Research article

Finite-time passivity of neutral-type complex-valued neural networks with time-varying delays

Haydar Akca[1,*]**, Chaouki Aouiti**[2,*]**, Farid Touati**[2,*] **and Changjin Xu**[3,*]

[1] Abu Dhabi University, College of Arts and Sciences, Department of Applied Sciences and Mathematics, Abu Dhabi, UAE

[2] University Carthage, Faculty of Sciences of Bizerta, Department of Mathematics, GAMA Laboratory LR21ES10, BP W, 7021 Zarzouna, Bizerta, Tunisia

[3] Guizhou Key Laboratory of Economics System Simulation, Guizhou University of Finance and Economics, Guiyang 550004, China

* **Correspondence:** Email: Haydar.Akca@adu.ac.ae, chaouki.aouiti@fsb.rnu.tn, farid.touati@fsb.rnu.tn, xcj403@126.com.

Abstract: In this work, we investigated the finite-time passivity problem of neutral-type complex-valued neural networks with time-varying delays. On the basis of the Lyapunov functional, Wirtinger-type inequality technique, and linear matrix inequalities (LMIs) approach, new sufficient conditions were derived to ensure the finite-time boundedness (FTB) and finite-time passivity (FTP) of the concerned network model. At last, two numerical examples with simulations were presented to demonstrate the validity of our criteria.

Keywords: complex-valued neural networks; finite-time boundedness; finite-time passivity; neutral-type delayes; time-varying delays

1. Introduction

Hopfield neural network (HNN) is in the powerful class of artificial neural networks (ANNs), which has been introduced in the literature by John Hopfield in 1982. Since then, its investigation has become a worldwide focus [1–5]. All of those models are real-valued neural networks (NNs). However, complex-valued neural networks (CVNNs) can handle problems that cannot be handled with real-valued parallel networks [6, 7]. Recently, some authors started the dynamical analysis of CVNNs. For example, in [8], Ali et al. studied the finite time stability analysis of delayed fractional-order memristive CVNNs by using the Gronwall inequality, H*ö*lder inequality and inequality scaling skills. In [9], Zhang and Cao investigated the existence and global exponential

stability of periodic solutions of neutral type CVNNs by collecting the Lyapunov functional method with the coincidence degree theory as well as graph theory. In article [21], the authors dealt with the problem stability of impulsive CVNNs with time delay.

In the functioning of ANNs, time-delays always exist in the signal transmission between neurons on account to the limited pace of signal exchange and transmission. Thus, it is one of the main reasons poor performance and instability in a system is the presence of delays. Therefore, in recent years, the analysis of the stability of delayed ANNs has been the great attention of researchers, and various results have been discussed in the literature [16–22].

ANNs with neutral-type delays were also very seldom discussed in the existing literature; such models are called neutral-type ANNs. They have been the subject of in-depth studies by many researchers, for instance, Guo and Du [17], concerned with the global exponential stability of periodic solution for neutral-type CVNNs.

The theory of passivity is a significant notion of automatic for the analysis and control of models whose certain input/output characteristics are established in terms of energy criteria. The notions of passivity are adapted to several scientific fields and are effective for the regulation of electrical, mechanical, and electromechanical systems present in several fields of engineering, such as robotics, power electronics, aeronautics, etc. Many results are concentrated in the research of the passivity of NN systems. In [11], Li and Zheng proved the global exponential passivity for delayed quaternion-valued memristor-based NNs. Ge et al. [12] studied the robust passivity analysis for a class of uncertain NNs subject to mixed delays. The problem of passivity analysis for uncertain bidirectional associative memory NNs in the presence of mixed delays are discussed in [13]. In [35], Khonchaiyaphum et al. studied FTP analysis of neutral-type NNs.

Many researchers have been interested in finite-time control problems of dynamical systems based on the Lyapunov theory of stability to give big attention to the asymptotic model of systems over an infinite time interval. Recently, many interested results have been published. In [14], Thuan et al. discussed the robust FTP for a fractional order NNs with uncertainties. In paper [19], Wei et al. are focused on a class of coupled quaternion-valued NNs with several delayed couplings to study fixed-time passivity.

The main objective of this manuscript is to deal with the FTP for the following neutral-type CVNNs:

$$\begin{cases} \dot{z}(t) &= -Az(t) + Bf(z(t)) + Cf(z(t - \tau(t))) + D\dot{z}(t - h(t)) + \omega(t), \\ u(t) &= K_1 z(t) + K_2 f(z(t)). \end{cases} \tag{1.1}$$

Here, we have

○ $z(\cdot) = [z_1(\cdot), z_2(\cdot), \cdots, z_n(\cdot)]^T \in \mathbb{C}^n$ is the complex valued state vector,

○ $\omega(\cdot) \in \mathbb{C}^n$ is the disturbance input which belongs to $L^2[0, +\infty)$,

○ $u(\cdot) \in \mathbb{C}^n$ is the control output,

○ $f(z(\cdot)) = [f(z_1(\cdot)), f(z_2(\cdot)), \cdots, f(z_n(\cdot))]^T \in \mathbb{C}^n$ is the complex-valued activation function,

○ $A = diag\{a_1, a_2, \cdots, a_n\} > 0 \in \mathbb{R}^{n \times n}$ is a diagonal matrix,

○ $B = (b_{jk}) \in \mathbb{C}^{n \times n}, \ C = (c_{jk}) \in \mathbb{C}^{n \times n}, \ D = (d_{jk}) \in \mathbb{C}^{n \times n}$ are respectively the connection weight matrix and the connection weight matrix with delays,

○ K_1 is a known real constant matrice with appropriate dimension,

○ K_2 is a known complex matrice with appropriate dimension.

The initial condition of (1.1) is given as

$$z(s) = \psi(s), \quad s \in [-\rho, \ 0], \tag{1.2}$$

where $\rho = \max\{\sup_{t \in \mathbb{R}}(\tau(t), \ \sup_{t \in \mathbb{R}}(h(t))\}$.

The main contributions of our article are :

○ We use a more adequate hypothesis for the complex-valued activation functions (CVAF) considered in our model. Based on this assumption, a controllable model is formed by divised the real and imaginary parts, and we give a sufficient condition to realize FTP. This result is more general than the existing passivity results on real-valued NNs [35–37].

○ A Lyapunov–Krasovskii function that support triple, four, and five integral terms is introduced, and the Wirtinger-type inequality technique and convex combination approach are espoused.

○ A new set of sufficient conditions in terms of LMIs is derived to guarantee FTB and FTP results. These conditions can be simply found by the Matlab LMI toolbox.

The main contents of the manuscript are outlined as follows: in Section 2, we established new assumptions, definitions, and lemmas for the dynamic systems (1.1), which will be used later. New sufficient conditions on the FTB and FTP are discussed in Section 3. Two examples are presented in Section 4 to verify our results. At last, in Section 5, we end with the conclusion and perspective.

2. Problem formulation and preliminaries

In this section, we present new necessary assumptions and some definitions and lemmas, which are utilized in the next section.

Assumption 1: The delays $\tau(\cdot)$ and $h(\cdot)$ are differentiable functions satisfies the inequalities below:

$$0 \le \tau(t) \le \bar{\tau}, \ \dot{\tau}(t) \le \mu,$$
$$0 \le h(t) \le \bar{h}, \ \dot{h}(t) \le h_1.$$

Assumption 2: The neuron activation function $f_k(\cdot)$ satisfy the below Lipschitz condition:

$$|f_k(z_1) - f_k(z_2)| \le l_k|z_1 - z_2|, \ l_k > 0 \ (k = 1, \ 2, \cdots, \ n). \tag{2.1}$$

Furthermore, by means of Assumption 2, it is clear to get

$$(f(z_1) - f(z_2))^*(f(z_1) - f(z_2)) \le (z_1 - z_2)^* L^T L(z_1 - z_2), \tag{2.2}$$

where $L = diag\{l_1, \ l_2, \cdots, l_n\}$.

Assumption 3: The neuron activation functions $f(\cdot)$ can be divided into two parts as follows:

$$f(z) = f^R(x, \ y) + if^I(x, \ y),$$

where $z = x + iy$, i shows the imaginary unit, and the real imaginary parts check the below conditions:

1) The partial derivatives $\dfrac{\partial f}{\partial x}$, $\dfrac{\partial f}{\partial y}$ exist and are continuous.

2) There exist positive constant numbers γ_k^{RR}, γ_k^{RI}, γ_k^{IR}, γ_k^{II} such that

$$\frac{\partial f_k^R}{\partial x} \le \gamma_k^{RR}, \ \frac{\partial f_k^R}{\partial y} \le \gamma_k^{RI},$$

$$\frac{\partial f_k^I}{\partial x} \le \gamma_k^{IR}, \ \frac{\partial f_k^I}{\partial y} \le \gamma_k^{II}.$$

Thus, one can obtain that for any $x_1, \ x_2, \ y_1, \ y_2 \in \mathbb{R}$,

$$|f_k^R(x_1, \ y_1) - f_k^R(x_2, \ y_2)| \le \gamma_k^{RR}|x_1 - x_2| + \gamma_k^{RI}|y_1 - y_2|,$$
$$|f_k^I(x_1, \ y_1) - f_k^I(x_2, \ y_2)| \le \gamma_k^{IR}|x_1 - x_2| + \gamma_k^{II}|y_1 - y_2|,$$

for all $k = 1, \ 2, \cdots, \ n$.

Remark 1: As we know, in many literatures [29–31], after the separation of the activation functions into real-imaginary type, $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ are still supposed to exist and be continuous and limited. Via these conditions, we can point that they checked the same inequalities given in hypothesis 3 by considering the mean value theorem for multivariable functions. Here, we delete these constraints of the partial derivatives and make the hypothesis into an appropriate one that imaginary and real parts of the activation functions just check the inequalities in hypothesis 3. Consequently, our study can help a broader class of CVAF.

Remark 2: By applying the modulus of a complex number to some simple inequalities, it is worth mentioning that hypothesis 2 is equivalent to hypothesis 3. Therefore, for writing convenience, we will assume that activation functions satisfy hypothesis 2 to study our main results.

Assumption 4: The neuron activation function $f_k(\cdot)$ may be divided into two parts according to the complex number z as follows

$$f_k(z) = f_k^R(Re(z)) + if_k^I(Im(z))$$

where $f_k^R(\cdot), \ f_k^I(\cdot) : \mathbb{R} \to \mathbb{R}$, then for any $k = 1, \ 2, \cdots, \ n$, there exist constants $\check{l}_k^R$, $\hat{l}_k^R$, $\check{l}_k^I$ $\hat{l}_k^I$ such that

$$\check{l}_k^R \le \frac{f_k^R(\kappa_1) - f_k^R(\kappa_2)}{\kappa_1 - \kappa_2} \le \hat{l}_k^R, \quad \check{l}_k^I \le \frac{f_k^I(\kappa_1) - f_k^I(\kappa_2)}{\kappa_1 - \kappa_2} \le \hat{l}_k^I, \tag{2.3}$$

where $f_k^R(0) = 0, \ f_k^I(0) = 0, \ \kappa_1, \ \kappa_2 \in \mathbb{R}, \ \kappa_1 \ne \kappa_2$.

For presentation convenience, we denote

$$\hat{L}^R = diag\{\hat{l}_1^R, \cdots, \hat{l}_n^R\}, \ \check{L}^R = diag\{\check{l}_1^R, \cdots, \check{l}_n^R\}, \ \hat{L}^I = diag\{\hat{l}_1^I, \cdots, \hat{l}_n^I\},$$

$$\check{L}^I = diag\{\check{l}_1^I, \cdots, \check{l}_n^I\}, \ L_1 = \hat{L}^R \check{L}^R, \ L_2 = \hat{L}^I \check{L}^I, \ L_3 = \frac{\hat{L}^R + \check{L}^R}{2},$$

$$L_4 = \frac{\hat{L}^I + \check{L}^I}{2}, \quad \bar{L}_1 = diag\{L_1, \ L_2\}, \quad \bar{L}_2 = diag\{L_3, \ L_4\}.$$

Remark 3: In [23–25], hypotheses 3 and 4 are given on the activation functions which separate between the real and imaginary parts. Moreover, we can even simply say that hypothesis 3 is quite strict and that it is a special case of hypothesis 2. This fact has already been mentioned in [26] and [27]. In addition, hypothesis 4 is also a strong constraint. For instance, if the activation functions $f_k(z)$ ($k = 1, \ 2, \cdots, \ n$) checked hypothesis 4, we have:

$$|f_k^R(x_1) - f_k^R(x_2)| \le l_k^R|x_1 - x_2|, \ |f_k^I(y_1) - f_k^I(y_2)| \le l_k^I|y_1 - y_2|$$

where $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$, $l_k^R = \max\{|\check{l}_k^R|, \ |\hat{l}_k^R|\}$, and $l_k^I = \max\{|\check{l}_k^I|, \ |\hat{l}_k^I|\}$, then one can have that

$$
\begin{aligned}
|f_k(z_1) - f_k(z_2)| &= |(f_k^R(x_1) + if_k^I(y_1)) - (f_k^R(x_2) + if_k^I(y_2))| \\
&\le |f_k^R(x_1) - (f_k^R(x_2)| + |f_k^I(y_1) - f_k^I(y_2)| \\
&\le l_k^R|x_1 - x_2| + l_k^I|y_1 - y_2| \\
&\le \sqrt{(l_k^R)^2 + (l_k^I)^2}|z_1 - z_2|.
\end{aligned}
$$

Thus, the activation function $f_k(z)$ ($k = 1, \ 2, \cdots, \ n$) satisfies the Lipschitz condition of hypothesis 2. Therefore, hypotheses 3 and 4 imply hypothesis 2, that is, the condition of hypothesis 2 is a special case of hypotheses 3 and 4.

