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Research article

The dynamics and control of a multi-lingual rumor propagation model with non-smooth inhibition mechanism

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Abstract: In this paper, the dynamic behaviors and control strategies of a rumor propagation model are studied in multi-lingual environment. First, an S2E2I2R rumor propagation model is proposed, which incorporates a non-smooth inhibition mechanism. Meanwhile, the existence and stability of the equilibrium are analyzed, grounded in the spreader threshold of the government intervention. Finally, the optimal control and the event-triggered impulsive control strategies are proposed to mitigate the spread of rumors, and the comparison of their effectiveness is further presented by the numerical simulation and a practical case.

Keywords: rumor propagation; multi-lingual environment; non-smooth inhibition mechanism; optimal control; event-triggered impulsive control

1. Introduction

With the rapid development of science and technology, the arrival of the era of self-media has been promoted, and a communication pattern in which everyone has a microphone in today's society has been formed. While the public has gained an unprecedented right to speak and spread, the Internet has become a hotbed for the breeding and dissemination of rumors. In particular, the vigorous development of social media has made it possible for everyone to freely publish, disseminate, and obtain information, which has facilitated the spread of rumors and resulted extremely bad social impacts. For example, during the outbreak of COVID-19, rumors such as prevention methods and epidemic dynamics emerged in an endless stream, which has caused many hazards to public physical and mental health and social stability. Therefore, exploring the rules and characteristics of rumor propagation plays an important role in formulating rumor control strategies.

Since the rumor propagation process is similar to the spread of infectious diseases in the population, most infectious disease theories can be used to study the spread of rumors [1–3]. The study of rumor propagation models began in the 1860s. Daley and Kendal [4] proposed the DK model considering the difference between rumors and infectious disease transmission. Based on the DK model, Maki and Thompson [5] established the MT model by changing the propagation mechanism. Later, many rumor propagation models were derived. Among them, SI [6], SIS [7], SIR [1,2], and SEIR [8] are the classical models for studying the dynamics of rumor propagation. However, with the in-depth study of rumor propagation, many scholars believe that the establishment of a rumor propagation model on complex networks can intuitively describe the propagation process of rumors [9–11]. Subsequently, the psychological factors [12], behavioral performance [13], and reaction time [14, 15] of the crowd were taken into account in the rumor propagation model.

However, the above research work mainly considers the spread of rumors in a single language environment. In fact, the actual spread process of rumors is more complicated, and there are cases of spreading through multiple languages. Therefore, Wang et al. [16] considered the SIR rumor propagation model with a cross-propagation mechanism in the bilingual environment, and analyzed its dynamic behaviors. Furthermore, Li et al. [17] established and analyzed the I2S2R rumor propagation model in complex networks. At present, the rumor propagation models established in multi-lingual environment are not comprehensive enough. In this paper, the exposed (E) will be introduced into the multi-lingual environment to establish the S2E2I2R model.

In addition, to effectively control the spread of rumors, more and more scholars have studied the impact of external control measures on the spread of rumors, such as, trust mechanism [18], feedback mechanism [19], and education mechanism [17]. Most of the above models considered control measures with linear or smooth functions, but there were few studies on rumor propagation with nonlinear non-smooth inhibition mechanism [20]. Before the rumor is widely spread on the Internet, the government and media will not intervene, so there is a spreader threshold to trigger the intervention mechanism, which is more in line with the actual situation. Therefore, we will introduce a non-smooth inhibition mechanism to analyze the process of multi-lingual rumor propagation.

In the process of rumor control, the implementation of control strategies requires a certain cost. In order to control the number of rumor spreaders within a certain range with less cost, many scholars use optimal control methods to study rumor spread control strategies [21,22]. In practical applications, it is very difficult to implement real-time optimal control of the rumor propagation process. As a powerful discontinuous control method, impulsive control only needs to control the system at some discrete sampling points, thus greatly reducing the control cost [23,24]. Therefore, it is of great practical value for us to study the event-triggered impulsive control strategy.

Motivated by the above analysis, this paper proposes an S2E2I2R model with non-smooth inhibition mechanism in multi-lingual environment. The main works of this paper include the following aspects:

1) A rumor propagation model with non-smooth inhibition mechanism is established, and the jump discontinuity points divide the model into two stages, which are consistent with reality.

2) Taking the spreader threshold I_c as the boundary, the existence and stability of the rumor-free equilibrium and rumor-spreading equilibrium are analyzed here.

3) Two different control strategies, optimal control and event-triggered impulsive control, are designed, and the advantages of these control strategies are fully illustrated by comparing different control

strategies in the numerical simulation.

The rest of the paper is organized as follows: In Section 2, the model formulation is presented. In terms of the spreader threshold I_c, the stability of an S2E2I2R rumor propagation model is explored in Section 3. In Section 4, both the optimal control and event-triggered impulsive control are designed to restrain the rumor's diffusion. In Section 5, numerical simulation is further addressed to display their validity. Finally, conclusions are given in Section 6.

2. Model formulation

In fact, network users can come from different regions and have diverse languages. Once a rumor is released, it can quickly spread via various languages. Besides, government and other institutions can control rumors, but when the number of rumor spreaders is below a certain threshold, it might not be essential to implement control measures. Hence, a non-smooth inhibition mechanism is proposed to establish the multi-lingual rumor propagation model (see Figure 1).

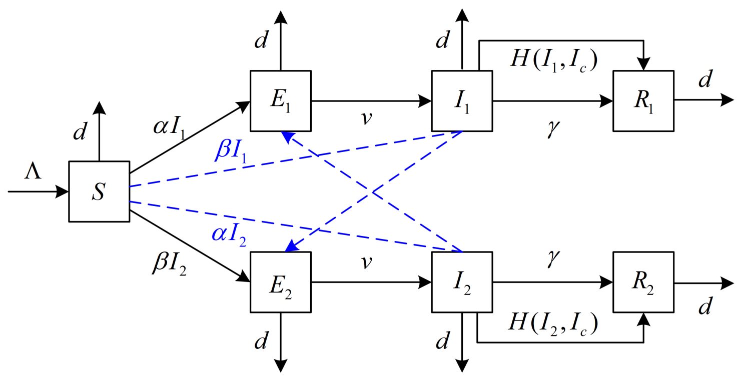


Figure 1. Flow chart of rumor propagation dynamics.

Table 1. Definitions of the relevant parameters.

Parameter	Definition
$\langle k \rangle$	Average degree of homogeneous network
Λ	The immigration rate of $S(t)$ per unit time
d	The removal rate of users per unit time
α	The probability of $S(t)$ contact $I_i(t)$ and become $E_1(t)$
β	The probability of $S(t)$ contact $I_i(t)$ and become $E_2(t)$
v	The transformation rate from $E_i(t)$ to $I_i(t)$
γ	The transformation rate from $I_i(t)$ to $R_i(t)$
I_c	The spreader threshold of government and media intervention
δ	The control rate of spreaders under government and media intervention
ξ	The effectiveness of inhibition mechanism

Assume that there are two languages in the networks, and the network users $N(t)$ can be divided into seven categories: ignorants $S(t)$ (users who do not know rumors but are easily infected), exposed $E_i(t)$ (users who have been infected, in hesitation state and do not spread rumor), spreaders $I_i(t)$ (users who master two languages, and use language i to spread rumors), and stiflers $R_i(t)$ (users who identify rumors and do not spread rumors), $i = 1, 2$.

Remark 1. When rumors reach a critical level of seriousness, government and official media intervention become necessary [18, 20, 25]. Therefore, the phased control measures align better with the actual situation. Therefore, a non-smooth inhibition function $H(I_i, I_c)$ ($i = 1, 2$) for multi-lingual system is proposed below:

$$H(I_i, I_c) = \begin{cases} 0, & 0 \le I_i \le I_c, \\ \dfrac{\delta(I_i - I_c)}{1 + \xi(I_1 + I_2)}, & I_i > I_c. \end{cases} \tag{2.1}$$

Based on the above analysis and Figure 1, the S2E2I2R rumor propagation model can be expressed as

$$\begin{cases} \dfrac{\mathrm{d}S(t)}{\mathrm{d}t} = \Lambda - (\alpha + \beta) \langle k \rangle S(t)(I_1(t) + I_2(t)) - dS(t), \\ \dfrac{\mathrm{d}E_1(t)}{\mathrm{d}t} = \alpha \langle k \rangle S(t)(I_1(t) + I_2(t)) - (d + v)E_1(t), \\ \dfrac{\mathrm{d}E_2(t)}{\mathrm{d}t} = \beta \langle k \rangle S(t)(I_1(t) + I_2(t)) - (d + v)E_2(t), \\ \dfrac{\mathrm{d}I_1(t)}{\mathrm{d}t} = vE_1(t) - (d + \gamma)I_1(t) - H(I_1, I_c), \\ \dfrac{\mathrm{d}I_2(t)}{\mathrm{d}t} = vE_2(t) - (d + \gamma)I_2(t) - H(I_2, I_c), \\ \dfrac{\mathrm{d}R_1(t)}{\mathrm{d}t} = H(I_1, I_c) + \gamma I_1(t) - dR_1(t), \\ \dfrac{\mathrm{d}R_2(t)}{\mathrm{d}t} = H(I_2, I_c) + \gamma I_2(t) - dR_2(t), \end{cases} \tag{2.2}$$

where parameters are shown in Table 1, and the initial condition is

$$S(0) > 0, E_i(0) > 0, I_i(0) > 0, R_i(0) > 0, i = 1, 2. \tag{2.3}$$

In order to make the model have realistic significance, we first give the nonnegativity and boundedness of all solutions of system (2.2) before analyzing the stability.

