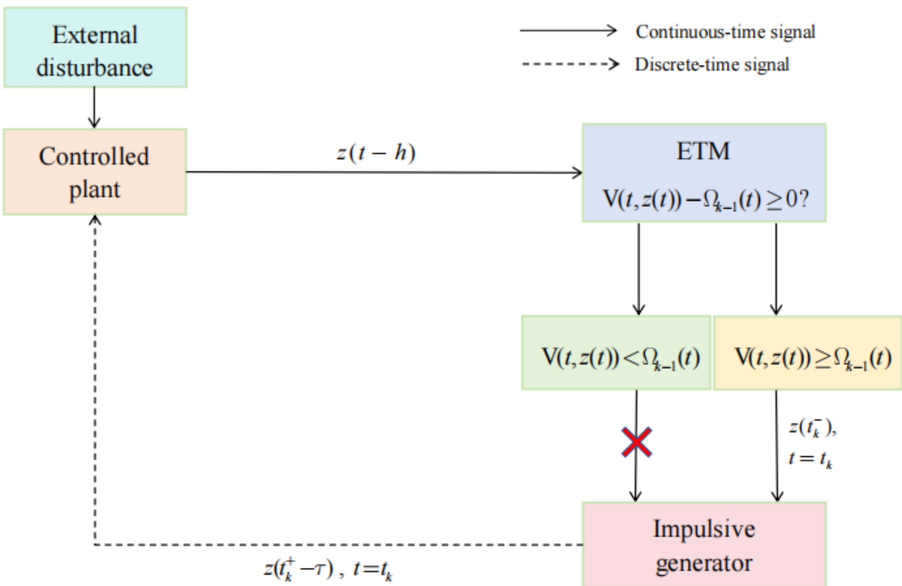

Figure 1. The block diagram of the considered system under ETIC.

Based on the designed ETM (2), the following result is given to exclude Zeno behavior.

Theorem 1. *Assume that there exist locally Lipschitz function* $V : \mathbb{R}^n \to \mathbb{R}_+$, ϕ_1, $\phi_2 \in \mathcal{K}_\infty$, *and constants* $\mu > 1, p, q, d, \iota, \sigma > 0$ *satisfying* $d - \ln\mu > \iota\sigma$ *such that for all* $t \in \mathbb{R}_+, k \in \mathbb{Z}_+$ *satisfy:*
(H₁) $D^+V(t, z(t)) \leq pV(t, z(t)) + qV(t+s, z(t+s)) + \phi_1(|u_c(t)|)$ *for all* $s \in [-h, 0]$*;*
(H₂) $V(t_k, g_k(z(t_k{}^- - \tau, u_d(t_k{}^- - \tau))) \leq e^{-d}V(t_k{}^- - \tau, z(t_k{}^- - \tau))) + \phi_2(|u_d(t_k{}^- - \tau)|)$*;*
(H₃) $V(t, z(t)) \geq \phi_2(\max_{t_0 \leq t_k \leq t} |u_d(t_k^- - \tau)|)$.

Then, system (1) does not exhibit Zeno behavior, and triggered impulse sequence $\{t_k\}_{k\in\mathbb{Z}_+}$ *satisfies*

$$t_k - t_{k-1} \geq \Theta = \min\left\{\frac{\ln(\mu \wedge \nu \wedge r)}{p + \iota + q\varepsilon e^{\iota h} + 1}, \sigma\right\}, \tag{4}$$

in which $\varepsilon = [\mu + \nu + r + e^d - \mu e^{\iota\sigma}]/[1 - \mu e^{\iota\sigma - d}]$.

Proof. According to the definition of ETM (2), three aspects will be considered.

Case 1. The full sequence of impulse times t_k includes only the instants at which impulses are enforced, i.e., $t_k = \tau_k$, based on $t_k - t_{k-1} = \sigma > 0$. Consequently, Zeno behavior is avoided.

Case 2. The full sequence of impulse times t_k includes only the instants at which impulses are event-triggered $t_k^\diamond$, that is, $t_k = t_k^\diamond, \forall k \in \mathbb{Z}_+$. Obviously, $t_k^\diamond$ can be defined as

$$t_k^\diamond = \inf\{t \geq t_{k-1} : e^{\iota(t-t_0)}V(t)) \geq \mu e^{\iota(t-t_0)}V(t_{k-1}) + \nu V_0 + re^{\iota(t-t_0)}\phi_1(\|u_c\|_{[t_0,t]})\}.$$

Let

$$\bar{V}(t) = \begin{cases} e^{\iota(t-t_0)}V(t), & t \in [t_0, \infty), \\ V(t), & t \in [t_0 - h, t_0). \end{cases}$$

Therefore, ETM (2) is transformed into the form presented below:

$$\begin{aligned} t_k &= \min\{t_k^\diamond, \tau_k\}, \\ t_k^\diamond &= \inf\left\{t \geq t_{k-1} : \bar{V}(t, z(t)) - \bar{\Omega}_{k-1}(t) \geq 0\right\}, \end{aligned} \tag{5}$$

in which

$$\bar{\Omega}_{k-1}(t) = \mu e^{\iota(t-t_0)}V(t_{k-1}) + \nu V_0 + re^{\iota(t-t_0)}\phi_1(\|u_c\|_{[t_0,t]}).$$

It should be noted that

$$\begin{aligned} \bar{V}(t_{k-1}) &\leq e^{\iota(t_k - t_{k-1})}\bar{V}(t_{k-1}) + V_0 + e^{\iota(t_k - t_0)}\phi_1(\| u_c \|_{[t_0,t_k]}) \\ &\leq \mu e^{\iota(t_k - t_{k-1})}\bar{V}(t_{k-1}) + \nu V_0 + re^{\iota(t_k - t_0)}\phi_1(\|u_c\|_{[t_0,t_k]}) \\ &= \bar{V}(t_k^-). \end{aligned} \tag{6}$$

Therefore, there exists

$$\tilde{t}_k = sup\left\{t \in [t_{k-1}, t_k) : \bar{V}(t) \leq e^{\iota(t_k - t_{k-1})}\bar{V}(t_{k-1}) + V_0 + e^{\iota(t_k - t_0)}\phi_1(\|u_c\|_{[t_0,t_k]})\right\}$$

such that

$$\bar{V}(\tilde{t}_k) = e^{\iota(t_k - t_{k-1})}\bar{V}(t_{k-1}) + V_0 + e^{\iota(t_k - t_0)}\phi_1(\|u_c\|_{[t_0,t_k]}) \tag{7}$$

and

$$\begin{aligned}\bar{V}(t) &\geq e^{\iota(t_k-t_{k-1})}\bar{V}(t_{k-1}) + V_0 + e^{\iota(t_k-t_0)}\phi_1(\|u_c\|_{[t_0,t_k]}) \\ &\geq V_0 + e^{\iota(t_k-t_0)}\phi_1(\|u_c\|_{[t_0,t_k]}), t \in [\tilde{t}_k, t_k).\end{aligned} \tag{8}$$