Remark 4: Indeed, when the activation function is given as follows: $f_k(z) = f_k^R(x, \ y) + if_k^I(x, \ y)$ includes $f_k(z) = f_k^R(Re(z)) + if_k^I(Im(z))$ as a special case. Therefore, in the following section, we will indicate that $f_k(z)$ is of the type of distinct real-imaginary activation functions. Let $z(t) = x(t) + iy(t)$, $z(t - \tau(t)) = z^\tau(t)$, $x(t - \tau(t)) = x^\tau(t)$, $y(t - \tau(t)) = y^\tau(t)$, $x(t - h(t)) = x^h(t)$, $y(t - h(t)) = y^h(t)$, $B = B^R + iB^I$, $C = C^R + iC^I$, $f(z(t)) = f^R(x(t), y(t)) + if^I(x(t), y(t))$, $f(z^\tau(t)) = f^R(x^\tau(t), y^\tau(t)) + if^I(x^\tau(t), y^\tau(t))$, $\omega(t) = \omega^R(t) + i\omega^I(t)$, and $u(t) = u^R(t) + iu^I(t)$, then model (1.1) can be divided into imaginary and real parts as the following form

$$
\begin{cases}
\dot{x}(t) &= -Ax(t) + B^R f^R(x(t), \ y(t)) - B^I f^I(x(t), \ y(t)) + C^R f^R(x^\tau(t), \ y^\tau(t)) \\
 &\quad - C^I f^I(x^\tau(t), \ y^\tau(t)) + D^R \dot{x}^h(t) - D^I \dot{y}^h(t) + \omega^R(t), \\
\\
\dot{y}(t) &= -Ay(t) + B^I f^R(x, \ y)(t)) + B^R f^I(x(t), \ y(t)) + C^I f^R(x^\tau(t), \ y^\tau(t)) \\
 &\quad + C^R f^I(x^\tau(t), \ y^\tau(t)) + D^R \dot{y}^h(t) + D^I \dot{x}^h(t) + \omega^I(t), \\
\\
u^R(t) &= K_1 x(t) + K_2^R f^R(x(t), \ y(t)) - K_2^I f^I(x(t), \ y(t)), \\
\\
u^I(t) &= K_1 y(t) + K_2^I f^R(x(t), \ y(t)) + K_2^R f^I(x(t), \ y(t)),
\end{cases}
\tag{2.4}
$$

The initial condition of (2.4) is given by:

$$
\begin{cases}
x(s) = \varphi^R(s), & s \in [-\rho, \ 0], \\
y(s) = \varphi^I(s), & s \in [-\rho, \ 0],
\end{cases}
$$

where $\varphi^R(s)$, $\varphi^I(s) \in C([-\rho, 0], \mathbb{R}^n)$.

Let $\theta = \begin{pmatrix} x \\ y \end{pmatrix}$, $\bar{f}(\theta) = \begin{pmatrix} f^R(x, y) \\ f^I(x, y) \end{pmatrix}$, $\bar{f}(\theta^\tau) = \begin{pmatrix} f^R(x^\tau, y^\tau) \\ f^I(x^\tau, y^\tau) \end{pmatrix}$, $\bar{\omega} = \begin{pmatrix} \omega^R \\ \omega^I \end{pmatrix}$, $\bar{A} = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$,

$\bar{B} = \begin{pmatrix} B^R & -B^I \\ B^I & B^R \end{pmatrix}$, $\bar{C} = \begin{pmatrix} C^R & -C^I \\ C^I & C^R \end{pmatrix}$, $\bar{D} = \begin{pmatrix} D^R & -D^I \\ D^I & D^R \end{pmatrix}$,

$\bar{u} = \begin{pmatrix} u^R \\ u^I \end{pmatrix}$, $\bar{K}_1 = \begin{pmatrix} K_1 & 0 \\ 0 & K_1 \end{pmatrix}$, $\bar{K}_2 = \begin{pmatrix} K_1^R & -K_1^I \\ K_1^I & K_1^R \end{pmatrix}$.

Model (2.4) can be rewritten as:

$$
\begin{cases}
\dot{\theta}(t) = -\bar{A}\theta(t) + \bar{B}\bar{f}(\theta(t)) + \bar{C}\bar{f}(\theta^\tau(t)) + \bar{D}\dot{\theta}^h(t) + \bar{\omega}(t), \\
\bar{u}(t) = \bar{K}_1\theta(t) + \bar{K}_2\bar{f}(\theta(t)),
\end{cases}
\tag{2.5}
$$

with

$$
\theta(s) = \phi(s), \ s \in [-\rho, 0],
$$

where $\phi(s) = \begin{pmatrix} \varphi^R(s) \\ \varphi^I(s) \end{pmatrix} \in C([-\rho, 0], \mathbb{R}^{2n})$.

The norm is defined as follows: For every $\phi(\cdot) \in C([-\rho, 0], \mathbb{R}^{2n})$, $\|\phi\|_{\hat{h}} = \max\{\|\phi\|, \|\phi'\|\} = \max\{ \sup_{\varsigma\in[-\rho,0]} |\phi(\theta)|, \sup_{\varsigma\in[-\rho,0]} |\phi'(\varsigma)|\}$.

It is clear from (2.3) that:

$$
\bar{L}_k^- \le \frac{\bar{f}_k(\kappa_1) - \bar{f}_k(\kappa_2)}{\kappa_1 - \kappa_2} \le \bar{L}_k^+,
\tag{2.6}
$$

where, $\bar{L}^- = \begin{pmatrix} \check{L}^R & 0 \\ 0 & \check{L}^I \end{pmatrix}$, $\bar{L}^+ = \begin{pmatrix} \hat{L}^R & 0 \\ 0 & \hat{L}^I \end{pmatrix}$.

Assumption 5: Given a positive value b^R, b^I, then $\omega^R(t)$, $\omega^I(t)$ checks

$$
\int_0^{T_1} (\omega^R)^T(t)\omega^R(t)dt \le b^R, \quad \int_0^{T_1} (\omega^I)^T(t)\omega^I(t)dt \le b^I, \ T_1 \ge 0, \ b^R \ge 0, \ b^I \ge 0,
$$

$$
\Rightarrow \int_0^{T_1} \bar{\omega}^T(t)\bar{\omega}(t)dt \le \bar{b},
\tag{2.7}
$$

where $\bar{b} = \begin{pmatrix} b^R \\ b^I \end{pmatrix}$.

Definition 1 (FTB): For a known constant $T_1 > 0$, system (1.1) is FTB with regard to $(\bar{c}_1, \quad \bar{c}_2, \quad T_1, \quad \bar{L}, \quad \bar{b})$, and $\bar{\omega}(\cdot)$ checked (2.7) if: $\sup_{t_0\in[-\bar{\tau},0]}\{\theta^T(t_0)\bar{L}\theta(t_0), \quad \dot{\theta}^T(t_0)\bar{L}\dot{\theta}(t_0)\} \le \bar{c}_1 \Rightarrow \theta^T(t)\bar{L}\theta(t) \le \bar{c}_2$, for $t \in [0, \ T_1]$, where $\bar{c}_1 = \begin{pmatrix} c_{11} \\ c_{12} \end{pmatrix}$, $\bar{c}_2 = \begin{pmatrix} c_{21} \\ c_{22} \end{pmatrix}$, $0 < c_{11} < c_{21}$, $0 < c_{12} < c_{22}$, $\bar{L} > 0$ is a matrix.

Definition 2 (FTP): For a known constant $T_1 > 0$, system (1.1) is said to be FTP, with regard to $(\bar{c}_1, \ \bar{c}_2, \ T_1, \ \bar{L}, \ \bar{b})$, if the beelow conditions are checked:

(i) Model (1.1) is FTB with regard to $(\bar{c}_1,\ \bar{c}_2,\ T_1,\ \bar{L},\ \bar{b})$.

(ii) For a known constant $\gamma > 0$, and since the zero initial condition, the below relation is true:

$$2 \int_0^{T_1} \bar{u}^T(t)\bar{\omega}(t)dt \geq -\gamma \int_0^{T_1} \bar{\omega}^T(t)\bar{\omega}(t)dt,$$

when $\bar{\omega}(\cdot)$ checked (2.7).

Lemma 1 [28]: For a symmetric matrix $\vartheta = \vartheta^T \geq 0$, scalars $e_1 > e_2 > 0$, such that the integrations below are well defined. We have

$$-(e_1 - e_2) \int_{t-e_1}^{t-e_2} \theta^T(s)\vartheta\theta(s)ds \leq -\left(\int_{t-e_1}^{t-e_2} \theta(s)ds\right)^T \vartheta\left(\int_{t-e_1}^{t-e_2} \theta(s)ds\right),$$

$$-\frac{(e_1^2 - e_2^2)}{2} \int_{-e_1}^{-e_2} \int_{t+s}^{t} \theta^T(u)\vartheta\theta(u)duds \leq -\left(\int_{-e_1}^{-e_2} \int_{t+s}^{t} \theta(u)duds\right)^T \vartheta\left(\int_{-e_1}^{-e_2} \int_{t+s}^{t} \theta(u)duds\right),$$

$$-\frac{(e_1^3 - e_2^3)}{6} \int_{-e_1}^{-e_2} \int_{s}^{0} \int_{t+u}^{t} \theta^T(v)\vartheta\theta(v)dvduds \leq -(\int_{-e_1}^{-e_2} \int_{s}^{0} \int_{t+u}^{t} \theta(v)dvdud\lambda ds)^T \theta$$

$$\times(\int_{-e_1}^{-e_2} \int_{s}^{0} \int_{t+u}^{t} \theta(v)dvdud\lambda ds). \qquad (2.8)$$

Lemma 2: For a symmetric matrix $\vartheta = \vartheta^T \geq 0$, scalars $e_1 > e_2 > 0$, such that the integrations below are well defined. We have

$$-\frac{(e_1^4 - e_2^4)}{24} \int_{-e_1}^{-e_2} \int_{s}^{0} \int_{\lambda}^{0} \int_{t+u}^{t} \theta^T(v)\vartheta\theta(v)dvdud\lambda ds \leq -(\int_{-e_1}^{-e_2} \int_{s}^{0} \int_{\lambda}^{0} \int_{t+u}^{t} \theta(v)dvdud\lambda ds)^T \vartheta$$

$$\times(\int_{-e_1}^{-e_2} \int_{s}^{0} \int_{\lambda}^{0} \int_{t+u}^{t} \theta(v)dvdud\lambda ds). \qquad (2.9)$$

Proof: The proof of Lemma 2 is inspired by the proof of Lemma 2 in [28].

For any symmetric matrix $\vartheta = \vartheta^T \geq 0$, we have

$$\begin{bmatrix} \theta^T(v)\vartheta\theta(v) & \theta^T(v) \\ \theta(v) & \vartheta^{-1} \end{bmatrix} \geq 0,$$

then after integration of it from $t + u$ to t, from λ to 0, from s to 0, and from $-e_1$ to $-e_2$ in turn, we can obtain

$$\begin{pmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12}^T & \Pi_{22} \end{pmatrix} \geq 0,$$

where

$$\Pi_{11} = \int_{-e_1}^{-e_2} \int_{s}^{0} \int_{\lambda}^{0} \int_{t+u}^{t} \theta^T(v)\vartheta\theta(v)dvdud\lambda ds,$$

$$\Pi_{12} = \int_{-e_1}^{-e_2} \int_{s}^{0} \int_{\lambda}^{0} \int_{t+u}^{t} \theta^T(v)dvdud\lambda ds,$$

$$\begin{aligned}
\Pi_{22} &= \int_{-e_1}^{-e_2} \int_s^0 \int_\lambda^0 \int_{t+u}^t \vartheta^{-1} dv du d\lambda ds \\
&= \int_{-e_1}^{-e_2} \int_s^0 \int_\lambda^0 -u\vartheta^{-1} du d\lambda ds \\
&= \int_{-e_1}^{-e_2} \int_s^0 [-\frac{u^2}{2}\vartheta^{-1}]_\lambda^0 d\lambda ds \\
&= \int_{-e_1}^{-e_2} [\frac{\lambda^3}{6}\vartheta^{-1}]_s^0 ds \\
&= [-\frac{s^4}{24}M^{-1}]_{-e_1}^{-e_2} = \frac{e_1^4 - e_2^4}{24}\vartheta^{-1}.
\end{aligned}$$

According to Schur complement, [16] is equivalent to the below condition: $\Pi_{22} > 0$, $\Pi_{11} - \Pi_{12}\Pi_{22}^{-1}\Pi_{12}^T \geq 0. \Rightarrow \Pi_{11} \geq \Pi_{12}\Pi_{22}^{-1}\Pi_{12}^T$, which is equivalent to inequality (2.9).

This completes the proof.

Lemma 3 [38]: For a given symmetric definite matrix $\xi > 0$ and for any differentiable function $\zeta(\cdot) : [e_1, \ e_2] \rightarrow \mathbb{R}^n$, the below inequality holds:

$$\int_{e_1}^{e_2} \dot{\zeta}^T(s)\xi\dot{\zeta}(s)ds \geq \frac{1}{e_2 - e_1}\begin{pmatrix} \zeta(e_2) \\ \zeta(e_1) \\ \chi \end{pmatrix}^T \times \Xi_2(\xi)\begin{pmatrix} \zeta(e_2) \\ \zeta(e_1) \\ \chi \end{pmatrix},$$

where $\chi = \frac{1}{e_2-e_1}\int_{e_1}^{e_2}\zeta(s)ds$, $\Xi_2(\xi) = \begin{pmatrix} \xi & -\xi & 0 \\ \star & \xi & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{\pi^2}{4}\begin{pmatrix} \xi & \xi & -2\xi \\ \star & \xi & -2\xi \\ \star & \star & 4\xi \end{pmatrix}$.

