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Research article

On the impossibility of increasing the MSY in a multisite Schaefer fishing model

Pierre Auger[1], **Tri Nguyen-Huu**[1,2] **and Doanh Nguyen-Ngoc**[3,*]

[1] IRD, Sorbonne Université, Unité de Modélisation Mathématique et Informatique des Systèmes Complexes, UMMISCO, F-93143, Bondy, France

[2] IXXI, ENS Lyon, 46 allée d'Italie, Lyon 69007, France

[3] College of Engineering and Computer Science and Center for Environmental Intelligence, VinUniversity, Vietnam

* **Correspondence:** Email: doanh.nn@vinuni.edu.vn; Tel: +842471089779.

Abstract: Here, we consider a multisite Schaefer fishing model. The fishery resource grows logistically on each site and is exploited with different fishing efforts. We showed that the Maximum Sustainable Yield (MSY) of the multisite network, when the sites are connected, is always less than or equal to the sum of the MSY of the isolated sites. Equality occurred when the fish population is spatially distributed according to the ideal free distribution (IFD). In this case, the fish had the same access to the resource at each site. We generalized the known result for two sites and the same fishing effort to any number of sites and different fishing efforts. We also discussed how the creation of Marine Protected Areas impacts the fishing efforts. We showed that to minimize the fishing effort to reach the MSY, it is necessary to deploy the entire fishing fleet to the site where the fish is most abundant, the other sites being Marine Protected Areas.

Keywords: multi-site fishery; maximum sustainable yield; ideal free distribution; fish habitat connectivity; marine protected area; aggregation of variables

1. Introduction

We focus on a discrete network of interconnected sites linked by migration. We consider various sites in which individuals in the population can grow logistically. Thus, each site is characterized by a growth rate and a carrying capacity corresponding to the population that can survive according to the local resource if that site is isolated. In the simple case of two sites with fast migrations of individuals, it is well known from the work of [1] and [2] that the total equilibrium population can be greater than the sum of the carrying capacities of the two sites. In other words, the total population that can survive at the two connected sites can be greater than the sum of the populations that can survive at each of the two isolated sites, that is to say, in the absence of migration. In [3], the authors showed the important role of heterogeneity between sites and asymmetry in migrations in generating excess populations when sites are connected. Moreover the study of the multisite model with logistic growth connected by fast or slow, symmetrical or asymmetrical migrations in a heterogeneous environment has received a lot of attention, such as that in [4–8]. Furthermore, an experimental approach consisting of using yeast culture has been conducted, showing that habitat connectivity in a heterogeneous multisite environment can favor a larger total population size [9].

In this contribution, we consider the capture of individuals by fishing at each site and are interested in the MSY which corresponds to the maximum catch associated with the fishery equilibrium point when it is positive and stable. Our aim of this work is to compare the catch at MSY when the sites are connected by fast migrations with the sum of catches at MSY when each site is isolated. In [10], the researchers showed, in the case of two logistic sites connected by fast fish migrations, that the total catch at the MSY at the two connected sites is always lower than the sum of the catches at the MSY of the two isolated sites. Our purpose of this work is to extend this study to the case of any number of sites greater than two.

In Section 2, we recall the case of two connected fishing sites. We present the two-site fishery model and perform model reduction using aggregation of variables methods as in [11] and [12]. We present, in more detail, the proof that the capture cannot be increased by connecting two fishing sites as that given in [10]. In Section 3, we generalize this result to the case of a network of connected fishing sites with any number of sites that can be greater than two. We illustrate our results with numerical simulations for three fishing sites and discuss the case of the creation of Marine Protected Areas (MPAs). The manuscript ends with a conclusion and some perspectives.

2. MSY in the two-site fishery model with a single harvested fish species and different fishing efforts

2.1. Presentation and reduction of the two-site Schaefer fishery model

We consider two fishing sites connected by fast fish migrations. Fish populations grow at each site, according to a logistic law, and are harvested at constant rates. $B_i(t)$ is the fish biomass at time t at site $i = 1, 2$. We assume that fish growth is logistic with growth rate r_i and carrying capacity K_i at site i. Let m_i be the migration rate of fish leaving site i. We assume that fish are captured according to a Schaefer catch at each site [13]. The complete model reads as follows:

$$\left\{\begin{array}{lcl} \frac{dB_1}{d\tau} & = & m_2B_2 - m_1B_1 + \epsilon\Big(r_1B_1\Big(1-\frac{B_1}{K_1}\Big) - qE_1B_1\Big) \\ \frac{dB_2}{d\tau} & = & m_1B_1 - m_2B_2 + \epsilon\Big(r_2B_2\Big(1-\frac{B_2}{K_2}\Big) - qE_2B_2\Big). \end{array}\right. \tag{2.1}$$

Parameter E_i is the constant fishing effort at site i, $i = 1, 2$, and parameter q is the catchability. We further assume that fish migration is fast in comparison to fish growth and fishing. This can be represented using a small dimensionless parameter $\epsilon << 1$ relative to the timescale difference between the fish growth and fishing processes (slow dynamics) and the migration processes (fast dynamics). We take benefits from this timescale difference to build a reduced model using aggregation of variables methods. The principle of those methods is to split the analysis of the main model into the analysis of two simpler systems, as described in [11, 12]. We first analyze the fast dynamics by neglecting the slow processes. The fast model is obtained by setting $\epsilon = 0$. It concerns only fish migration:

$$\begin{cases} \frac{dB_1}{d\tau} & = & m_2 B_2 - m_1 B_1 \\ \frac{dB_2}{d\tau} & = & m_1 B_1 - m_2 B_2. \end{cases} \tag{2.2}$$

The fast model is conservative, and the total fish biomass $B = B_1 + B_2$ is constant on the fast time scale. After a short time, the fish are distributed in constant proportions at the two sites corresponding to the fast equilibrium, $B_1^* = uB$ and $B_2^* = (1 - u)B$, where $u = \frac{m_2}{m_1 + m_2}$. Parameter u represents the proportion of fish at site 1 at fast equilibrium.