Moreover, based on $d - \ln\mu > \iota\sigma$ and condition (H_2), one can get the following inequalities:

$$\begin{aligned}\bar{V}(t_1^-) &= \mu e^{\iota(t_1-t_0)}V_0 + \nu V_0 + re^{\iota(t_1-t_0)}\phi_1(\|u_c\|_{[t_0,t_1]}) \\ &\leq \left(e^{\iota\sigma+\ln\mu} + \nu\right)V_0 + re^{\iota(t_1-t_0)}\phi_1(\|u_c\|_{[t_0,t_1]}) \\ &\leq \left(e^{d} + \nu\right)V_0 + re^{\iota(t_1-t_0)}\phi_1(\|u_c\|_{[t_0,t_1]}),\end{aligned}$$

$$\begin{aligned}\bar{V}(t_2^-) &= \mu e^{\iota(t_2-t_1)}\bar{V}(t_1) + \nu V_0 + re^{\iota(t_2-t_0)}\phi_1(\|u_c\|_{[t_0,t_2]}) \\ &\leq \mu e^{\iota(t_2-t_1)-d}\bar{V}(t_1^- - \tau) + \mu e^{\iota(t_2-t_1)}\phi_2(|u_d(t_1^- - \tau)|) + \nu V_0 + re^{\iota(t_2-t_0)}\phi_1(\|u_c\|_{[t_0,t_2]}) \\ &\leq \begin{cases}\left(\mu(\mu e^{-d})e^{\iota(t_2-\tau-t_0)} + \nu + \nu\mu e^{\iota(t_2-t_1)-d}\right)V_0 + \left(r(\mu e^{-d})e^{\iota(t_2-\tau-t_0)} + re^{\iota(t_2-t_0)}\right)\phi_1(\|u_c\|_{[t_0,t_2]}) \\ +\mu e^{\iota(t_2-t_1)}\phi_2(|u_d(t_1^- - \tau)|),\ t_0 \leq t_1 - \tau \leq t_1, \\ \left(\mu^2 e^{\iota(t_2-t_1)-d} + \nu\mu e^{\iota(t_2-t_1)-d} + \nu\right)V_0 + re^{\iota(t_2-t_0)}\phi_1(\|u_c\|_{[t_0,t_2]}) \\ +\mu e^{\iota(t_2-t_1)}\phi_2(|u_d(t_1^- - \tau)|),\ t_1 - \tau \leq t_0,\end{cases} \\ &\leq \left(\mu(\mu e^{-d})e^{\iota(t_2-\tau-t_0)} + \nu + \nu\mu e^{\iota(t_2-t_1)-d}\right)V_0 + \left(r(\mu e^{-d})e^{\iota(t_2-\tau-t_0)} + re^{\iota(t_2-t_0)}\right)\phi_1(\|u_c\|_{[t_0,t_2]}) \\ &\quad + \mu e^{\iota(t_2-t_1)}\phi_2(|u_d(t_1^- - \tau)|) \\ &\leq \left(\mu(\mu e^{-d})e^{\iota\sigma} + \nu + \nu\mu e^{\iota\sigma-d}\right)V_0 + \left(r(\mu e^{-d})e^{\iota(t_2-t_0)} + re^{\iota(t_2-t_0)}\right)\phi_1(\|u_c\|_{[t_0,t_2]}) \\ &\quad + \mu e^{\iota(t_2-t_1)}\phi_2(|u_d(t_1^- - \tau)|) \\ &\leq \left(e^{d} + \frac{\nu}{1-\mu e^{\iota\sigma-d}}\right)V_0 + \frac{re^{\iota(t_2-t_0)}}{1-\mu e^{-d}}\phi_1(\|u_c\|_{[t_0,t_2]}) + \mu e^{\iota(t_2-t_1)}\phi_2(|u_d(t_1^- - \tau)|),\end{aligned}$$

and

$$\begin{aligned}\bar{V}(t_3^-) &= \mu e^{\iota(t_3-t_2)}\bar{V}(t_2) + \nu V_0 + re^{\iota(t_3-t_0)}\phi_1(\|u_c\|_{[t_0,t_3]}) \\ &\leq \mu e^{\iota(t_3-t_2)-d}\bar{V}(t_2^- - \tau) + \mu e^{\iota(t_3-t_2)}\phi_2(|u_d(t_2^- - \tau)|) + \nu V_0 + re^{\iota(t_3-t_0)}\phi_1(\|u_c\|_{[t_0,t_3]}) \\ &\leq \begin{cases}\left(\mu(\mu e^{-d})^2 e^{\iota(t_3-2\tau-t_0)} + \nu + \nu(\mu e^{-d})e^{\iota(t_3-t_2)} + \nu(\mu e^{-d})^2 e^{\iota(t_3-\tau-t_1)}\right)V_0 \\ + \left(r(\mu e^{-d})^2 e^{\iota(t_3-2\tau-t_0)} + r(\mu e^{-d})e^{\iota(t_3-\tau-t_0)} + re^{\iota(t_3-t_0)}\right)\phi_1(\|u_c\|_{[t_0,t_3]}) \\ +\mu(\mu e^{-d})e^{\iota(t_3-\tau-t_1)}\phi_2(|u_d(t_1^- - \tau)|) + \mu e^{\iota(t_3-t_2)}\phi_2(|u_d(t_2^- - \tau)|),\ t_1 \leq t_2 - \tau \leq t_2, \\ \left(\mu(\mu e^{-d})e^{\iota(t_3-\tau-t_0)} + \nu(\mu e^{-d})e^{\iota(t_3-t_2)} + \nu\right)V_0 + \left(r(\mu e^{-d})e^{\iota(t_3-\tau-t_0)} + re^{\iota(t_3-t_0)}\right)\phi_1(\|u_c\|_{[t_0,t_3]}) \\ +\mu e^{\iota(t_3-t_2)}\phi_2(|u_d(t_2^- - \tau)|), t_0 \leq t_2 - \tau \leq t_1, \\ (\mu(\mu e^{-d})e^{\iota(t_3-t_2)} + \nu(\mu e^{-d})e^{\iota(t_3-t_2)} + \nu)V_0 + re^{\iota(t_3-t_0)}\phi_1(\|u_c\|_{[t_0,t_3]}) \\ +\mu e^{\iota(t_3-t_2)}\phi_2(|u_d(t_2^- - \tau)|),\ t_2 - \tau \leq t_0,\end{cases} \\ &\leq \left(e^{3(\iota\sigma+\ln\mu-d)+d} + \nu + \nu e^{\iota\sigma+\ln\mu-d} + \nu e^{2(\iota\sigma+\ln\mu-d)}\right)V_0 \\ &\quad + \left(r(\mu e^{-d})^2 e^{\iota(t_3-2\tau-t_0)} + r(\mu e^{-d})e^{\iota(t_3-\tau-t_0)} + re^{\iota(t_3-t_0)}\right)\phi_1(\|u_c\|_{[t_0,t_3]}) \\ &\quad + \mu(\mu e^{-d})e^{\iota(t_3-\tau-t_1)}\phi_2(|u_d(t_1^- - \tau)|) + \mu e^{\iota(t_3-t_2)}\phi_2(|u_d(t_2^- - \tau)|) \\ &\leq \left(e^{d} + \frac{\nu}{1-\mu e^{\iota\sigma-d}}\right)V_0 + \frac{re^{\iota(t_3-t_0)}}{1-\mu e^{-d}}\phi_1(\|u_c\|_{[t_0,t_2]}) + \frac{\mu e^{\iota(t_3-t_1)}}{1-\mu e^{-d}}\phi_2\left(|u_d(t_1^- - \tau)| \vee |u_d(t_2^- - \tau)|\right).\end{aligned}$$