Remark 5: In this manuscript, our goal is to know how to determine the sufficient condition to study the FTB and FTP of the proposed model. So, to get the desired objectives, a processable whole model (2.5) is first formed by dividing the initial model into imaginary and real parts and founding an equivalent real-valued model. Second, using the Lyapunov function approach, the design procedure can be easily performed by checked the LMIs, so in the next section the expected conditions will be obtained.

3. Main results

In this section, we will concentrate on the problem of FTB and FTP.

3.1. FTB analysis

In this first part, our goal is to study the FTB in the following system:

$$\dot{\theta}(t) = -\bar{A}\theta(t) + \bar{B}\bar{f}(\theta(t)) + \bar{C}\bar{f}(\theta^\tau(t)) + \bar{D}\dot{\theta}^h(t) + \bar{\omega}(t). \tag{3.1}$$

Theorem 1: Suppose that Assumptions 1–5 hold. Let $\bar{\tau}$, μ, $\bar{h}$, h_1, and δ be positive scalars, then system (3.1) is FTB, with respect to $(\bar{c}_1, \bar{c}_2, T_1, \bar{L}, \bar{b})$, if there exists symmetric positive definite matrices

P, Q_1, Q_2, R_{01}, R_1, R_{02}, R_2, R_3, R_4 R_5 R_6, T_{01}, T_2, T_3 and diagonal matrices $M_1 > 0$, $M_2 > 0$, $M_3 > 0$, such that the following LMIs hold:

$$\Omega = \begin{bmatrix} \eta_{1,1} & \bar{L}_1 M_3 & \eta_{1,3} & \eta_{1,4} & \eta_{1,5} & G_1 D & \eta_{1,7} & \eta_{1,8} & 0 & \eta_{1,11} & \frac{\Pi^2}{2\bar{h}} R_6 & \frac{\bar{\tau}^2}{2} T_2 & \frac{\bar{\tau}^3}{6} T_3 & G_1 \\ \star & \eta_{2,2} & 0 & 0 & 0 & 0 & -M_3^T F_2^T & \eta_{2,8} & 0 & 0 & 0 & 0 & 0 & 0 \\ \star & \star & \eta_{3,3} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\Pi^2}{2\bar{\tau}} R_4 & 0 & 0 & 0 & 0 \\ \star & \star & \star & \eta_{4,4} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\Pi^2}{2\bar{h}} R_6 & 0 & 0 & 0 \\ \star & \star & \star & \star & \eta_{5,5} & G_2 \bar{D} & \eta_{5,7} & G_2 \bar{C} & 0 & 0 & 0 & 0 & 0 & G_2 \\ \star & \star & \star & \star & \star & \eta_{6,6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & \star & \eta_{7,7} & M_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & \star & \star & \eta_{8,8} & 0 & 0 & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & \star & \star & \star & \eta_{9,9} & 0 & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & \star & \star & \star & \star & \eta_{10,10} & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \eta_{11,11} & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & -T_2 & 0 & 0 \\ \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & -T_3 & 0 \\ \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & -\delta I \end{bmatrix} < 0, \quad (3.2)$$

$$\bar{c}_1 \Gamma + \bar{b}(1 - e^{-\delta T_1}) < \bar{c}_2 \lambda_1 e^{-\delta T_1}, \tag{3.3}$$

where

$\eta_{1,1} = R_{01} + R_1 - R_4 - \frac{\Pi^2}{4} R_4 + \bar{\tau}^2 R_5 - R_6 - \frac{\Pi^2}{4} R_6 - \bar{\tau}^2 T_{01} - \frac{\bar{\tau}^4}{4} T_2 - \frac{\bar{\tau}^6}{6} T_3 - \bar{L}_1 M_1 - \bar{L}_1 M_3 - G_1 \bar{A} - \bar{A}^T G_1^T - \delta P$,

$\eta_{1,3} = R_4 - \frac{\pi^2}{4} R_4$, $\eta_{1,4} = R_6 - \frac{\Pi^2}{4} R_6$,

$\eta_{1,5} = P - \bar{L}^- Q_1 + \bar{L}^+ Q_2 - G_1 - \bar{A}^T G_2^T$,

$\eta_{1,7} = \bar{L}_2 M_1 + \bar{L}_2 M_3 + G_1 \bar{B}$, $\eta_{1,11} = \frac{\Pi^2}{2\bar{\tau}} R_4 + \bar{\tau} T_{01}$,

$\eta_{2,2} = -(1 - \mu) R_{01} - \bar{L}_1 M_2 - \bar{L}_1 M_3$, $\eta_{2,8} = \bar{L}_2 M_2 + M_3^T L_2^T$,

$\eta_{3,3} = -R_4 - \frac{\pi^2}{4} R_4 - R_1$, $\eta_{4,4} = -R_6 - \frac{\Pi^2}{4} R_6$,

$\eta_{5,5} = R_3 + \bar{\tau}^2 R_4 + \bar{h}^2 R_6 + \frac{\bar{\tau}^4}{4} T_{01} + \frac{\bar{\tau}^6}{36} T_2 + \frac{\bar{\tau}^8}{576} T_3 - G_2 - G_2^T$,

$\eta_{5,7} = Q_1^T - Q_2^T + G_2 \bar{B}$, $\eta_{6,6} = -(1 - h_1) R_3$, $\eta_{7,7} = R_{02} + R_2 - M_1 - M_3$,

$\eta_{8,8} = -(1 - \mu) R_{02} - M_2 - M_3$, $\eta_{9,9} = -R_2$, $\eta_{10,10} = \frac{-\Pi^2}{\bar{\tau}^2} R_4 - R_5 - T_{01}$,

$\eta_{11,11} = -\frac{\Pi^2}{\bar{h}^2} R_6$.

Proof: We consider the Lyapunov functional below:

$$V(\theta(t)) = \sum_{i=1}^{5} V_i(\theta(t)), \tag{3.4}$$

where

$$V_1(\theta(t)) = \theta^T(t) P \theta(t),$$
$$V_2(\theta(t)) = 2 \int_0^{\theta(t)} Q_1(\bar{f}(s) - \bar{L}^- s) + Q_2(\bar{L}^+ s - \bar{f}(s)) ds,$$

$$V_3(\theta(t)) = \int_{t-\tau(t)}^{t} \theta^T(s)R_{01}\theta(s)ds + \int_{t-\tau}^{t} \theta^T(s)R_1\theta(s)ds$$

$$+ \int_{t-\tau(t)}^{t} \bar{f}^T(\theta(s))R_{02}\bar{f}(\theta(s))ds + \int_{t-\bar{\tau}}^{t} \bar{f}^T(\theta(s))R_2\bar{f}(\theta(s))ds,$$

$$V_4(\theta(t)) = \int_{t-h(t)}^{t} \dot{\theta}^T(s)R_3\dot{\theta}(s)ds + \bar{\tau}\int_{-\bar{\tau}}^{0}\int_{t+\beta}^{t} \dot{\theta}^T(s)R_4\dot{\theta}(s)dsd\beta$$

$$+ \bar{\tau}\int_{-\bar{\tau}}^{0}\int_{t+\beta}^{t} \theta^T(s)R_5\theta(s)dsd\beta + \bar{h}\int_{-\bar{h}}^{0}\int_{t+\beta}^{t} \dot{\theta}^T(s)R_6\dot{\theta}(s)dsd\beta,$$

$$V_5(\theta(t)) = \frac{\bar{\tau}^2}{2}\int_{-\bar{\tau}}^{0}\int_{\gamma}^{0}\int_{t+\beta}^{t} \dot{\theta}^T(s)T_{01}\dot{\theta}(s)dsd\beta d\gamma$$

$$+ \frac{\bar{\tau}^3}{6}\int_{-\bar{\tau}}^{0}\int_{\lambda}^{0}\int_{\gamma}^{0}\int_{t+\beta}^{t} \dot{\theta}^T(s)T_2\dot{\theta}(s)dsd\beta d\gamma d\lambda$$

$$+ \frac{\bar{\tau}^4}{24}\int_{-\bar{\tau}}^{0}\int_{\lambda}^{0}\int_{\alpha}^{0}\int_{\gamma}^{0}\int_{t+\beta}^{t} \dot{\theta}^T(s)T_3\dot{\theta}(s)dsd\beta d\gamma d\alpha d\lambda.$$

Calculating the time-derivative of $V(\theta(\cdot))$ along any trajectory of model (3.1), we obtain

$$\dot{V}(\theta(t)) = \sum_{i=1}^{5} \dot{V}_i(\theta(t)), \tag{3.5}$$

where

$$\dot{V}_1(\theta(t)) = 2\theta^T(t)P\dot{\theta}(t),$$

$$\dot{V}_2(\theta(t)) = 2(\bar{f}(\theta(t)) - \bar{L}^-\theta(t))^T Q_1\dot{\theta}(t) + 2(\bar{L}^+\theta(t) - \bar{f}(\theta(t)))^T Q_2\dot{\theta}(t),$$

$$= 2\bar{f}(\theta(t))^T Q_1\dot{\theta}(t) - 2\theta(t)^T\bar{L}^- Q_1\dot{\theta}(t) + 2\theta^T(t)\bar{L}^+ Q_2\dot{\theta}(t)$$

$$- 2\bar{f}(\theta(t))^T Q_2\dot{\theta}(t),$$

$$\dot{V}_3(\theta(t)) \leq \theta^T(t)[R_{01} + R_1]\theta(t) + \bar{f}^T(\theta(t))[R_{02} + R_2]\bar{f}(\theta(t))$$

$$- (1-\mu)\bar{f}^T(\theta(t-\tau(t)))R_{02}\bar{f}(\theta(t-\tau(t))) - \theta^T(t-\bar{\tau})R_1\theta(t-\bar{\tau})$$

$$- (1-\mu)\theta^T(t-\tau(t))R_{01}\theta(t-\tau(t)) - \bar{f}^T(\theta(t-\bar{\tau}))R_2\bar{f}(\theta(t-\bar{\tau})),$$

$$\dot{V}_4(\theta(t)) = \dot{\theta}^T(t)R_3\dot{\theta}(t) - (1-\dot{h}(t))\dot{\theta}^T(t-h(t))R_3\dot{\theta}(t-h(t))$$

$$+ \bar{\tau}\int_{-\bar{\tau}}^{0} \dot{\theta}^T(t)R_4\dot{\theta}(t) - \dot{\theta}^T(t+\beta)R_4\dot{\theta}(t+\beta)d\beta$$

$$+ \bar{\tau}\int_{-\bar{\tau}}^{0} \theta^T(t)R_5\theta(t) - \theta^T(t+\beta)R_5\theta(t+\beta)d\beta$$

$$+ \bar{h}\int_{-\bar{h}}^{0} \dot{\theta}^T(t)R_6\dot{\theta}(t) - \dot{\theta}^T(t+\beta)R_6\dot{\theta}(t+\beta)d\beta$$

$$\leq \dot{\theta}^T(t)R_3\dot{\theta}(t) - (1-h_1)\dot{\theta}^T(t-h(t))R_3\dot{\theta}(t-h(t))$$

$$+ \bar{\tau}^2\dot{\theta}^T(t)R_4\dot{\theta}(t) - \bar{\tau}\int_{t-\bar{\tau}}^{t} \dot{\theta}^T(s)R_4\dot{\theta}(s)ds + \bar{\tau}^2\theta^T(t)R_5\theta(t)$$

$$- \quad \bar{\tau} \int_{t-\bar{\tau}}^{t} \theta^{T}(s) R_{5}\theta(s)ds + \bar{h}^{2}\dot{\theta}^{T}(t)R_{6}\dot{\theta}(t) - \bar{h}\int_{t-\bar{h}}^{t} \dot{\theta}^{T}(s)R_{6}\dot{\theta}(s)ds.$$

Applying Lemma 3 to the above inequality, we get

$$-\bar{\tau}\int_{t-\bar{\tau}}^{t} \dot{\theta}^{T}(s)R_{4}\dot{\theta}(s)ds \leq - \begin{pmatrix} \theta(t) \\ \theta(t-\bar{\tau}) \\ \frac{1}{\bar{\tau}}\int_{t-\bar{\tau}}^{t}\theta(s)ds \end{pmatrix}^{T} \times \Xi_{2}(R_{4}) \begin{pmatrix} \theta(t) \\ \theta(t-\bar{\tau}) \\ \frac{1}{\bar{\tau}}\int_{t-\bar{\tau}}^{t}\theta(s)ds \end{pmatrix},$$

where

$$\Xi_{2}(R_{4}) = \begin{pmatrix} R_{4} & -R_{4} & 0 \\ \star & R_{4} & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{\Pi^{2}}{4} \begin{pmatrix} R_{4} & R_{4} & -2R_{4} \\ \star & R_{4} & -2R_{4} \\ \star & \star & 4R_{4} \end{pmatrix},$$

$$\begin{aligned}
-\bar{\tau}\int_{t-\bar{\tau}}^{t} \dot{\theta}^{T}(s)R_{4}\dot{\theta}(s)ds &\leq -\Big[\theta^{T}(t)(R_{4}+\frac{\Pi^{2}}{4}R_{4})\theta(t) + 2\theta^{T}(t)(-R_{4}+\frac{\Pi^{2}}{4}R_{4})\theta(t-\bar{\tau}) \\
&+ 2\theta^{T}(t)(-\frac{\Pi^{2}}{2\bar{\tau}}R_{4})\int_{t-\bar{\tau}}^{t}\theta(s)ds + \theta^{T}(t-\bar{\tau})(R_{4}+\frac{\Pi^{2}}{4}R_{4})\theta(t-\bar{\tau}) \\
&+ 2\theta^{T}(t-\bar{\tau})(-\frac{\Pi^{2}}{2\bar{\tau}}R_{4})\int_{t-\bar{\tau}}^{t}\theta(s)ds + (\int_{t-\bar{\tau}}^{t}\theta^{T}(s)ds)\frac{\Pi^{2}}{\bar{\tau}^{2}}R_{4}\int_{t-\bar{\tau}}^{t}\theta(s)ds\Big].
\end{aligned}$$