Following the aggregation of variables methods, we substitute the fast equilibrium into the complete model (2.1). We obtain an aggregated model that describes the dynamics of the system at the slow time scale $t = \epsilon\tau$, which reads as follows:

$$\frac{dB}{dt} = rB\left(1 - \frac{B}{\tilde{K}}\right) - qEB, \tag{2.3}$$

in which $E = E_1 u + E_2(1 - u)$ will be called the *virtual global fishing effort*: The aggregated model behaves as if the fishing effort is E. Note that it differs from the *effective total fishing effort* $E_t = E_1 + E_2$, which is the sum of the fishing efforts that has to be effectively deployed at each site.

We have now obtained an aggregated model that is much simpler to analyze since it has fewer equations. The main benefit is that it enables us to provide explicit expressions for the equilibria and their stability, which will be very useful for the further generalization to N sites. The properties of the aggregated model are guaranteed to hold for the complete model when the scale parameter is very small ($\epsilon << 1$). This means that the fish migration dynamics needs to be fast compared to the population growth and mortality, which is quite realistic in the context of fisheries. In practice, solutions of complete and aggregated models tend to remain close for ϵ values that are not so small, as illustrated in [10].

The overall fish growth rate of the reduced model reads:

$$r = ur_1 + (1 - u)r_2. \tag{2.4}$$

The overall fish carrying capacity checks expression

$$\frac{r}{\tilde{K}} = \frac{r_1 u^2}{K_1} + \frac{r_2(1 - u)^2}{K_2}, \tag{2.5}$$

and is expressed as follows:

$$\tilde{K} = \frac{K_1 K_2(ur_1 + (1 - u)r_2)}{u^2 K_2 r_1 + (1 - u)^2 K_1 r_2}, \tag{2.6}$$

which is in general different from the total carrying capacity noted $K = K_1 + K_2$.

The reduced model has two equilibria, the origin and a fish biomass B^* at equilibrium, which is:

$$B^* = \tilde{K}\left(1 - \frac{qE}{r}\right) \tag{2.7}$$

, which is positive and stable when $E < \frac{r}{q}$ while the origin is unstable. The global catch at equilibrium can be seen as a function of the fishing effort, as it is:

$$Y^*(E) = qB^*(E)E = q\tilde{K}\left(1 - \frac{qE}{r}\right)E. \tag{2.8}$$

This is a negative parabola with respect to E, with a maximum at $E^*_{MSY} = \frac{r}{2q}$. The maximum of the catch is called the MSY. It is obtained by replacing E by its value at MSY in the catch, leading to:

$$Y^*_{MSY} = \frac{r\tilde{K}}{4}. \tag{2.9}$$

It is notable that the maximum catch at MSY is independent of catchability. Furthermore, note that the biomass of fish at MSY is half its carrying capacity, i.e., $B^*_{MSY} = \frac{K}{2}$.

2.2. Comparison of MSY catches when fishing sites are connected by migrations to those when they are isolated

The global yield at MSY when the two sites are connected is thus expressed as follows:

$$Y^*_{MSY} = \frac{r\tilde{K}}{4}, \tag{2.10}$$

where parameters are those of the aggregated model, while the local yield at MSY when two sites are isolated is given by:

$$Y^*_{MSYi} = \frac{r_i K_i}{4}, \tag{2.11}$$

for i=1,2.

We define the excess yield at MSY as the difference between captures at MSY of connected sites and isolated sites in the absence of migration:

$$\Delta Y^*_{MSY} = \frac{r\tilde{K}}{4} - \frac{r_1 K_1}{4} - \frac{r_2 K_2}{4}. \tag{2.12}$$

As shown in [10], the excess yield is always negative or zero. In other words, it is impossible to increase the catch at MSY by connecting two fishing sites through fast migrations. As the proof of $\Delta Y^*_{MSY} \leq 0$ was a bit overlooked in [10], we provide a more detailed version. After substituting the expressions of the global growth rates r and the global carrying capacity $\tilde{K}$ of the aggregated model, the previous inequality becomes :

$$K_1K_2(r_1^2u^2 + r_2^2(1-u)^2 + 2r_1r_2u(1-u)) \leq (r_1K_2u^2 + r_2K_1(1-u)^2)(r_1K_1 + r_2K_2), \tag{2.13}$$

or after simplification:

$$(K_2u - K_1(1-u))^2 \geq 0, \tag{2.14}$$

which is verified. Furthermore, we can show that the equality occurs when the fish distributes among the fishing sites according to the resource, i.e., in proportion of the local carrying capacity, $u = \dfrac{K_1}{K}$. In this case, the fish distributes itself at sites with the IFD. This can occur when the fish migration rates are inversely proportional to the local carrying capacity : $m_1 = \frac{\alpha}{K_1}$ and $m_2 = \frac{\alpha}{K_2}$ where α is a positive parameter. Let us show that at IFD, $\Delta Y^*_{MSY} = 0$. First, let us show that at IFD, $\tilde{K} = K$. Indeed, $\tilde{K}$ is defined by the next equation:

$$\tilde{K} = \frac{K_1 K_2 (u r_1 + (1-u) r_2)}{u^2 K_2 r_1 + (1-u)^2 K_1 r_2}. \tag{2.15}$$

