Mathematical Biosciences and Engineering

http://www.aimspress.com/journal/mbe

MBE, 21(3): 4501–4520.
DOI: 10.3934/mbe.2024199
Received: 20 December 2023
Revised: 04 February 2024
Accepted: 19 February 2024
Published: 28 February 2024

Research article

Synchronization of time-delay systems with impulsive delay via an average impulsive estimation approach

Biwen Li and Qiaoping Huang*

School of Mathematics and Statistics, Hubei Normal University, Huangshi 435002, China

* **Correspondence:** Email: hqp15572910919@163.com; Tel: +8615572910919; Fax: +8615572910919.

Abstract: We investigated synchronization of dynamic systems with mixed delays and delayed impulses. Using impulsive control method and the average impulsive interval approach, several Lyapunov sufficient conditions were given for ensuring synchronization in terms of impulsive perturbation and impulsive control, respectively. The derived conditions indicated that delays in continuous dynamical systems were flexible under impulsive perturbation and were not strictly dependent on the size of impulsive delays, and they may have a potential impact on synchronization of the considered system. In addition, applying the proposed concepts of average positive impulsive estimation and average impulsive estimation, we integrated the information in impulsive delay into the rate coefficient to eliminate the limitation of having the same threshold at each impulse point, while the impulsive delay maintained the synchronization effect. This was an improvement on the previous results obtained. Finally, we provided two numerical examples to illustrate the validity of our results.

Keywords: impulsive delay; time-delay systems; synchronization; average impulsive estimation

1. Introduction

When control systems produce a state change at one discrete instant suddenly, system evolutions exhibit impulsive behavior, and such systems can also be called impulsive dynamical systems, which belong to a special class of hybrid systems. Impulsive systems have attracted the attention of many researchers due to their important applications in various fields such as network control systems [1], financial interest rate adjustment [2], and pharmaceutical management [3]. It is well known that delays are inevitable in signal transmission and impulsive input, so the study of impulsive delay is very necessary. Furthermore, time delay systems have been widely studied in many fields, such as neural networks [4]. Recently, impulsive time-delay systems have been studied in some studies [5–9]. Specially, reference [9] obtained an implicit function related time delay and synchronization rate of impulsive systems. This function was used to reveal the potential influence of delay in continuous dynamical systems on the synchronization of the systems. However, there are few studies on synchronization of hybrid time-delay systems, which is a topic worth investigating.

In addition, impulsive control is an effective method of regulating the synchronization of time-delay systems, as it has a simple structure and makes it possible to change the state of the systems in a fraction of a second and transmit information at discrete times. This feature of impulsive control reduces communication costs to a certain extent and has been used effectively in secure communication, epidemiological model control, and biomedical systems [10–13]. Unstable or asynchronous systems tend to be stable or synchronous under impulsive control. For instance, reference [14] proposed a class of event-triggered impulsive controllers with time delay for system stability. The synchronization of linear dynamical systems with impulsive delay was discussed in [15] by pinning control. However, the interaction between the system time delay and the impulsive delay can increase the difficulty of analyzing stability and synchronization, and there are fewer relevant studies. The literature [16] concluded that the impulsive delay had a beneficial effect on the stability of time-delay systems through a Razumikhin-type inequality with impulsive delay. Based on the idea of average impulsive delay (AID), reference [17] addressed the influence of impulsive delay on time-delay systems. However, the impulsive control in most researches usually does not take into account delay, i.e., $\Delta z(t_u) = z(t_u) - z(t_u^-) = A_u z(t_u^-)$ [18,19], or just consider "pure" delay, that is, $\Delta z(t_u) = f(z(t_u - \rho_u)^-)$ [16,17,20]. We consider a more universal impulsive delay: $z(t_u) = D_u z(t_u^-) + E_u z((t_u - \rho_u)^-)$. Furthermore, many related works imposed strict limitations on impulsive size or impulsive interval length [17,19,21]. For instance, reference [21] required that input delays must be small enough, whereas time delays may even be longer than impulsive intervals. Reference [19] considered more general input delays, but required that the length of impulsive interval is subject to a common upper or lower bound. Therefore, obtaining relaxing criterion for the synchronization of impulsive time-delay systems is crucial.

On the other hand, if the time-delay system is stable or synchronous but the impulses are unstable, we refer to this situation as an impulsive perturbation [22,23]. However, in the vast majority of current studies on impulsive perturbation problems usually in the Lyapunov sense, requires that $V(t_u) \leq \exp\{\sigma\} V((t_u - \rho_u)^-)$ with $\sigma > 0$ [17,20]. However, due to the fact that impulsive delays may not be invariable forever, it is more practical to set a flexible σ_u rather than setting the same threshold at every impulsive moment. Additionally impulsive delays may affect the characteristics of original system [20,24]. It is natural to consider the question: Can we extract the delay information of the impulse term and then integrate it into the rate coefficient σ_u so to ensure the influence of impulsive perturbation even though a number of $\sigma_u < 0$?

Based on the above proposed problems, we employ the Razumikhin-type Lyapunov function to give sufficient conditions for synchronization of the impulsive system with mixed time delays, and the key contributions of this paper are as follows:

1). Differ from [9] proposed an implicit function, we consider the relationship between the system time delay and the rate of synchronization through a display inequality $\epsilon = q_1 exp\left\{\rho^{\frac{\lambda\alpha\bar{t} - p\rho^* + \hat{\sigma}_+^*}{\bar{t}}}\right\} + q_2 exp\left\{\mu^{\frac{\lambda\alpha\bar{t} - p\rho^* + \hat{\sigma}_+^*}{\bar{t}}}\right\}$ which shows that the system time delay has a positive or negative effect on synchronization in this paper.

2). The size relationship between impulsive delay and impulsive interval length has no strict limitation, which was restricted in [16,20]. In addition, the time delay in continuous dynamics is smaller or greater than impulsive delay.

3). Through the presented concept of average impulsive estimation (AIE), the effect of impulsive perturbation is ensured by integrating the delay information obtained in impulse into impulsive estimation σ_u, even if some σ_u are negative. Furthermore, the limitation of having an universal threshold for impulsive estimates at every impulsive point is eliminated. Compared with recent relevant studies [16,17,20,25], the results in this paper are less conservative.

2. Model description and preliminaries

2.1. Notations

Let $\mathcal{R}$ and $\mathcal{R}_+$ denote the set of real numbers and positive real numbers, respectively. Denote the set of positive integers by $\mathcal{Z}_+$, the set of nonnegative integers by $\mathcal{Z}_+^0$, the set of n-dimensional real-valued vectors by $\mathcal{R}^n$, and the set of $n \times m$-dimensional real matrices by $\mathcal{R}^{n\times m}$. $\|\cdot\|$ denotes the vector Euclidean norm. Let $PC_v := PC([-v, 0], \mathcal{R}^n)$ is the set of piece-wise right-continuous function $\phi : [-v, 0] \to \mathcal{R}_+$ with the norm $\|\phi\|_v := \sup_{-v\leq\theta\leq 0} \|\phi(\theta)\|$. $D^+\hbar(\cdot)$ denotes the upper-right Dini-derivative of $\hbar(\cdot)$.

2.2. Model

Consider a class of delayed system involved finite distributed delay:

$$\begin{cases} \dot{\xi}(t) = C\xi(t) + Af(\xi(t)) + B\bar{f}(\xi(t-\rho)) + \hat{B}\int_0^{\mu} k(s)\xi(t-\rho)ds + J, \quad t \geq t_0, \\ \xi(t_0 + s) = \psi(s), \quad s \in [-v, 0], \end{cases} \tag{2.1}$$

where $\xi(t) = (\xi_1(t), \cdots, \xi_n(t))^T \in \mathcal{R}^n$ is the state vector, C, A, B, $\hat{B} \in \mathcal{R}^{n\times n}$ are constant system matrices, $f(\cdot)$, $\bar{f}(\cdot) \in \mathcal{R}^n$ represent the neuron activation functions and $f(0) \equiv 0$ and $\bar{f}(0) \equiv 0$. ρ and μ denote time delay and distributed delay, respectively. $k(\cdot) : [0, \mu] \to \mathcal{R}_+$ is the delay kernel satisfying $\int_0^{\mu} k(s)ds = 1$, $\int_0^{\mu} sk(s)ds < +\infty$. $v = max\{\rho, \mu\}$. Let the initial condition is $\psi \in PC_v$. Denote external input by J.