Similary,

$$\begin{aligned}
-\bar{h}\int_{t-\bar{h}}^{t} \dot{\theta}^{T}(s)R_{6}\dot{\theta}(s)ds &\leq -\Big[\theta^{T}(t)(R_{6}+\frac{\Pi^{2}}{4}R_{6})\theta(t) + 2\theta^{T}(t)(-R_{6}+\frac{\Pi^{2}}{4}R_{6})\theta(t-\bar{h}) \\
&+ 2\theta^{T}(t)(-\frac{\Pi^{2}}{2\bar{h}}R_{6})\int_{t-\bar{h}}^{t}\theta(s)ds + \theta^{T}(t-\bar{h})(R_{6}+\frac{\Pi^{2}}{4}R_{6})\theta(t-\bar{h}) \\
&+ 2\theta^{T}(t-\bar{h})(-\frac{\Pi^{2}}{2\bar{h}}R_{6})\int_{t-\bar{h}}^{t}\theta(s)ds \\
&+ \int_{t-\bar{h}}^{t}\theta^{T}(s)ds(\frac{\Pi^{2}}{\bar{h}^{2}}R_{6})\int_{t-\bar{h}}^{t}\theta(s)ds\Big],
\end{aligned}$$

and by applying Lemma 1,

$$-\bar{\tau}\int_{t-\bar{\tau}}^{t} \theta^{T}(s)R_{5}\theta(s)ds \leq -(\int_{t-\bar{\tau}}^{t}\theta^{T}(s)ds)^{T}R_{5}(\int_{t-\bar{\tau}}^{t}\theta(s)ds).$$

$$\begin{aligned}
\dot{V}_{5}(\theta(t)) &= \frac{\bar{\tau}^{4}}{4}\dot{\theta}^{T}(t)T_{01}\dot{\theta}(t) - \frac{\bar{\tau}^{2}}{2}\int_{-\bar{\tau}}^{0}\int_{t+\gamma}^{t}\dot{\theta}^{T}(s)T_{01}\dot{\theta}(s)dsd\gamma \\
&+ \frac{\bar{\tau}^{6}}{36}\dot{\theta}^{T}(t)T_{2}\dot{\theta}(t) - \frac{\bar{\tau}^{3}}{6}\int_{-\bar{\tau}}^{0}\int_{\lambda}^{0}\int_{t+\gamma}^{t}\dot{\theta}^{T}(s)T_{2}\dot{\theta}(s)dsd\gamma d\lambda \\
&+ \frac{\bar{\tau}^{8}}{576}\dot{\theta}^{T}(t)T_{3}\dot{\theta}(t) - \frac{\bar{\tau}^{4}}{24}\int_{-\bar{\tau}}^{0}\int_{\lambda}^{0}\int_{\alpha}^{0}\int_{t+\gamma}^{t}\dot{\theta}^{T}(s)T_{3}\dot{\theta}(s)dsd\gamma d\alpha d\lambda.
\end{aligned}$$

Applying Lemma 1 and 3 to the above inequality, we get

$$\mathbf{A}_1 = -\frac{\bar{\tau}^2}{2}\int_{-\bar{\tau}}^{0}\int_{t+\gamma}^{t}\dot{\theta}^T(s)T_{01}\dot{\theta}(s)dsd\gamma$$

$$\mathbf{A}_1 \leq -(\int_{-\bar{\tau}}^{0}\int_{t+\gamma}^{t}\dot{\theta}^T(s)ds)T_{01}(\int_{-\bar{\tau}}^{0}\int_{t+\gamma}^{t}\dot{\theta}(s)dsd\gamma)$$

$$= -[\bar{\tau}\theta^T(t) - \int_{t-\bar{\tau}}^{t}\theta^T(s)ds]T_1[\bar{\tau}\theta(t) - \int_{t-\bar{\tau}}^{t}\theta(s)ds]$$

$$= -\bar{\tau}^2\theta^T(t)T_{01}\theta(t) + 2\bar{\tau}\theta^T(t)T_{01}\int_{t-\bar{\tau}}^{t}\theta(s)ds - (\int_{t-\bar{\tau}}^{t}\theta^T(s)ds)T_1(\int_{t-\bar{\tau}}^{t}\theta(s)ds).$$

$$\mathbf{A}_2 = -\frac{\bar{\tau}^3}{6}\int_{-\bar{\tau}}^{0}\int_{\lambda}^{0}\int_{t+\gamma}^{t}\dot{\theta}^T(s)T_2\dot{\theta}(s)dsd\gamma d\lambda \leq (\int_{-\bar{\tau}}^{0}\int_{\lambda}^{0}\int_{t+\gamma}^{t}\dot{\theta}^T(s)dsd\gamma d\lambda)$$

$$\times \; T_2(\int_{-\bar{\tau}}^{0}\int_{\lambda}^{0}\int_{t+\gamma}^{t}\dot{\theta}(s)dsd\gamma d\lambda)$$

$$= -[\frac{\bar{\tau}^2}{2}\theta^T(t) - \int_{-\bar{\tau}}^{0}\int_{t+\lambda}^{t}\theta^T(s)dsd\lambda]T_2[\frac{\bar{\tau}^2}{2}\theta(t) - \int_{-\bar{\tau}}^{0}\int_{t+\lambda}^{t}\theta(s)dsd\lambda]$$

$$= -\frac{\bar{\tau}^4}{4}\theta^T(t)T_2\theta(t) + 2\theta^T(t)(\frac{\bar{\tau}^2}{2}T_2)\int_{-\bar{\tau}}^{0}\int_{t+\lambda}^{t}\theta(s)dsd\lambda$$

$$- \; (\int_{-\bar{\tau}}^{0}\int_{t+\lambda}^{t}\theta^T(s)dsd\lambda)T_2(\int_{-\bar{\tau}}^{0}\int_{t+\lambda}^{t}\theta(s)dsd\lambda).$$

$$\mathbf{A}_3 = -\frac{\bar{\tau}^4}{24}\int_{-\bar{\tau}}^{0}\int_{\lambda}^{0}\int_{\alpha}^{0}\int_{t+\gamma}^{t}\dot{\theta}^T(s)T_3\dot{\theta}(s)dsd\gamma d\alpha d\lambda$$

$$\mathbf{A}_3 \leq -\Big(\int_{-\bar{\tau}}^{0}\int_{\lambda}^{0}\int_{\alpha}^{0}\int_{t+\gamma}^{t}\dot{\theta}^T(s)dsd\gamma d\alpha d\lambda\Big)T_3\Big(\int_{-\bar{\tau}}^{0}\int_{\lambda}^{0}\int_{\alpha}^{0}\int_{t+\gamma}^{t}\dot{\theta}(s)dsd\gamma d\alpha d\lambda\Big)$$

$$= -[\frac{\bar{\tau}^3}{6}\theta^T(t) - \int_{-\bar{\tau}}^{0}\int_{\lambda}^{0}\int_{t+\alpha}^{t}\theta^T(s)dsd\alpha d\lambda]T_3[\frac{\bar{\tau}^3}{6}\theta(t) - \int_{-\bar{\tau}}^{0}\int_{\lambda}^{0}\int_{t+\alpha}^{t}\theta(s)dsd\alpha d\lambda]$$

$$= -\frac{\bar{\tau}^6}{36}\theta^T(t)T_3\theta(t) + 2\theta^T(t)(\frac{\bar{\tau}^3}{6}T_3)\Big(\int_{-\bar{\tau}}^{0}\int_{\lambda}^{0}\int_{t+\alpha}^{t}\theta(s)dsd\alpha d\lambda\Big)$$

$$-\Big(\int_{-\bar{\tau}}^{0}\int_{\lambda}^{0}\int_{t+\alpha}^{t}\theta^T(s)dsd\alpha d\lambda\Big)T_3\Big(\int_{-\bar{\tau}}^{0}\int_{\lambda}^{0}\int_{t+\alpha}^{t}\theta(s)dsd\alpha d\lambda\Big). \tag{3.6}$$

For any $\rho_{1k} > 0$, $\rho_{2k} > 0$, $\rho_{3k} > 0$, $k = 1, 2, \cdots, n$, it follows from (2.3) that

$$[\bar{f}_k(\theta_k(t)) - \bar{L}_k^-\theta_k(t)]^T\rho_{1k}[\bar{L}_k^+\theta_k(t) - \bar{f}_k(\theta_k(t))] \geq 0,$$

$$[\bar{f}_k(\theta_k(t - \tau(t))) - \bar{L}_k^-\theta_k(t - \tau(t))]^T\rho_{2k}[\bar{L}_k^+\theta_k(t - \tau(t)) - \bar{f}_k(\theta_k(t - \tau(t)))] \geq 0,$$

$$[\bar{f}_k(\theta_k(t)) - \bar{f}_k(\theta_k(t - \tau(t))) - \bar{L}_k^-(\theta_k(t) - \theta_k(t - \tau(t)))]^T \rho_{3k}[\bar{L}_k^+(\theta_k(t) - \theta_k(t - \tau(t))) - (\bar{f}_k(\theta_k(t))$$
$$+ \bar{f}_k(\theta_k(t - \tau(t))))] \geq 0,$$

which implies

$$\begin{pmatrix} \theta^T(t) \\ \bar{f}^T(\theta(t)) \end{pmatrix} \begin{pmatrix} -\bar{L}_1 M_1 & \bar{L}_2 M_1 \\ \star & -M_1 \end{pmatrix} \times \begin{pmatrix} \theta(t) \\ \bar{f}(\theta(t)) \end{pmatrix} \geq 0, \tag{3.7}$$

$$\begin{pmatrix} \theta^T(t - \tau(t)) \\ \bar{f}^T(\theta(t - \tau(t))) \end{pmatrix} \begin{pmatrix} -\bar{L}_1 M_2 & \bar{L}_2 M_2 \\ \star & -M_2 \end{pmatrix} \times \begin{pmatrix} \theta(t - \tau(t)) \\ \bar{f}(\theta(t - \tau(t))) \end{pmatrix} \geq 0, \tag{3.8}$$

and

$$\begin{pmatrix} \theta^T(t) \\ \bar{f}^T(\theta(t)) \\ \theta^T(t - \tau(t)) \\ \bar{f}^T(\theta(t - \tau(t))) \end{pmatrix} \times \begin{pmatrix} -\bar{L}_1 M_3 & \bar{L}_2 M_3 & \bar{L}_1 M_3 & -\bar{L}_1 M_3 \\ \star & -M_3 & -\bar{L}_2 M_3 & \bar{L}_3 \\ \star & \star & -\bar{L}_1 M_3 & \bar{L}_1 M_3 \\ \star & \star & \star & -M_3 \end{pmatrix} \begin{pmatrix} \theta(t) \\ \bar{f}(\theta(t)) \\ \theta(t - \tau(t)) \\ \bar{f}(\theta(t - \tau(t))) \end{pmatrix} \geq 0, \tag{3.9}$$

where $M_1 = diag\{\rho_{11}, \rho_{12}, \cdots \rho_{1n}\}$, $M_2 = diag\{\rho_{21}, \rho_{22}, \cdots \rho_{2n}\}$, $M_3 = diag\{\rho_{31}, \rho_{32}, \cdots \rho_{3n}\}$.