Similarly,

$$
\begin{aligned}
\bar{V}(t_k^-) &= \mu e^{\iota(t_k-t_{k-1})}\bar{V}(t_{k-1}) + \nu V_0 + re^{\iota(t_k-t_0)}\phi_1(\|u_c\|_{[t_0,t_k]}) \\
&\leq \mu e^{\iota(t_k-t_{k-1})-d}\bar{V}(t_{k-1}^- - \tau) + \mu e^{\iota(t_k-t_{k-1})}\phi_2(|u_d(t_{k-1}^- - \tau)|) + \nu V_0 + re^{\iota(t_k-t_0)}\phi_1(\|u_c\|_{[t_0,t_k]}) \\
&\leq \left(\mu(\mu e^{-d})^{k-1}e^{\iota(t_k-(k-1)\tau-t_0)} + \nu + \nu(\mu e^{-d})e^{\iota(t_k-t_{k-1})} + \cdots + \nu(\mu e^{-d})^{k-1}e^{\iota(t_k-(k-2)\tau-t_1)}\right)V_0 \\
&\quad + \left(re^{\iota(t_k-t_0)} + r(\mu e^{-d})e^{\iota(t_k-\tau-t_0)} + \cdots + r(\mu e^{-d})^{k-1}e^{\iota(t_k-(k-1)\tau-t_0)}\right)\phi_1(\|u_c\|_{[t_0,t_3]}) \\
&\quad + \left(\mu e^{\iota(t_k-t_{k-1})} + \mu(\mu e^{-d})e^{\iota(t_k-\tau-t_{k-2})} + \cdots + \mu(\mu e^{-d})^{k-2}e^{\iota(t_k-(k-2)\tau-t_0)}\right)\phi_2(\max_{t_0\leq t_k\leq t}|u_d(t_k^- - \tau)|) \\
&\leq \left(e^{k(\iota\sigma+\ln\mu-d)+d} + \sum_{i=0}^{k-1}\nu(\mu e^{-d})^i e^{\iota(t_k-t_{k-i})}\right)V_0 + \sum_{i=0}^{k-1}r(\mu e^{-d})^i e^{\iota(t_k-t_0)}\phi_1(\|u_c\|_{[t_0,t_k]}) \\
&\quad + \sum_{i=0}^{k-2}\mu(\mu e^{-d})^i e^{\iota(t_k-t_1)}\phi_2(\max_{t_0\leq t_k\leq t}|u_d(t_k^- - \tau)|) \\
&\leq \left(e^d + \frac{\nu}{1-\mu e^{\iota\sigma-d}}\right)V_0 + \frac{re^{\iota(t_k-t_0)}}{1-\mu e^{-d}}\phi_1(\|u_c\|_{[t_0,t_k]}) + \frac{\mu e^{\iota(t_k-t_1)}}{1-\mu e^{-d}}\phi_2(\max_{t_0\leq t_k\leq t}|u_d(t_k^- - \tau)|).
\end{aligned}
$$

As such, for $t \in [\tilde{t}_k, t_k)$, using (H_3) and (8), one has

$$
\begin{aligned}
\bar{V}(t+s) &\leq \begin{cases} \bar{V}(t_k^-), \text{ if } t+s \in [t_{k-1}, t_k), \\ \bar{V}(t_{k-1}^-), \text{ if } t+s \in [t_{k-2}, t_{k-1}), \\ \quad\vdots \\ \bar{V}(t_1^-), \text{ if } t+s \in [t_0, t_1), \\ V_0, \text{ if } t+s \in [t_0-h, t_0), \end{cases} \\
&\leq \frac{\nu + e^d - \mu e^{\iota\sigma}}{1-\mu e^{\iota\sigma-d}}V_0 + \frac{re^{\iota(t_k-t_0)}}{1-\mu e^{-d}}\phi_1(\|u_c\|_{[t_0,t_k]}) + \frac{\mu e^{\iota(t_k-t_1)}}{1-\mu e^{-d}}\phi_2(\max_{t_0\leq t_k\leq t}|u_d(t_k^- - \tau)|) \\
&\leq \frac{\mu+\nu+r+e^d-\mu e^{\iota\sigma}}{1-\mu e^{\iota\sigma-d}}\bar{V}(t).
\end{aligned}
$$

According to condition (H_1), one obtains

$$
\begin{aligned}
D^+\bar{V}(t) &= D^+e^{\iota(t-t_0)}V(t) \\
&\leq \iota e^{\iota(t-t_0)}V(t) + e^{\iota(t-t_0)}\left(pV(t) + qV(t+s) + \phi_1(\|u_c\|_{[t_0,t]})\right) \\
&= (\iota+p)\bar{V}(t) + qe^{\iota(t+s-t_0)}V(t+s)e^{-\iota s} + \phi_1(\|u_c\|_{[t_0,t]}) \\
&= (\iota+p)\bar{V}(t) + qe^{-\iota s}\bar{V}(t+s) + \phi_1(\|u_c\|_{[t_0,t]}) \\
&\leq (\iota+p)\bar{V}(t) + qe^{\iota h}\bar{V}(t+s) + \phi_1(\|u_c\|_{[t_0,t]}) \\
&\leq (\iota+p+q\varepsilon e^{\iota h}+1)\bar{V}(t).
\end{aligned} \tag{9}
$$

Then, from (9), it can be inferred that

$$
\bar{V}(t_k^-) \leq \bar{V}(\tilde{t}_k)e^{(\iota+p+q\varepsilon e^{\iota h}+1)(t_k-\tilde{t}_k)} \leq \bar{V}(\tilde{t}_k)e^{(\iota+p+q\varepsilon e^{\iota h}+1)(t_k-t_{k-1})},
$$

and combined $\bar{V}(t_k^-) \geq (\mu\wedge\nu\wedge r)\bar{V}(\tilde{t}_k)$, it follows that

$$
t_k - t_{k-1} \geq \frac{\ln(\mu\wedge\nu\wedge r)}{p+\iota+q\varepsilon e^{\iota h}+1} > 0. \tag{10}
$$

Case 3. The impulse time sequence t_k contains both forced impulse instant τ_k and event-triggered impulse instant $t_k^\diamond$. Over the time interval $[t_{k-1}, t_k]$, two cases should be taken into consideration. When $t_k = t_k^\diamond$, the proof is similar to Case 2, and one can know that (10) holds regardless of $t_{k-1} = t_{k-1}^\diamond$ or $t_{k-1} = \tau_{k-1}$. When $t_k = \tau_k$, that is, $t_k - t_{k-1} = \sigma$.

Thus, Zeno behavior does not occur under ETM (2) in either case. In addition, it is evident from the above discussion that

$$t_k - t_{k-1} \geq \min\left\{\frac{\ln(\mu \wedge \nu \wedge r)}{p + \iota + q\varepsilon e^{\iota h} + 1}, \sigma\right\}.$$

The proof is finished.

Theorem 2. *Under conditions in Theorem 1, there exist functions $\alpha_1,\ \alpha_2 \in \mathcal{K}_\infty$ such that*
(H_4) $\alpha_1(|z|) \leq V(t,z) \leq \alpha_2(|z|)$ for all $t \geq t_0 - h, z \in \mathbb{R}^n$,
then system (1) achieves ISS via ETM (2).