Consider Eq (2.1) as drive system, then corresponding response system with delayed impulsive control can be

$$\begin{cases} \dot{\zeta}(t) = C\zeta(t) + Af(\zeta(t)) + B\bar{f}(\zeta(t-\rho)) + \hat{B}\int_0^{\mu} k(s)\zeta(t-\rho)ds + J, \quad t \geq t_0 \neq t_u, \\ \zeta(t) - \zeta(t^-) = H(t), \quad t = t_u, \\ \zeta(t_0 + s) = \varphi(s), \quad s \in [-v, 0], \end{cases} \tag{2.2}$$

in which $\{t_u\}$ is a strictly increasing sequence such that $\lim_{u\to+\infty} t_u = +\infty$ and t_0 is the initial time. $\zeta(t^-)$ and $\zeta(t^+)$ denote the left limit and right limit at time t, respectively. For this paper, let $\zeta(t)$ is right-continuous at every t_u, that is, $\zeta(t_u^+) = \zeta(t_u)$. ρ_u, $u \in \mathcal{Z}_+$, is impulse input delays satisfying $0 \leq \rho_u \leq t_u - t_{u-1}$, $\rho_0 = 0$ and $\rho_u \leq v$. Let synchronization error is $z(t) = \zeta(t) - \xi(t)$ and impulsive control input is $H(t_u) = D_u z(t_u^-) + E_u z((t_u - \rho_u)^-) - z(t_u^-)$, such that $z(t_u)$ can be expressed by

$$\begin{aligned}
z(t_u) &= h_u(z(0), z((-\tau_u)^-)) \\
&= \zeta(t_u) - \xi(t_u) \\
&= \zeta(t_u) - \zeta(t_u^-) + \zeta(t_u^-) - \xi(t_u) \\
&= \zeta(t_u^-) + D_u z(t_u^-) + E_u z((t_u - \rho_u)^-) - z(t_u^-) - \xi(t_u^-) \\
&= D_u z(t_u^-) + E_u z((t_u - \rho_u)^-) - z(t_u^-) + z(t_u^-) \\
&= D_u z(t_u^-) + E_u z((t_u - \rho_u)^-).
\end{aligned}$$

Then it is easy to obtain error system as follows:

$$\begin{cases}
\dot{z}(t) = Cz(t) + Ag(z(t)) + B\bar{g}(z(t-\rho)) + \hat{B}\int_0^{\mu} k(s)z(t-\rho)ds, & t \neq t_u, \\
z(t) = D_u z(t^-) + E_u z((t-\rho_u)^-), & t = t_u, \\
z(t_0 + s) = \varphi(s) - \psi(s), & s \in [-v, 0],
\end{cases} \tag{2.3}$$

where $g(z(t)) = f(\zeta(t)) - f(\xi(t))$, $\bar{g}(z(t-\rho)) = \bar{f}(\zeta(t-\rho)) - \bar{f}(\xi(t-\rho))$.

2.3. Properties and definitions

Definition 1. [26] Suppose that there are positive numbers N_0 and $\bar{t}$, such that

$$\frac{\hat{t} - \check{t}}{\bar{t}} - N_0 \leq N(\hat{t}, \check{t}) \leq \frac{\hat{t} - \check{t}}{\bar{t}} + N_0, \tag{2.4}$$

where $N(\hat{t}, \check{t})$ is the number of impulsive times $\{t_u\}$ occurring in $(\hat{t}, \check{t}]$, $\check{t} > \hat{t} > t_0$. Then $\bar{t}$ is the average impulsive interval (AII) of impulsive instant sequence $\{t_u\}$ and N_0 is the chatter bound.

Definition 2. [27] Suppose that there exist positive numbers $\hat{\rho}_0$ and ρ_* such that

$$\sum_{j=1}^{N(t,t_0)} \rho_j \leq \rho^* N(t, t_0) + \hat{\rho}_0, \tag{2.5}$$

where $N(t, t_0)$ is the number of impulses on the interval $(t_0, t]$, then ρ^* is the AID of impulsive delay sequence $\{\rho_u\}$.

Let $\mathscr{H}\,[\{t_u, \rho_u\}]$ is the class consisting of impulse time sequence $\{t_u\}$ satisfying AII condition Eq (2.4) and impulsive delay sequence $\{\rho_u\}$ satisfying AID condition Eq (2.5).

Definition 3. [28] Response system (2.2) can achieve exponential synchronization with drive system (2.1) if there exist positive scalars χ, λ satisfy

$$\|\delta(t)\| \leq \chi(\|\varphi - \psi\|_v \exp(-\lambda(t - t_0))), \forall t \geq t_0. \tag{2.6}$$

Definition 4. [29] Function $V : [t_0 - v, +\infty) \times \mathcal{R}^n \longrightarrow \mathcal{R}_+$ belong to the class $\mathcal{V}^*$ when following conditions are met:

(1). V is continuous on each set $[t_{u-1}, t_u) \times \mathcal{R}^{n_z}$ and $\lim_{(t,z)\to(t_u^-,z)} V(t, z) = V(t_u^-, z)$ exists;

(2). $V(t,z)$ is locally Lipschitz in z and $V(t,0)\equiv 0$, $\forall t\in\mathcal{R}_+$;

(3). $V(t,z)$ satisfies $l_1\|z\|^{\alpha}\le V(t,z)\le l_2\|z\|^{\alpha}$, where l_1,l_2,α are positive scalars.

If $V\in\mathcal{V}^*$ is a locally Lipschitz function, then $D^+V(t,z(0))$ along with the state trajectory of system (2.3) is defined by

$$D^+V(t,z(0))=\lim_{r\to 0^+}\sup\frac{1}{r}\left[V(t+r,z(0)+rf)-V(t,z(0))\right],$$

in which $(t,z)\in[t_0,+\infty)\times PC_v$.

Assumption 1: For any $s\in\mathcal{R}$, $z\in\mathcal{R}$, there exist Lipschitz constants $r_i>0$ and $\bar{r}_i>0$, such that

$$|f_i(s)-f_i(z)|\le r_i|s-z|,$$
$$|\bar{f}_i(s)-\bar{f}_i(z)|\le \bar{r}_i|s-z|,$$

where $i=1,2,\cdots,n$ and $f_i(0)=\bar{f}_i(0)=0$.

3. Main results

In this section, we obtain some sufficient conditions for exponential synchronization of systems (2.1) and (2.2).

3.1. The case of impulsive perturbation

From impulsive perturbations point of view, we establish some criteria for exponential synchronization.

Theorem 1: Considering system (2.3) under Assumption 1. Suppose that there exists a function $V\in\mathcal{V}^*$, scalars $p>0$, c with $p>c>0$, $q_1=\sum_{i=1}^{n}r_i\max_j\left|g_{ij}\right|$, $q_2=\sum_{i=1}^{n}\bar{r}_i\max_j\left|\hat{g}_{ij}\right|$, $\epsilon=q_1\exp\{c\rho\}+q_2\exp\{c\mu\}$, $\sigma_u=\ln\left(\sum_{i=1}^{n}\max_j\left|d_{ij}^{(u)}\right|+\sum_{i=1}^{n}\max_j\left|e_{ij}^{(u)}\right|\right)$, $\Gamma_u=\frac{\sum_{i=1}^{n}\max_j\left|d_{ij}^{(u)}\right|}{\exp\{\sigma_u\}}$, $\hat{\Gamma}_u=\frac{\sum_{i=1}^{n}\max_j\left|e_{ij}^{(u)}\right|}{\exp\{\sigma_u\}}$, $\hat{\sigma}_+^0>0$, and $\sigma_+^*>0$ such that for every $t\in\mathcal{R}_+$, we have

$$D^+V\left(t,z\left(0\right)\right)\le -pV(t,z(0)),\text{ whenever}$$

$$q_1V(t-\rho,z(-\rho))+q_2\int_0^{\mu}k(s)V(t-s,z(-s))ds\le\epsilon V(t,z(0)),\quad t\ne t_u,\tag{3.1}$$

$$V(t_u,h_u(z(0),z((-\tau_u)^-)))\le\exp\{\sigma_u\}\left(\Gamma_uV(t_u^-,z(0^-))+\hat{\Gamma}_uV((t_u-\rho_u)^-,z((-\rho_u)^-))\right),\tag{3.2}$$

$$\sum_{j=0,\sigma_j>0}^{N(t,t_0)}\sigma_j\le\sigma_+^*N(t,t_0)+\hat{\sigma}_+^0,\tag{3.3}$$

where $N(t,t_0)$ is the same as in Definition 1, and

$$-c\bar{t}+p\rho^*+\sigma_+^*<0.\tag{3.4}$$

Then drive system (2.1) can achieve exponential synchronization with response system (2.2) over the class $\mathscr{H}\left[\{t_u,\rho_u\}\right]$.

Proof: Define $V(t) = V(t, z(t)) = \|z(t)\| = \sum_{i=1}^{n} |z_i(t)|$, and $V_0 = \sup_{u\in[t_0-\nu,t_0]} V(u)$.
The proof is divided into the next three steps.