Lemma 1. *Under the initial conditions (2.3), the solutions of system (2.2) are positive for $t > 0$, and a positively invariant set of system (2.2) is*

$$\Omega = \left\{ (S(t), E_i(t), I_i(t), R_i(t)) \in \mathbb{R}_+^7 | S(t) + \sum_{i=1}^{2} [E_i(t) + I_i(t) + R_i(t)] \le \frac{\Lambda}{d} \right\}, i = 1, 2.$$

Proof. Let

$$q(t) = \min_t \{S(t), E_i(t), I_i(t), R_i(t), i = 1, 2\}.$$

To prove that the solution of system (2.2) is positive, we only need to prove $q(t) > 0$ for any $t > 0$. By using reduction to absurdity, assume that there exists $t_1 > 0$ such that $q(t_1) = 0$, $q(t) > 0$ for $t \in [0, t_1)$. Subsequently, we need to discuss seven cases. First, if $q(t_1) = S(t_1)$, then $S(t_1) = 0$, $E_i(t_1) > 0, I_i(t_1) > 0, R_i(t_1) > 0$, and $q(t) > 0$ for $t \in [0, t_1)$. According to system (2.2), it yields $\dot{S}(t_1) = \Lambda > 0$, which contradicts $S(t) > 0 = S(t_1)$ for $t \in [0, t_1)$. Moreover, if $q(t_1) = R_1(t_1)$, then

$$\dot{R}_1(t_1) = \begin{cases} \gamma I_1(t_1) > 0, & 0 \leq I_1(t_1) \leq I_c, \\ H(I_1(t_1), I_c) + \gamma I_1(t_1) > 0, & I_1(t_1) \geq I_c, \end{cases}$$

which is a contradiction to the fact that $R_1(t) > 0 = R_1(t_1), t \in [0, t_1)$. Further, the other cases can be discussed similarly. Hence, we have $S(t) > 0, E_i(t) > 0, I_i(t) > 0$, and $R_i(t) > 0$ for any $t > 0$.

Next, let

$$N(t) = S(t) + E_1(t) + E_2(t) + I_1(t) + I_2(t) + R_1(t) + R_2(t).$$

According to system (2.2), we have

$$\frac{\mathrm{d}N(t)}{\mathrm{d}t} = \Lambda - dN(t),$$

and it follows that

$$\lim_{t\to+\infty} \sup N(t) \leq \frac{\Lambda}{d}.$$

Therefore, the positive invariant set of system (2.2) can be obtained.

Since the first five equations of the model are independent of $R_i(t)$, we only consider the following system in the later discussion.

$$\begin{cases} \dfrac{\mathrm{d}S(t)}{\mathrm{d}t} = \Lambda - (\alpha + \beta) \langle k \rangle S(t)(I_1(t) + I_2(t)) - dS(t), \\ \dfrac{\mathrm{d}E_1(t)}{\mathrm{d}t} = \alpha \langle k \rangle S(t)(I_1(t) + I_2(t)) - (d + v)E_1(t), \\ \dfrac{\mathrm{d}E_2(t)}{\mathrm{d}t} = \beta \langle k \rangle S(t)(I_1(t) + I_2(t)) - (d + v)E_2(t), \\ \dfrac{\mathrm{d}I_1(t)}{\mathrm{d}t} = vE_1(t) - (d + \gamma)I_1(t) - H(I_1, I_c), \\ \dfrac{\mathrm{d}I_2(t)}{\mathrm{d}t} = vE_2(t) - (d + \gamma)I_2(t) - H(I_2, I_c). \end{cases} \tag{2.4}$$

3. Stability analysis with the spreader threshold I_c

3.1. Stability analysis of rumor-free equilibrium

In this section, the next generation matrix method is used to calculate the basic reproduction number R_0 of the system, and the stability of the rumor-free equilibrium is discussed.

Let $\mathcal{X} = (E_1(t), E_2(t), I_1(t), I_2(t), S(t))^T$, then system (2.4) can be written as

$$\frac{\mathrm{d}\mathcal{X}}{\mathrm{d}t} = \mathcal{F}(\mathcal{X}) - \mathcal{V}(\mathcal{X}),$$

where

$$\mathcal{F}(\mathcal{X}) = \begin{pmatrix} \alpha \langle k \rangle S(t)(I_1(t) + I_2(t)) \\ \beta \langle k \rangle S(t)(I_1(t) + I_2(t)) \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$\mathcal{V}(\mathcal{X}) = \begin{pmatrix} (d+v)E_1(t) \\ (d+v)E_2(t) \\ -vE_1(t) + (d+r)I_1(t) + H(I_1, I_c) \\ -vE_2(t) + (d+r)I_2(t) + H(I_2, I_c) \\ -\Lambda + (\alpha+\beta)\langle k \rangle S(t)(I_1(t) + I_2(t)) + dS(t) \end{pmatrix}.$$

Since the spreader threshold I_c divides the system into two stages, we discuss the existence and stability of the equilibrium in the case of $0 \leq I_i \leq I_c$ and $I_i > I_c$, respectively.

Case 1. $0 \leq I_i \leq I_c$

When $0 \leq I_i \leq I_c$, we have $H(I_i, I_c) = 0$. Obviously, for system (2.4) there exists a rumor-free equilibrium $E_0 = (0, 0, 0, 0, \frac{\Lambda}{d})$. The Jacobian matrices of $\mathcal{F}(\mathcal{X})$ and $\mathcal{V}(\mathcal{X})$ at E_0 are

$$D\mathcal{F}(E_0) = \begin{pmatrix} F & 0 \\ 0 & 0 \end{pmatrix}, \quad D\mathcal{V}(E_0) = \begin{pmatrix} V & 0 \\ J_1 & J_2 \end{pmatrix},$$

where

$$F = \begin{pmatrix} 0 & 0 & \frac{\alpha\langle k\rangle\Lambda}{d} & \frac{\alpha\langle k\rangle\Lambda}{d} \\ 0 & 0 & \frac{\beta\langle k\rangle\Lambda}{d} & \frac{\beta\langle k\rangle\Lambda}{d} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} d+v & 0 & 0 & 0 \\ 0 & d+v & 0 & 0 \\ -v & 0 & d+r & 0 \\ 0 & -v & 0 & d+r \end{pmatrix},$$

and $J_1 = \begin{pmatrix} 0 & 0 & \frac{(\alpha+\beta)\langle k\rangle\Lambda}{d} & \frac{(\alpha+\beta)\langle k\rangle\Lambda}{d} \end{pmatrix}$, $J_2 = d$. According to the next generation matrix method, it follows that

$$R_0 = \rho(FV^{-1}) = \frac{(\alpha+\beta)\langle k \rangle \Lambda v}{d(d+\gamma)(d+v)}.$$

Theorem 1. *If $R_0 < 1$, the rumor-free equilibrium E_0 of system (2.4) is locally asymptotically stable for $0 \leq I_i \leq I_c$.*

Proof. The Jacobian matrix of system (2.4) at the rumor-free equilibrium E_0 is

$$J(E_0) = \begin{pmatrix} -(d+v) & 0 & \frac{\alpha\langle k\rangle\Lambda}{d} & \frac{\alpha\langle k\rangle\Lambda}{d} & 0 \\ 0 & -(d+v) & \frac{\beta\langle k\rangle\Lambda}{d} & \frac{\beta\langle k\rangle\Lambda}{d} & 0 \\ v & 0 & -(d+\gamma) & 0 & 0 \\ 0 & v & 0 & -(d+\gamma) & 0 \\ 0 & 0 & -\frac{(\alpha+\beta)\langle k\rangle\Lambda}{d} & -\frac{(\alpha+\beta)\langle k\rangle\Lambda}{d} & -d \end{pmatrix}.$$

By calculation, the characteristic equation is

$$(\lambda + d)(\lambda + d + \gamma)(\lambda + d + v)\left[\lambda^2 + (2d + \gamma + v)\lambda + (d+\gamma)(d+v)(1 - R_0)\right] = 0.$$

Hence, the characteristic roots are given as $\lambda_1 = -d < 0, \lambda_2 = -(d+\gamma) < 0$, and $\lambda_3 = -(d+v) < 0$, and the rest of the characteristic roots are determined by

$$\lambda^2 + (2d+\gamma+v)\lambda + (d+\gamma)(d+v)(1-R_0) = 0. \tag{3.1}$$

According to the Routh-Hurwitz criterion, if $R_0 < 1$, all roots of Eq (3.1) have negative real parts. Therefore, the rumor-free equilibrium E_0 of system (2.4) is locally asymptotically stable.

Theorem 2. *If $R_0 < 1$, the rumor-free equilibrium E_0 of system (2.4) is globally asymptotically stable for $0 \le I_i \le I_c$.*

Proof. Consider the following Lyapunov function

$$V(t) = E_1(t) + E_2(t) + \frac{(\alpha+\beta)\langle k\rangle \Lambda}{d(d+\gamma)}(I_1(t) + I_2(t)).$$

By calculating the derivative of $V(t)$ along system (2.4), we can obtain

$$\begin{aligned}
\frac{\mathrm{d}V(t)}{\mathrm{d}t} &= (\alpha+\beta)\langle k\rangle S(t)(I_1(t)+I_2(t)) - (d+v)(E_1(t)+E_2(t)) \\
&\quad + \frac{(\alpha+\beta)\langle k\rangle \Lambda}{d(d+\gamma)}[v(E_1(t)+E_2(t)) - (d+\gamma)(I_1(t)+I_2(t))] \\
&\le (\alpha+\beta)\langle k\rangle \frac{\Lambda}{d}(I_1(t)+I_2(t)) + (R_0-1)(d+v)(E_1(t)+E_2(t)) \\
&\quad - \frac{(\alpha+\beta)\langle k\rangle \Lambda}{d}(I_1(t)+I_2(t)) \\
&= (R_0-1)(d+v)(E_1(t)+E_2(t)).
\end{aligned}$$

If $R_0 < 1$, we have $\frac{\mathrm{d}V(t)}{\mathrm{d}t} \le 0$. In addition, $\frac{\mathrm{d}V(t)}{\mathrm{d}t} = 0$ if and only if $E_1(t) = E_2(t) = 0$.