Proof. With ETM (2), one derives

$$V(t) \leq \mu V_0 + \nu e^{-\iota(t-t_0)} V_0 + r\phi_1(\|u_c\|_{[t_0,t]}), t \in [t_0, t_1).$$

Using condition (H_2), for triggering instant t_1, one gains

$$\begin{aligned}
V(t_1) &\leq e^{-d} V(t_1^- - \tau) + \phi_2(|u_d(t_1^- - \tau)|)\\
&\leq \begin{cases} e^{-d}\left(\mu + \nu e^{-\iota(t_1-\tau-t_0)}\right) V_0 + r\phi_1(\|u_c\|_{[t_0,t_1]})) + \phi_2(|u_d(t_1^- - \tau)|),\ t_0 \leq t_1 - \tau \leq t_1,\\ e^{-d}\left(\mu V_0 + \nu V_0\right) + \phi_2(|u_d(t_1^- - \tau)|),\ t_1 - \tau \leq t_0, \end{cases}\\
&\leq (\mu e^{-d} + \nu e^{-\iota(t_1-\tau-t_0)-d}) V_0 + re^{-d}\phi_1(\|u_c\|_{[t_0,t_1]}) + \phi_2(|u_d(t_1^- - \tau)|).
\end{aligned}$$

and

$$\begin{aligned}
V(t) &\leq \mu V(t_1) + \nu e^{-\iota(t-t_0)} V_0 + r\phi_1(\|u_c\|_{[t_0,t]})\\
&\leq \mu\left((\mu e^{-d} + \nu e^{-\iota(t_1-\tau-t_0)-d} V_0 + re^{-d}\phi_1(\|u_c\|_{[t_0,t_1]}) + \phi_2(|u_d(t_1^- - \tau)|)\right)\\
&\quad + \nu e^{-\iota(t-t_0)} V_0 + r\phi_1(\|u_c\|_{[t_0,t]})\\
&\leq \left(\mu(\mu e^{-d}) + \nu e^{-\iota(t-t_0)} + \nu(\mu e^{-d}) e^{-\iota(t_1-\tau-t_0)}\right) V_0 + (r\mu e^{-d} + r)\phi_1(\|u_c\|_{[t_0,t]})\\
&\quad + \mu\phi_2(|u_d(t_1^- - \tau)|),\ t \in [t_1, t_2).
\end{aligned}$$

Analogously, at triggering instant t_2,

$$\begin{aligned}
V(t_2) &= e^{-d} V(t_2^- - \tau) + \phi_2(|u_d(t_2^- - \tau)|)\\
&\leq \begin{cases} \left((\mu e^{-d})^2 + \nu\mu e^{-\iota(t_1-\tau-t_0)-2d} + \nu e^{-\iota(t_2-\tau-t_0)-d}\right) V_0 + (r\mu e^{-2d} + re^{-d})\phi_1(\|u_c\|_{[t_0,t_2]})\\ +\mu e^{-d}\phi_2(|u_d(t_1^- - \tau)|) + \phi_2(|u_d(t_2^- - \tau)|),\ t_1 \leq t_2 - \tau \leq t_2,\\ \left(\mu e^{-d} + \nu e^{-\iota(t_2-\tau-t_0)-d}\right) V_0 + re^{-d}\phi_1(\|u_c\|_{[t_0,t_1]}) + \phi_2(|u_d(t_2^- - \tau)|),\ t_0 \leq t_2 - \tau \leq t_1,\\ e^{-d}\left(\mu V_0 + \nu V_0\right) + \phi_2(|u_d(t_2^- - \tau)|),\ t_2 - \tau \leq t_0, \end{cases}\\
&\leq \left((\mu e^{-d})^2 + \nu\mu e^{-\iota(t_1-\tau-t_0)-2d} + \nu e^{-\iota(t_2-\tau-t_0)-d}\right) V_0 + (r\mu e^{-2d} + re^{-d})\phi_1(\|u_c\|_{[t_0,t_2]})\\
&\quad + \mu e^{-d}\phi_2(|u_d(t_1^- - \tau)|) + \phi_2(|u_d(t_2^- - \tau)|).
\end{aligned}$$

Likewise, one can also acquire

$$
\begin{aligned}
V(t) &\le \mu V(t_2) + \nu e^{-\iota(t-t_0)}V_0 + r\phi_1(\|u_c\|_{[t_0,t]}) \\
&\le \mu(((\mu e^{-d})^2 + \nu\mu e^{-\iota(t_1-\tau-t_0)-2d} + \nu e^{-\iota(t_2-\tau-t_0)-d})V_0 + (r\mu e^{-2d} + re^{-d})\phi_1(\|u_c\|_{[t_0,t_2]}) \\
&\quad + \mu e^{-d}\phi_2(|u_d(t_1^- - \tau)|) + \phi_2(|u_d(t_2^- - \tau)|)) + \nu e^{-\iota(t-t_0)}V_0 + r\phi_1(\|u_c\|_{[t_0,t]}) \\
&\le \left(\mu(\mu e^{-d})^2 + \nu e^{-\iota(t-t_0)} + \nu(\mu e^{-d})e^{-\iota(t_2-\tau-t_0)} + \nu(\mu e^{-d})^2 e^{-\iota(t_1-\tau-t_0)}\right)V_0 \\
&\quad + \left(r(\mu e^{-d})^2 + r\mu e^{-d} + r\right)\phi_1(\|u_c\|_{[t_0,t]}) + \mu(\mu e^{-d})\phi_2(|u_d(t_1^- - \tau)|) + \mu\phi_2(|u_d(t_2^- - \tau)|),\ t \in [t_2, t_3).
\end{aligned}
$$

Repeating the above steps, one can infer that

$$
\begin{aligned}
V(t) &\le \mu V(t_{k-1}) + \nu e^{-\iota(t-t_0)}V_0 + r\phi_1(\|u_c\|_{[t_0,t]}) \\
&\le \mu\left(e^{-d}(\mu V(t_{k-2}) + \nu e^{-\iota(t_{k-1}-\tau-t_0)}V_0 + r\phi_1(\|u_c\|_{[t_0,t_{k-1}]})) + \phi_2(|u_d(t_{k-1}^- - \tau)|)\right) \\
&\quad + \nu e^{-\iota(t-t_0)}V_0 + r\phi_1(\|u_c\|_{[t_0,t]}) \\
&\le \cdots \\
&\le \left(\mu(\mu e^{-d})^{k-1} + \nu e^{-\iota(t-t_0)} + \sum_{i=0}^{k-1}\nu(\mu e^{-d})^i e^{-\iota(t_{k-i}-\tau-t_0)}\right)V_0 + \sum_{i=0}^{k-1} r(\mu e^{-d})^i\phi_1(\|u_c\|_{[t_0,t]}) \\
&\quad + \mu\phi_2(|u_d(t_{k-1}^- - \tau)|) + \cdots + \mu(\mu e^{-d})^{k-2}\phi_2(|u_d(t_1^- - \tau)|) \\
&\le \left(\mu(\mu e^{-d})^{k-1} + \nu e^{-\iota(t-t_0)} + \sum_{i=0}^{k-1}\nu e^{-\iota\tau}e^{i(\ln\mu-d)-(k-i)\iota\Theta}\right)V_0 + \sum_{i=0}^{k-1} r(\mu e^{-d})^i\phi_1(\|u_c\|_{[t_0,t]}) \\
&\quad + \sum_{i=0}^{k-2}\mu(\mu e^{-d})^i\phi_2(\max_{t_0\le t_k\le t}|u_d(t_k^- - \tau)|) \\
&\le \left(e^{d-k\iota\sigma} + \nu e^{-\iota(t-t_0)} + \nu k e^{-\iota\tau}e^{-k\iota\Theta}\right)V_0 + \frac{r}{1-\mu e^{-d}}\phi_1(\|u_c\|_{[t_0,t]}) + \frac{\mu}{1-\mu e^{-d}}\phi_2(\max_{t_0\le t_k\le t}|u_d(t_k^- - \tau)|) \\
&\le \left(\nu e^{-\iota(t-t_0)} + (\nu k e^{-\iota\tau} + e^d)e^{-k\iota\Theta}\right)V_0 + \frac{r}{1-\mu e^{-d}}\phi_1(\|u_c\|_{[t_0,t]}) \\
&\quad + \frac{\mu}{1-\mu e^{-d}}\phi_2(\max_{t_0\le t_k\le t}|u_d(t_k^- - \tau)|),\ t \in [t_{k-1}, t_k).
\end{aligned}
$$