Step 1: We firstly need to prove that for some $t \in [t_0, t_u)$ one has

$$V(t) \leq \Omega_u V_0 \exp(-c(t-t_0)),\ t \in [t_0, t_u)\,,\ u \in \mathcal{Z}_+, \tag{3.5}$$

where $\Omega_u = \exp\left\{\sum_{j=0, p\rho_j+\sigma_j>0}^{u-1} (p\rho_j + \sigma_j)\right\}$.

In order to prove Eq (3.5), we construct the function

$$\Lambda(t) = \begin{cases} V(t)\exp\{c(t-t_0)\}, & t \geq t_0, \\ V(t), \quad t_0 - \nu \leq t \leq t_0. \end{cases}$$

From Eq (3.5), it yields that

$$\Lambda(t) \leq \Omega_u V_0, t \in [t_0, t_u)\,,\ \ u \in \mathcal{Z}_+. \tag{3.6}$$

We will show that Eq (3.6) holds for $u = 1$, i.e.,

$$\Lambda(t) \leq \Omega_1 V_0 = V_0,\ \ t \in [t_0, t_1)\,. \tag{3.7}$$

It is easy for us to get $\Lambda(t) \leq V_0$ for $t \in [t_0 - \nu, t_0]$, which indicates that $\Lambda(t_0) \leq V_0$. If (3.7) is not true, then there exists $t^* \in (t_0, t_1)$ such that $\Lambda(t^*) > V_0$, $\Lambda(t) \leq V_0$ for $t \in (t_0 - \nu, t^*)$ and $D^+\Lambda(t)|_{t=t^*} \geq 0$. Obviously, $\Lambda(t^*) > \Lambda(t)$ for $t \in (t^* - \mu, t^*)$, then we have

$$\begin{aligned} & q_1 V(t^*-\rho) + q_2 \int_0^{\mu} k(s)V(t^*-s)ds \\ &= \sum_{i=1}^{n} r_i \max_j \left|b_{ij}\right| V(t^*-\rho) + \sum_{i=1}^{n} \bar{r}_i \max_j \left|\hat{b}_{ij}\right| \int_0^{\mu} k(s)V(t^*-s)ds \\ &< \sum_{i=1}^{n} r_i \max_j \left|b_{ij}\right| \exp\{c\rho\} V(t^*) + \sum_{i=1}^{n} \bar{r}_i \max_j \left|\hat{b}_{ij}\right| \int_0^{\mu} k(s)\exp\{cs\} V(t^*)ds \\ &\leq \sum_{i=1}^{n} r_i \max_j \left|b_{ij}\right| \exp\{c\rho\} V(t^*) + \sum_{i=1}^{n} \bar{r}_i \max_j \left|\hat{b}_{ij}\right| \int_0^{\mu} k(s)\exp\{c\mu\} V(t^*)ds \\ &= \sum_{i=1}^{n} r_i \max_j \left|b_{ij}\right| \exp\{c\rho\} V(t^*) + \sum_{i=1}^{n} \bar{r}_i \max_j \left|\hat{b}_{ij}\right| \exp\{c\mu\} V(t^*) \\ &= \epsilon V(t^*). \end{aligned}$$

From Eq (3.1) we can obtain

$$D^+V(t)|_{t=t^*} \leq -pV(t^*). \tag{3.8}$$

Hence, there is

$$\begin{aligned} D^+\Lambda(t)|_{t=t^*} &= [D^+V(t)|_{t=t^*} + cV(t^*)]\exp\{c(t^*-t_0)\} \\ &\leq (c-p)V(t^*)\exp\{c(t^*-t_0)\} \\ &< 0, \end{aligned}$$

which contradicts $D^+\Lambda(t)|_{t=t^*} \geq 0$. Then, provided that (3.6) is true when $t \in [t_0, t_m)$, $1 \leq m \leq u-1$, that is $\Lambda(t) \leq \Omega_m V_0$, $t \in [t_0, t_m)$. Next, we will show that (3.6) is true when $t \in [t_0, t_{m+1})$, i.e., we need to prove that $\Lambda(t) \leq \Omega_{m+1} V_0$, for $t \in [t_m, t_{m+1})$.

When $t = t_m$, from (3.2) we can get

$$
\begin{aligned}
\Lambda(t_m) =& V(t_m)\exp\{c(t_m - t_0)\}\\
=& \sum_{i=1}^{n} |\delta_i(t)| \exp\{c(t_m - t_0)\}\\
\leq& \exp\{\sigma_m\}\left(\frac{\sum_{i=1}^{n}\max_j \left|d_{ij}^{(u)}\right|}{\exp(\sigma_u)} V(t_m^-) + \frac{\sum_{i=1}^{n}\max_j \left|e_{ij}^{(u)}\right|}{\exp(\sigma_u)} V((t_m - \rho_m)^-)\right)\exp\{c(t_m - t_0))\}\\
=& \exp\{\sigma_m\}\left(\Gamma_m V(t_m^-) + \hat{\Gamma}_m (V(t_m - \rho_m)^-)\right)\exp\{c(t_m - t_0)\}\\
\leq& \exp\{\sigma_m\}\left(\Gamma_m V(t_m^-)\exp\{c(t_m - t_0)\} + \hat{\Gamma}_m V((t_m - \rho_m)^-)\exp\{c(t_m - \rho_m - t_0)\}\exp\{c\rho_m\}\right)\\
\leq& \exp\{\sigma_m\}\left(\Gamma_m \Omega_m V_0 + \hat{\Gamma}_m \Omega_m V_0 \exp(c\rho_m)\right)\\
\leq& \exp\{\sigma_m + p\rho_m\}\,\Omega_m V_0\\
\leq& \Omega_{m+1} V_0.
\end{aligned}
$$

Therefore, Eq (3.6) holds for $t = t_m$. Provided that Eq (3.6) is not true for $t \in (t_m, t_{m+1})$, then there exists $t_* \in (t_m, t_{m+1})$ has $\Lambda(t_*) > \Omega_{m+1} V_0$, $\Lambda(t) \leq \Omega_{m+1} V_0$ for $t \in (t_0 - v, t_*)$ and $D^+\Lambda(t)|_{t=t_*} \geq 0$. Similar with the argument used in Eq (3.7), we can obtain $D^+\Lambda(t)|_{t=t_*} < 0$, it contradicts $D^+\Lambda(t)|_{t=t_*} \geq 0$. Hence, we can found Eq (3.6) holds through using mathematical induction, in which $t \in [t_0, t_u)$, which implies Eq (3.5) holds for $t \in [t_0, t_u)$, $u \in \mathcal{Z}_+$.

Step 2: According to Eq (3.5), condition Eqs (2.4), (2.5) and (3.3), we have

$$
\begin{aligned}
V(t) \leq& V_0 \exp\{-c(t - t_0)\}\cdot\exp\left\{\sum_{j=0,p\rho_j+\sigma_j>0}^{u-1} (p\rho_j + \sigma_j)\right\}\\
\leq& V_0 \exp\left\{-c(t - t_0) + p\sum_{j=0}^{u-1}\rho_j + \sum_{j=0,\sigma_j>0}^{u-1}\sigma_j\right\}\\
\leq& V_0 \exp\left\{-c(t - t_0) + p(\rho^*(u-1) + \hat{\rho}_0) + \sigma_+^*(u-1) + \hat{\sigma}_+^0\right\}\\
\leq& V_0 \exp\left\{-c(t - t_0) + p(\rho^* N(t, t_0) + \hat{\rho}_0) + \sigma_+^* N(t, t_0) + \hat{\sigma}_+^0\right\}\\
\leq& V_0 \exp\left\{-c(t - t_0) + p\left(\frac{\rho^*(t - t_0)}{\bar{t}} + N_0\rho^* + \hat{\rho}_0\right) + \frac{\sigma_+^*(t - t_0)}{\bar{t}} + N_0\sigma_+^* + \hat{\sigma}_+^0\right\}\\
\leq& V_0 \exp\left\{pN_0\rho^* + p\hat{\rho}_0 + N_0\sigma_+^* + \hat{\sigma}_+^0\right\}\exp\left\{\left(-c + \frac{p\rho^* + \sigma_+^*}{\bar{t}}\right)(t - t_0)\right\},
\end{aligned}
\tag{3.9}
$$

where $t \geq t_0$.

Step 3: Based on condition Eqs (3.4), (3.9) and Assumption $l_1\|z\|^\alpha \leq V(t, z) \leq l_2\|z\|^\alpha$, which can derive that

$$
\|z(t)\| \leq \chi(\|\varphi - \psi\|_v \exp(-\lambda(t - t_0))), \quad \forall t \geq t_0,
$$

where $\chi = \left(\frac{l_2}{l_1}\exp\left\{pN_0\rho^* + p\hat{\rho}_0 + N_0\sigma_+^* + \hat{\sigma}_+^0\right\}\right)^{\frac{1}{\alpha}}$, $\lambda = \frac{c\bar{t}-p\rho^*-\hat{\sigma}_+^*}{\alpha\bar{t}}$, which implies the system (2.2) can be exponentially synchronized with System (2.1) over the class $\mathscr{H}\left[\{t_u, \rho_u\}\right]$.