According to LaSalle's invariance principle, the rumor-free equilibrium E_0 of system (2.4) is globally asymptotically stable.

Remark 2. Note that if $0 \le I_i \le I_c$, the rumor-free equilibrium E_0 of system (2.4) is locally and globally asymptotically stable with $R_0 < 1$ from the above theorem. Meanwhile, when $I_i > I_c$, the system has no rumor-free equilibrium E_0 since the density of spreaders is always greater than the spread threshold I_c. Hence, government and media intervention can adjust I_c for dissemination via the information credibility, thereby reducing the impact and speed of rumor propagation.

3.2. Stability analysis of rumor-spreading equilibrium

The existence and stability of the rumor-spreading equilibrium $E^* = (E_1^*, E_2^*, I_1^*, I_2^*, S^*)$ in system (2.4) are further discussed under the spreader threshold I_c below.

Case 1. $0 \le I_i \le I_c$

Theorem 3. *If $R_0 > 1$, system (2.4) has a unique rumor-spreading equilibrium E^* for $0 \le I_i \le I_c$.*

Proof. Let $\eta = I_1^* + I_2^* \geq 0$ and substitute it into system (2.4). We have

$$S^* = \frac{\Lambda}{(\alpha+\beta)\langle k\rangle \eta + d},\ E_1^* = \frac{\alpha\langle k\rangle \eta S^*}{d+v},\ E_2^* = \frac{\beta\langle k\rangle \eta S^*}{d+v}.$$

From further calculation, one can obtain

$$I_1^* + I_2^* = \frac{v(E_1^*+E_2^*)}{d+\gamma} = \frac{(\alpha+\beta)\langle k\rangle \Lambda v\eta}{(d+\gamma)(d+v)[(\alpha+\beta)\langle k\rangle \eta + d]} = \eta.$$

Construct the following function

$$f(\eta) = 1 - \frac{(\alpha+\beta)\langle k\rangle \Lambda v}{(d+\gamma)(d+v)[(\alpha+\beta)\langle k\rangle \eta + d]}.$$

Since $\dot{f}(\eta) > 0$ and $\lim\limits_{\eta\to+\infty} f(\eta) = 1$, it yields that $f(\eta) = 0$ has a unique solution if and only if $f(0) = 1 - R_0 < 0$. Hence, when $0 \leq I_i \leq I_c$, system (2.4) has a unique rumor-spreading equilibrium E^* with $R_0 < 1$.

Theorem 4. *Suppose* $P = \frac{(\alpha+\beta)\langle k\rangle(I_1^*+I_2^*)+d}{d}$, *if* $1 < R_0 < P$, *then the rumor-spreading equilibrium* E^* *of system (2.4) is locally asymptotically stable for* $0 \leq I_i \leq I_c$.

Proof. The Jacobian matrix of the linearized system of system (2.4) at the rumor-spreading equilibrium E^* is

$$J(E^*) = \begin{pmatrix} -(d+v) & 0 & \alpha\langle k\rangle S^* & \alpha\langle k\rangle S^* & \alpha\langle k\rangle \eta \\ 0 & -(d+v) & \beta\langle k\rangle S^* & \beta\langle k\rangle S^* & \beta\langle k\rangle \eta \\ v & 0 & -(d+\gamma) & 0 & 0 \\ 0 & v & 0 & -(d+\gamma) & 0 \\ 0 & 0 & -(\alpha+\beta)\langle k\rangle S^* & -(\alpha+\beta)\langle k\rangle S^* & -(\alpha+\beta)\langle k\rangle \eta - d \end{pmatrix}.$$

By calculation, the characteristic equation of $J(E^*)$ is

$$(\lambda + d + v)(\lambda + d + \gamma)[\lambda^3 + A_2(\eta)\lambda^2 + A_1(\eta)\lambda + A_0(\eta)] = 0,$$

where

$$\begin{aligned} A_2(\eta) &= 2d + v + \gamma + (\alpha+\beta)\langle k\rangle \eta + d > 0, \\ A_1(\eta) &= (d+v)(d+\gamma)[1 - R_0\tfrac{d}{(\alpha+\beta)\langle k\rangle\eta+d}] + (2d+v+\gamma)[(\alpha+\beta)\langle k\rangle \eta + d], \\ A_0(\eta) &= (d+v)(d+\gamma)[(\alpha+\beta)\langle k\rangle \eta + d)][1 - R_0\tfrac{d^2}{((\alpha+\beta)\langle k\rangle\eta+d)^2}]. \end{aligned}$$

Obviously, it yields $\lambda_1 = -(d+v) < 0, \lambda_2 = -(d+\gamma) < 0$, and the rest of the characteristic roots are determined by

$$\lambda^3 + A_2(\eta)\lambda^2 + A_1(\eta)\lambda + A_0(\eta) = 0. \tag{3.2}$$

In fact, if $1 < R_0 < \frac{(\alpha+\beta)\langle k\rangle(I_1^*+I_2^*)+d}{d}$, it follows that $A_1(\eta) > 0, A_0(\eta) > 0$. Ulteriorly, one has

$$
\begin{aligned}
& A_2(\eta)A_1(\eta) - A_0(\eta) \\
= \ & (2d+v+\gamma)[(\alpha+\beta)\langle k\rangle\eta+d]^2 + \frac{(\alpha+\beta)\langle k\rangle\Lambda vd}{(\alpha+\beta)\langle k\rangle\eta+d} \\
& +(2d+v+\gamma)^2[(\alpha+\beta)\langle k\rangle\eta+d] - (\alpha+\beta)\langle k\rangle\Lambda v \\
& +(2d+v+\gamma)(d+v)(d+\gamma)[1-R_0\frac{d}{(\alpha+\beta)\langle k\rangle\eta+d}] \\
> \ & (2d+v+\gamma)[(\alpha+\beta)\langle k\rangle\eta+d]^2 + \frac{(\alpha+\beta)\langle k\rangle\Lambda vd}{(\alpha+\beta)\langle k\rangle\eta+d} \\
& +(d+v)(d+\gamma)[(\alpha+\beta)\langle k\rangle\eta+d][1-R_0\frac{d}{(\alpha+\beta)\langle k\rangle\eta+d}] \\
& +(2d+v+\gamma)(d+v)(d+\gamma)[1-R_0\frac{d}{(\alpha+\beta)\langle k\rangle\eta+d}] \\
> \ & 0.
\end{aligned}
$$

According to the Routh-Hurwitz criterion, all roots of Eq (3.2) have negative real parts. Therefore, the rumor-spreading equilibrium E^* of system (2.4) is locally asymptotically stable.

Case 2. $I_i > I_c$

Similar to the case of $0 \le I_i \le I_c$, the cubic equation for η by calculation is present as

$$g(\eta) = B_3\eta^3 + B_2\eta^2 + B_1\eta + B_0,$$

where

$$
\begin{aligned}
B_3 &= (d+v)(d+\gamma)(\alpha+\beta)\langle k\rangle\xi > 0, \\
B_2 &= (d+v)(\alpha+\beta)\langle k\rangle\delta + (d+v)(d+\gamma)(\alpha+\beta)\langle k\rangle + d(d+v)(d+\gamma)\xi \\
&\quad - (\alpha+\beta)\langle k\rangle\Lambda v\xi, \\
B_1 &= d(d+v)\delta + d(d+v)(d+\gamma) - 2(d+v)(\alpha+\beta)\langle k\rangle\delta I_c - (\alpha+\beta)\langle k\rangle\Lambda v, \\
B_0 &= -2d(d+v)\delta I_c < 0.
\end{aligned}
$$

For convenience, let $\Delta = 4B_2^2 - 12B_3B_1$, η_{11}, η_{12} $(\eta_{11} < \eta_{12})$ be two roots of $g'(\eta)$, and η_1–η_3 be three roots of $g(\eta)$, then we can acquire following conclusion:

(H_1) Suppose $\Delta > 0$, if $B_2 > 0, B_1 > 0$ (or $B_2 > 0, B_1 < 0$ or $B_2 < 0, B_1 < 0$) holds, then the system (2.4) has a unique positive equilibrium.

(H_2) Suppose $\Delta \le 0$, then the system (2.4) has a unique positive equilibrium.

Theorem 5. *Under condition* (H_1) *or* (H_2)*, let*

$$\Theta = \frac{(\alpha+\beta)\langle k\rangle\Lambda v}{d(d+v)(d+\gamma+\hat{H}_{12}^*+\hat{H}_{22}^*)} < P,$$

then the rumor-spreading equilibrium E^* *of system (2.4) is locally asymptotically stable for* $I_i > I_c$.

Proof. The Jacobian matrix of the linearized system of system (2.4) at the rumor-spreading equilibrium E^* is

$$J(E^*) = \begin{pmatrix} -(d+v) & 0 & \alpha\langle k\rangle S^* & \alpha\langle k\rangle S^* & \alpha\langle k\rangle\eta \\ 0 & -(d+v) & \beta\langle k\rangle S^* & \beta\langle k\rangle S^* & \beta\langle k\rangle\eta \\ v & 0 & -(d+\gamma)-\hat{H}^*_{11} & -\hat{H}^*_{12} & 0 \\ 0 & v & -\hat{H}^*_{21} & -(d+\gamma)-\hat{H}^*_{22} & 0 \\ 0 & 0 & -(\alpha+\beta)\langle k\rangle S^* & -(\alpha+\beta)\langle k\rangle S^* & -(\alpha+\beta)\langle k\rangle\eta - d \end{pmatrix},$$

where $\hat{H}^*_{ij}(i,j=1,2)$ is the value of $\frac{\partial H(I_i,I_c)}{\partial I_j}$ at E^*.