Together with (H_4), one gets

$$
\begin{aligned}
|z(t)| \le \alpha_1^{-1}\left((\nu e^{-\iota(t-t_0)} + (\nu k e^{-\iota\tau} + e^d)e^{-k\iota\Theta})\alpha_2(\|\varrho\|_h)\right) + \frac{r\alpha_1^{-1}}{1-\mu e^{-d}}\phi_1(\|u_c\|_{[t_0,t]}) \\
+ \frac{\mu\alpha_1^{-1}}{1-\mu e^{-d}}\phi_2(\max_{t_0\le t_k\le t}|u_d(t_k^- - \tau)|), t \in [t_{k-1}, t_k).
\end{aligned}
$$

which confirms system (1) is ISS under ETM (2).

Remark 2. *The stability analysis of complex systems often requires certain conditions to be met, ensuring the system's ability to maintain bounded and stable behavior under various circumstances. In this paper, (H_1) describes the rate of change during the continuous flow process of the system, limiting the growth rate of the solution; (H_2) describes the state change characteristics at discrete jump instants t_k, ensuring that the system can maintain stability through a certain contraction* *mechanism at these discrete moments; (H_3) provides a lower bound for the Lyapunov function based on historical disturbance inputs; (H_4) ensures the positive definiteness and boundedness of the Lyapunov function, guaranteeing its effectiveness in describing the variation magnitude of the system state.*

Remark 3. *The traditional ETIC strategy limits the growth rate of continuous dynamics, ensuring it doesn't grow too rapidly [34]. In Theorem 1 of this paper, condition (H_1) defines the continuous evolution of the system's state, comprising both delay-dependent and delay-independent components, and condition (H_2) characterizes the discrete dynamics of the system. It should be noted that the results of this paper do not imply any direct link between the continuous and discrete dynamics. In addition, the impulse intensity plays a key role in system performance. To ensure the convergence of system state in the sense of ISS, we give the restriction condition $d - \ln \mu > \iota\sigma$ in Theorem 1. It is worth noting that in Theorem 1, the coefficients are constant, implying linear rates. However, there have been studies using nonlinear rates [37, 38], as these can better capture the complex dynamic characteristics of the system and thus provide less conservative stability conditions. However, compared to nonlinear rates, linear rates simplify mathematical derivation while maintaining broad applicability.*

Remark 4. *Zeno behavior implies that within a finite time period, impulsive control tasks take place infinitely times. The ETM (2) can effectively eliminate Zeno behavior, and from the demonstration of Theorem 1, it is evident that introducing forced triggering moments in ETM (2) is essential to ensure ISS. Furthermore, it is clear from (3) that there is no upper bound on forced impulse instants in this paper, that is, the value of σ in this paper is arbitrary. In simpler terms, σ may be chosen to a comparatively large value (in this situation, the other parameters need to be adjusted accordingly, according to condition $d - \ln \mu > \iota\sigma$), then ETM (2) is essentially event triggered. Additionally, as shown in (4), the triggering intervals are bounded by the triggering parameters μ, ν, and r. Specifically, a larger μ, ν, or r will result in a lower event-trigger frequency, while a smaller μ, ν, or r will result in a higher event-trigger frequency.*

Remark 5. *Due to the influence of external interference, $\phi_1(\|u_c\|_{[t_0,t]})$ is introduced in this paper to consider the potential effect on the system, which is different from previous studies (e.g., [39]), resulting in differences in the proof in this paper. It should be emphasized that $\phi_1(\|u_c\|_{[t_0,t]})$ cannot be substituted by just $\phi_1(\|u_c(t)\|)$, as the sizes of the two are not comparable. Therefore, the value of ϕ_1 must be given explicitly over the interval $[t_0, t]$.*

Remark 6. *In comparison to the existing ISS results based on time-dependent control scheme [40], this paper presents a more flexible and efficient Lyapunov-based ETM to ensure ISS for impulsive time-delay systems, where the release time of the impulsive control signal is dictated by the event-triggering rule that relies on the system state. It effectively decreases the frequency of impulsive control and optimizes the utilization of resources. In addition, it is notable that various ETIC strategies are used in the existing papers [26, 27] to ensure the ISS of nonlinear systems, but the obtained results are difficult to apply to delayed systems with delayed pulses. This paper overcomes the limitations of previous methods by introducing a combined exponential attenuation term $\nu e^{-\iota(t-t_0)}V_0$ and non-decay term $\mu V(t_{k-1})$ in the event-triggered rule and utilizing a Lyapunov–Razumikhin-like method to address time-delay effects. This method solves the difficulty of extracting delayed state information while avoiding Zeno behavior, and ISS for time-delay systems is realized under ETIC .*

Remark 7. *It should be mentioned that there are many similar results in the existing literature that use different approaches, in particular, the dwell-time approach [38,41,42]. Specifically, the works in [41] and [42] provide dwell-time stability conditions for nonlinear impulsive systems, even allowing for the potential instability of both continuous and discrete dynamics. These studies mainly focus on infinite-dimensional hybrid systems and adopt Lyapunov-like methods for stability analysis. In contrast, the ETIC method used in this paper is more suitable for systems where control actions are performed only at discrete moments under specific events or conditions. This method enhances the robustness of the system to external perturbations by incorporating the external input information into the threshold adjustment of the ETM. Moreover, there is no upper-bound restriction on the delay, which makes it more flexible compared to the dwell-time method, and is able to deal with the situation in which the interval between impulses is larger than the delay. The ETIC method also reduces the frequency of release of the control signals, thus, reducing the consumption of communication and computational resources.*

4. Applications

In this section, our presented event-triggered impulse control tactics are applied to nonlinear systems to achieve ISS and impulsive control gain is given in terms of LMIs.

Considering the following nonlinear delay systems subject to external disturbance:

$$\dot{z}(t) = \mathfrak{A}z(t) + \mathfrak{B}g(z(t-h)) + \mathfrak{C}u(t) + v(t),\ t \neq t_r,\ t \geq t_0, \tag{11}$$

where $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C} \in \mathbb{R}^n$ are given real matrices. $u(t) \in \mathbb{R}^m$ is the locally bounded exogenous disturbance. g satisfies $g(0) = 0$ and is globally Lipschitz, with the Lipschitz matrix being L, and $v(t)$ is the Dirac control input.