Substituting $u = \frac{K_1}{K}$ into this expression, a simple calculation shows that

$$\tilde{K} = K_1 + K_2 = K. \tag{2.16}$$

Furthermore, at IFD the yield is defined as follows:

$$\frac{r\tilde{K}}{4} = \frac{(r_1 u + r_2(1-u))(K_1 + K_2)}{4}. \tag{2.17}$$

After substitution of $u = \frac{K_1}{K}$, we finally get:

$$\frac{r\tilde{K}}{4} = \frac{r_1 K_1}{4} + \frac{r_2 K_2}{4}. \tag{2.18}$$

Consequently, in the case of two fishing sites, the optimal overall catch at MSY when the sites are connected is always lower than the sum of the catches of the isolated sites. Equality occurs when fish are distributed at IFD between the two sites. In any case, this result shows that it is impossible to increase the maximum sustainable yield of the fishery by establishing connections between fishing areas.

2.3. Sensitivity analysis

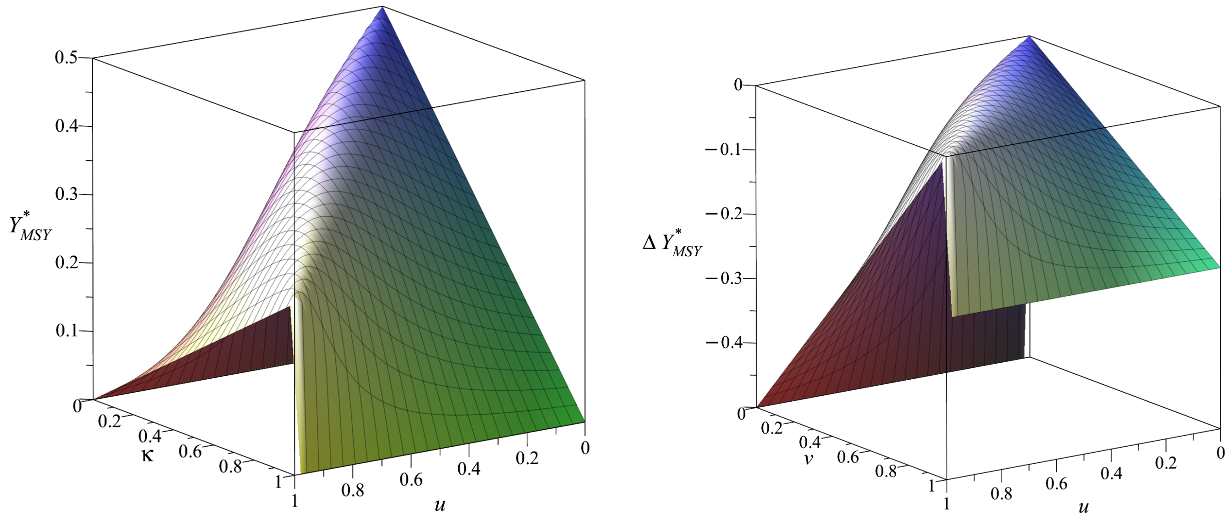
We now perform a sensitivity analysis. Since the number of parameter is small, we use a Two At A Time approach, which enables us to discuss the effect of the fish distribution u relative to the growth rates and carrying capacities. The catch at MSY reads:

$$Y^*_{MSY} = \frac{(u r_1 + (1-u) r_2)^2 K_1 K_2}{4(u^2 K_2 r_1 + (1-u)^2 K_1 r_2)}, \tag{2.19}$$

which is linear in the average growth rate $\bar{r} = mean(r_1, r_2)$ and in the average carrying capacity $\bar{K} = mean(K_1, K_2)$. We can thus omit their effect to focus on the heterogeneity of the growth rate and the carrying capacity at both sites. To do so, we rewrite $r_1 = 2\rho\bar{r}$, $r_2 = 2(1-\rho)\bar{r}$, $K_1 = 2\kappa\bar{K}$ and $K_2 = 2(1-\kappa)\bar{K}$. Parameters ρ and κ represent the growth rate and carrying capacity heterogeneity, respectively.