By calculation, the characteristic equation of $J(E^*)$ is

$$(\lambda+d+v)(\lambda+d+\gamma+\hat{H}^*_{11}-\hat{H}^*_{12})[\lambda^3+C_2(\eta)\lambda^2+C_1(\eta)\lambda+C_0(\eta)]=0, \tag{3.3}$$

where

$$C_2(\eta) = 2d+v+\gamma+\hat{H}^*_{12}+\hat{H}^*_{22}+(\alpha+\beta)\langle k\rangle\eta+d>0,$$

$$\begin{aligned} C_1(\eta) = {} & \frac{d(d+v)(d+\gamma+\hat{H}^*_{12}+\hat{H}^*_{22})}{(\alpha+\beta)\langle k\rangle\eta+d}\Big[\frac{(\alpha+\beta)\langle k\rangle\eta+d}{d}-\Theta\Big] \\ & +(2d+v+\gamma+\hat{H}^*_{12}+\hat{H}^*_{22})[(\alpha+\beta)\langle k\rangle\eta+d], \end{aligned}$$

$$C_0(\eta) = \frac{d^2(d+v)(d+\gamma+\hat{H}^*_{12}+\hat{H}^*_{22})}{(\alpha+\beta)\langle k\rangle\eta+d}\Big[\frac{[(\alpha+\beta)\langle k\rangle\eta+d]^2}{d^2}-\Theta\Big].$$

Similarly, note that if $\Theta < P$, then the eigenvalues of $J(E^*)$ have negative real parts . Therefore, the rumor-spreading equilibrium E^* of system (2.4) is locally asymptotically stable.

Remark 3. In fact, Theorem 5 only discusses the stability of system (2.4) when it has a unique positive equilibrium. However, when $\Delta > 0$, and $B_2 < 0, B_1 > 0, g(\eta_{11}) > 0, g(\eta_{12}) < 0$, the system (2.4) has three positive equilibrium points. Besides, when $\Delta > 0$, the system (2.4) has two positive equilibrium points with $B_2 < 0, B_1 > 0, g(\eta_{11}) = 0, g(\eta_{12}) < 0$ or $B_2 < 0, B_1 > 0, g(\eta_{11}) > 0, g(\eta_{12}) = 0$. Regarding the issue of multiple equilibria, we will conduct further research in future.

4. Control strategies for the system

4.1. Optimal control

In order to further curb the spread of rumors while simultaneously minimizing the associated control costs, an optimal control rooted in a non-smooth inhibition mechanism is proposed in this section. Now, let the control set $U = \{(u_1(t), u_2(t)) \mid 0 \le u_1(t) \le u_1^{\max}, 0 \le u_2(t) \le u_2^{\max}, t \in [0,T]\}$, where $u_1^{\max} \in (0,1]$

and $u_2^{\max} \in (0, 1]$ are the upper bounds of $u_1(t)$ and $u_1(t)$, and the optimal control system is

$$\begin{cases} \dfrac{\mathrm{d}S(t)}{\mathrm{d}t} = \Lambda - (\alpha + \beta) \langle k \rangle S(t)(I_1(t) + I_2(t)) - dS(t), \\ \dfrac{\mathrm{d}E_1(t)}{\mathrm{d}t} = \alpha \langle k \rangle S(t)(I_1(t) + I_2(t)) - (d + v)E_1(t), \\ \dfrac{\mathrm{d}E_2(t)}{\mathrm{d}t} = \beta \langle k \rangle S(t)(I_1(t) + I_2(t)) - (d + v)E_2(t), \\ \dfrac{\mathrm{d}I_1(t)}{\mathrm{d}t} = vE_1(t) - (d + \gamma + u_1(t))I_1(t) - H(I_1, I_c), \\ \dfrac{\mathrm{d}I_2(t)}{\mathrm{d}t} = vE_2(t) - (d + \gamma + u_2(t))I_2(t) - H(I_2, I_c), \\ \dfrac{\mathrm{d}R_1(t)}{\mathrm{d}t} = H(I_1, I_c) + (\gamma + u_1(t))I_1(t) - dR_1(t), \\ \dfrac{\mathrm{d}R_2(t)}{\mathrm{d}t} = H(I_2, I_c) + (\gamma + u_2(t))I_2(t) - dR_2(t), \end{cases} \tag{4.1}$$

where $u_1(t)$ and $u_2(t)$ represent the degree of control over the spreader, and the control system (4.1) satisfies the initial conditions. In order to describe the relationship between control cost and spreads, we use θ_i and ϕ_i $(i = 1, 2)$ to denote the weight coefficients between $I_1(t)$, $I_2(t)$ and the degree of control, respectively. Assume that the termination time is T and the objective function is

$$J(u_1(t), u_2(t)) = \int_0^T \theta_1 I_1(t) + \theta_2 I_2(t) + \phi_1 u_1^2(t) + \phi_2 u_2^2(t)\mathrm{d}t. \tag{4.2}$$

Moreover, the optimal controls $u_1^*(t)$ and $u_2^*(t)$ satisfy

$$J(u_1^*(t), u_2^*(t)) = \min \{J(u_1(t), u_2(t)) : (u_1(t), u_2(t)) \in U\}.$$

To further describe the optimization problem, the Lagrange function is defined as

$$L(I_1(t), I_2(t), u_1(t), u_2(t)) = \theta_1 I_1(t) + \theta_2 I_2(t) + \phi_1 u_1^2(t) + \phi_2 u_2^2(t),$$

and the Hamiltonian function is

$$H(S(t), E_i(t), I_i(t), R_i(t), u_i(t), \lambda_j(t)) = L(I_i(t), u_i(t)) + \lambda^T(t) f(S(t), E_i(t), I_i(t), R_i(t)),$$

where $i = 1, 2$, $j = 1, 2, \cdots, 7$, and $\lambda(t)$ denotes the adjoint function. According to the Pontryagin's maximum principle, the following conclusions can be obtained.

Theorem 6. *Let $(\bar{S}, \bar{E}_1, \bar{E}_2, \bar{I}_1, \bar{I}_2, \bar{R}_1, \bar{R}_2)$ be the optimal state solution of system (4.1) under the optimal* *control* $(u_1^*(t), u_2^*(t))$. *Then, the adjoint variables satisfy the equation*

$$\begin{cases} \dfrac{d\lambda_1(t)}{dt} = & \alpha \langle k \rangle (\bar{I}_1(t) + \bar{I}_2(t))(\lambda_1(t) - \lambda_2(t)) \\ & + \beta \langle k \rangle (\bar{I}_1(t) + \bar{I}_2(t))(\lambda_1(t) - \lambda_3(t)) + d\lambda_1(t), \\ \dfrac{d\lambda_2(t)}{dt} = & (d+v)\lambda_2(t) - v\lambda_4(t), \\ \dfrac{d\lambda_3(t)}{dt} = & (d+v)\lambda_3(t) - v\lambda_5(t), \\ \dfrac{d\lambda_4(t)}{dt} = & -\theta_1 + \alpha \langle k \rangle \bar{S}(t)(\lambda_1(t) - \lambda_2(t)) + \beta \langle k \rangle \bar{S}(t)(\lambda_1(t) - \lambda_3(t)) \\ & + d\lambda_4(t) + (\gamma + u_1(t) + \hat{H}_{11})(\lambda_4(t) - \lambda_6(t)) + \hat{H}_{21}(\lambda_5(t) - \lambda_7(t)), \\ \dfrac{d\lambda_5(t)}{dt} = & -\theta_2 + \alpha \langle k \rangle \bar{S}(t)(\lambda_1(t) - \lambda_2(t)) + \beta \langle k \rangle \bar{S}(t)(\lambda_1(t) - \lambda_3(t)) \\ & + d\lambda_5(t) + (\gamma + u_2(t) + \hat{H}_{22})(\lambda_5(t) - \lambda_7(t)) + \hat{H}_{12}(\lambda_4(t) - \lambda_6(t)), \\ \dfrac{d\lambda_6(t)}{dt} = & d\lambda_6(t), \\ \dfrac{d\lambda_7(t)}{dt} = & d\lambda_7(t), \end{cases}$$

with transversal conditions $\lambda_j(T) = 0,\ j = 1, 2, \cdots, 7$, *and the optimal controls* $u_1^*(t)$ *and* $u_2^*(t)$ *are*

$$u_1^*(t) = \min\left\{\max\left\{\frac{(\lambda_4(t) - \lambda_6(t))\bar{I}_1(t)}{2\phi_1}, 0\right\}, u_1^{\max}\right\},$$
$$u_2^*(t) = \min\left\{\max\left\{\frac{(\lambda_5(t) - \lambda_7(t))\bar{I}_2(t)}{2\phi_2}, 0\right\}, u_2^{\max}\right\}. \tag{4.3}$$

Proof. According to the Pontryagin's maximum principle, let

$$S(t) = \bar{S},\ E_i(t) = \bar{E}_i,\ I_i(t) = \bar{I}_i,\ R_i(t) = \bar{R}_i.$$