Under the influence of $v(t)$, at each impulse time t_k, the state $z(t)$ undergoes a discrete jump, which is captured by the term $(I + \mathfrak{D})z(t^- - \tau)$. Then, system (11) can be written as follows:

$$\begin{aligned} &\dot{z}(t) = \mathfrak{A}z(t) + \mathfrak{B}g(z(t-h)) + \mathfrak{C}u(t),\ t \geq t_0,\ t \neq t_k, \\ &z(t) = (I + \mathfrak{D})z(t^- - \tau),\ t = t_k, k \in \mathbb{Z}_+, \end{aligned} \tag{12}$$

where $\{t_k\}_{k\in\mathbb{Z}_+}$ is the impulse instant. $\mathfrak{D}$ is the impulsive control gain matrix to be given later.

According to Theorems 1 and 2, the LMI-based ISS result is outlined below.

Theorem 3. *If positive constants $\mu, \nu, r > 1, \sigma, \iota > 0$, and positive constants p, q, d satisfying $d > \ln\mu + \iota\sigma$, $n \times n$ matrices $P, Q > 0$, $n \times n$ diagonal matrix $K > 0$, $n \times n$ real matrix M, such that $LKL < qP$ and*

$$\begin{pmatrix} \mathfrak{A}^T P + P\mathfrak{A} - pP & P\mathfrak{B} & P\mathfrak{C} \\ \star & -K & 0 \\ \star & \star & -Q \end{pmatrix} \leq 0, \tag{13}$$

$$\begin{pmatrix} -e^{-d}P & P + M \\ \star & -P \end{pmatrix} \leq 0, \tag{14}$$

then, the ISS of system (13) is ensured with impulsive control gain $\mathfrak{D} = P^{-1}M^T$ and ETM:

$$\begin{aligned} t_k &= \min\{t_k^\diamond, \tau_k\},\ k \in \mathbb{Z}_+, \\ t_k^\diamond &= \inf\{t \geq t_{k-1} : \Xi(t) \geq 0\}, \end{aligned} \tag{15}$$

with event generator function

$$\Xi(t) = z^T(t)Pz(t) - \mu z^T(t_{k-1})Pz(t_{k-1}) - \nu e^{-\iota(t-t_0)}V_0 - r\lambda_{max}(Q)\|u\|^2_{[t_0,t]},$$

where $V_0 = \sup_{s\in[-h,0]} x^T(t_0+s)Px(t_0+s)$.

Proof. Select $V(t) = z^T(t)Pz(t)$. Then, on the basis of (13), (14), the Schur complement, and Lemma 1, one can obtain that

$$\begin{aligned} V(t_k) &= z^T(t_k)Pz(t_k) \\ &= z^T(t_k^- - \tau)(I+\mathfrak{D})^T P(I+\mathfrak{D})z(t_k^- - \tau) \\ &\le e^{-d}V^T(t_k^- - \tau), \end{aligned}$$

and

$$\begin{aligned} D^+V^T(t) = &\ (\mathfrak{A}z(t) + \mathfrak{B}g(z(t-h)) + \mathfrak{C}u(t))\,Pz(t) + z^T(t)P\,(\mathfrak{A}z(t) + \mathfrak{B}g(z(t-h)) + \mathfrak{C}u(t)) \\ &= z^T(t)(P\mathfrak{A} + \mathfrak{A}^TP)z(t) + 2z^T(t)P\mathfrak{B}g(z(t-h)) + 2z^T(t)P\mathfrak{C}u(t) \\ &\le z^T(t)(P\mathfrak{A} + \mathfrak{A}^TP)z(t) + z^T(t)P\mathfrak{B}K^{-1}\mathfrak{B}^TPz(t) + g^T(z(t-h))Kg(z(t-h)) \\ &\quad + z^T(t)P\mathfrak{C}Q^{-1}\mathfrak{C}^TPz(t) + u^T(t)Qu(t) \\ &= z^T(t)(P\mathfrak{A} + \mathfrak{A}^TP + P\mathfrak{B}K^{-1}\mathfrak{B}^TP + P\mathfrak{C}Q^{-1}\mathfrak{C}^TP)z(t) + z^T(t-h)L^TKLz(t-h) + u^T(t)Qu(t) \\ &\le pV(t) + qV(t-h) + \lambda_{max}(Q)|u|^2. \end{aligned}$$

Hence, analogous to the demonstration in Theorem 2, the ISS of system (12) is guaranteed.

In practical operations, in order to implement ETIM and achieve ISS while avoiding Zeno behavior, we select and verify the parameters according to the following steps: First, select an appropriate Lyapunov function and determine the parameters μ, ν, r, ι, and σ in ETM to meet the conditions in Theorems 1 and 2, especially ensuring that $d - ln\mu > \iota\sigma$. Next, software is used to solve the LMIs in Theorem 3 to obtain the matrices P, Q, R, and W, then determine the impulsive gain matrix $\mathfrak{D} = P^{-1}M^T$. Finally, the conditions of ISS are verified and validate the effectiveness of the design through numerical simulations. Based on the simulation results, we adjust the parameters as necessary to achieve the desired performance. This process can effectively tune the parameters and gains, ensuring that the system achieves the intended ISS properties while avoiding Zeno behavior.

Remark 8. *In contrast to the existing results, Theorem 3 fully considers the time delays in both the continuous and the impulsive part, and it accounts for different external disturbances affecting these two components. When applying Theorem 3, it is required to set the parameters p and d in advance to satisfy the LMIs, then impulse control gain matrix* $\mathfrak{D}$ *can be determined. The result indicates that the state trajectory can be confined within a bounded region under the predefined event trigger impulsive control.*

5. Examples

In this section, we demonstrate the acquired results through two examples. These examples are simulated and verified by MATLAB software.

Example 1. Consider the following nonlinear delayed impulsive system [43]:

$$
\begin{aligned}
\dot{z}(t) &= \frac{1.1|cost|z^3(t)}{1+z^2(t)} - 0.8|sint|z(t) + 0.8costz(t-7) + u_c(t),\ t \neq t_k,\\
z(t) &= e^{-1.2}z(t^- - 1.6) + u_d(t^-),\ t = t_k.
\end{aligned} \tag{16}
$$

Figure 2 shows the evolution of the state of system (16) over time, the upper figure shows the state trajectory when the external input $u_c(t) = sin(t)$, and the lower figure shows the trajectory when $u_c(t) = 0$. It can be seen that the upper trajectory exhibits greater divergence compared to the lower figure, suggesting that the external input affects the system, and it can be seen from the trajectory that neither of the two cases is ISS. The impulse sequence $\{t_k\}_{k\in\mathbb{Z}_+}$ is dictated by several predefined events. In order to investigate the ISS of system (16), an appropriate function $V(z(t)) = |z(t)|$ is chosen, with the triggering parameters $\mu = 1.8$, $\nu = 3.5$, $r = 3.5$, $\iota = 0.03$, $V_0 = sup_{s\in[-7,0]}|z(t_0+s)|$ satisfying $\ln\mu + \iota\sigma \in (0, 1.2)$. Considering the forced impulse $\tau_k = t_{k-1} + \sigma, \sigma = 15$, ETM is designed in the following way:

$$
\begin{aligned}
t_k &= min\left\{t_k^{\diamond}, \tau_k\right\},\ r \in \mathbb{Z}_+,\\
t_k^{\diamond} &= inf\left\{t \geq t_{k-1} : |z(t)| \geq \Omega_{k-1}(t) = 1.8V(t_{k-1}) + 3.5e^{-0.03(t-t_0)}V_0 + 3.5\phi_1(\|u_c\|_{[t_0,t]})\right\}.
\end{aligned} \tag{17}
$$

According to Theorem 2, system (16) with external input $u_c(t) = sin(t), u_d(t^-) = \frac{1}{10}cos(t^-)$ is ISS under ETM (17), as shown in Figure 3. The initial condition is set to $z(t) = 1.5$ for $t \in [-7, 0]$. The blue curve and blue dots in the figure represent the trajectory of the state and the triggered instants of system (16), respectively. Under the same conditions, if the influence of exogenous disturbances is disregarded, the corresponding system is globally asymptotically stable (GAS), as depicted in Figure 3, where the red curve and red dots denote the state trajectory and triggered instants. Comparing the two cases, we can find that time delays will have a negative impact on the stability of system (16).