Moreover, for any matrices G_1 and G_2 with appropriate dimensions, it is true that,

$$2[\theta^T(t)G_1 + \dot{\theta}^T(t)G_2][-\dot{\theta}(t) - \bar{A}\theta(t) + \bar{B}\bar{f}(\theta(t)) + \bar{C}\bar{f}(\theta(t - \tau(t))) + \bar{D}\dot{\theta}(t - h(t)) + \bar{\omega}(t)] = 0$$
$$= -2\theta^T(t)G_1\dot{\theta}(t) - 2\theta^T(t)G_1\bar{A}\theta(t) + 2\theta^T(t)G_1\bar{B}\bar{f}(\theta(t)) + 2\theta^T(t)G_1\bar{C}\bar{f}(\theta(t - \tau(t)))$$
$$+ 2\theta^T(t)G_1\bar{D}\dot{\theta}(t - h(t)) + 2\theta^T(t)G_1\bar{\omega}(t) - 2\dot{\theta}^T(t)G_2\dot{\theta}(t) - 2\dot{\theta}^T(t)G_2\bar{A}\theta(t)$$
$$+ 2\dot{\theta}^T(t)G_2\bar{B}\bar{f}(\theta(t)) + 2\dot{\theta}^T(t)G_2\bar{C}f(\theta(t - \tau(t))) + 2\dot{\theta}^T(t)G_2\bar{D}\dot{\theta}(t - h(t))$$
$$+ 2\dot{\theta}^T(t)G_2\bar{\omega}(t). \tag{3.10}$$

We can say that

$$\dot{V}(\theta(t)) - \delta V_1(\theta(t)) - \delta\bar{\omega}^T(t)\bar{\omega}(t) \leq \xi^T(t)\Omega\xi(t) < 0, \tag{3.11}$$

where

$$\xi(t) = \begin{bmatrix} \theta^T(t) & \theta^T(t - \tau(t)) & \theta^T(t - \bar{\tau}) & \theta^T(t - \bar{h}) & \dot{\theta}^T(t) & \dot{\theta}^T(t - h(t)) \end{bmatrix}$$

$$\bar{f}^T(\theta(t)) \quad \bar{f}^T(\theta(t - \tau(t))) \quad \bar{f}^T(\theta(t - \bar{\tau})) \quad \int_{t-\bar{\tau}}^t \theta^T(s)ds \quad \int_{t-\bar{h}}^t \theta^T(s)ds$$

$$\int_{-\bar{\tau}}^0 \int_{t+\lambda}^t \theta^T(s)dsd\lambda \quad \int_{-\bar{\tau}}^0 \int_\lambda^0 \int_{t+\alpha}^t \theta^T(s)dsd\alpha d\lambda \quad \bar{\omega}(t) \end{bmatrix}^T,$$

and Ω is given in (3.2).

$$\dot{V}(\theta(t)) \leq \delta V_1(\theta(t)) + \delta\bar{\omega}^T(t)\bar{\omega}(t)$$

$$< \quad \delta V(\theta(t)) + \delta \bar{\omega}^T(t)\bar{\omega}(t) \tag{3.12}$$

Multiplying by $e^{-\delta t}$, we can obtain

$$e^{-\delta t}\dot{V}(\theta(t)) - \delta e^{-\delta t}V(\theta(t)) < \delta e^{-\delta t}\bar{\omega}^T(t)\bar{\omega}(t),$$
$$\frac{d}{dt}(e^{-\delta t}V(\theta(t))) < \delta e^{-\delta t}\bar{\omega}^T(t)\bar{\omega}(t). \tag{3.13}$$

By integrating (3.13] between 0 to t, such as $t \in [0, \ T_1]$, we can write

$$e^{-\delta t}V(\theta(t)) - V(x(0)) < \delta \int_0^t e^{-\delta s}\bar{\omega}^T(s)\bar{\omega}(s)ds,$$
$$V(\theta(t)) < e^{\delta t}[V(\theta(0)) + \int_0^t e^{-\delta s}\bar{\omega}^T(s)\omega(s)ds]. \tag{3.14}$$

So,

$$V(\theta(0)) = \theta^T(0)P\theta(0) + 2\int_0^{\theta(0)} Q_1(\bar{f}(s) - \bar{L}^- s) + Q_2(\bar{L}^+ s - \bar{f}(s))ds$$

$$+ \int_{-\tau(t)}^0 \theta^T(s)R_{01}\theta(s)ds + \int_{-\bar{\tau}}^0 \theta^T(s)R_1\theta(s)ds + \int_{-\bar{\tau}}^0 \bar{f}^T(\theta(s))R_2\bar{f}(\theta(s))ds$$

$$+ \int_{-\tau(t)}^0 \bar{f}^T(\theta(s))R_{02}\bar{f}(\theta(s))ds + \int_{-h(t)}^0 \dot{\theta}^T(s)R_3\dot{\theta}(s)ds + \bar{\tau}\int_{-\bar{\tau}}^0 \int_\beta^t \dot{\theta}^T(s)R_4\dot{\theta}(s)dsd\beta$$

$$+ \bar{\tau}\int_{-\bar{\tau}}^0 \int_\beta^0 \theta^T(s)R_5\theta(s)dsd\beta + \bar{h}\int_{-\bar{h}}^0 \int_\beta^0 \dot{\theta}^T(s)R_6\dot{\theta}(s)dsd\beta$$

$$+ \frac{\bar{\tau}^2}{2}\int_{-\bar{\tau}}^0 \int_\gamma^0 \int_\beta^0 \dot{\theta}^T(s)T_{01}\dot{\theta}(s)dsd\beta d\gamma + \frac{\bar{\tau}^3}{6}\int_{-\bar{\tau}}^0 \int_\lambda^0 \int_\gamma^0 \int_\beta^0 \dot{\theta}^T(s)T_2\dot{\theta}(s)dsd\beta d\gamma d\lambda$$

$$+ \frac{\bar{\tau}^4}{24}\int_{-\bar{\tau}}^0 \int_\lambda^0 \int_\alpha^0 \int_\gamma^0 \int_\beta^0 \dot{\theta}^T(s)T_3\dot{\theta}(s)dsd\beta d\gamma d\alpha d\lambda.$$

Letting

$$\hat{P} = \bar{L}^{-\frac{1}{2}}P\bar{L}^{-\frac{1}{2}}, \ \hat{Q}_1 = \bar{L}^{-\frac{1}{2}}Q_1\bar{L}^{-\frac{1}{2}}, \ \hat{Q}_2 = \bar{L}^{-\frac{1}{2}}Q_2\bar{L}^{-\frac{1}{2}}, \ \hat{R}_{01} = \bar{L}^{-\frac{1}{2}}R_{01}\bar{L}^{-\frac{1}{2}},$$
$$\hat{R}_1 = \bar{L}^{-\frac{1}{2}}R_1\bar{L}^{-\frac{1}{2}}, \ \hat{R}_{02} = \bar{L}^{-\frac{1}{2}}R_{02}\bar{L}^{-\frac{1}{2}}, \ \hat{R}_2 = \bar{L}^{-\frac{1}{2}}R_2\bar{L}^{-\frac{1}{2}}, \ \hat{R}_3 = \bar{L}^{-\frac{1}{2}}R_3\bar{L}^{-\frac{1}{2}},$$
$$\hat{R}_4 = \bar{L}^{-\frac{1}{2}}R_4\bar{L}^{-\frac{1}{2}}, \ \hat{R}_5 = \bar{L}^{-\frac{1}{2}}R_5\bar{L}^{-\frac{1}{2}}, \ \hat{R}_6 = \bar{L}^{-\frac{1}{2}}R_6\bar{L}^{-\frac{1}{2}}, \ \hat{T}_{01} = \bar{L}^{-\frac{1}{2}}T_{01}\bar{L}^{-\frac{1}{2}},$$
$$\hat{T}_2 = \bar{L}^{-\frac{1}{2}}T_2\bar{L}^{-\frac{1}{2}}, \hat{T}_3 = \bar{L}^{-\frac{1}{2}}T_3\bar{L}^{-\frac{1}{2}}, \ \lambda_1 = \lambda_{\min}(\hat{P}), \ \lambda_2 = \lambda_{\max}(\hat{P}),$$
$$\lambda_3 = \lambda_{\max}(\hat{Q}_1), \ \lambda_4 = \lambda_{\max}(\hat{Q}_2), \ \lambda_5 = \lambda_{\max}(\hat{R}_{01}), \ \lambda_6 = \lambda_{\max}(\hat{R}_1),$$
$$\lambda_7 = \lambda_{\max}(\hat{R}_{02}), \ \lambda_8 = \lambda_{\max}(\hat{R}_2), \ \lambda_9 = \lambda_{\max}(\hat{R}_3), \ \lambda_{10} = \lambda_{\max}(\hat{R}_4),$$
$$\lambda_{11} = \lambda_{\max}(\hat{R}_5), \ \lambda_{12} = \lambda_{\max}(\hat{R}_6), \ \lambda_{13} = \lambda_{\max}(\hat{T}_{01}), \ \lambda_{14} = \lambda_{\max}(\hat{T}_2),$$
$$\lambda_{15} = \lambda_{\max}(\hat{T}_3),$$

we obtain

$$V(\theta(0)) \quad = \quad \theta^T(0)\bar{L}^{\frac{1}{2}}\hat{P}\bar{L}^{\frac{1}{2}}\theta(0) + 2\int_0^{\theta(0)} \bar{L}^{\frac{1}{2}}\hat{Q}_1\bar{L}^{\frac{1}{2}}(\bar{f}(s) - \bar{L}^- s)$$

$$+ \quad \bar{L}^{\frac{1}{2}}\hat{Q}_2\bar{L}^{\frac{1}{2}}(\bar{L}^+s - \bar{f}(s))ds + \int_{-\tau(t)}^{0} \theta^T(s)\bar{L}^{\frac{1}{2}}\hat{R}_{01}\bar{L}^{\frac{1}{2}}\theta(s)ds$$

$$+ \quad \int_{-\bar{\tau}}^{0} \theta^T(s)\bar{L}^{\frac{1}{2}}\hat{R}_1\bar{L}^{\frac{1}{2}}\theta(s)ds + \int_{-\bar{\tau}}^{0} ((\bar{L}^+)^2\theta^T(s)\bar{L}^{\frac{1}{2}}\hat{R}_2\bar{L}^{\frac{1}{2}}\theta(s)ds$$

$$+ \quad \int_{-\tau(t)}^{0} ((\bar{L}^+)^2\theta^T(s)\bar{L}^{\frac{1}{2}}\hat{R}_{02}\bar{L}^{\frac{1}{2}}\theta(s)ds + \int_{-h(t)}^{0} \dot{\theta}^T(s)\bar{L}^{\frac{1}{2}}\hat{R}_3\bar{L}^{\frac{1}{2}}\dot{\theta}(s)ds$$

$$+ \quad \bar{\tau}\int_{-\bar{\tau}}^{0}\int_{\beta}^{0} \dot{\theta}^T(s)\bar{L}^{\frac{1}{2}}\hat{R}_4\bar{L}^{\frac{1}{2}}\dot{\theta}(s)dsd\beta + \bar{\tau}\int_{-\bar{\tau}}^{0}\int_{\beta}^{0} \theta^T(s)\bar{L}^{\frac{1}{2}}\hat{R}_5\bar{L}^{\frac{1}{2}}\theta(s)dsd\beta$$

$$+ \quad \bar{h}\int_{-\bar{h}}^{0}\int_{\beta}^{0} \dot{\theta}^T(s)\bar{L}^{\frac{1}{2}}\hat{R}_6\bar{L}^{\frac{1}{2}}\dot{\theta}(s)dsd\beta + \frac{\bar{\tau}^2}{2}\int_{-\bar{\tau}}^{0}\int_{\gamma}^{0}\int_{\beta}^{0} \dot{\theta}^T(s)\bar{L}^{\frac{1}{2}}\hat{T}_1\bar{L}^{\frac{1}{2}}\dot{\theta}(s)dsd\beta d\gamma$$

$$+ \quad \frac{\bar{\tau}^3}{6}\int_{-\bar{\tau}}^{0}\int_{\lambda}^{0}\int_{\gamma}^{0}\int_{\beta}^{0} \dot{\theta}^T(s)\bar{L}^{\frac{1}{2}}\hat{T}_2\bar{L}^{\frac{1}{2}}\dot{\theta}(s)dsd\beta d\gamma d\lambda$$

$$+ \quad \frac{\bar{\tau}^4}{24}\int_{-\bar{\tau}}^{0}\int_{\lambda}^{0}\int_{\alpha}^{0}\int_{\gamma}^{0}\int_{\beta}^{0} \dot{\theta}^T(s)\bar{L}^{\frac{1}{2}}\hat{T}_3\bar{L}^{\frac{1}{2}}\dot{\theta}(s)dsd\beta d\gamma d\alpha d\lambda$$

$$\leq \quad \left\{ \lambda_{\max}(\hat{P})\theta^T(0)\bar{L}\theta(0) + 2\lambda_{\max}(\hat{Q}_1)[\max\{|\bar{L}^+,\ \bar{L}^-|^2 - \bar{L}^-\}] \right.$$

$$+ \quad 2\lambda_{\max}(\hat{Q}_2)[\max\{\bar{L}^+ - |\bar{L}^+,\ \bar{L}^-|^2\}] + \bar{\tau}[\lambda_{\max}(\hat{R}_{01}) + \lambda_{\max}(\hat{R}_1)]$$

$$+ \quad \bar{\tau}[\max\{|\bar{L}^+,\ \bar{L}^-|^2\}][\lambda_{\max}(\hat{R}_{02}) + \lambda_{\max}(\hat{R}_2)] + \bar{h}\lambda_{\max}(\hat{R}_3) + \frac{\bar{\tau}^3}{2}\lambda_{\max}(\hat{R}_4)$$

$$+ \quad \frac{\bar{\tau}^3}{2}\lambda_{\max}(\hat{R}_5) + \frac{\bar{h}^3}{2}\lambda_{\max}(\hat{R}_6) + \frac{\bar{\tau}^5}{12}\lambda_{\max}(\hat{T}_1) + \frac{\bar{\tau}^7}{144}\lambda_{\max}(\hat{T}_2) + \frac{\bar{\tau}^9}{2880}\lambda_{\max}(\hat{T}_3) \left. \right\}$$

$$\times \quad \sup_{t_0\in[-\bar{\tau},0]} \{\theta^T(t_0)\bar{L}\theta(t_0),\ \dot{\theta}^T(t_0)\bar{L}\dot{\theta}(t_0)\},$$

$$V(\theta(0)) \leq \bar{c}_1\Gamma,$$

where

$$\Gamma = [\lambda_2 + \lambda_3[\tilde{L}^2 - \bar{L}^-] + \lambda_4[\bar{L}^+ - \tilde{L}^2] + \bar{\tau}[\lambda_5 + \lambda_6] + \bar{\tau}\tilde{L}^2[\lambda_7 + \lambda_8] + \bar{h}\lambda_9 + \frac{\bar{\tau}^3}{2}[\lambda_{10}$$

$$+ \quad \lambda_{11}] + \frac{\bar{h}^3}{2}\lambda_{12} + \frac{\bar{\tau}^5}{12}\lambda_{13} + \frac{\bar{\tau}^7}{144}\lambda_{14} + \frac{\bar{\tau}^9}{2880}\lambda_{15},$$

where $\tilde{L} = \max\{|\bar{L}^+,\ \bar{L}^-|^2\}$.