Under the transversal condition, it follows that

$$\begin{cases} \dfrac{d\lambda_1(t)}{dt} = -\dfrac{\partial H}{\partial S(t)}, \dfrac{d\lambda_2(t)}{dt} = -\dfrac{\partial H}{\partial E_1(t)}, \dfrac{d\lambda_3(t)}{dt} = -\dfrac{\partial H}{\partial E_2(t)}, \\ \dfrac{d\lambda_4(t)}{dt} = -\dfrac{\partial H}{\partial I_1(t)}, \dfrac{d\lambda_5(t)}{dt} = -\dfrac{\partial H}{\partial I_2(t)}, \dfrac{d\lambda_6(t)}{dt} = -\dfrac{\partial H}{\partial R_1(t)}, \dfrac{d\lambda_7(t)}{dt} = -\dfrac{\partial H}{\partial R_2(t)}, \end{cases}$$

and from the optimal control condition

$$\begin{aligned} \left.\frac{\partial H(t)}{\partial u_1(t)}\right|_{u_1(t)=u_1^*(t)} &= 2\phi_1 u_1^*(t) - \bar{I}_1(t)\lambda_4(t) + \bar{I}_1(t)\lambda_6(t) = 0, \\ \left.\frac{\partial H(t)}{\partial u_2(t)}\right|_{u_2(t)=u_2^*(t)} &= 2\phi_2 u_2^*(t) - \bar{I}_2(t)\lambda_5(t) + \bar{I}_2(t)\lambda_7(t) = 0. \end{aligned} \tag{4.4}$$

Then, we can obtain

$$u_1^*(t) = \frac{(\lambda_4(t) - \lambda_6(t))\bar{I}_1(t)}{2\phi_1}, u_2^*(t) = \frac{(\lambda_5(t) - \lambda_7(t))\bar{I}_2(t)}{2\phi_2}.$$

Hence, combined with the range of control set U, $u_1^*(t)$ and $u_2^*(t)$ of (4.1) can be acquired here.

4.2. Event-triggered impulsive control

In practical applications, continuous control of rumor propagation, while feasible, is resource-intensive. However, impulsive control, as a potent discontinuous control method, offers a solution by necessitating control interventions only at discrete sampling points. Ulteriorly, in order to optimize resource utilization and cost-effectively intervene in rumor dissemination when necessary, this paper adopts event-triggered impulsive control to suppress the spread of rumors.

The event-triggered impulsive control system is shown as follows.

For $t \in [t_{k-1}, t_k)$,

$$\begin{cases} \dfrac{\mathrm{d}S(t)}{\mathrm{d}t} = \Lambda - (\alpha+\beta)\langle k\rangle S(t)(I_1(t)+I_2(t)) - dS(t), \\ \dfrac{\mathrm{d}E_1(t)}{\mathrm{d}t} = \alpha\langle k\rangle S(t)(I_1(t)+I_2(t)) - (d+v)E_1(t), \\ \dfrac{\mathrm{d}E_2(t)}{\mathrm{d}t} = \beta\langle k\rangle S(t)(I_1(t)+I_2(t)) - (d+v)E_2(t), \\ \dfrac{\mathrm{d}I_1(t)}{\mathrm{d}t} = vE_1(t) - (d+\gamma)I_1(t), \\ \dfrac{\mathrm{d}I_2(t)}{\mathrm{d}t} = vE_2(t) - (d+\gamma)I_2(t), \\ \dfrac{\mathrm{d}R_1(t)}{\mathrm{d}t} = \gamma I_1(t) - dR_1(t), \\ \dfrac{\mathrm{d}R_2(t)}{\mathrm{d}t} = \gamma I_2(t) - dR_2(t), \end{cases} \tag{4.5}$$

and for $t = t_k$,

$$\begin{cases} S(t_k^+) = S(t_k^-), \\ E_1(t_k^+) = (1-\varepsilon\mu)E_1(t_k^-), \\ E_2(t_k^+) = (1-\varepsilon\mu)E_2(t_k^-), \\ I_1(t_k^+) = (1-\varepsilon\mu)I_1(t_k^-), \\ I_2(t_k^+) = (1-\varepsilon\mu)I_2(t_k^-), \\ R_1(t_k^+) = \varepsilon\mu(I_1(t_k^+) + E_1(t_k^-)) + R_1(t_k^-), \\ R_2(t_k^+) = \varepsilon\mu(I_2(t_k^+) + E_2(t_k^-)) + R_2(t_k^-). \end{cases} \tag{4.6}$$

Let $A(t) = (E_1(t), E_2(t), I_1(t), I_2(t))^T$, and construct the following auxiliary variables

$$\begin{cases} B(t) = -\mu A(t), & t \in [t_{k-1}, t_k), \\ \Delta A(t) = \varepsilon B(t), & t = t_k, \end{cases}$$

where $B(t)$ indicates the reduction of the exposed and spreader in $[t_{k-1}, t_k)$ compared to the previous time period due to the control, and $\Delta A(t) = A(t_k^+) - A(t_k^-), A(t_k) = A(t_k^+)$. t_k is the impulsive sequence which satisfies $0 = t_0 < t_1 < \cdots < t_k < \cdots, \lim_{k\to\infty} t_k = \infty$. Moreover, $\varepsilon \in (0,1)$ and $\mu \in (0,1)$ represent impulsive strength and control strength, respectively.

Next, define the following event-triggered function

$$h(x) = \|e(t)\|^2 - \frac{a}{2}\|B(t_k)\|^2 - \frac{b}{(t-t_0)^2}, \tag{4.7}$$

where $a > 0, b > 0, e(t) = B(t_k) - B(t)$, and the $k+1$-th impulsive time is $t_{k+1} = \inf\{t | h(t) \geq 0, t > t_k\}$.

Theorem 7. *For the event-triggered impulsive control system (4.5), if there is a positive definite matrix U such that*

$$\lim_{t\to\infty}\left[k\ln(1-\varepsilon\mu)+\frac{k+1}{2}\ln(\frac{\lambda_{\max}(U)}{\lambda_{\min}(U)})+\frac{\lambda_{\max}(\varphi)}{2\lambda_{\min}(U)}(t-t_0)\right]\to-\infty, \tag{4.8}$$

where $\varphi = KU + U^T K$, *then* $\lim_{t\to\infty}\|A(t)\| = 0$, *that is, the rumor will disappear.*

Proof. For $t \in [t_k, t_{k+1})$, we study the equation

$$\frac{\mathrm{d}A(t)}{\mathrm{d}t} = KA(t),$$

where

$$K = \begin{pmatrix} -(d+v) & 0 & \alpha\langle k\rangle S(t) & \alpha\langle k\rangle S(t) \\ 0 & -(d+v) & \beta\langle k\rangle S(t) & \beta\langle k\rangle S(t) \\ v & 0 & -(d+\gamma) & 0 \\ 0 & v & 0 & -(d+\gamma) \end{pmatrix}.$$

Since $S(t) \le \frac{\Lambda}{d}$, $S(t)$ can be replaced by $\frac{\Lambda}{d}$. Now, construct the Lyapunov function

$$V(t) = A^T(t)UA(t),$$

where U is a positive definite matrix. The derivative of $V(t)$ is

$$\dot{V}(t) = A^T(t)(KU + U^T K)A(t) \le \lambda_{\max}(\varphi)A^T(t)A(t),$$

where $\lambda_{\max}(\varphi)$ denotes the maximum eigenvalue of $\varphi = KU + U^T K$.

For $t \in [t_0, t_1)$, based on Gronwall's inequality [26], one has

$$\begin{aligned}\|A(t)\|^2 &\le \frac{1}{\lambda_{\min}(U)}[V(t_0) + \int_{t_0}^{t}\lambda_{\max}(\varphi)\|A(s)\|^2\mathrm{d}s] \\ &\le \frac{\lambda_{\max}(U)}{\lambda_{\min}(U)}\|A(t_0)\|^2\exp\left\{\frac{\lambda_{\max}(\varphi)}{\lambda_{\min}(U)}(t-t_0)\right\}.\end{aligned}$$

When $t = t_1$, it can yield that

$$\|A(t_1)\|^2 \le (1-\varepsilon\mu)^2\frac{\lambda_{\max}(U)}{\lambda_{\min}(U)}\|A(t_0)\|^2\exp\left\{\frac{\lambda_{\max}(\varphi)}{\lambda_{\min}(U)}(t-t_0)\right\}.$$

Based on mathematical induction, for any $t \in [t_k, t_{k+1})$, the following inequality holds.

$$\|A(t)\|^2 \le (1-\varepsilon\mu)^{2k}(\frac{\lambda_{\max}(U)}{\lambda_{\min}(U)})^{k+1}\|A(t_0)\|^2\exp\left\{\frac{\lambda_{\max}(\varphi)}{\lambda_{\min}(U)}(t-t_0)\right\}.$$

Further, we obtain $\lim_{t\to\infty}\|A(t)\| = 0$ by condition (4.8).

Remark 4. Differing from existing discontinuous control methods, the impulsive instants are determined by the event-triggered function (4.7), and the Zeno phenomenon can be excluded via the term $\frac{b}{(t-t_0)^2}$. In addition, note that the extinction of rumors depends on the selection of parameters ε and μ , which means that government and relevant agencies can provide more cost-effective interventions.