Remark 1: System under consideration with hybrid delayed impulses is discussed in Theorem 1 using the Lyapunov–Razumikhin method. Condition Eq (3.1) describes the continuous evolution of the considered system, and it follows from $a > 0$ that the continuous dynamics are stabilizing. Condition Eq (3.2) overviews the impulsive effect. In the case where $\Gamma_u = 0$, $\hat{\Gamma}_u = 1$, Eq (3.2) simplifies to $U(t_u, h(z)) \le \exp\{d\}\, U(\pi, z)$, where $\pi = t_u - \rho_u$ [20]. In the case where $\Gamma_u = 1$, $\hat{\Gamma}_u = 0$, Eq (3.2) simplifies to $U(t_u, h_u(z(0))) \le \exp\{d\}\, U(t_u^-, z(0))$ [7].

Remark 2: According to the derivation condition $\epsilon = q_1 \exp\{c\rho\} + q_2 \exp\{c\mu\}$ of Theorem 1, it can be learnt that the parameter c will increase with the decrease of ρ and μ. Furthermore, the synchronization rate $\lambda = \frac{c\bar{t}-p\rho^*-\hat{\sigma}_+^*}{\alpha\bar{t}}$ will increase. Therefore we have $\epsilon = q_1 exp\left\{\rho\frac{\lambda\alpha\bar{t}-p\rho^*+\hat{\sigma}_+^*}{\bar{t}}\right\} + q_2 exp\left\{\mu\frac{\lambda\alpha\bar{t}-p\rho^*+\hat{\sigma}_+^*}{\bar{t}}\right\}$. This means that, in some cases, system delays ρ might have potentially negative impact on the synchronization between the systems (2.1) and (2.2).

Remark 3: According to condition Eq (3.2), σ_u is called as impulsive estimate. In order to analyze the function of the impulsive estimation sequence $\{\sigma_u\}$, the concepts of average positive impulsive estimation (APIE) and AIE are introduced in this paper.

Assuming the existence of scalars $\hat{\sigma}_+^0 > 0$ and $\sigma_+^* > 0$ satisfying condition Eq (3.3), σ_+^* is called as APIE.

Supposing that there are some $\hat{\sigma}^0 > 0$ and $\sigma^* > 0$ such that

$$\sigma^* N(t, t_0) - \hat{\sigma}^0 \le \sum_{j=1}^{N(t,t_0)} \sigma_j \le \sigma^* N(t, t_0) + \hat{\sigma}^0, \tag{3.10}$$

where $N(t, t_0)$ is given in Definition 2, thus σ^* is referred to as AIE of the impulsive estimation sequence $\{\sigma_u\}$. When $\sigma_u > 0$, there is $\sigma^* = \sigma_+^*$. Since t_0 does not act as an impulse point, we assume that $\sigma_0 = 0$.

Remark 4: Actually, there exist some results about impulsive delays. In [30,31], the time delays had to have strict upper and lower bounds, or to be smaller than the length of impulse interval [16,20]. Even if [19,21,32] relaxed impulsive delays, impulsive interval should meet that $\inf_{u\in\mathcal{Z}_+}\{t_u - t_{u-1}\} \ge \varrho$ or $\sup_{u\in\mathcal{Z}_+}\{t_u - t_{u-1}\} \le \varrho$ for some $\varrho \ge 0$. The length of impulsive interval is flexible in Theorem 1.

When the delays of the continuous dynamics are not to be considered, systems (2.1) and (2.2) can be represented as

$$\begin{cases} \dot{\xi}(t) = C\xi(t) + Af(\xi(t)) + B\bar{f}(\xi(t)) + J, & t \ne t_u, \\ \xi(t_0 + s) = \psi(s), & s \in [-v, 0], \end{cases} \tag{3.11}$$

$$\begin{cases} \dot{\zeta}(t) = C\zeta(t) + Af(\zeta(t)) + B\bar{f}(\zeta(t)) + J, & t \ne t_u, \\ \zeta(t) - \zeta(t^-) = H(t), & t = t_u, \\ \zeta(t_0 + s) = \varphi(s), & s \in [-v, 0], \end{cases} \tag{3.12}$$

and the resulting error system for systems (1) and (2) follows as

$$\begin{cases} \dot{z}(t) = Cz(t) + Ag(z(t)) + B\bar{g}(z(t)), & t \ne t_u, \\ z(t) = D_u z(t^-) + E_u z((t - \rho_u)^-), & t = t_u, \\ z(t_0 + s) = \varphi(s) - \psi(s), & s \in [-v, 0], \end{cases} \tag{3.13}$$

Corollary 1: Suppose that Assumption 1 holds. If there exists a function $V \in \mathcal{V}^*$, scalars $p > 0$, c with $p > c > 0$, $\sigma_u = \ln\left(\sum_{i=1}^{n} \max_j \left|d_{ij}^{(u)}\right| + \sum_{i=1}^{n} \max_j \left|e_{ij}^{(u)}\right|\right)$, σ_u with $p\rho_u + \sigma_u > 0$, $\sigma^* > 0$ satisfying condition Eq (3.10), $\Gamma_u = \frac{\sum_{i=1}^{n} \max_j \left|d_{ij}^{(u)}\right|}{\exp(\sigma_u)}$, $\hat{\Gamma}_u = \frac{\sum_{i=1}^{n} \max_j \left|e_{ij}^{(u)}\right|}{\exp(\sigma_u)}$. In that case the exponential synchronization of systems (3.11) and (3.12) can be achieved while Eq (3.1) holds with $\epsilon = q_1 = q_2 = 0$ and Eq (3.2) holds, and

$$-c\bar{t} + p\rho^* + \sigma^* < 0.$$

Remark 5: Corollary 1 offers some criteria for exponential synchronization between system (2.1) and system (2.2) from the point of view of impulsive perturbation, which lowers the limitation on $\sigma_u > 0$. Most of previous works [17,20,25] need $\sigma_u < 0$ in the impulsive control case and $\sigma_u > 0$ in the impulsive perturbation case. Corollary 1 presents condition $p\rho_u + \sigma_u > 0$ which makes σ_u is flexible. If $\sigma_u > 0$, $p\rho_u + \sigma_u > 0$ always holds. If $\sigma_u < 0$, we just need $\rho_u > -\frac{\sigma_u}{p}$ which ensures above condition to hold. It's worth noting that, in the impulsive perturbation problem, the smaller σ_u is, ρ_u must be larger to compensate.

3.2. *The case of impulsive control*

In this subsection, from the perspective of impulsive control, we establish a number of criteria of exponential synchronization based on the concepts of AID and AIE. Moreover, we assume that $t_u - t_{u-1} \geq \rho > \rho_u$ and $t_u - t_{u-1} \geq \eta > \rho_u$, $u \in \mathcal{Z}_+$.

Theorem 2: Considering system (2.3) under Assumption 1. Suppose that there exists a function $V \in \mathcal{V}^*$, scalars $b_1 = \max_i c_{ii} + \sum_{i=1}^{n} \max_{j,j\neq i} \left|c_{ij}\right| + \sum_{i=1}^{n} \max_j \left|a_{ij}\right|$, $b_2 = \sum_{i=1}^{n} r_i \max_j \left|g_{ij}\right|$ and $b_3 = \sum_{i=1}^{n} \bar{r}_i \max_j \left|\hat{g}_{ij}\right|$, $\sigma_u = -\ln\left(\sum_{i=1}^{n} \max_j \left|d_{ij}^{(u)}\right| + \sum_{i=1}^{n} \max_j \left|e_{ij}^{(u)}\right|\right)$ with $\bar{\sigma} = \sup_{u\in\mathcal{Z}_+} \sigma_u > 0$, $\gamma > 0$ such that $\gamma > b_1 + b_2 \exp\{\bar{\sigma}\} + b_3 \exp\{\bar{\sigma}\}$, $\Gamma_u = \frac{\sum_{i=1}^{n} \max_j \left|d_{ij}^{(u)}\right|}{\exp\{\sigma_u\}}$, $\hat{\Gamma}_u = \frac{\sum_{i=1}^{n} \max_j \left|e_{ij}^{(u)}\right|}{\exp\{\sigma_u\}}$, the impulsive estimation sequence $\{\sigma_u\}$ satisfies the condition Eq (3.10), for every $t > 0$, the following inequalities hold:

$$D^+V(t, z(0)) \leq b_1 V(t, z(0)) + b_2 V(t-\rho, z(-\rho)) + b_3 \int_0^{\mu} k(s) V(t-s, z(-s)) ds, t \neq t_u,$$

$$V(t_u, h_u(-z(0), z(-\tau_u)) \leq \exp(\sigma_u)\left(\Gamma_u V(t_u^-, z(0)) + \hat{\Gamma}_u V((t_u - \rho_u)^-, z(-\rho_u))\right),$$

$$\gamma\bar{t} - \sigma^* < 0.$$

Then, drive system (2.1) can achieve exponential synchronization with response system (2.2) over the class $\mathscr{H}\left[\{t_u, \rho_u\}\right]$.