Furthermore, it follows from (3.14) that

$$V(\theta(t)) \geq \theta^T(t)P\theta(t) \geq \lambda_{min}(\hat{P})\theta^T(t)\bar{L}\theta(t) = \lambda_1\theta^T(t)\bar{L}\theta(t). \tag{3.15}$$

Because of inequalities (3.14) and (3.15), we obtain

$$\lambda_1\theta^T(t)P\theta(t) \leq e^{\delta t}[V(\theta(0)) + \int_0^t e^{-\delta t}\bar{\omega}^T(s)\bar{\omega}(s)ds],$$

$$\Rightarrow \theta^T(t)\bar{L}\theta(t) \leq \frac{e^{\delta T_1}[\bar{c}_1\Gamma + b(1 - e^{(-\delta T_1)})]}{\lambda_1}. \tag{3.16}$$

From condition (3.3), we arrive at $\theta^T(t)L\theta(t) < \bar{c}_2$. From Definition 1, the model (3.1) is FTB with regard to $(\bar{c}_1, \bar{c}_2, T_1, \bar{L}, \bar{b})$.

This allowed the proof to be obtained.

Remark 6: In Theorem 1, sufficient conditions are met to verify that the model (3.1) is FTB, then Theorem 2 will present FTP conditions.

3.2. FTP analysis

In this second part, we study the FTP analysis for the below model

$$\begin{cases} \dot{\theta}(t) = -\bar{A}\theta(t) + \bar{B}\bar{f}(\theta(t)) + \bar{C}\bar{f}(\theta(t - \tau(t))) + \bar{D}\dot{\theta}(t - h(t)) + \bar{\omega}(t), \\ \bar{u}(t) = \bar{K}_1\theta(t) + \bar{K}_2\bar{f}(\theta(t)). \end{cases} \quad (3.17)$$

Theorem 2: Suppose that Assumptions 1–5 hold. Let $\bar{\tau}$, μ, $\bar{h}$, h_1, and δ be scalars, then system (3.17) is FTP with keeping the parameter $(\bar{c}_1, \bar{c}_2, T_1, \bar{L}, \bar{b})$, if there exists symmetric positive definite matrices P, Q_1, Q_2, R_{01}, R_1, R_{02}, R_2, R_3, R_4, R_5, T_{01}, T_2, T_3 and diagonal matrices $M_1 > 0$, $M_2 > 0$, $M_3 > 0$, and scalar $\beta > 0$ such that the following LMIs (3.18) hold:

$$\tilde{\Omega} = \begin{bmatrix} \eta_{1,1} & \bar{L}_1 M_3 & \eta_{1,3} & \eta_{1,4} & \eta_{1,5} & G_1 D & \eta_{1,7} & \eta_{1,8} & 0 & \eta_{1,10} & \frac{\Pi^2}{2h}R_6 & \frac{\bar{\tau}^2}{2}T_2 & \frac{\bar{\tau}^3}{6}T_3 & G_1 - \beta I - \bar{K}_1 \\ \star & \eta_{2,2} & 0 & 0 & 0 & 0 & -M_3^T \bar{L}_2^T & \eta_{2,8} & 0 & 0 & 0 & 0 & 0 & 0 \\ \star & \star & \eta_{3,3} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\Pi^2}{2\bar{\tau}}R_4 & 0 & 0 & 0 & 0 \\ \star & \star & \star & \eta_{4,4} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\Pi^2}{2h}R_6 & 0 & 0 & 0 \\ \star & \star & \star & \star & \eta_{5,5} & G_2\bar{D} & \eta_{5,7} & G_2\bar{C} & 0 & 0 & 0 & 0 & 0 & G_2 \\ \star & \star & \star & \star & \star & \eta_{6,6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & \star & \eta_{7,7} & M_3 & 0 & 0 & 0 & 0 & 0 & -\bar{K}_2 \\ \star & \star & \star & \star & \star & \star & \star & \eta_{8,8} & 0 & 0 & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & \star & \star & \star & \eta_{9,9} & 0 & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & \star & \star & \star & \star & \eta_{10,10} & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \eta_{11,11} & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & -T_2 & 0 & 0 \\ \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & -T_3 & 0 \\ \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & -\beta I \end{bmatrix} < 0, \quad (3.18)$$

$$\bar{c}_1\Gamma + \bar{b}(1 - e^{-\delta T_1}) < \bar{c}_2\lambda_1 e^{-\delta T_1},$$

where the parameters are kept the same as Theorem 3.1.

Proof: Select the Lyapunov function used in the Theorem 3.1. We get

$$\dot{V}(\theta(t)) - \delta V_1(\theta(t)) - 2\bar{u}^T(t)\bar{\omega}(t) - \beta\bar{\omega}^T(t)\bar{\omega}(t) \leq \xi^T(t)\tilde{\Omega}\xi(t) < 0,$$
$$\Rightarrow \dot{V}(\theta(t)) - \delta V(\theta(t)) - 2\bar{u}^T(t)\bar{\omega}(t) - \beta\bar{\omega}^T(t)\bar{\omega}(t) < 0,$$

where $\tilde{\Omega}$ is given in (3.18), then

$$\dot{V}(\theta(t)) - \delta V(\theta(t)) \leq 2\bar{u}^T(t)\bar{\omega}(t) + \beta\bar{\omega}^T(t)\bar{\omega}(t). \quad (3.19)$$

Multiplying (3.19) by $e^{-\delta t}$, we can get

$$e^{-\delta t}\dot{V}(\theta(t)) - \delta e^{-\delta t}V(\theta(t)) \leq 2e^{-\delta t}\bar{u}^T(t)\bar{\omega}(t) + \beta e^{-\delta t}\bar{\omega}^T(t)\bar{\omega}(t),$$

$$\frac{d}{dt}(e^{-\delta t}V(\theta(t))) \leq 2e^{-\delta t}\bar{u}^T(t)\bar{\omega}(t) + \beta e^{-\delta t}\bar{\omega}^T(t)\bar{\omega}(t).$$

Integrating the above inequality between 0 to T_1, we can write

$$e^{-\delta T_1}V(\theta(t)) - V(\theta(0)) \leq 2\int_0^{T_1} e^{-\delta t}\bar{u}^T(t)\bar{\omega}(t)dt + \beta \int_0^{T_1} e^{-\delta t}\bar{\omega}^T(t)\bar{\omega}(t)dt.$$

Consider the zero initial condition for $\theta_0 = 0$ and we have $V(\theta(0)) = 0$, then

$$e^{-\delta T_1}V(\theta(t)) \leq 2\int_0^{T_1} e^{-\delta t}\bar{u}^T(t)\bar{\omega}(t)dt + \beta \int_0^{T_1} e^{-\delta t}\bar{\omega}^T(t)\bar{\omega}(t)dt,$$

$$V(\theta(t)) \leq e^{\delta T_1}\int_0^{T_1} [2\bar{u}^T(t)\bar{\omega}(t) + \beta\bar{\omega}^T(t)\bar{\omega}(t)]dt. \tag{3.20}$$

Since $V(\theta(t)) \geq 0$, we can say from (3.20) that

$$2\int_0^{T_1} \bar{u}^T(t)\bar{\omega}(t)dt \geq -\beta \int_0^{T_1} \bar{\omega}^T(t)\bar{\omega}(t)dt. \tag{3.21}$$

Finally, we can say that the system (3.17) is FTP. This allowed the proof to be obtained.

Remark 7: It should be emphasized that the LMI approach proposed in this paper is more useful for reducing the conservatism of the delay system, which may lead to obtain less conservative results. This proves the advantage of our proposed method.

Remark 8: We note that the optimal value of $\bar{c}_2$ depends on the parameter δ, then the optimal minimum value of $\bar{c}_2$ is determined from the minimum value of δ, such that the LMIs matrix solution remains feasible.

Remark 9: In the majority of published work, the Lyapunov function theory is the most effective approach to investigating the problem of stability and passivity for divers dynamical models. Moreover, it can be seen that the existing literature [11–13, 15, 19, 32–35] contains the Lyapunov function with single, double, tripe, and four integral terms. However, in this article, we give the Lyapunov–Krasovskii function with five integral terms such as $\frac{\bar{\tau}^4}{24}\int_{-\bar{\tau}}^0 \int_\lambda^0 \int_\alpha^0 \int_\gamma^0 \int_{t+\beta}^t \dot{\theta}^T(s)T_3\dot{\theta}(s)ds\,d\beta\,d\gamma\,d\alpha\,d\lambda$. Different from the published work, this is the first time studying the problem of FTP of neutral-type CVNNs. Via a Lyapunov–Krasovskii function with triple, four and five integral terms, by utilizing Jensons inequality and the Wirtinger-type inequality technique, new sufficient conditions for FTB and FTP are taken in terms of LMIs, which is effective on reducing conservatism.

4. Numerical examples

In this section, two examples are given to show the feasibility of our results.

Example 1: Consider the following model

$$\begin{cases} \dot{z}(t) = -Az(t) + Bf(z(t)) + Cf(z(t - \tau(t))) + D\dot{z}(t - h(t)) + \omega(t), \\ z(s) = \psi(s), \quad s \in [-\rho, \, 0], \end{cases} \tag{4.1}$$

where

$$A = \begin{pmatrix} 1.9 & 0 \\ 0 & 1.2 \end{pmatrix}, \ B = \begin{pmatrix} 0.5 + 0.5i & 0.03 + 0.03i \\ -0.4 - 0.4i & -0.1 - 0.1i \end{pmatrix}, \ C = \begin{pmatrix} 1.1 + 1.1i & 0.03 + 0.03i \\ 0.07 + 0.07i & 0.1 + 0.1i \end{pmatrix},$$

$$D = \begin{pmatrix} 0.1 + 0.1i & 0 \\ 0 & 0.1 + 0.1i \end{pmatrix}, \ \bar{A} = \begin{pmatrix} 1.9 & 0 & 0 & 0 \\ 0 & 1.2 & 0 & 0 \\ 0 & 0 & 1.9 & 0 \\ 0 & 0 & 0 & 1.2 \end{pmatrix}, \ \bar{B} = \begin{pmatrix} 0.5 & 0.03 & -0.5 & -0.03 \\ -0.4 & -0.1 & 0.4 & 0.1 \\ 0.5 & 0.03 & 0.5 & 0.03 \\ -0.4 & -0.1 & -0.4 & -0.1 \end{pmatrix},$$

$$\bar{C} = \begin{pmatrix} 1.1 & 0.03 & -1.1 & -0.03 \\ 0.07 & 0.1 & -0.07 & -0.1 \\ 1.1 & 0.03 & 1.1 & 0.03 \\ 0.07 & 0.1 & 0.07 & 0.1 \end{pmatrix}, \ \bar{D} = \begin{pmatrix} 0.1 & 0 & -0.1 & 0 \\ 0 & 0.1 & 0 & -0.1 \\ 0.1 & 0 & 0.1 & 0 \\ 0 & 0.1 & 0 & 0.1 \end{pmatrix},$$