5. Numerical simulation

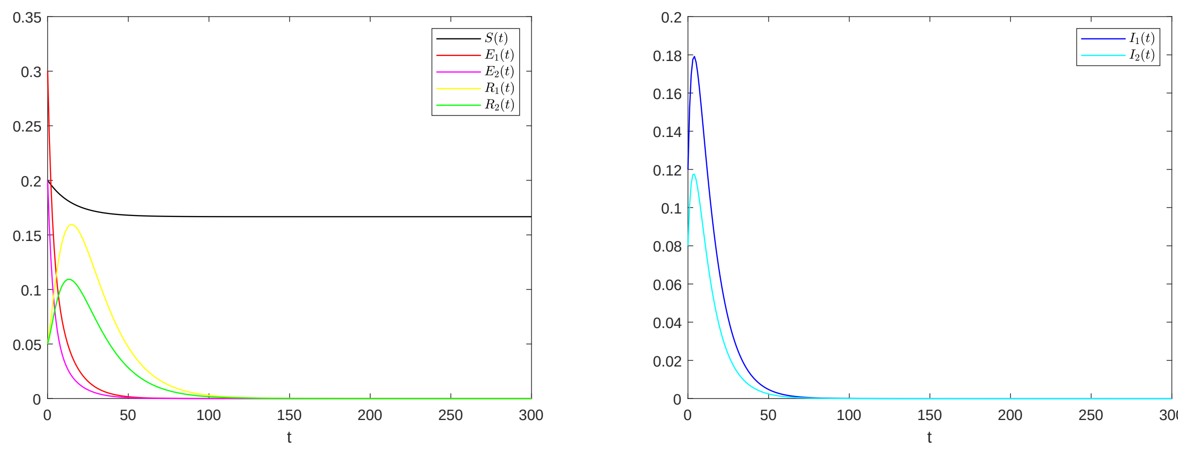
In this section, different parameters are selected to verify the correctness of the theoretical analysis and the effectiveness of the optimal control. In addition, the actual rumor data is simulated. Several sets of parameters of system (2.2) are shown in Table 2.

Table 2. System parameters.

Parameter	$\langle k \rangle$	Λ	d	α	β	v	γ	δ	ξ	I_c
data 1	12	0.01	0.06	0.02	0.01	0.2	0.1	-	-	1
data 2	12	0.02	0.06	0.06	0.04	0.2	0.1	-	-	1
data 3	12	0.02	0.06	0.06	0.04	0.2	0.1	0.3	0.1	0.03
data 4	12	0.01	0.06	0.02	0.01	0.02	0.01	-	0.1	0.01
data 5	12	0.01	0.06	0.05	0.03	0.4	0.06	0.3	0.1	0.01
data 6	12	0.02	0.06	0.06	0.04	0.2	0.1	0.8	0.1	0.02

5.1. Stability simulation

In this part, we will numerically simulate the stability of the equilibrium for the cases of $0 \leq I_i \leq I_c$ and $I_i > I_c$, respectively.



(a) (b)

Figure 2. The local stability of the rumor-free equilibrium E_0 with $R_0 < 1$.

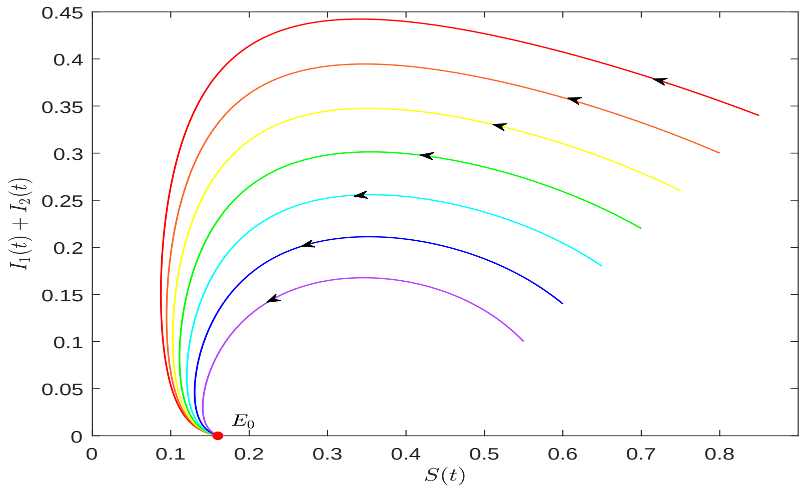
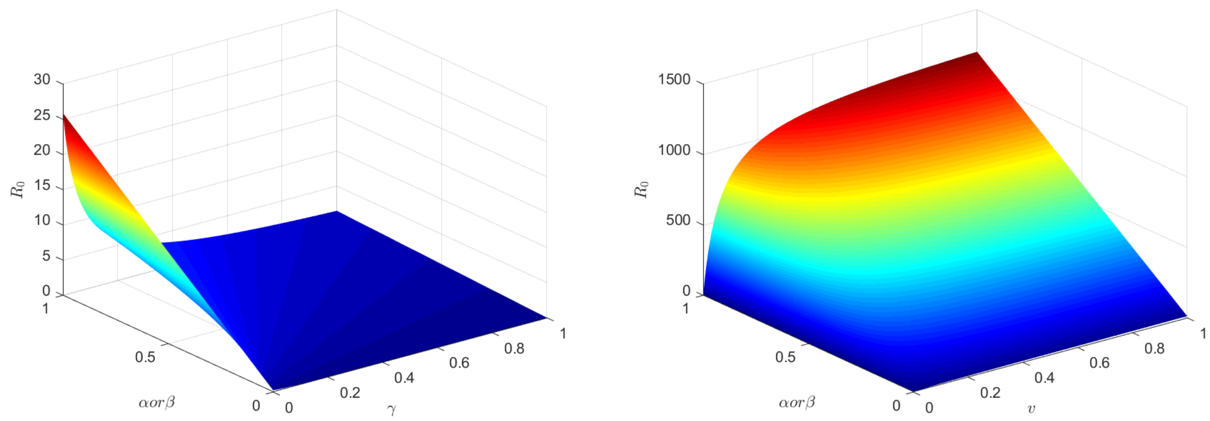


Figure 3. The global stability of rumor-free equilibrium E_0 with $R_0 < 1$.



(a) (b)

Figure 4. The relationship between the basic reproduction number R_0, $\alpha(\beta)$, γ, and v.

Case 1. $0 \leq I_i \leq I_c$

Choose the system parameter data 1 and the initial densities $S(0) = 0.2$, $E_1(0) = 0.3$, $E_2(0) = 0.2$, $I_1(0) = 0.12$, $I_2(0) = 0.08$, $R_1(0) = 0.05$, $R_2(0) = 0.05$. Then, $R_0 = 0.2885 < 1$. From Theorem 1, the rumor-free equilibrium E_0 is locally asymptotically stable, as shown in Figure 2. To verify the global stability of E_0, we select different initial values to simulate in Figure 3. Obviously, the density of the ignorant $S(t)$ stabilizes to 0.167, and the density of $E_i(t)$, $I_i(t)$, and $R_i(t)$ tend to 0 eventually. Meanwhile, Figure 4 shows the influence of model parameters on the basic reproduction number R_0. That is, the higher propagation rate α or β and the conversion rate v, the greater R_0, and the R_0 decreases with the increase of the recovery rate γ.

In addition, consider the parameter data 2 and the initial densities $S(0) = 0.3$, $E_1(0) = 0.15$, $E_2(0) = 0.1$, $I_1(0) = 0.12$, $I_2(0) = 0.08$, $R_1(0) = 0.05$, $R_2(0) = 0.05$. Then, it yields $R_0 = 1.9231 > 1$.

From Theorems 3 and 4, the system exists a unique rumor-spreading equilibrium E^* for $R_0 < P = 1.924$, which E^* is locally asymptotically stable in Figure 5.

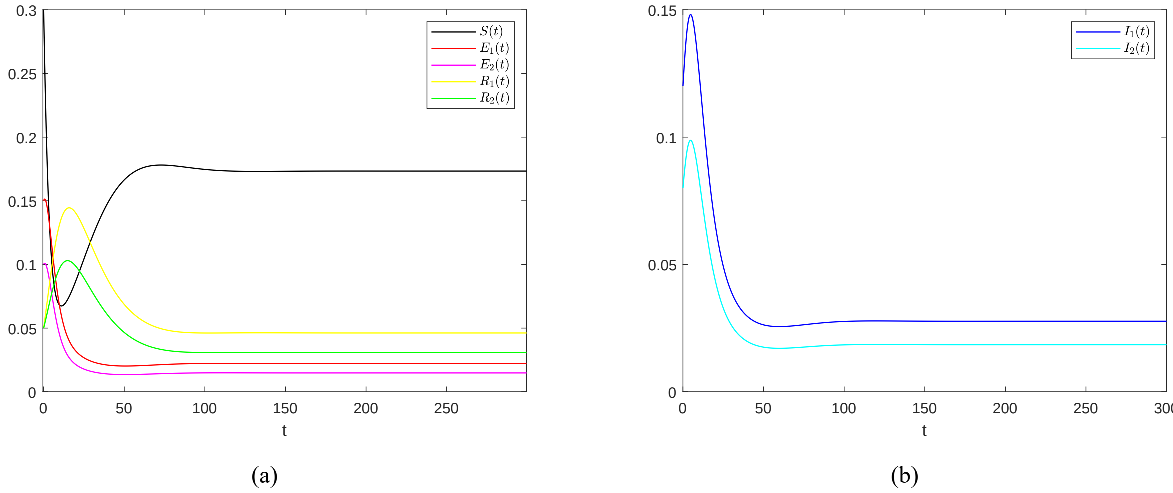


Figure 5. The local stability of rumor-spreading equilibrium E^* with $1 < R_0 < P$.

Case 2. $I_i > I_c$

To compare the influence of rumor propagation with and without the non-smooth inhibition mechanism, consider the system parameter data 3. From Figure 6, note that under the non-smooth inhibition mechanism ($NSIM$), the peak value of spreader density $I_i(t)$ is lower than that without the non-smooth inhibition mechanism, and stabilizes faster. The valley value of ignorant density $S(t)$ is higher than that without non-smooth inhibition mechanism, and stabilizes faster. This shows that the non-smooth inhibition mechanism can effectively suppress the spread of rumors.