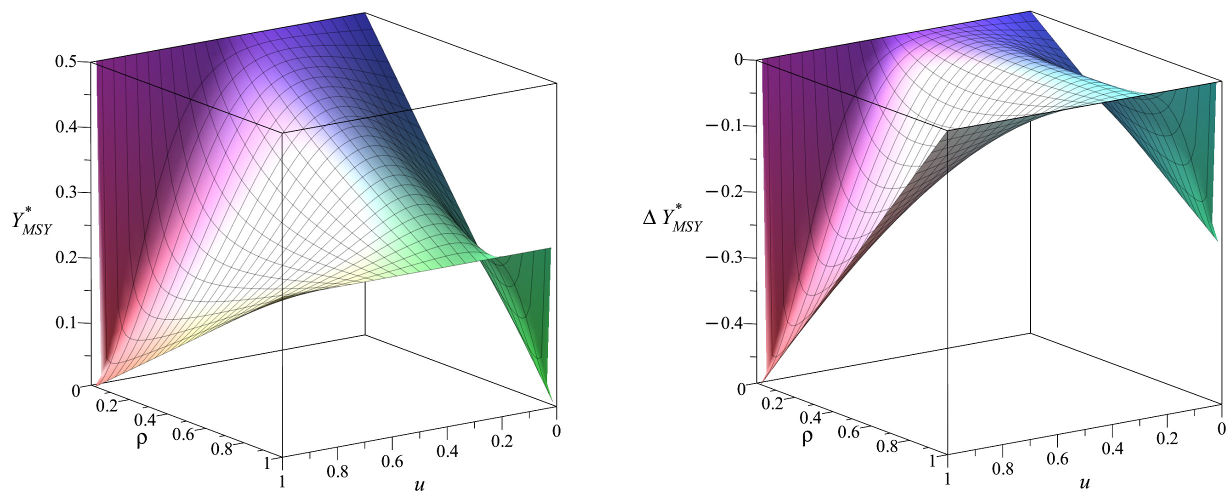
We now illustrate the variations of Y^*_{MSY} and ΔY^*_{MSY} as a function of ρ, κ and u. Figure 1 represents the effects of κ and u on the catch and excess yield at MSY while ρ is kept constant with a value of 1/3.

We observe that, for a given value of κ, the catch at MSY is higher when $u = \kappa$, i.e., at Ideal Free Distribution. The growth rate is higher at site 2, so the MSY is higher when both carrying capacity and fish proportion are higher at site 2. Figure 1b illustrates that keeping u close to κ value enables to avoid too much loss at MSY when connecting the two sites.



(a) Y_{MSY} **(b)** ΔY_{MSY}

Figure 1. (a) Catch at MSY and (b) excess yield at MSY for different values of carrying capacity heterogeneity κ and fish distribution u for $\rho = 1/3$.



(a) Y_{MSY} **(b)** ΔY_{MSY}

Figure 2. (a) Catch at MSY and (b) excess yield at MSY for different values of carrying capacity heterogeneity κ and fish distribution u for $\kappa = 1/3$.

Figure 2 represents the effects of ρ and u on the catch and excess yield at MSY while κ is kept constant with a value of 1/3. The catch at MSY is higher when u is close to $\kappa = 1/3$, and the loss of MSY for connected sites compared to the sum of MSY is smaller. We observe that a strong heterogeneity in growth rate (when ρ is close to 0 or 1) enhances the global MSY and reduces the loss.

It appears that the most favorable condition is a combination of a strong consistency between carrying capacity and fish distribution (close to IFD), and a strong heterogeneity in growth rate.

3. MSY in the multi site fishery model with a single harvested fish species and different fishing effort

We now generalize this result to any number of fishing sites $N > 2$.

3.1. Presentation and reduction of the multisite fishery model

We consider the same model as before with fish logistic growth at each fishing site and a Schaefer catch term in a multisite version. The equation for each site reads:

$$\frac{dB_i}{d\tau} = \sum_{j=1, j\neq i}^{N} m_{ij}B_j - \sum_{k=1, k\neq i}^{N} m_{ki}B_i + \epsilon\left(r_iB_i\left(1 - \frac{B_i}{K_i}\right) - qE_iB_i\right), \tag{3.1}$$

where B_i is the fish biomass on site $i = 1, N$. All biological parameters are the same as before with index i. Parameter m_{ij} represents the migration rate form site j to site i. The first migration term corresponds to the entries in the site i of fish coming from a site j and the second term to the outputs of the site i to all the other sites k. The fast model comes when we set $\epsilon = 0$:

$$\frac{dB_i}{d\tau} = \sum_{j=1, j\neq i}^{N} m_{ij}B_j - \sum_{k=1, k\neq i}^{N} m_{ki}B_i. \tag{3.2}$$

The fast model is conservative, ensuring that fish leaving one site arrive at another. Under the fast dynamics, if the migration matrix $M = (m_{ij})$ is irreducible, the model exhibits a unique positive and stable equilibrium.

Indeed, the interpretation of the irreducibility of a matrix in this context is that the connections between sites enable us to establish a path joining any site to any other. The movements dynamics leave the total stock $B = \sum_{j=1}^{N} B_j$ are invariant. Moreover, it makes the proportions of the stock at each site to rapidly tend to an equilibrium. Let $\bar{u} = (u_1, ..., u_N)$ be these equilibrium proportions for fish biomass. Vector $\bar{u}$ is the right eigenvector of matrix M and associated to eigenvalue 0 and whose entries sum up to 1. The fast equilibrium point $(B_1^*, B_2^*, ..., B_N^*)$ is thus characterized by spatial constant frequency u_i as follows:

$$B_i^* = u_iB. \tag{3.3}$$

In the general case, using the method of aggregation of variables [11, 12], we obtain the same expression of the aggregated model as in the case of two sites (Eq (2.3)) that we recall now:

$$\frac{dB}{dt} = rB\left(1 - \frac{B}{\tilde{K}}\right) - qEB, \tag{3.4}$$

where $E = \sum_{i=1}^{N} E_i u_i$ is the virtual global fishing effort. As in the previous section, it differs from the effective total fishing effort $E_t = \sum_{i=1}^{N} E_i$. The overall fish growth rate of the aggregated model reads:

$$r = \sum_{i=1}^{N} u_i r_i \tag{3.5}$$

and the overall fish carrying capacity checks the next expression:

$$\frac{r}{\tilde{K}} = \sum_{i=1}^{N} \frac{r_i u_i^2}{K_i}. \tag{3.6}$$

3.2. Comparison of MSY catches when fishing sites are connected by migrations to those when they are isolated in the multisite case