Proof: Define $V(t) = V(t, z(t)) = \|z(t)\| = \sum_{i=1}^{n} |z_i(t)|$, and $V_0 = \sup_{u\in[t_0-v,t_0]} V(u)$, such that

$$
\begin{aligned}
D^+V(t) =& D^+ \|z(t)\| = D^+ \sum_{i=1}^{n} |z_i(t)| \\
\leq& \left(\max_i c_{ii} + \sum_{i=1}^{n} \max_{j,j\neq i} \left|c_{ij}\right| + \sum_{i=1}^{n} \max_j \left|a_{ij}\right| \right) V(t) + \sum_{i=1}^{n} r_i \max_j \left|g_{ij}\right| V(t-\rho) \\
&+ \sum_{i=1}^{n} \bar{r}_i \max_j \left|\hat{g}_{ij}\right| \int_0^{\mu} k(s)V(t-s)ds \\
=& b_1 V(t, z(0)) + b_2 V(t-\rho, z(-\rho)) + b_3 \int_0^{\mu} k(s)V(t-s, z(-s))ds, \quad t \neq t_u.
\end{aligned}
$$

The proof is divided into three steps.

Step 1 : We will demonstrate that

$$
V(t) \leq \Omega_u V_0 \exp\{\gamma(t-t_0)\}, \tag{3.14}
$$

where $\Omega_u = \exp\left\{-\sum_{j=0}^{u} \sigma_j\right\}$.

First, we need show that the following two situations for $u \in \mathcal{Z}_+^0$.

(i). If $t^{\triangle} \in [t_0, t_1)$, one has

$$
\Theta(s) \leq \Theta(t^{\triangle}), \quad t_0 - v \leq s \leq t^{\triangle}. \tag{3.15}
$$

(ii). If $t^{\triangle} \in [t_u, t_{u-1})$, $u \in Z_+$, one can obtain that

$$
\Theta(s) \leq \Theta(t^{\triangle}), \quad t_u \leq s \leq t^{\triangle}, \tag{3.16}
$$

and

$$
\Theta(s)\exp\{\gamma(t_u - t_{u-1})\} \leq \Theta(t^{\triangle})\exp\{\bar{\sigma}\}, \quad t_{u-1} \leq s \leq t_u. \tag{3.17}
$$

Then $D^+\Theta(t)|_{t=t^{\triangle}} < 0$, where

$$
\Theta(t) = \begin{cases} V\exp\{-\gamma(t-t_u)\}, & t \in [t_u, t_{u+1}), u \in \mathcal{Z}_+, \\ V(t), & t_0 - v \leq t \leq t_0. \end{cases}
$$

Construct an auxiliary function with $\iota > 0$

$$
\Theta_\iota(t) = \begin{cases} V\exp\{-(\gamma+\iota)(t-t_u)\}, & t \in [t_u, t_{u+1}), u \in \mathcal{Z}_+, \\ V(t), & t_0 - v \leq t \leq t_0. \end{cases}
$$

Without loss of generality, we assume $\rho \geq \eta$. First, provided that Eq (3.15) holds.

If $t^{\triangle} - \eta \geq t^{\triangle} - \rho \geq t_0$, based on Eq (3.16), one has

$$\begin{aligned}
&\exp\{\iota(t^{\triangle} - t_0)\} D^{+}\Theta_{\iota}(t)|_{t=t^{\triangle}} \\
&= (D^{+}V(t)|_{t=t^{\triangle}} - (\gamma + \iota)V(t^{\triangle})) \exp\{-\gamma(t^{\triangle} - t_0)\} \\
&\leq (b_1 - \gamma - \iota)V(t^{\triangle}) \exp\{-\gamma(t^{\triangle} - t_0)\} + b_2 V(t^{\triangle} - \rho) \exp\{-\gamma(t^{\triangle} - \rho - t_0)\} \exp\{-\gamma\rho\} \\
&\quad + b_3 \int_0^{\mu} k(s)V(t^{\triangle} - s) \exp(-\gamma(t^{\triangle} - s - t_0)) \exp\{-\gamma s\}\, ds \\
&\leq (b_1 - \gamma - \iota)\Theta(t^{\triangle}) + b_2\Theta(t^{\triangle} - \rho) \exp\{-\gamma\rho\} + b_3 \int_0^{\mu} k(s)\Theta(t^{\triangle} - s) \exp\{-\gamma s\}\, ds \\
&\leq (b_1 - \gamma - \iota + b_2 + b_3)\Theta(t^{\triangle}).
\end{aligned}$$

If $t^{\triangle} - \rho < t_0 < t^{\triangle} - \eta$ or $t^{\triangle} - \rho \leq t^{\triangle} - \eta < t_0$, we can derive similarly that

$$\begin{aligned}
&\exp\{\iota(t^{\triangle} - t_0)\} D^{+}\Theta_{\iota}(t)|_{t=t^{\triangle}} \\
&\leq (b_1 - \gamma - \iota)V(t^{\triangle}) \exp\{-\gamma(t^{\triangle} - t_0)\} + b_2 V(t^{\triangle} - \rho) \exp\{-\gamma(t^{\triangle} - t_0)\} \\
&\quad + b_3 \int_0^{\mu} k(s)V(t^{\triangle} - s)ds \exp\{-\gamma(t^{\triangle} - t_0)\} \\
&\leq (b_1 - \gamma - \iota + b_2 + b_3)\Theta(t^{\triangle}).
\end{aligned}$$

Next, suppose that Eqs (3.16) and (3.17) hold.
If $t^{\triangle} - \eta \geq t^{\triangle} - \rho \geq t_u$, using Eq (3.16) we can found

$$\begin{aligned}
&\exp\{\iota(t^{\triangle} - t_u)\} D^{+}\Theta_{\iota}(t)|_{t=t^{\triangle}} \\
&\leq (b_1 - \gamma - \iota)V(t^{\triangle}) \exp\{-\gamma(t^{\triangle} - t_u)\} + b_2 V(t^{\triangle} - \rho) \exp\{-\gamma(t^{\triangle} - \rho - t_u)\} \exp\{-\gamma\rho\} \\
&\quad + b_3 \int_0^{\mu} k(s)V(t^{\triangle} - s) \exp\{-\gamma(t^{\triangle} - s - t_u)\} \exp\{-\gamma s\}\, ds \\
&\leq (b_1 - \gamma - \iota)\Theta(t^{\triangle}) + b_2\Theta(t^{\triangle} - \rho) \exp\{-\gamma\rho\} + b_3 \int_0^{\mu} k(s)\Theta(t^{\triangle} - s) \exp\{-\gamma s\}\, ds \\
&\leq (b_1 - \gamma - \iota + b_2 + b_3)\Theta(t^{\triangle}).
\end{aligned}$$

If $t^{\triangle} - \rho < t_u \leq t^{\triangle} - \eta$, by the fact $t_{u-1} \leq t^{\triangle} - \rho < t_u$, Eqs (3.16) and (3.17), it leads to

$$\begin{aligned}
&\exp\{\iota(t^{\triangle} - t_0)\} D^{+}\Theta_{\iota}(t)|_{t=t^{\triangle}} \\
&\leq (b_1 - \gamma - \iota)V(t^{\triangle}) \exp\{-\gamma(t^{\triangle} - t_u)\} + b_2 V(t^{\triangle} - \rho) \exp\{-\gamma(t^{\triangle} - \rho - t_{u-1})\} \exp\{\gamma(t_u - t_{u-1} - \rho)\} \\
&\quad + b_3 \int_0^{\mu} k(s)V(t^{\triangle} - s) \exp\{-\gamma(t^{\triangle} - s - t_u)\} \exp\{-\gamma s\}\, ds \\
&\leq (b_1 - \gamma - \iota)\Theta(t^{\triangle}) + b_2\Theta(t^{\triangle} - \rho) \exp\{\gamma(t_u - t_{u-1} - \rho)\} + b_3 \int_0^{\mu} k(s)\Theta(t^{\triangle} - s) \exp\{-\gamma s\}\, ds \\
&\leq (b_1 - \gamma - \iota + b_2 \exp\{\bar{\sigma}\} + b_3)\Theta(t^{\triangle}).
\end{aligned}$$