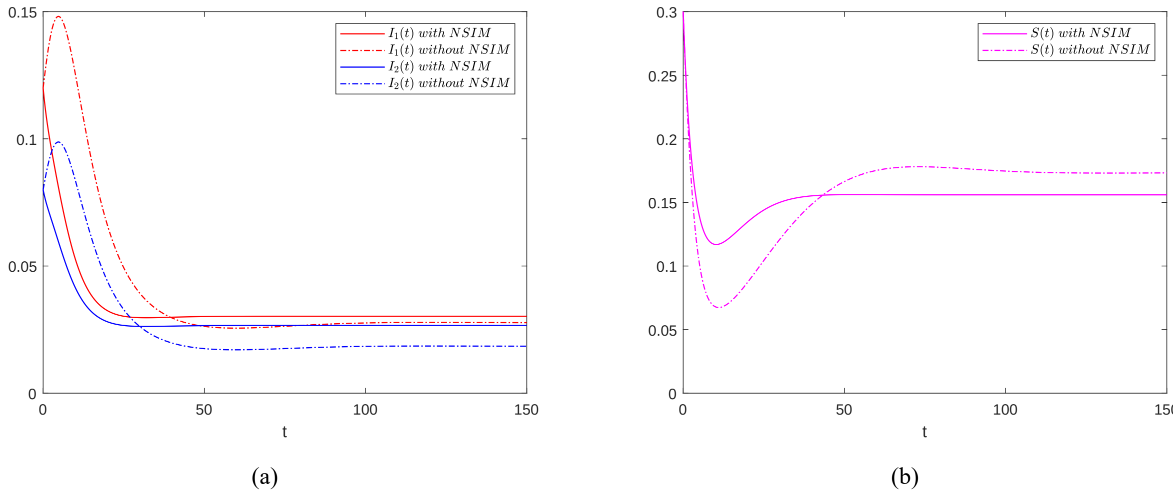


Figure 6. The influence of non-smooth inhibition mechanism on rumor propagation.

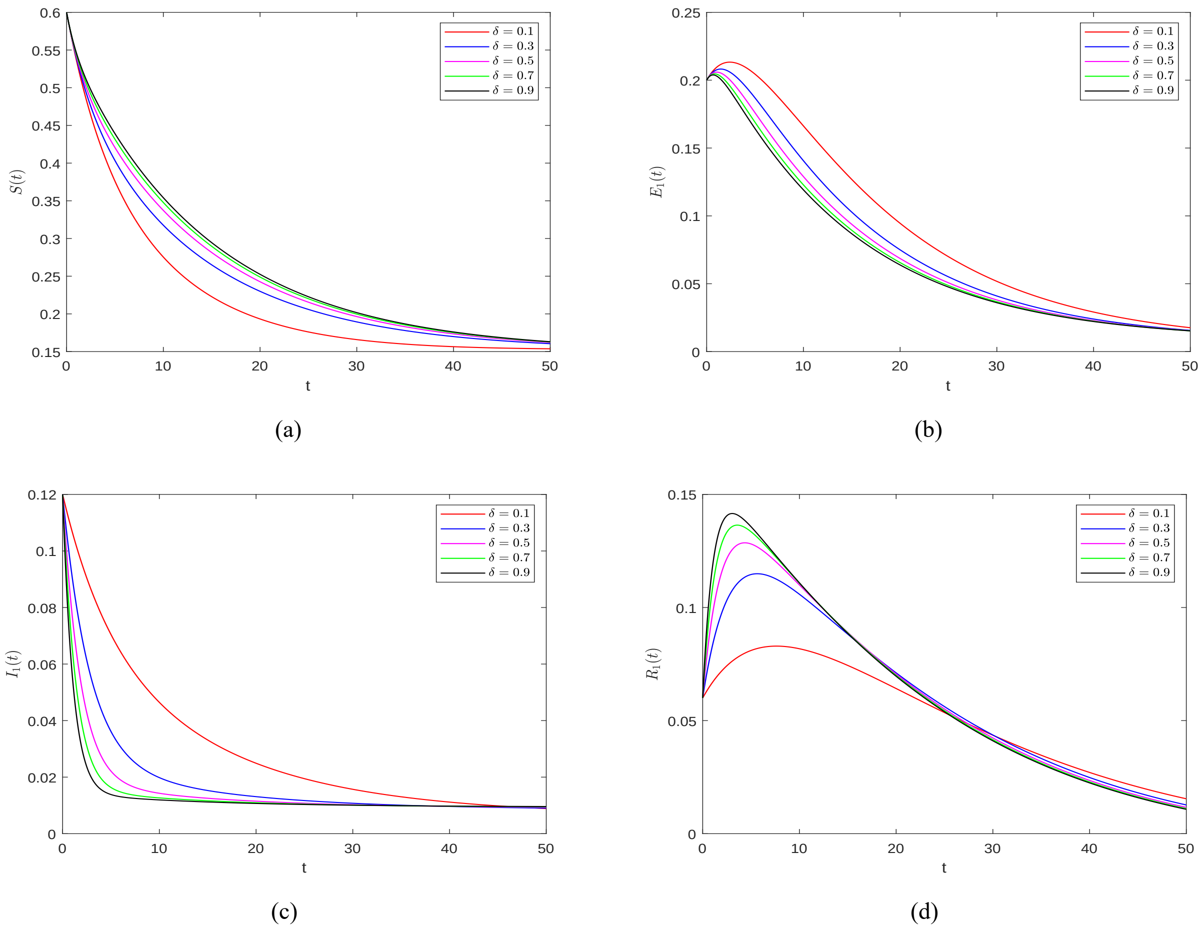


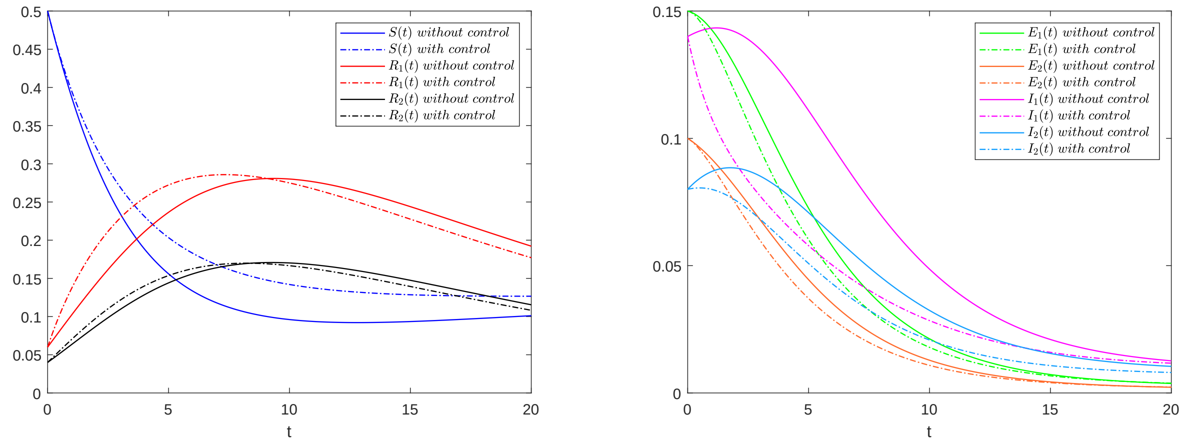
Figure 7. The density variation of $S(t)$, $E_1(t)$, $I_1(t)$, $R_1(t)$ when δ takes different values.

Next, to further show the influence on different government and media control intensity, we choose the parameter data 4 here. For better visual effects, the figures of $E_2(t), I_2(t)$, and $R_2(t)$ are omitted here. Evidently, it indicates that the larger the value of δ, the faster the densities of $E_1(t)$ and $I_1(t)$ tend to balance (see Figure 7(b),(c)), the higher the peak density of $R_1(t)$ (see Figure 7(d)), and the greater the density of $S(t)$ (see Figure 7(a)). That is, if the density of spreader exceeds the threshold, the greater the intervention is.

Remark 5. In fact, Figure 6 shows that the non-smooth inhibition mechanism can effectively suppress rumor propagation. Besides, in Figure 7, the greater δ (representing the intensity of intervention by the government and related media), the more effective the control of rumor propagation, which implies that we can adjust the value of parameter δ in conjunction with the severity of rumor propagation to achieve effective intervention in rumor dissemination.

5.2. The effectiveness of the optimal control

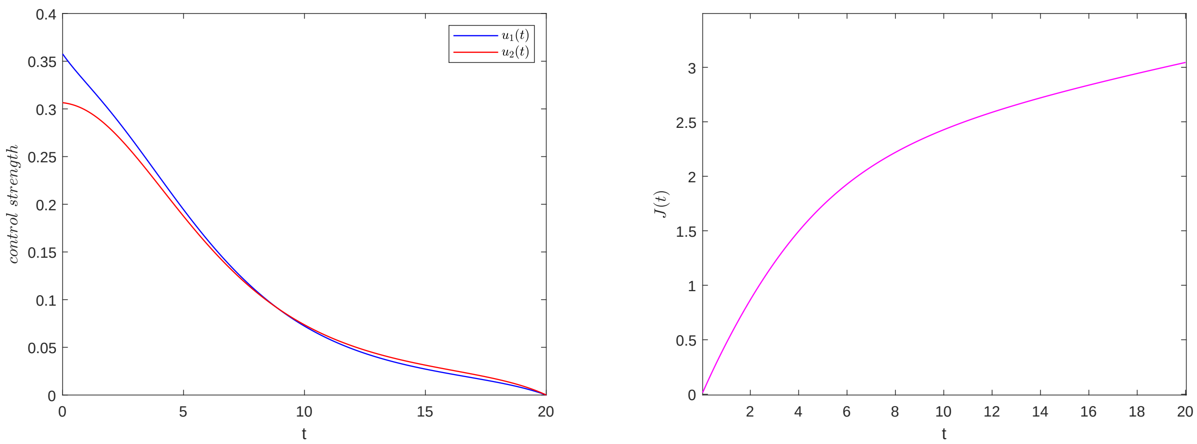
Consider the optimal control system (4.1), where the parameters are selected as data 5, and take $\theta_1 = 1, \theta_2 = 1, \phi_1 = 2, \phi_2 = 3$. Figure 8 shows the density changes of each population with control and without control. Note that under the optimal control, the density of the ignorant $S(t)$ and the stifler $R_i(t)$ increase, while the density of the spreader $I_i(t)$ and the exposed $E_1(t)$ decreases rapidly and stabilizes faster.