The global yield at MSY when the sites are connected is still expressed as follows:

$$Y_{MSY}^{*} = \frac{r\tilde{K}}{4}, \tag{3.7}$$

where parameters are those of the previous aggregated model. The local yields at MSY when the isolated sites are given by:

$$Y_{MSYi}^{*} = \frac{r_i K_i}{4}, \tag{3.8}$$

for $i = 1, N$. We also define the excess yield at MSY as the difference between the capture at MSY for connected sites and the one at MSY for isolated sites (in the absence of fish migration):

$$\Delta Y_{MSY}^{*} = \frac{r\tilde{K}}{4} - \sum_{i=1}^{N} \frac{r_i K_i}{4}. \tag{3.9}$$

Theorem 1. *The excess yield is always non-positive, i.e.,* $\Delta Y_{MSY}^{*} \leq 0$.

Proof. By applying the Cauchy-Schwarz inequality to $r_i u_i = \sqrt{r_i K_i}.\sqrt{r_i/K_i} u_i$, we obtain

$$\left(\sum_{i=1}^{N} \left(\sqrt{r_i K_i}.\sqrt{r_i/K_i} u_i\right)\right)^2 \leq \left(\sum_{i=1}^{N} \left(\sqrt{r_i K_i}\right)^2\right)\left(\sum_{i=1}^{N} \left(\sqrt{\frac{r_i}{K_i}} u_i\right)^2\right), \tag{3.10}$$

which simplifies to

$$\left(\sum_{i=1}^{N} r_i u_i\right)^2 \leq \left(\sum_{i=1}^{N} r_i K_i\right)\left(\sum_{i=1}^{N} \frac{r_i u_i^2}{K_i}\right). \tag{3.11}$$

Using Eqs (3.5) and (3.6), the inequality becomes

$$r^2 \leq \left(\sum_{i=1}^{N} r_i K_i\right) \frac{r}{\tilde{K}}, \tag{3.12}$$

or

$$r\tilde{K} \leq \sum_{i=1}^{N} r_i K_i. \tag{3.13}$$

From Eq (3.9), we deduce that $\Delta Y^*_{MSY} \leq 0$.

□

Corollary 1. *The equality happens if and only if* $u_i = K_i/K$.

Proof. According to the Cauchy Schwarz inequality, the equality happens if and only if

$$\frac{u_1}{K_1} = \frac{u_2}{K_2} = ... = \frac{u_N}{K_N} = \frac{u_1 + u_2 + ... + u_N}{K_1 + K_2 + ... + K_N} = \frac{1}{K}. \tag{3.14}$$

This means that $u_i = K_i/K$. □

This occurs when the fish migration rates are inversely proportional to the local carrying capacity $m_{ji} = \alpha/K_i$ for some positive α. Thus, it means that the fish distributes at the IFD. Indeed, as the average time at a site is inversely proportional to the migration rate from this site, the higher the carrying capacity of the site, the longer the residence time at the site. This assumption makes it possible to obtain a similar access to the resource for each individual.

Corollary 2. *At IFD, the aggregated carrying capacity is equal to the sum of local carrying capacities, i.e.,* $\tilde{K} = K$.

Proof. We know that $\tilde{K}$ is defined by the next equation:

$$\tilde{K} = \frac{\sum_{i=1}^{N} r_i u_i}{\sum_{i=1}^{N} \frac{r_i u_i^2}{K_i}} \tag{3.15}$$

Substituting $u_i = \frac{K_i}{K}$ into this expression, a simple calculation shows that:

$$\tilde{K} = \sum_{i=1}^{N} K_i = K. \tag{3.16}$$

□

Consequently, in the case of any number of fishing sites, the optimal overall catch at MSY when the sites are connected is always lower than the sum of the catches of the isolated sites. Equality occurs when fish are distributed at IFD.

3.3. Numerical simulation and discussion of results

Figure 3 depicts the sites' connectivity and the fish migration rates in the case of three fishing sites connected by asymmetric migrations. Figure 4 illustrates the captures at the three isolated sites according to the fishing effort as well as the total catch when the sites are connected based on global virtual fishing effort. The figure shows that the MSY of site 1 is found at $E^*_{MSY1} = r_1/2 = 0.05$, at $E^*_{MSY2} = r_2/2 = 0.1$ for site 2, and $E^*_{MSY3} = r_3/2 = 0.03$ for site 3 and for the global MSY at

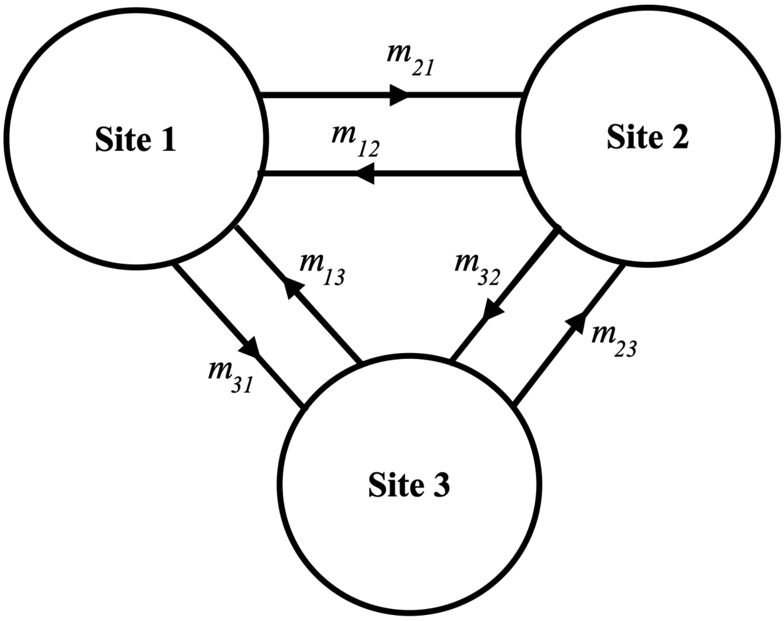


Figure 3. Example of three fishing sites connected by asymmetric migration.