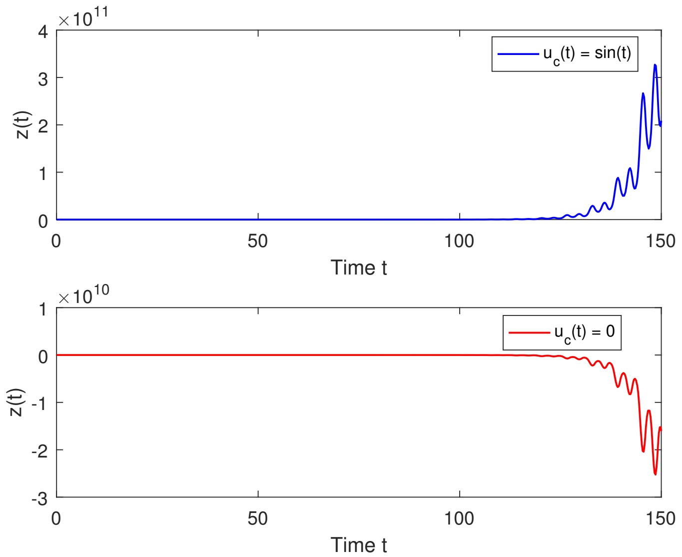

Figure 2. The trajectories of system (16) without impulse control.

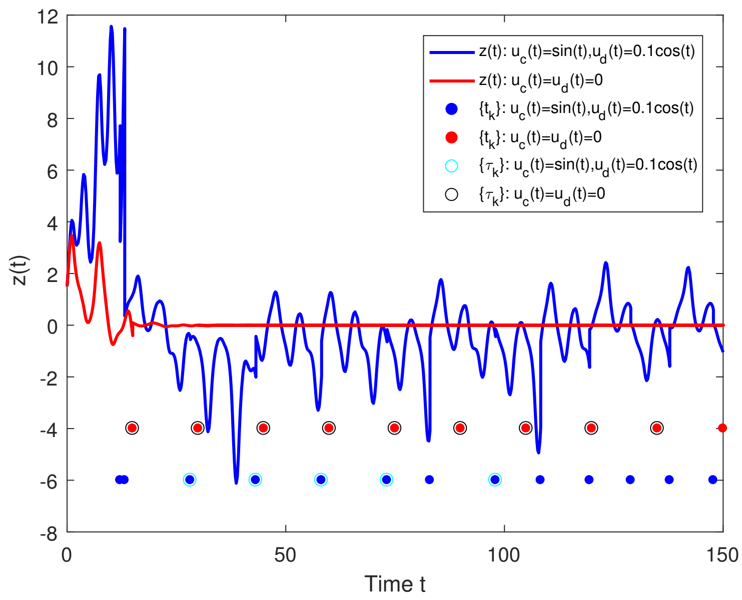

Figure 3. State trajectory of system (16) under ETM (18).

Example 2. Now we consider system (12) with

$$\mathfrak{A} = \begin{pmatrix} 1 & -1 \\ 1.5 & -1 \end{pmatrix}, \ \mathfrak{B} = \begin{pmatrix} 0.4 & -0.5 \\ 0.4 & 0.3 \end{pmatrix}, \ \mathfrak{C} = \begin{pmatrix} 0.5 & 0.4 \\ 0.5 & 0.6 \end{pmatrix},$$

$g_1(z(t)) = g_2(z(t)) = tanh(2z(t))$, $u(t) = (sin0.5t, arctan2t)^T$. In the simulation results, it is clearly evident that system (12) without impulse control cannot reach ISS, as demonstrated in Figure 4, where $z0 = (-1, 2)^T, h = 2.5$. To stabilize system (12) with respect to ISS, we design an ETM with relevant parameters $\mu = 1.2, \nu = 2, r = 2.5, p = 1.2, d = 0.9, \iota = 0.09, \sigma = 7, \tau = 1$. By using Matlab to solve LMIs (13) and (14), the ETM is presented as shown below:

$$\begin{aligned} t_k &= \min\{t_k^\diamond, \tau_k\}, \ k \in \mathbb{Z}_+, \\ t_k^\diamond &= \inf\left\{t \geq t_{k-1} : z^T(t)Pz(t) \geq 1.2z^T(t_{k-1})Pz(t_{k-1}) - 2e^{-\iota(t-t_0)}V_0 - 2.5\lambda_{max}(Q)\|u\|^2_{[t_0,t]}\cdot\right\}, \end{aligned} \tag{18}$$

where

$$P = \begin{pmatrix} 0.9152 & -0.4715 \\ -0.4715 & 0.6638 \end{pmatrix}, \ Q = \begin{pmatrix} 1.1594 & 0.0480 \\ 0.0480 & 1.1425 \end{pmatrix}, \ M = \begin{pmatrix} -0.7082 & 0.4419 \\ 0.4419 & -0.4771 \end{pmatrix},$$

and impulsive control gain

$$\mathfrak{D} = P^{-1}M^T = \begin{pmatrix} -0.6794 & 0.1775 \\ 0.1832 & -0.5623 \end{pmatrix}. \tag{19}$$

Then based on Theorem 3, system (12) is ISS with the ETM (18) and impulsive control gain $\mathfrak{D}$ in (19), and this conclusion is verified by the numerical results and graphical presentations in Figure 5, which proves the effectiveness of the proposed control strategy in ensuring the stability of the system.

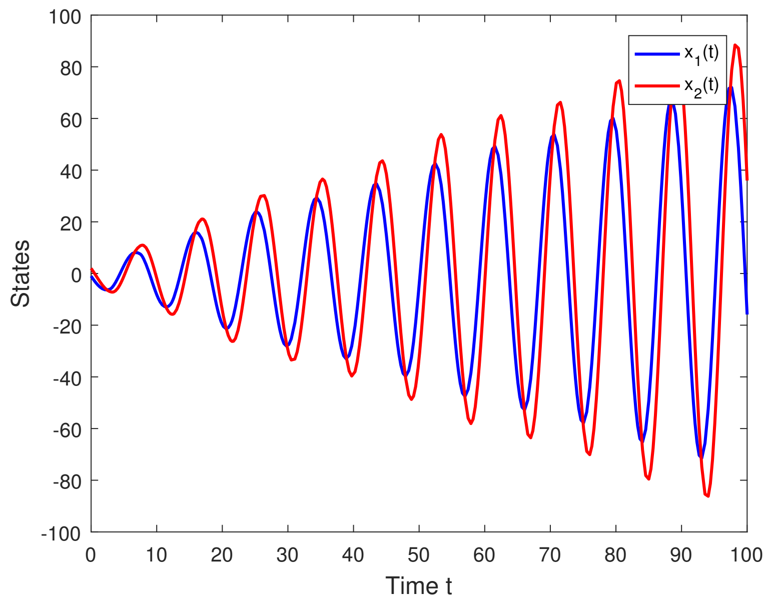

Figure 4. The trajectories of system (12) without control input.