If $t^{\triangle}-\rho \leq t^{\triangle}-\eta < t_u$, because of $t_{u-1} \leq t^{\triangle}-\rho < t_u$, Eqs (3.16) and (3.17), one has

$$
\begin{aligned}
&\exp\{\iota(t^{\triangle}-t_0)\}\, D^{+}\Theta_{\iota}(t)|_{t=t^{\triangle}} \\
&\leq (b_1-\gamma-\iota)\Theta(t^{\triangle}) + b_2\Theta(t^{\triangle}-\rho)\exp\{\gamma(t_u-t_{u-1}-\rho)\} \\
&\quad + b_3\int_0^{\mu} k(s)V(t^{\triangle}-s)\exp\{-\gamma(t^{\triangle}-s-t_u)\})\exp\{-\gamma s\}\, ds \\
&\leq (b_1-\gamma-\iota)\Theta(t^{\triangle}) + b_2\Theta(t^{\triangle}-\rho)\exp\{\gamma(t_u-t_{u-1}-\rho)\} + b_3\int_0^{t^{\triangle}-t_u} k(s)\Theta(t^{\triangle}-s)\exp\{-\gamma s\}\, ds \\
&\quad + b_3\int_{t^{\triangle}-t_u}^{\eta} k(s)\Theta(t^{\triangle}-s)\exp\{\gamma(t_u-t_{u-1}-s)\}\, ds \\
&\leq (b_1-\gamma-\iota+b_2\exp\{\bar{\sigma}\})\Theta(t^{\triangle}) + b_3\exp\{\bar{\sigma}\}\,\Theta(t^{\triangle})\left(\int_0^{t^{\triangle}-t_u} k(s)ds + \int_{t^{\triangle}-t_u}^{\eta} k(s)ds\right) \\
&\leq (b_1-\gamma-\iota+b_2\exp\{\bar{\sigma}\}+b_3\exp\{\bar{\sigma}\})\Theta(t^{\triangle}).
\end{aligned}
$$

According to the above situations and $\gamma > b_1 + b_2\exp(\bar{\sigma}) + b_3\exp(\bar{\sigma})$, one can obtain that

$$
\begin{aligned}
\exp\{\iota(t^{\triangle}-t_u)\}\, D^{+}\Theta_{\iota}(t)|_{t=t^{\triangle}} &\leq (b_1-\gamma-\iota+b_2\exp\{\bar{\sigma}\}+b_3\exp\{\bar{\sigma}\})\Theta(t^{\triangle}) \\
&< -\iota\Theta(t^{\triangle}).
\end{aligned}
$$

It can be further deduced that

$$
\begin{aligned}
D^{+}\Theta(t)|_{t=t^{\triangle}} &= \exp\{\iota(t^{\triangle}-t_u)\}\, D^{+}\Theta_{\iota}(t)|_{t=t^{\triangle}} + \iota\exp\{\iota(t^{\triangle}-t_u)\}\,\Theta_{\iota}(t)|_{t=t^{\triangle}} \\
&< \iota\Theta(t^{\triangle}) - \iota\Theta(t^{\triangle}) \\
&= 0.
\end{aligned}
$$

Then, we shall show that

$$
\Theta(t) \leq \Omega_u V_0 \exp(\gamma(t_u-t_0)), t \in [t_u, t_{u-1}), u \in \mathcal{Z}^{+}. \tag{3.18}
$$

We can easily get $\Theta(t) \leq V_0$ when $t \in [t_0-\nu, t_0]$, so that $\Theta(t_0) \leq V_0$. Suppose Eq (3.18) is false for u=0, then there exists $t_{\triangledown} \in (t_0, t_1)$ makes $\Theta(t_{\triangledown}) > V_0$, $\Theta(t) \leq V_0$ for $t \in (t_0-\nu, t_{\triangledown})$ and $D^{+}\Theta(t)|_{t=t_{\triangledown}} \geq 0$, which is contrary to $D^{+}\Theta(t)|_{t=t_{\triangledown}} < 0$ in previous discussion. Provided that (3.18) holds for $u \leq U$, next we will prove that Eq (3.18) holds for $u = U+1$. Based on $\Theta(t) \leq \Omega_U V_0 exp\{\gamma(t_U-t_0)\}$, $t \in [t_U, t_{U+1})$, we have

$$
\begin{aligned}
\Theta(t_{U+1}) &= V(t_{U+1}) \\
&\leq \exp\{-\sigma_{U+1}\}\left(\Gamma_{U+1}V(t_{U+1}^{-}) + \hat{\Gamma}_{U+1}V((t_{U+1}-\rho_{U+1})^{-})\right) \\
&\leq \exp\{-\sigma_{U+1}\}\left(\Gamma_{U+1}\Omega_U V_0\exp\{\gamma(t_{U+1}-t_0)\} + \hat{\Gamma}_{U+1}\Omega_U V_0\exp\{\gamma(t_{U+1}-\rho_{U+1}-t_0)\}\right) \\
&\leq \exp\{-\sigma_{U+1}\}\,\Omega_U V_0\exp\{\gamma(t_{U+1}-t_0)\} \\
&\leq \Omega_{U+1}V_0\exp\{\gamma(t_{U+1}-t_0)\}.
\end{aligned}
$$

Therefore, Eq (3.18) holds for $t = t_{U+1}$. Suppose that there are some $t \in (t_{U+1}, t_{U+2})$ which leads to $\Theta(t) \geq \Omega_U V_0\exp\{\gamma(t_u-t_0)\}$, through the continuity of $V(t)$ in (t_{U+1}, t_{U+2}), we can get $\hat{t} \in$

(t_{U+1}, t_{U+2}) such that $\Theta(\hat{t}) = \Omega_{U+1} V_0 \exp\{\gamma(t_{U+1} - t_0)\}$, $\Theta(t) < \Omega_{U+1} V_0 \exp\{\gamma(t_{U+1} - t_0)\}$, $t \in (t_{U+1}, \hat{t}\,)$ and $D^{+}\Theta(t)|_{t=\hat{t}} \geq 0$. When $s \in [t_U, t_{U+1})$, it leads to

$$\begin{aligned}
&\Theta(s) \exp\{\gamma(t_{U+1} - t_U)\} \\
\leq &\Omega_U V_0 \exp\{\gamma(t_U - t_0)\} \exp\{\gamma(t_{U+1} - t_U)\} \\
= &\Omega_U V_0 \exp\{\gamma(t_{U+1} - t_0)\} \\
= &\Omega_{U+1} \exp\{\sigma_{U+1}\} V_0 \exp\{\gamma(t_{U+1} - t_0)\} \\
= &\Theta(\hat{t}) \exp\{\bar{\sigma}\}.
\end{aligned}$$

Thus, it follows from Eqs (3.16) and (3.17) that $D^{+}\Theta(t)|_{t=\hat{t}} < 0$, which is a contradiction with $D^{+}\Theta(t)|_{t=\hat{t}} \geq 0$.

Then we can conclude that Eq (3.18) is true through using mathematical induction.

Step 2 : From Eqs (3.8) and (3.10), it can be derived that

$$\begin{aligned}
V(t) \leq &V_0 exp\left\{-\sum_{j=0}^{N(t,t_0)} \sigma_j\right\} \exp\{\gamma(t - t_0)\} \\
\leq &V_0 \exp\left\{-(\sigma^* N(t, t_0) - \hat{\sigma}^0)\right\} \exp\{\gamma(t - t_0)\} \\
\leq &V_0 \exp\left\{-\left(\sigma^*\left(\frac{t - t_0}{\bar{t}} - N_0\right) - \hat{\sigma}^0\right)\right\} \exp\{\gamma(t - t_0)\} \\
\leq &V_0 \exp\left\{\left(\gamma - \frac{\sigma^*}{\bar{t}}\right)(t - t_0)\right\} \exp\left\{\sigma^* N_0 + \hat{\sigma}^0\right\}.
\end{aligned} \tag{3.19}$$

Based on condition $\gamma\bar{t} - \sigma^* < 0$, inequality (3.19) and Assumption $l_1 \|z\|^{\alpha} \leq V(t, z) \leq l_2 \|z\|^{\alpha}$, one has

$$\|z(t)\| \leq \chi \|\varphi - \psi\|_v \exp(-\lambda(t - t_0)), \quad t \geq t_0,$$

where $\chi = \left(\frac{l_2}{l_1} \exp\left\{\sigma^* N_0 + \hat{\sigma}^0\right\}\right)^{\frac{1}{\alpha}}$, $\lambda = \frac{\sigma^* - \gamma\bar{t}}{\alpha\bar{t}}$, which implies the system (2.2) can be exponentially synchronized with system (2.1) over the class $\mathscr{H}[\{t_m, \rho_m\}]$.

Remark 6: Theorem 2 presents some criteria for exponential synchronization between systems (2.1) and (2.2) from the point of view of impulsive control. Compared to previous studies [16–18,20], which have a common threshold for $\sigma > 0$ at each impulsive point, the results in this paper are less conservative and the rate coefficient σ_u is flexible here through the proposed concept of AIE.