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$E^*_{MSY} = r/2 = 0.046$ with connectivity. In this example, $\tilde{K}$ is almost equal to 101.55, which is greater than $K = 100$. Regarding the catches at MSY, the sum of local MSY is equal to 2.75, and the global catch at MSY when sites are connected is almost equal to 2.34. Therefore, it shows that even if the global carrying capacity exceeds the sum of local ones, the global maximum catch is smaller than the sum of catches at MSY for the three isolated sites.

Figures 5 and 6 illustrate Theorem 1 in the case of three fishing sites. We use the same parameters set for all these figures: $K_1 = 20$, $r_1 = 0.1$, $K_2 = 30$, $r_2 = 0.2$, $K_3 = 50$, $r_3 = 0.06$, $q = 1$.

Figure 5 shows the global carrying capacity $\tilde{K}$ with respect to the proportions of fish at fast equilibrium at sites 1 and 2 knowing that the proportion at site 3 is fixed by the following relationship: $u_3 = 1 - u_1 - u_2$. The blue plane corresponds to the sum of local carrying capacities K. It shows that the overall carrying capacity $\tilde{K}$ is equal to the sum of the carrying capacities of the isolated sites K when the fish distributes according to IFD, red dot. It also shows that part of the $\tilde{K}$ curve lies above the plane with constant K, which verifies an already known result that $\tilde{K} > K$ under some heterogeneity and spatial distribution conditions (see [1–3]).

Figure 6 shows the global catch at MSY, Y^*_{MSY}, when the sites are connected and the sum of local captures at MSY for isolated sites with respect to the proportions of fish at fast equilibrium at sites 1 and 2. The blue plane represents the sum of catches for isolated sites. The curve below shows the overall catch at MSY when sites are connected by fast migrations. This shows that the MSY catch curve in the case of connected sites is always below the blue plane. In accordance with Theorem 1, the excess yield is strictly negative for any spatial distributions of fish at the sites except when the fish are distributed between sites at the IFD, where it is zero. The catch at IFD is represented by a red dot at position $(0.2, 0.3)$.

Figure 7 depicts the domain of local fishing effort points for which the system is at MSY: Any point in the triangle enables maximizing the global catch when sites are connected. This triangle is located

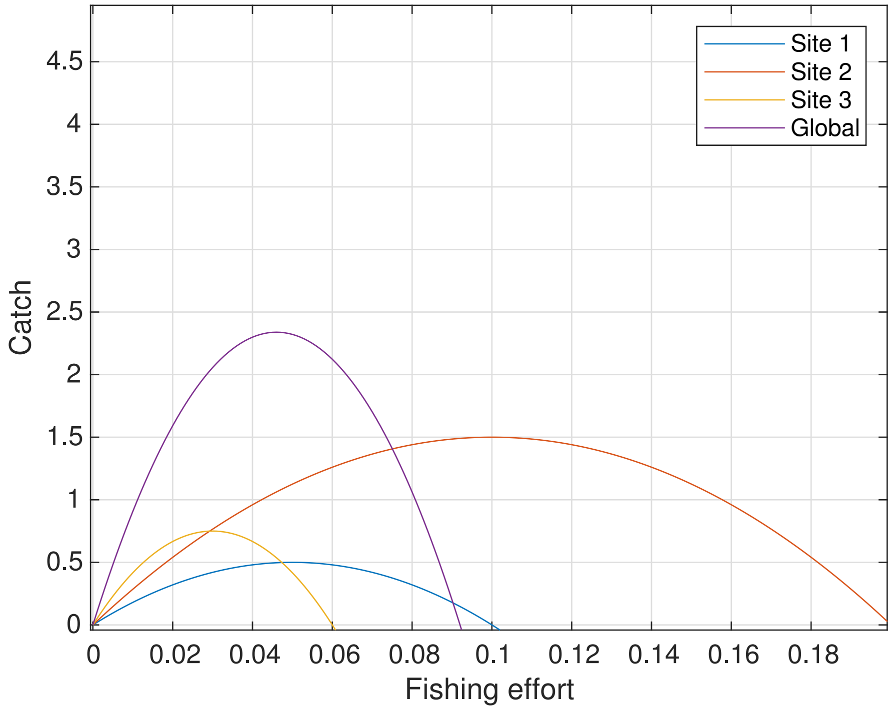

Figure 4. The figure shows the catches of isolated sites and global catch when connected with respect to fishing effort. The parameters are as follows: $K_1 = 20$, $r_1 = 0.1$, $K_2 = 30$, $r_2 = 0.2$, $K_3 = 50$, $r_3 = 0.06$, $u_1 = 0.1$, $u_2 = 0.2$, $u_3 = 0.7$, and $q = 1$.