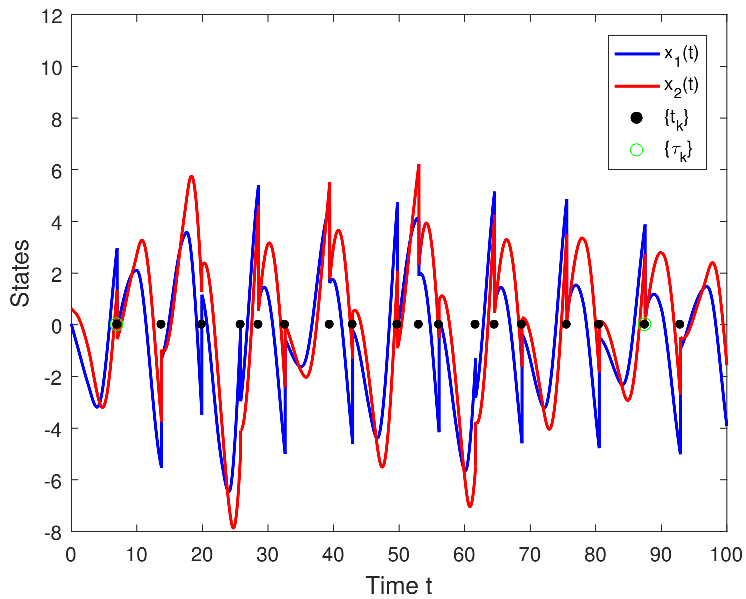

Figure 5. The trajectories of system (12) with ETM (18).

From the other side, with other parameters fixed, we modify impulsive control gain $\mathfrak{D}$ so that it does not satisfy control strategy, such as $\bar{\mathfrak{D}} = \begin{pmatrix} -0.6794 & 0.8775 \\ 0.8832 & -0.0027 \end{pmatrix}$. As can be seen from Figure 6, system (12) is non-ISS, which shows the effectiveness of our proposed ETIC method.

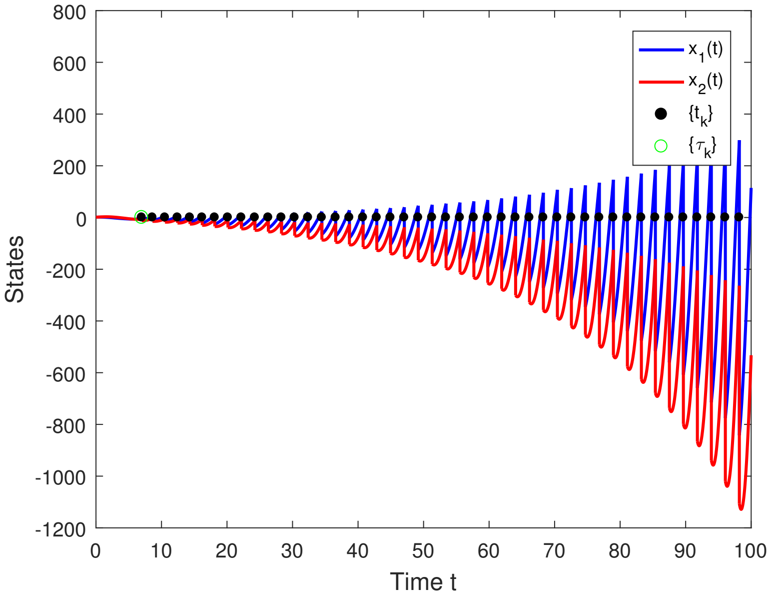

Figure 6. The trajectories of system (12) with inappropriate impulsive control gain $\bar{\mathfrak{D}}$.

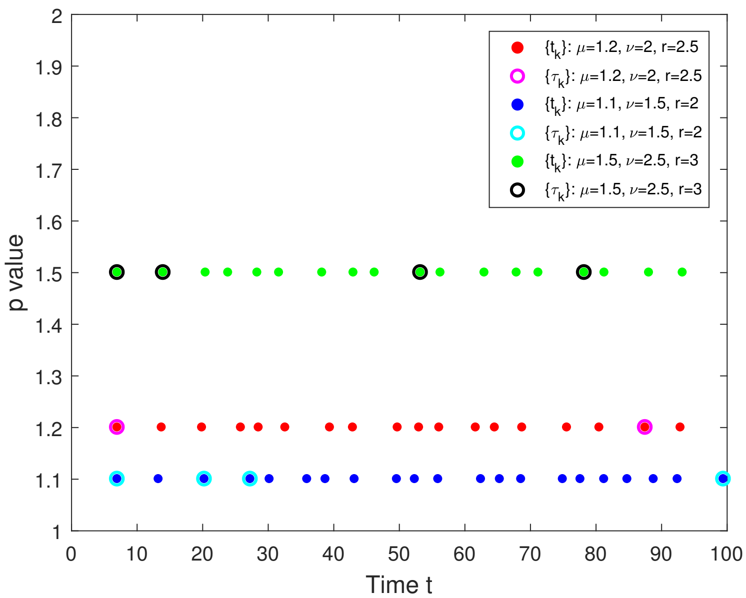

Figure 7. Triggered instants of system (12) with different triggering parameters μ, ν, and r.

Figure 7 depicts the distribution of triggered impulse instants of system (12) for different triggering parameters μ, ν, and r in the time interval from 0 to 100. It should be emphasized that, as explained in Remark 4, the lower bound of the triggering interval increases as the triggering parameters μ, ν, and r increase. In practical applications, the triggered impulse instants can be adjusted by selecting appropriate triggering parameters μ, ν, and r, thereby ensuring that the system achieves the desired performance.

Remark 9. *Compared to the existing papers on ISS that only consider time delay in either the system [31] or the impulse part [30], or even neglect time delay [28, 29] altogether, this paper simultaneously takes into account time delays in both the system and the impulse part. Moreover, the external inputs in these two parts are different. The simulation results demonstrate that the system achieves ISS under the proposed ETIC, which validates the effectiveness of the proposed method.*

6. Conclusions

The ISS properties of nonlinear delayed systems with delayed impulses are examined within the ETIC framework in this paper. An ETM is obtained by using Lyapunov theory, and the ISS criterion of the considered system is derived from the ETM, where Zeno behavior is excluded. The theoretical results are subsequently employed in nonlinear systems, and some sufficient conditions of ETM and impulse control gain are obtained using the support of LMIs. Finally, the validity of the obtained results is confirmed through two simulation examples. An important direction for the future is to design a simpler and easier to implement ETM so that the argumented system still maintains the ISS even when the forced time sequence is removed. In addition, the application of nonlinear rates can be explored to obtain more optimized stability conditions and reduce conservatism while simplifying mathematical derivations.

Author contributions

Yilin Tu: writing-original draft; Jin-E Zhang: supervision, writing-review and editing. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used artificial intelligence (AI) tools in the creation of this article.

Acknowledgments

This work is supported by the National Natural Science Foundation of China under Grant 62476082.

Conflict of interest

The authors declare no conflicts of interest.

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