(a) (b)

Figure 8. The influence of optimal control on the rumor propagation.

The strength change of optimal control and control cost with time is shown in Figure 9. From Figure 9(a), the control strength is the largest in the initial stage, but gradually decreases to 0 with the increase of time. At the same time, it can be found from Figure 9(b) that as the control time increases, the control cost gradually increases, and the increase rate is faster in the early stage than in the later stage.

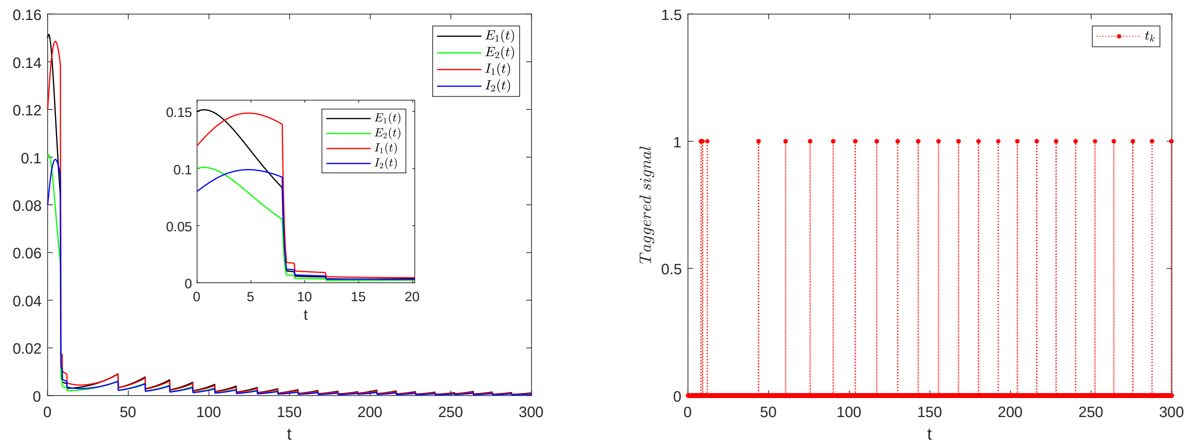


(a) (b)

Figure 9. The trajectory of optimal control $u_i(t), i = 1, 2$ and objective function $J(t)$.

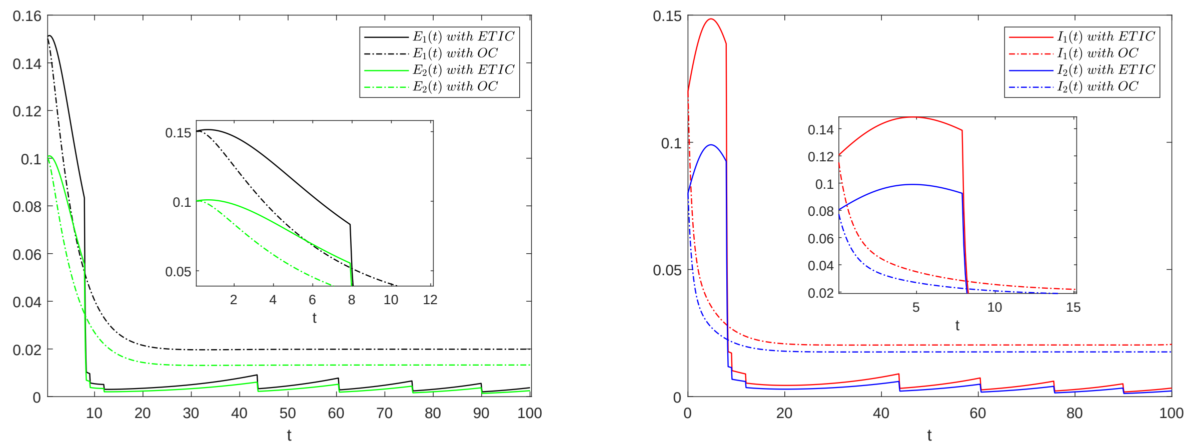
5.3. *The effectiveness of event-triggered impulsive control*

For system (4.5), choose $\mu = 0.5$, $\varepsilon = 0.8$, $a = 1$, $b = 0.008$, and data 6. Then, from Theorem 7, Figure 10(a) shows the trajectories of $E_i(t)$ and $I_i(t)$ under the event-triggered impulsive control ($ETIC$), and the rumor will eventually disappear. In Figure 10(b), 0 means not triggered and 1 means triggered. In addition, the comparison of optimal control (OC) and the $ETIC$ is further presented in Figure 11.



(a) (b)

Figure 10. The influence of $ETIC$ on rumor propagation and the triggered time.



(a) (b)

Figure 11. The comparison of $ETIC$ and OC on the rumor propagation.

Remark 6. From the comparison of $ETIC$ and OC on the rumor propagation in Figure 11, it shows that the OC combined with non-smooth inhibition mechanism more effectively suppresses the spread of rumors in a short time. Besides, $ETIC$ only requires to control at some discrete moments which can reduce resource consumption, and can let the rumors die out via adjusting parameters ε and μ . Therefore, we can judiciously choose which control measures to implement based on the severity of rumor propagation and available resources.

5.4. Example verification

In this part, we use an actual example to verify the validity of the theoretical analysis with the data provided in Zhiwei Data [27]. The background of the actual rumor is as follows. At 16:00 on January 29, 2020, some netizens on social media broke the news through video on social media that SF Express Courier privately opened a user's parcel by intercepting the goods and publicly hawked it in the video. At 23:00 on January 29, SF Group responded that the video was taken by the sender himself, and there was no behavior of intercepting others' express. At 15:00 on January 30, the Kunming Panlong Public Security Bureau of Yunnan Province issued a notice to refute this information. Table 3 shows the hourly propagation data of the event within 48 hours, where "T" is the propagation time and "N" is the number of rumor propagation.

Table 3. The data of rumor propagation.

T	N	T	N	T	N	T	N	T	N	T	N
1	25	9	134	17	292	25	291	33	39	41	39
2	26	10	72	18	221	26	191	34	20	42	51
3	27	11	46	19	196	27	136	35	16	43	59
4	50	12	35	20	173	28	114	36	12	44	47
5	110	13	29	21	108	29	94	37	7	45	56
6	225	14	38	22	79	30	120	38	18	46	64
7	517	15	140	23	183	31	82	39	23	47	45
8	321	16	242	24	272	32	73	40	42	48	32

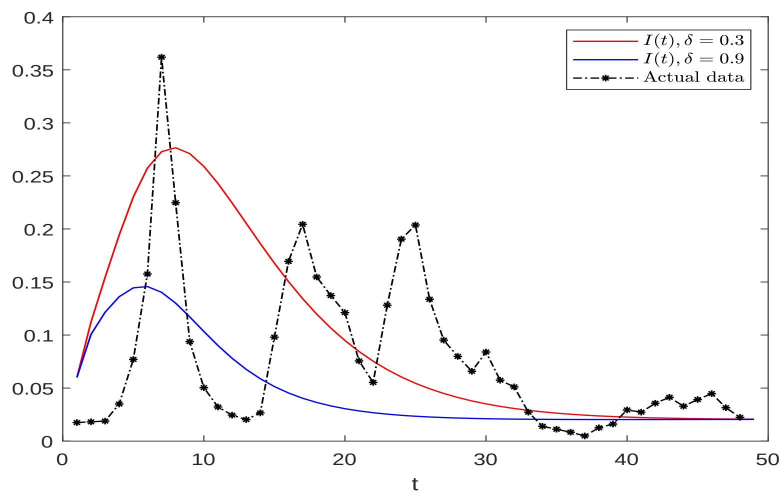


Figure 12. Actual data fitting and the influence of non-smooth inhibition mechanism.

In order to make the model closer to the actual data, we choose these parameters $\langle k \rangle = 12, \Lambda = 0.002, d = 0.001, \alpha = 0.4, \beta = 0.1, v = 0.5, \gamma = 0.06, \delta = 0.3, \xi = 0.1, I_c = 0.01$. Figure 12 describes the actual data trajectory and the spreader trajectory with $\delta = 0.3$ and $\delta = 0.9$. From Figure 12, the density trajectory of rumor spreader is basically consistent with the overall trend of actual data, which means that our system can roughly reflect the spread process of the rumor. Moreover, the greater the intensity of government and media intervention, the lower the peak of the density of spreader, indicating that the non-smooth inhibition mechanism can effectively suppress the spread of rumors.

6. Conclusions

In this paper, we have discussed the stability problem and control strategies of the rumor propagation S2E2I2R model with non-smooth inhibition mechanism in a multi-lingual environment. First, considering the spreader threshold I_c of government and media intervention as the boundary, we have given the basic reproduction number R_0 and analyzed its stability. In addition, both optimal control and event-triggered impulsive control have been proposed to suppress the spread of rumors, and the correctness of the theory was analyzed via numerical simulation. Due to the complexity of rumor propagation in social networks, there are many factors that affect rumor propagation. In future work, we will introduce more factors into the rumor propagation model.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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