4. Numerical examples

In this section, we give illustrative examples to show the effectiveness of the obtained results.

Example 1: We consider the error system (2.3) with parameters as

$$C = \begin{bmatrix} -0.9 & 0.01 \\ 0.02 & -0.9 \end{bmatrix},\ A = \begin{bmatrix} 0.11 & -0.15 \\ -0.2 & 0.1 \end{bmatrix}, B = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, \hat{B} = \begin{bmatrix} 0.29 & -0.31 \\ -0.32 & 0.28 \end{bmatrix},$$

$k(s) = \frac{1}{\mu}$, $0 \leq \rho_u \leq \bar{\rho}$, $v := \max\{\bar{\rho}, \rho, \mu\}$, $f(\cdot) = \bar{f}(\cdot) = 0.3tanh(\cdot)$, $\rho = 0.1$, $\mu = 0.4$, $t_u = 0.7u$ and

$$\rho_u = \begin{cases} 0, & u = 3m - 2, \\ 0.1, & u = 3m - 1, \\ 1.1, & u = 3m, \end{cases}$$

where $m \in \mathcal{Z}_+$, the matrices D_u, E_u can be chosen by

$$D_u = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, E_u = \begin{bmatrix} 0.85 & 0 \\ 0 & 0.85 \end{bmatrix}, \text{ when } \rho_u = 0;$$

$$D_u = \begin{bmatrix} 0.28 & 0 \\ 0 & 0.28 \end{bmatrix}, E_u = \begin{bmatrix} 0.85 & 0 \\ 0 & 0.85 \end{bmatrix}, \text{ when } \rho_u = 0.1;$$

$$D_u = \begin{bmatrix} 0.35 & 0 \\ 0 & 0.35 \end{bmatrix}, E_u = \begin{bmatrix} 0.45 & 0 \\ 0 & 0.45 \end{bmatrix}, \text{ when } \rho_u = 1.1.$$

Let $V(t) = \|z(t)\|$ in system (2.3), $c = 0.2$, $\epsilon = 0.7519$, it can be derived that $p = 0.2641$, and

$$\sigma_u = \begin{cases} -0.0512, & u = 3m-2, \\ 0.1222, & u = 3m-1, \\ -0.2231, & u = 3m. \end{cases}$$

Therefore, we can get $\bar{t} = 0.7$, $\rho^* = 0.4$, $\sigma_+^* = 0.0407$. Furthermore it can be obtained that $-c\bar{t} + p\rho^* + \sigma_+^* = -0.08166 < 0$. By Theorem 1, drive system (2.1) can achieve exponential synchronization with response system (2.2) over the class $\mathscr{H}\,[\{t_u, \rho_u\}]$, see Figure 1.

If the delays ρ, μ and the rate coefficient ρ are chosen as $\rho = 0.8$, $\mu = 1.4$ and $\rho = 0.3122$. We can find that all conditions in Theorem 1 are satisfied so that drive system (2.1) can achieve exponential synchronization with response system (2.2), see Figure 2. According to Remark 3, synchronization rate drops as the delay ρ or μ grows, which agrees well with the simulation result in Figure 2.

If D_u, E_u are selected as

$$D_u = \begin{bmatrix} 0.36 & 0 \\ 0 & 0.35 \end{bmatrix}, E_u = \begin{bmatrix} 1.31 & 0 \\ 0 & 1.46 \end{bmatrix}, \text{ when } \rho_u = 0;$$

$$D_u = \begin{bmatrix} 0.51 & 0 \\ 0 & 0.51 \end{bmatrix}, E_u = \begin{bmatrix} 0.86 & 0 \\ 0 & 0.86 \end{bmatrix}, \text{ when } \rho_u = 0.1;$$

$$D_u = \begin{bmatrix} 0.31 & 0 \\ 0 & 0.31 \end{bmatrix}, E_u = \begin{bmatrix} 0.82 & 0 \\ 0 & 0.82 \end{bmatrix}, \text{ when } \rho_u = 1.1,$$

where $u \in Z_+$, then by calculation we have

$$\sigma_u = \begin{cases} 0.5988, & u = 3m-2, \\ 0.3148, & u = 3m-1, \\ 0.1222, & u = 3m, \end{cases}$$

and $\sigma_u = 0.3452$. Under this circumstance, $-c\bar{t} + p\rho^* + \sigma_+^* = 0.2228 > 0$, this is contrary to condition Eq (3.4). Therefore, system (2.2) may not be able to achieve exponential synchronization with system (2.1), see Figure 3.

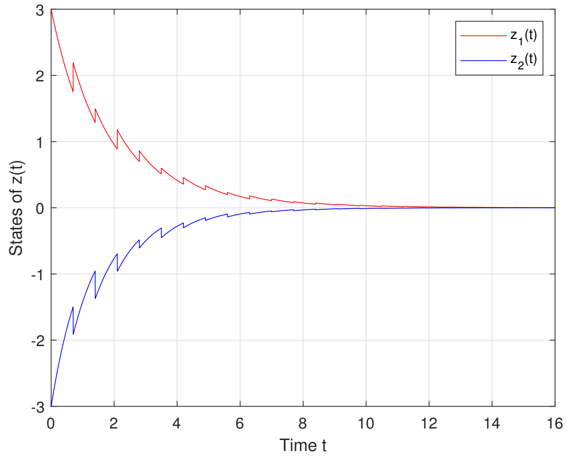

Figure 1. State trajectories of error system (2.3).

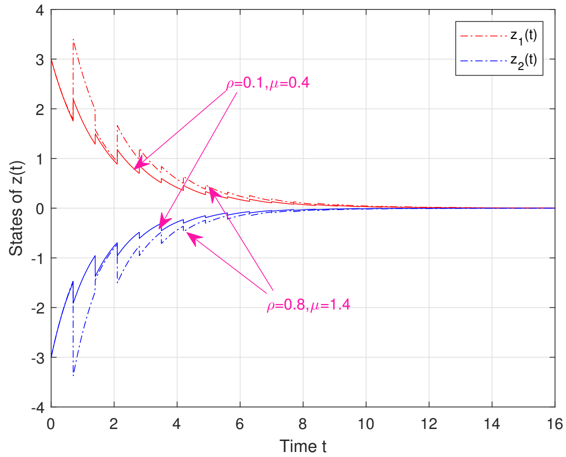

Figure 2. State trajectories of error system (2.3).

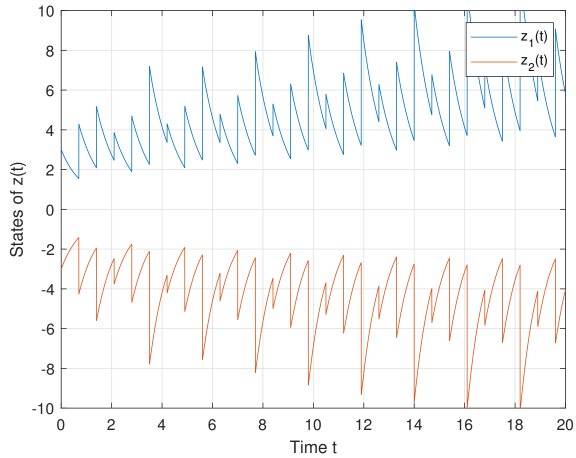

Figure 3. State trajectories of error system (2.3).

Example 2: We consider the following error system:

$$\begin{cases} \dot{z}(t) = 0.3z(t) + 0.03g(z(t)) + 0.08\int_0^{\mu} k(s)z(t-\rho)ds, & t \neq t_u, \\ z(t_u) = d_u z(t_u^-) + e_u z((t_u - \rho_u)^-), & t = t_u, \\ \delta(t_0 + s) = \varphi(s) - \psi(s), & s \in [-v, 0], \end{cases} \tag{4.1}$$

where $k(s) = \frac{1}{1-e^{-\mu}}e^{-s}$, $\rho = 0.8$, $\mu = 0.4$, $t_u = u$ and τ_u, c_u, d_u can be selected by

$$\rho_u = \begin{cases} 0, & u = 3m-2, \\ 0.1, & u = 3m-1, \\ 0.9, & u = 3m, \end{cases} \quad d_u = \begin{cases} 0.15, & \rho_u = 0, \\ 0.1, & \rho_u = 0.1, \\ 0.15, & \rho_u = 0.9, \end{cases} \quad e_u = \begin{cases} 0.5, & \rho_u = 0, \\ 0.3, & \rho_u = 0.1, \\ 0.9, & \rho_u = 0.9. \end{cases}$$

Let $V(t) = \|z(t)\|$ in system (4.1), there are

$$\sigma_u = \begin{cases} 0.4307, & u = 3m-2, \\ 0.916, & u = 3m-1, \\ -0.0487, & u = 3m, \end{cases}$$

and $\sigma^* = 0.4326$, $\bar{t} = 1$. Then we can choose $\gamma = 0.42 > 0.3 + 0.03 + 0.08$ makes condition $\gamma\bar{t} - \sigma^* = -0.0126 < 0$ is established. From Theorem 1, drive system (2.1) can achieve exponential synchronization with response system (2.2) over the class $\mathscr{H}[\{t_u, \rho_u\}]$, see Figure 4 (solid red line). It is clear that, owing to the idea of AIE, it is not necessary that σ_u is positive for every $u \in \mathcal{Z}_+$.