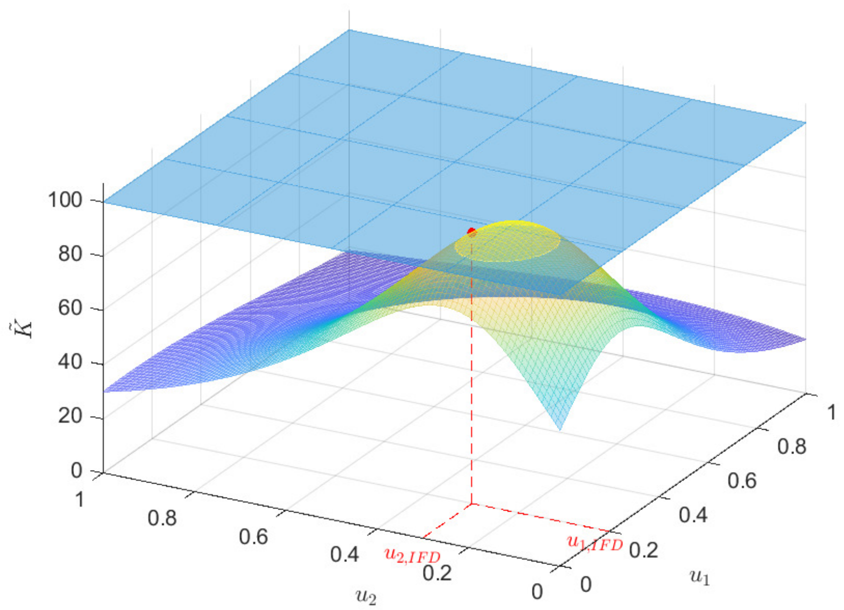

Figure 5. Global carrying capacity $\tilde{K}$ with respect to u_1 and u_2. The blue plane represents the sum of local carrying capacities of the disconnected sites. The red point corresponds to the fish population being at IFD. The parameters are given by $K_1 = 20$; $K_2 = 30$; $K_3 = 50$; $r_1 = 0.1$; $r_2 = 0.2$; $r_3 = 0.06$, and $q = 1$.

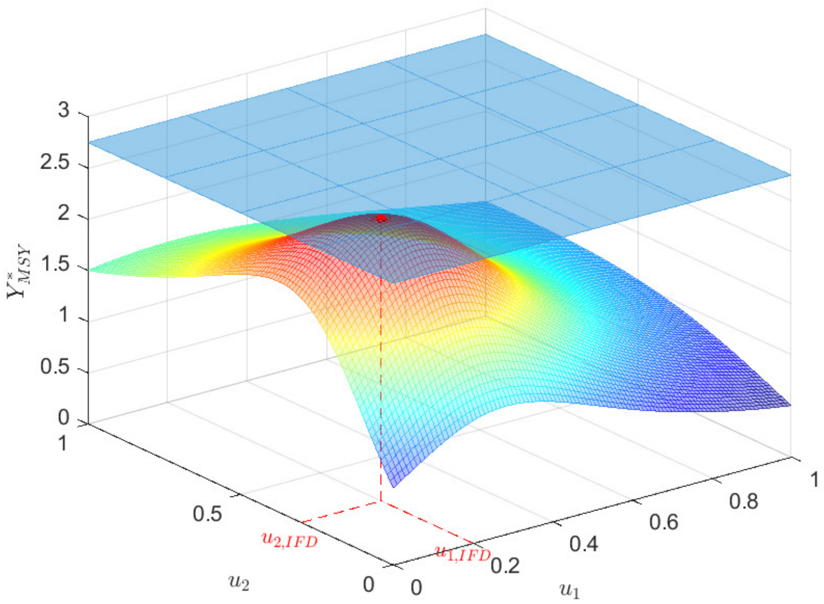

Figure 6. Global catch at MSY (Y^*_{MSY}) for the connected sites with respect to u_1 and u_2. The sum of the catches at MSY of the disconnected sites is represented by the blue plane and is always larger than the global catch for connected sites, with the equality holding at IFD (red point). The parameters are given by $K_1 = 20$; $K_2 = 30$; $K_3 = 50$; $r_1 = 0.1$; $r_2 = 0.2$; $r_3 = 0.06$, and $q = 1$.

on a hyperplane of equation $\psi(E_1, E_2, E_3) = \frac{r}{2q}$, where $\psi(E_1, E_2, E_3) = E_1u_1 + E_2u_2 + E_3u_3$. From the shipowner's point of view, it is preferable to minimize the fishing effort to achieve the MSY. This problem can be formalized as minimizing the total fishing effort $\varphi(E_1, E_2, E_3) = E_1 + E_2 + E_3$ under the constraint $\psi(E_1, E_2, E_3) = \frac{r}{2q}$ on domain $D = \{(E_1, E_2, E_3) \,|\, E_1 \geq 0, E_2 \geq 0, E_3 \geq 0\}$. Interior extrema can be found using the method of Lagrange multipliers with Lagrangian $\mathcal{L}(E_1, E_2, E_3, \lambda) = \varphi(E_1, E_2, E_3) + \lambda\psi(E_1, E_2, E_3)$. At interior extrema, the differential of $\mathcal{L}$ is null, and so

$$\frac{\partial \mathcal{L}}{\partial E_i} = 1 + \lambda u_i = 0 \tag{3.17}$$

for all $i \in \{1, 2, 3\}$. This condition can only hold if $u_1 = u_2 = u_3$; in that case, the fishing effort is constant on the triangle. In the most general case, there is no extrema in the interior of the triangle, and the minimum is located on one of the vertices.