$f_j(z) = 0.3(|x_j+1| - |x_j-1|) + i0.3(|y_j+1| - |y_j-1|)$. Thus, $L_1 = L_2 = 0_{2\times2}$ and $L_3 = L_4 = \begin{pmatrix} 0.3 & 0 \\ 0 & 0.3 \end{pmatrix}, \Rightarrow \bar{L}_1 = 0_{4\times4}$ and $\bar{L}_2 = 0.3I_4$ $\tau(t) = 0.15(1 - \sin(2t))$, $h(t) = 0.15(1 - \cos(2t))$, $\omega(t) = 0.9\sin(\pi t)e^{-0.5t} + \sin(\pi t)e^{-0.3t}i$, $\delta = 1$, $\bar{c}_1 = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}0.5$, $\bar{b} = \begin{pmatrix} 0.81 \\ 0.60 \end{pmatrix}$, $T_1 = 15$, matrix $\bar{L} = I$. Using the Matlab LMI toolbox to solve the LMIs (3.2) and (3.3) in Theorem 1, we obtained the feasible solutions for an optimal minimum value of $\bar{c}_2 = \begin{pmatrix} 9.8208 \\ 9.4072 \end{pmatrix}$. The trajectories of the solution of system (4.1) in Example 1 are shown in Figures 1 and 2, and the time history of $(Re(z))^T L(Re(z))$ and $(Im(z))^T L(Im(z))$ are given in Figure 3. Hence, it can be concluded that the system (4.1) is FTB.
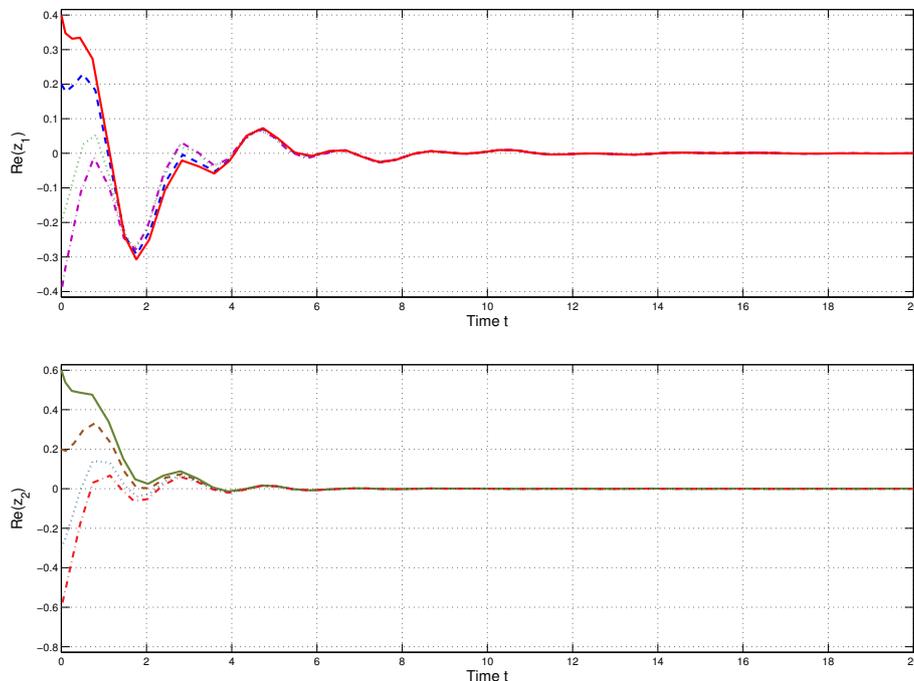
Figure 1. The trajectories of the real parts of the solution of model (4.1) in Example 1 with 4 initial conditions.
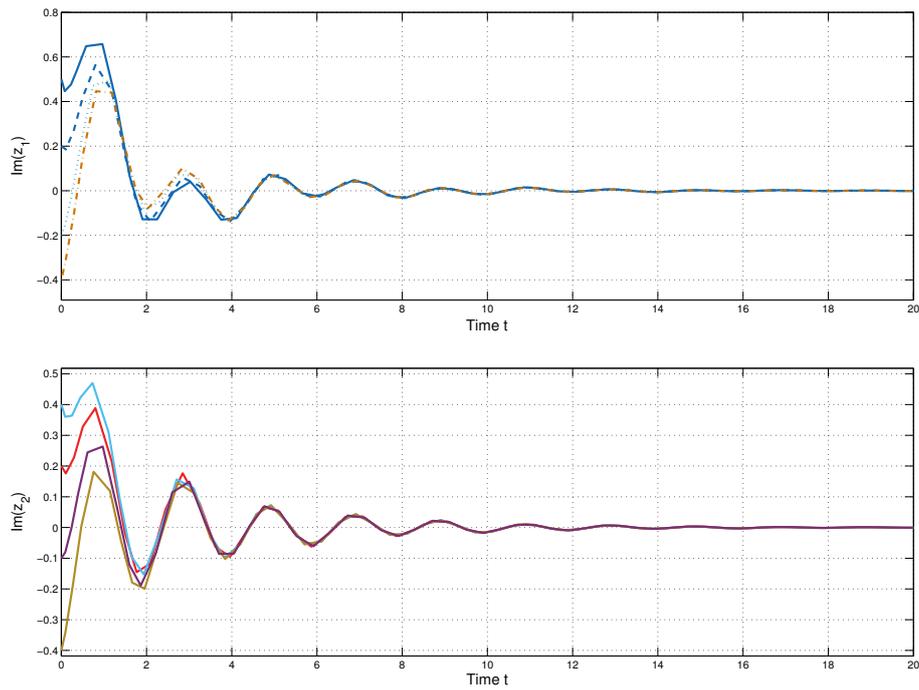
Figure 2. The trajectories of the imaginary parts of the solution of model (4.1) in Example 1 with 4 initial conditions.
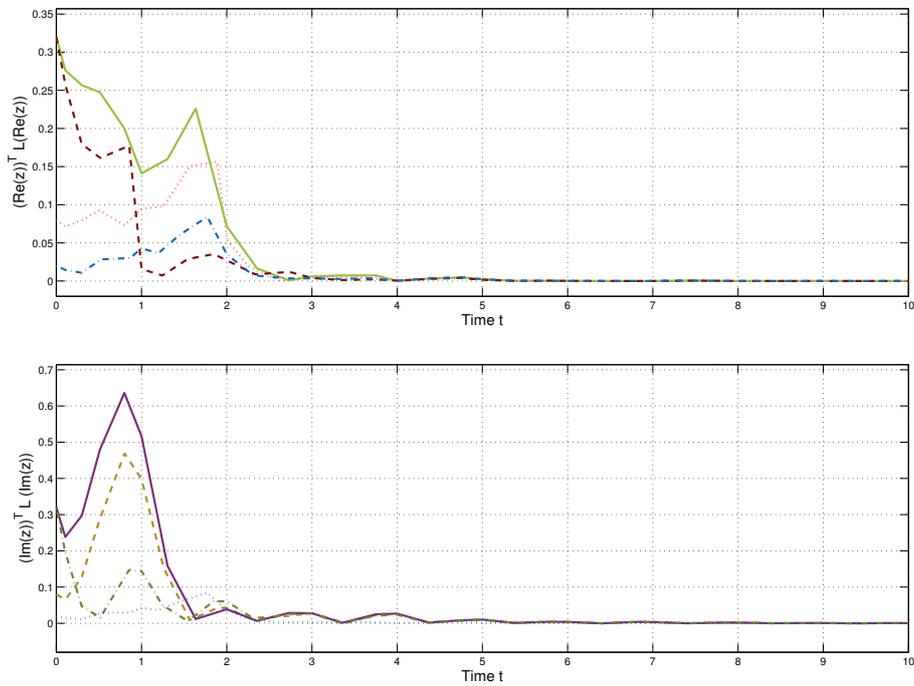
Figure 3. Time history of $(Re(z))^T L(Re(z))$ and $(Im(z))^T L(Im(z))$ of model (4.1) in Example 1 with 4 initial conditions.

Example 2: Consider the following system :

$$\begin{cases} \dot{z}(t) = -Az(t) + Bf(z(t)) + Cf(z(t - \tau(t))) + D\dot{z}(t - h(t)) + \omega(t), \\ u(t) = K_1 z(t) + K_2 f(z(t)), \\ z(s) = \psi(s), \quad s \in [-\rho, \ 0], \end{cases} \quad (4.2)$$

where

$$A = \begin{pmatrix} 3.8 & 0 \\ 0 & 2.4 \end{pmatrix}, \ B = \begin{pmatrix} 1+i & 0.06+0.06i \\ -0.8-0.8i & -0.2-0.2i \end{pmatrix}, \ C = \begin{pmatrix} 2.2+2.2i & 0.06+0.06i \\ 0.14+0.14i & 0.2+0.2i \end{pmatrix},$$

$$D = \begin{pmatrix} 0.2+0.2i & 0 \\ 0 & 0.2+0.2i \end{pmatrix}, \ K_1 = \begin{pmatrix} 1.2 & 1.6 \\ 1.25 & 1 \end{pmatrix}, \ K_2 = \begin{pmatrix} 1+i & 0.5+0.6i \\ 0.5-0.4i & 1-i \end{pmatrix},$$

$$\bar{A} = \begin{pmatrix} 3.8 & 0 & 0 & 0 \\ 0 & 2.4 & 0 & 0 \\ 0 & 0 & 3.8 & 0 \\ 0 & 0 & 0 & 2.4 \end{pmatrix}, \ \bar{B} = \begin{pmatrix} 1 & 0.06 & -1 & -0.06 \\ -0.8 & -0.2 & 0.8 & 0.2 \\ 1 & 0.06 & 1 & 0.06 \\ -0.8 & -0.2 & -0.8 & -0.2 \end{pmatrix}, \ \bar{C} = \begin{pmatrix} 2.2 & 0.06 & -2.2 & -0.06 \\ 0.14 & 0.2 & -0.14 & -0.2 \\ 2.2 & 0.06 & 2.2 & 0.06 \\ -0.14 & 0.2 & 0.14 & 0.2 \end{pmatrix},$$

$$\bar{D} = \begin{pmatrix} 0.2 & 0 & -0.2 & 0 \\ 0 & 0.2 & 0 & -0.2 \\ 0.2 & 0 & 0.2 & 0 \\ 0 & 0.2 & 0 & 0.2 \end{pmatrix}, \ \bar{K}_1 = \begin{pmatrix} 1.2 & -1.6 & 0 & 0 \\ -1.25 & 1 & 0 & 0 \\ 0 & 0 & 1.2 & -1.6 \\ 0 & 0 & -1.25 & 1 \end{pmatrix}, \ \bar{K}_2 = \begin{pmatrix} 1 & 0.5 & 1 & -0.6 \\ 0.5 & 1 & 0.4 & 1 \\ -1 & 0.6 & 1 & 0.5 \\ -0.4 & -1 & 0.5 & 1 \end{pmatrix},$$

$f_j(z) = 0.6\tanh(x_j) + i0.7\tanh(y_j)$. Thus, $L_1 = L_2 = 0_{2\times2}$ and $L_3 = \begin{pmatrix} 0.3 & 0 \\ 0 & 0.3 \end{pmatrix}$, $L_4 = \begin{pmatrix} 0.35 \\ 0.35 \end{pmatrix} \Rightarrow$

$\bar{L}_1 = 0_{4\times4}$ and $\bar{L}_2 = \begin{pmatrix} 0.3 & 0 & 0 & 0 \\ 0 & 0.3 & 0 & 0 \\ 0 & 0 & 0.35 & 0 \\ 0 & 0 & 0 & 0.35 \end{pmatrix}$, $\tau(t) = 0.15(1 - \cos(2t))$, $h(t) = 0.15(1 - \sin(2t))$, $\omega(t) =$

$0.2e^{-0.5t} + 0.3e^{-0.3t}i$, $\delta = 1$, $\bar{c}_1 = \begin{pmatrix} 0.1 \\ 0.1 \end{pmatrix}0.5$, $\bar{b} = \begin{pmatrix} 0.019 \\ 0.053 \end{pmatrix}$, $T_1 = 12$, matrix $\bar{L} = I$. Using the Matlab LMI toolbox to solve the LMIs (3.2) and (3.3) in Theorem 2, we obtained the feasible solutions for an optimal minimum value of $\bar{c}_2 = \begin{pmatrix} 12.4431 \\ 15.0562 \end{pmatrix}$.

Table 1. The maximum authorized limits of $\bar{\tau}$ for different values $\mu = 0.8$ and $\mu = 0.9$ in Example 2.

μ	0.8	0.9
[32]	3.9212	2.3901
[33]	5.2403	3.5211
[34]	5.6384	3.7718
[35]	6.5411	4.5074
$\bar{\tau}$ of our result	6.9307	5.2113

From Table 1, The results presented in this manuscript are significantly better than those of [32–35], which confirms the validity of our work.

The trajectories of the solution of model (4.2) in Example 4 are given in Figure 4 and Figure 5. The time history of $(Re(z))^T L(Re(z))$ and $(Im(z))^T L(Im(z))$ are given in Figure 6. Hence, it can be concluded that the system (4.2) is FTP.

5. Conclusions

This paper is focused on the FTP problem of neutral-type complex-valued NNs in the presence of time-varying delays. By using Lyapunov functionals, the Wirtinger inequality, and LMIs, new sufficient conditions are gotten to guarantee the finite-time boundedness and finite-time passivity of our model. Finally, two examples are presented to prove the effectiveness of our main results. In the future work, we will take the challenge to study the finite-time dissipativity of stochastic complex NNs with mixed delays via a non-separation approach and the fixed-time passivity of coupled clifford-valued NNs subject to multiple delayed couplings.
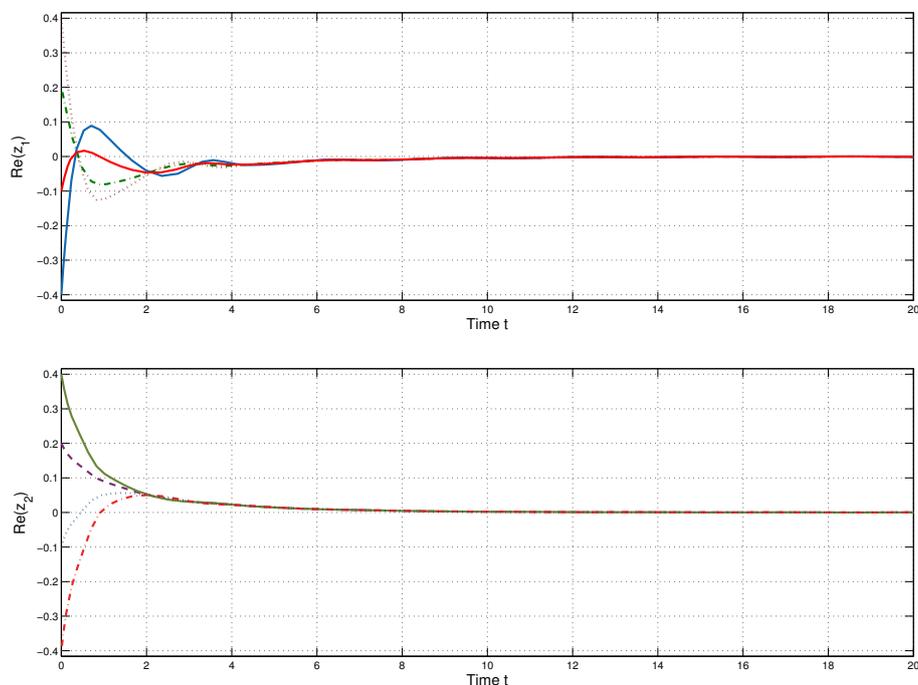
Figure 4. The trajectories of the real parts of the solution of model (4.2) in Example 2 with 4 initial conditions.
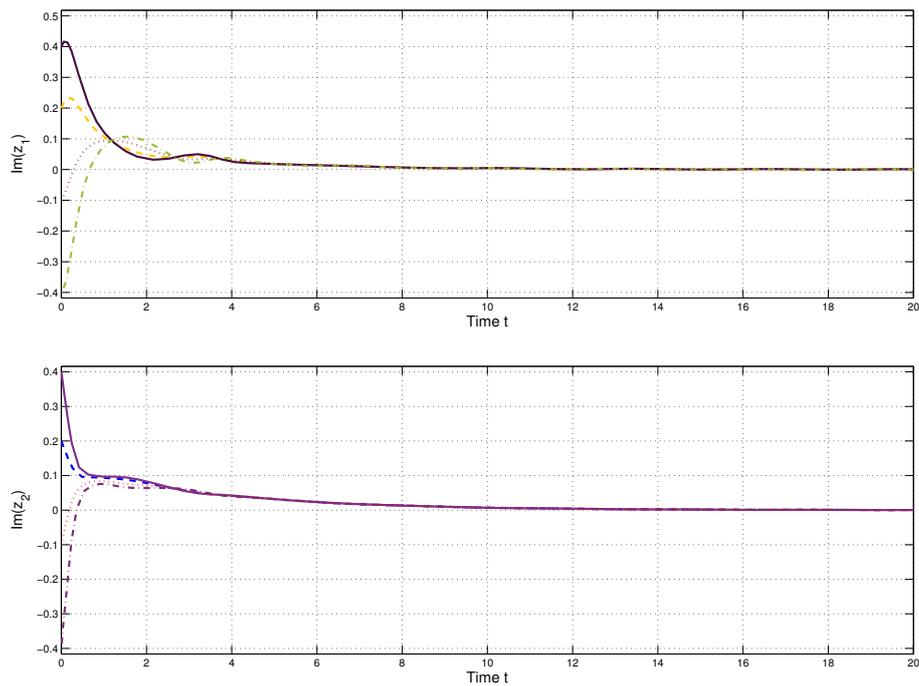
Figure 5. The trajectories of the imaginary parts of the solution of model (4.2) in Example 2 with 4 initial conditions.
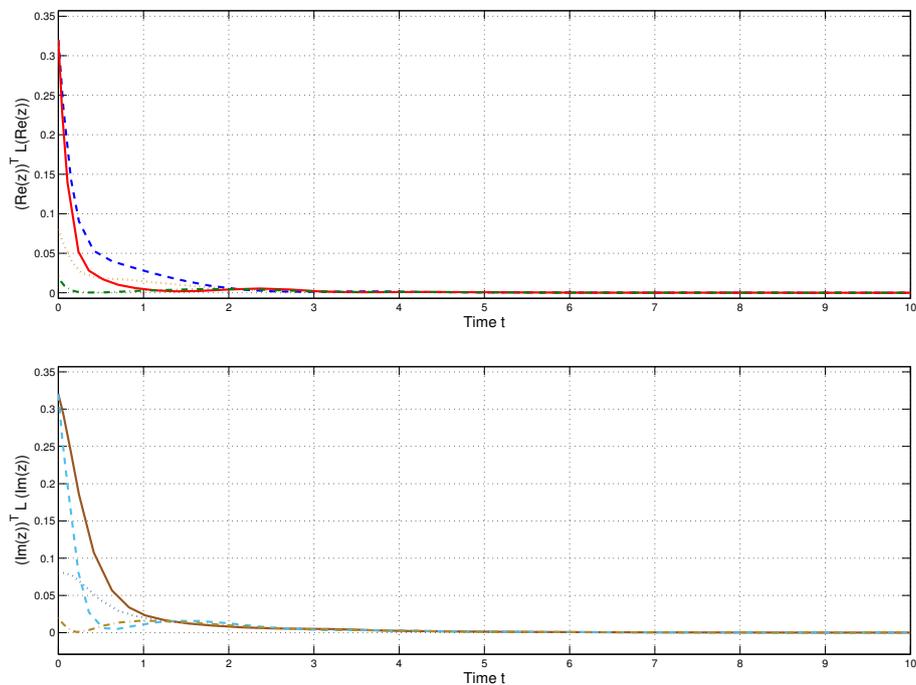
Figure 6. Time history of $(Re(z))^T L(Re(z))$ and $(Im(z))^T L(Im(z))$ of system (4.2) in Example 4 with 4 initial conditions.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare there is no conflict of interest.

References

1. C. Aouiti, R. Sakthivel, F. Touati, Global dissipativity of fuzzy bidirectional associative memory neural networks with proportional delays, *Iran. J. Fuzzy Syst.*, **18** (2021), 65–80. https://doi.org/10.22111/ijfs.2021.5914

2. F. Wu, T. Kang, Y. Shao, Q. Wang, Stability of Hopfield neural network with resistive and magnetic coupling, *Chaos Solitons Fractals*, **172** (2023), 113569. https://doi.org/10.1016/j.chaos.2023.113569

3. B. B. He, H. C. Zhou, Asymptotic stability and synchronization of fractional order Hopfield neural networks with unbounded delay, *Math. Methods Appl. Sci.*, **46** (2023), 3157–3175. https://doi.org/10.1002/mma.8000

4. R. P. Agarwal, S. Hristova, Stability of delay Hopfield neural networks with generalized proportional Riemann-Liouville fractional derivative, *AIMS Math.*, **8** (2023), 26801–26820. https://doi.org/10.3934/math.20231372

5. P. Li, R. Gao, C. Xu, J. Shen, S. Ahmad, Y. Li, Exploring the impact of delay on Hopf bifurcation of a type of BAM neural network models concerning three nonidentical delays, *Neural Process. Lett.*, **55** (2023), 1595–11635. https://doi.org/10.1007/s11063-023-11392-0

6. B. Zhou, Q. Song, Boundedness and complete stability of complex-valued neural networks with time delay, *IEEE Trans. Neural Networks Learn. Syst.*, **24** (2013), 1227–1238. https://doi.org/10.1109/TNNLS.2013.2247626

7. Z. Zhang, X. Liu, J. Chen, R. Guo, S. Zhou, Further stability analysis for delayed complex-valued recurrent neural networks, *Neurocomputing*, **251** (2017), 81–89. https://doi.org/10.1016/j.neucom.2017.04.013

8. M. S. Ali, G. Narayanan, Z. Orman, V. Shekher, S. Arik, Finite time stability analysis of fractional-order complex-valued memristive neural networks with proportional delays, *Neural Process. Lett.*, **51** (2020), 407–426. https://doi.org/10.1007/s11063-019-10097-7

9. Z. Zhang, J. Cao, Periodic solutions for complex-valued neural networks of neutral type by combining graph theory with coincidence degree theory, *Adv. Differ. Equations*, **2018** (2018), 1–23. https://doi.org/10.1186/s13662-018-1716-6

10. Y. Shi, X. Chen, P. Zhu, Dissipativity for a class of quaternion-valued memristor-based neutral-type neural networks with time-varying delays, *Math. Methods Appl. Sci.*, **46** (2023), 18166–18184. https://doi.org/10.1002/mma.9551

11. N. Li, W. X. Zheng, Passivity analysis for quaternion-valued memristor-based neural networks with time-varying delay, *IEEE Trans. Neural Networks Learn. Syst.*, **31** (2019), 639–650. https://doi.org/10.1109/TNNLS.2019.2908755

12. C. Ge, J. H. Park, C. Hua, C. Shi, Robust passivity analysis for uncertain neural networks with discrete and distributed time-varying delays, *Neurocomputing*, **364** (2019), 330–337. https://doi.org/10.1016/j.neucom.2019.06.077

13. S. Chandran, R. Ramachandran, J. Cao, R. P. Agarwal, G. Rajchakit, Passivity analysis for uncertain BAM neural networks with leakage, discrete and distributed delays using novel summation inequality, *Int. J. Control Autom. Syst.*, **17** (2019), 2114–2124. https://doi.org/10.1007/s12555-018-0513-z

14. M. V. Thuan, D. C. Huong, D. T. Hong, New results on robust finite-time passivity for fractional-order neural networks with uncertainties, *Neural Process. Lett.*, **50** (2019), 1065–1078. https://doi.org/10.1007/s11063-018-9902-9

15. J. Han, Finite-time passivity and synchronization for a class of fuzzy inertial complex-valued neural networks with time-varying delays, *Axioms*, **13** (2024), 39. https://doi.org/10.3390/axioms13010039 https://doi.org/10.3390/axioms13010039

16. A. Chaouki, F. Touati, Global dissipativity of Clifford-valued multidirectional associative memory neural networks with mixed delays, *Comput. Appl. Math.*, **39** (2020), 1–21. https://doi.org/10.1007/s40314-020-01367-5

17. S. Guo, B. Du, Global exponential stability of periodic solution for neutral-type complex-valued neural networks, *Discrete Dyn. Nat. Soc.*, **2016** (2016). https://doi.org/10.1155/2016/1267954

18. Y. P. Liu, L. H. Zhao, Y. W. Shi, S. Y. Ren, J. L. Wang, Finite-time passivity and synchronisation of complex networks with multiple output couplings, *Int. J. Control*, **96** (2023), 1470–1490. https://doi.org/10.1080/00207179.2022.2053208

19. R. Wei, J. Cao, F. E. Alsaadi, Fixed-time passivity of coupled quaternion-valued neural networks with multiple delayed couplings, *Soft Comput.*, **27** (2023), 8959–8970. https://doi.org/10.1007/s00500-022-07500-2

20. X. Chen, Q. Song, Global stability of complex-valued neural networks with both leakage time delay and discrete time delay on time scales, *Neurocomputing*, **121** (2013), 254–264. https://doi.org/10.1016/j.neucom.2013.04.040

21. Z. Wang, X. Liu, Exponential stability of impulsive complex-valued neural networks with time delay, *Math. Comput. Simul.*, **156** (2019), 143–157. https://doi.org/10.1016/j.matcom.2018.07.006

22. M. Chinnamuniyandi, S. Chandran, C. Xu, Fractional order uncertain BAM neural networks with mixed time delays: An existence and Quasi-uniform stability analysis, *J. Intell. Fuzzy Syst.*, **46** (2024), 4291–4313. https://doi.org/10.3233/JIFS-234744 https://doi.org/10.3233/JIFS-234744

23. G. Velmurugan, R. Rakkiyappan, S. Lakshmanan, Passivity analysis of memristor-based complex-valued neural networks with time-varying delays, *Neural Process. Lett.*, **42** (2015), 517–540. https://doi.org/10.1007/s11063-014-9371-8

24. Z. Zhang, S. Yu, Global asymptotic stability for a class of complex-valued Cohen-Grossberg neural networks with time delays, *Neurocomputing*, **171** (2016), 1158–1166. https://doi.org/10.1016/j.neucom.2015.07.051

25. H. Wang, S. Duan, T. Huang, L. Wang, C. Li, Exponential stability of complex-valued memristive recurrent neural networks, *IEEE Trans. Neural Networks Learn. Syst.*, **28** (2016), 766–771. https://doi.org/10.1109/TNNLS.2015.2513001

26. Z. Zhang, C. Lin, B. Chen, Global stability criterion for delayed complex-valued recurrent neural networks, *IEEE Trans. Neural Networks Learn. Syst.*, **25** (2013), 1704–1708. https://doi.org/10.1109/TNNLS.2013.2288943

27. J. Pan, X. Liu, W. Xie, Exponential stability of a class of complex-valued neural networks with time-varying delays, *Neurocomputing*, **164** (2015), 293–299. https://doi.org/10.1016/j.neucom.2015.02.024

28. W. Qian, S. Cong, T. Li, S. Fei, Improved stability conditions for systems with interval time-varying delay, *Int. J. Control Autom. Syst.*, **10** (2012), 1146–1152. https://doi.org/10.1007/s12555-012-0609-9

29. J. Hu, J. Wang, Global stability of complex-valued recurrent neural networks with time-delays, *IEEE Trans. Neural Networks Learn. Syst.*, **23** (2012), 853–865. https://doi.org/10.1109/TNNLS.2012.2195028

30. T. Dong, X. Liao, A. Wang, Stability and Hopf bifurcation of a complex-valued neural network with two time delays, *Nonlinear Dyn.*, **82** (2015), 173–184. https://doi.org/10.1007/s11071-015-2147-5

31. X. Xu, J. Zhang, J. Shi, Dynamical behaviour analysis of delayed complex-valued neural networks with impulsive effect, *Int. J. Syst. Sci.*, **48** (2017), 686–694. https://doi.org/10.1080/00207721.2016.1206988

32. R. Samidurai, S. Rajavel, J. Cao, A. Alsaedi, F. Alsaadi, B. Ahmad, Delay-partitioning approach to stability analysis of state estimation for neutral-type neural networks with both time-varying delays and leakage term via sampled-data control, *Int. J. Syst. Sci.*, **48** (2017), 1752–1765. https://doi.org/10.1080/00207721.2017.1282060

33. M. S. Ali, S. Saravanan, Q. Zhu, Finite-time stability of neutral-type neural networks with random time-varying delays, *Int. J. Syst. Sci.*, **48** (2017), 3279–3295. https://doi.org/10.1080/00207721.2017.1367434

34. C. Hua, Y. Wang, S. Wu, Stability analysis of neural networks with time-varying delay using a new augmented Lyapunov—Krasovskii functional, *Neurocomputing*, **332** (2019), 1–9. https://doi.org/10.1016/j.neucom.2018.08.044

35. I. Khonchaiyaphum, N. Samorn, T. Botmart, K. Mukdasai, Finite-time passivity analysis of neutral-type neural networks with mixed time-varying delays, *Mathematics*, **9** (2021), 3321. https://doi.org/10.3390/math9243321

36. J. Xiao, Z. Zeng, Finite-time passivity of neural networks with time varying delay, *J. Franklin Inst.*, **357** (2020), 2437–2456. https://doi.org/10.1016/j.jfranklin.2020.01.023

37. S. Saravanan, M. S. Ali, A. Alsaedi, B. Ahmad, Finite-time passivity for neutral-type neural networks with time-varying delays—via auxiliary function-based integral inequalities, *Nonlinear Anal. Modell. Control*, **25** (2020), 206–224. https://doi.org/10.15388/namc.2020.25.16513

38. A. Seuret, F. Gouaisbaut, On the use of the Wirtinger inequalities for time-delay systems, *IFAC Proc. Vol.*, **45** (2012), 260–265. https://doi.org/10.3182/20120622-3-US-4021.00035