The vertices of the triangle correspond to the cases for which fishing occurs only at a single site $i \in \{1, 2, 3\}$, the other two sites being Marine Protected Areas (MPAs), i.e., $E_j = 0$ for $j \neq i$. The edges of the triangle relate to the case of a single MPA and two fishing sites. Interior points of the triangle correspond to the case where fishing occurs at all sites. This means that the fishing effort is minimal when introducing MPAs: One has to fish at only one site i with fishing effort $E_i = r/2qu_i$, the other two sites being set as MPAs. The best site to choose is the one with largest u_i value. In other words, it is advantageous to create a MPA in the site where fish are less abundant. This result generalizes to any number of sites. In conclusion, to reach the MSY with the minimum total fishing effort, it is necessary to put the entire fishing fleet in the site where the fish are the most abundant.

We do not present a sensitivity analysis for $N > 2$ since the results are similar to the case $N = 2$.

4. Concluding remarks and perspectives

In this work, we have shown that it is impossible to increase the maximum yield of a multisite fishery by connecting heterogeneous sites regardless of the number of sites. We have thus generalized the result obtained in the case of two sites to any number of sites greater than two [10]. This result may appear negative but it concerns only the Schaefer model, that is to say, it corresponds only to a single species of fish caught. The opposite result is obtained in the case of a fish prey-predator community whose predator is fished [10] and [14]. These contradictory results are interesting because they show that the structure of the marine community and the choice of the targeted harvested species play a primordial role in the possibility of increasing the MSY of a network of heterogeneous fishing sites connected to each other.

In [10], the same fishing effort was implemented at each of the two connected sites. We generalize different fishing efforts at each site. In the aggregated model with connected sites, a virtual global fishing effort that is equal to the sum of local fishing efforts weighted by the spatial distribution of fish at the fast equilibrium. The remarkable point is that the global maximum sustainable yield occurs when $u_1E_1 + ... + u_NE_N = \dfrac{r}{2q}$. This is a hyperplane in space $\mathbb{R}_+^N$ of N variables $E_1, E_2, ..., E_N$.

Our results can be applied to some real cases including MPAs ([15, 16]). We show that in the case of several connected fishing sites, it is useful to deploy the entire fishing fleet at a single site that has the most fish. This minimizes the total fishing effort to achieve the overall MSY. In addition, it leads to the establishment of MPAs that are known to increase fish productivity locally. Furthermore, when the

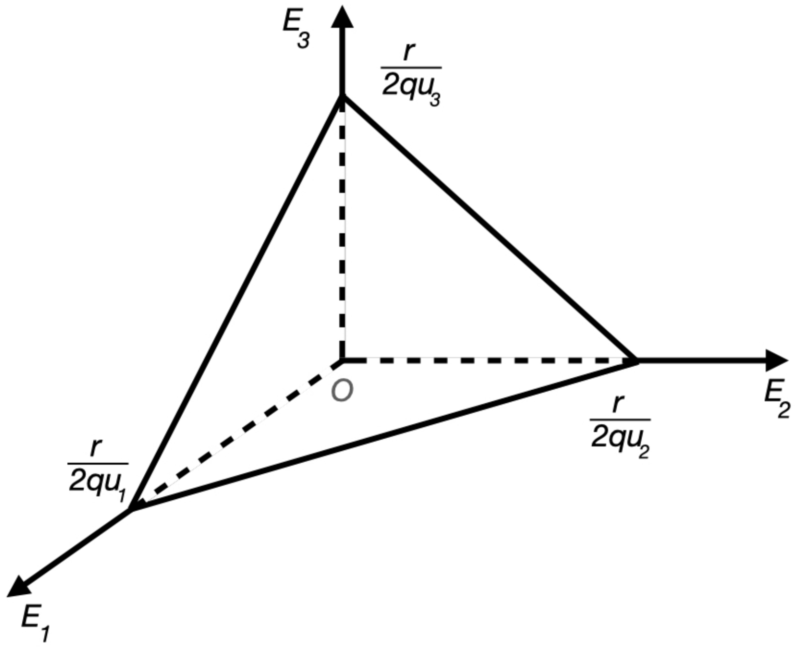

Figure 7. Domain of local fishing efforts at which the capture is maximal (Hyperplane of equation $E_1u_1 + E_2u_2 + E_3u_3 = r/2q$) for the system of three connected sites.

fisheries manager sets up MPAs, the method enables one to know the total fishing effort that must be deployed at the most abundant site. Our result could also extend to Fish Aggregation Devices (FAD) (see [17, 18]).

In our approach, the economic aspect is not taken into account; in particular, the costs of fishing. In the Gordon-Schaefer model, an equation governing fishing effort is added. Fishing effort becomes a variable that increases when the fishery is profitable and decreases otherwise. We would like to mention the work of [19] regarding a multisite fishery in the case of a single harvested fish species showing that it is possible to increase the overall MSY by connecting heterogeneous sites adequately. In this work, the researchers considered any number of fishing sites like us. This result was obtained with a restrictive but realistic conditions, which assumes that fishing costs are the same at all fishing sites. This latest work shows how very important it is to consider economic aspects in multisite fisheries models. In the future, we will study whether it is possible to increase the profit at Maximum Economic Yield (MEY) in a multisite fishery.

Moreover, the increase in the carrying capacity in a heterogeneous environment with diffusion has been verified experimentally using yeast cultures [9]. In the future, we plan to discuss, with marine biologists, an experiment that would enable highlighting the increase in MSY in a heterogeneous multisite environment with connectivity.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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