If d_u, e_u are

$$d_u = \begin{cases} 0.44, & \rho_u = 0, \\ 0.5, & \rho_u = 0.1, \\ 0.4, & \rho_u = 0.9, \end{cases} \quad e_u = \begin{cases} 0.4, & \rho_u = 0, \\ 0.4, & \rho_u = 0.1, \\ 0.5, & \rho_u = 0.9, \end{cases}$$

then it can be figured out that

$$\sigma_u = \begin{cases} 0.1743, & u = 3m-2, \\ 0.1053, & u = 3m-1, \\ 0.0943, & u = 3m, \end{cases}$$

therefore, $\sigma^* = 0.1246$. Under this circumstance, $\gamma\bar{t} - \sigma^* = 0.2954 > 0$, which contradicts one of the conditions of Theorem 2, i.e., $\gamma\bar{t} - \sigma^* < 0$. Consequently, system (2.2) may not be able to achieve exponential synchronization with system (2.1), see Figure 4 (dotted blue line).

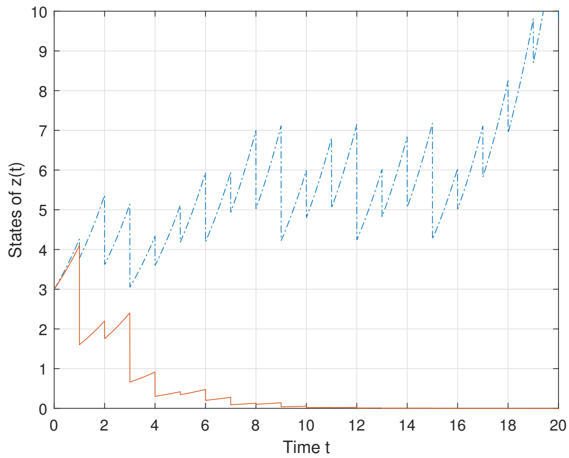

Figure 4. State trajectories of error system (4.1).

5. Conclusions

In this paper, we have explored the issue of synchronization for a class of impulsive systems with discrete and distributed delay. Sufficient Lyapunov conditions for the synchronization of the considered system under impulse perturbation and impulse control are established, respectively. It is worth noting that the concepts of AIE and APIE, which are presented in this paper, make impulsive estimation more flexible and relax the constraint of a common threshold. Theoretical results show that time delay size of continuous dynamics is variable and does not have a strict magnitude relationship with impulsive delay. In addition, the obtained display inequalities indicate that time delay of a continuous system may have a potential effect on synchronization. However, from the perspective of impulsive control, there exists a limitation between system time delay size and impulsive interval length, which is an issue to be discussed in subsequent work.

Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare that there are no conflicts of interest.

References

1. S. Dashkovskiy, P. J. Feketa, Input-to-state stability of impulsive systems and their networks, *Nonlinear Anal. Hybrid Syst.*, **26** (2017), 190–200. https://doi.org/10.1016/j.nahs.2017.06.004
2. G. Bertola, W. J. Runggaldier, K. Yasuda, On classical and restricted impulse stochastic control for the exchange rate, *Appl. Math. Optim.*, **74** (2016), 423–454. https://doi.org/10.1007/s00245-015-9320-6

3. C. Treesatayapun, Impulsive optimal control for drug treatment of influenza A virus in the host with impulsive-axis equivalent model, *Inf. Sci.*, **576** (2021), 122–139. https://doi.org/10.1016/j.ins.2021.06.051
4. T. Ensari, S. Arik, Global stability analysis of neural networks with multiple time varying delays, *IEEE Trans. Autom. Control*, **50** (2005), 1781–1785. https://doi.org/10.1109/TAC.2005.858634
5. M. A. Davo, A. Banos, F. Gouaisbaut, S. Tarbouriech, A. Seuret, Stability analysis of linear impulsive delay dynamical systems via looped-functionals, *Automatica*, **81** (2017), 107–114. https://doi.org/10.1016/j.automatica.2017.03.029
6. W. Chen, K. Zhang, X. Lu, A refined discretized timer-dependent Lyapunov functional for impulsive delay systems, *Automatica*, **134** (2021), 109929. https://doi.org/10.1016/j.automatica.2021.109929
7. Z. Tang, J. H. Park, J. Feng, Impulsive effects on quasi-synchronization of neural networks with parameter mismatches and time-varying delay, *IEEE Trans. Neural Networks Learn. Syst.*, **29** (2017), 908–919. https://doi.org/10.1109/TNNLS.2017.2651024
8. W. Ren, J. Xiong, Stability analysis of impulsive switched time-delay systems with state-dependent impulses, *IEEE Trans. Autom. Control*, **64** (2019), 3928–3935. https://doi.org/10.1109/TAC.2018.2890768
9. J. Lu, B. Jiang, W. Zheng, Potential impacts of delay on stability of impulsive control systems, *IEEE Trans. Autom. Control*, **67** (2021), 5179–5190. https://doi.org/10.1109/TAC.2021.3120672
10. A. Khadra, X. Z. Liu, X. Shen, Impulsively synchronizing chaotic systems with delay and applications to secure communication, *Automatica*, **41** (2005), 1491–1502. https://doi.org/10.1016/j.automatica.2005.04.012
11. K. P. Wijaya, J. P. Chavez, T. Gotz, A dengue epidemic model highlighting vertical-sexual transmission and impulsive control strategies, *Appl. Math. Modell.*, **95** (2021), 279–296. https://doi.org/10.1016/j.apm.2021.02.008
12. F. Cacace, V. Cusimano, P. Palumbo, Optimal impulsive control with application to antiangiogenic tumor therapy, *IEEE Trans. Control Syst. Technol.*, **28** (2018), 106–117. https://doi.org/10.1109/TCST.2018.2861410
13. P. S. Rivadeneira, C. H. Moog, X. Zhao, F. E. Alsaadi, Observability criteria for impulsive control systems with applications to biomedical engineering processes, *Automatica*, **55** (2015), 125–131. https://doi.org/10.1016/j.automatica.2015.02.042
14. X. Li, W. Liu, S. Gorbachev, J. Cao, Event-Triggered impulsive control for Input-to-State stabilization of nonlinear Time-Delay systems, *IEEE Trans. Cybern.*, 2023. https://doi.org/10.1109/TCYB.2023.3270487
15. X. Li, K. Zhang, Synchronization of linear dynamical networks on time scales: Pinning control via delayed impulses, *Automatica*, **72** (2016), 147–152. https://doi.org/10.1016/j.automatica.2016.06.001
16. X. Li, P. Li, Stability of time-delay systems with impulsive control involving stabilizing delays, *Automatica*, **124** (2021), 109336. https://doi.org/10.1016/j.automatica.2020.109336
17. B. Jiang, J. Lu, Y. Liu, Exponential stability of delayed systems with average-delay impulses, *SIAM J. Control Optim.*, **58** (2020), 3763–3784. https://doi.org/10.1137/20M1317037

18. X. Li, S. Song, Stabilization of delay systems: delay-dependent impulsive control, *IEEE Trans. Autom. Control*, **62** (2016), 406–411. https://doi.org/10.1109/TAC.2016.2530041

19. X. Liu, K. Zhang, Input-to-state stability of time-delay systems with delay-dependent impulses, *IEEE Trans. Autom. Control*, **65** (2019), 1676–1682. https://doi.org/10.1109/TAC.2019.2930239

20. X. Li, S. Song, J. Wu, Exponential stability of nonlinear systems with delayed impulses and applications, *IEEE Trans. Autom. Control*, **64** (2019), 4024–4034. https://doi.org/10.1109/TAC.2019.2905271

21. W. Chen, W. Zheng, Exponential stability of nonlinear time-delay systems with delayed impulse effects, *Automatica*, **47** (2011), 1075–1083. https://doi.org/10.1016/j.automatica.2011.02.031

22. T. Huang, C. Li, S. Duan, J. A. Starzyk, Robust exponential stability of uncertain delayed neural networks with stochastic perturbation and impulse effects, *IEEE Trans. Neural Networks Learn. Syst.*, **23** (2012), 866–875. https://doi.org/10.1109/TNNLS.2012.2192135

23. J. Lu, D. W. C. Ho, J. Cao, J. Kurths, Exponential synchronization of linearly coupled neural networks with impulsive disturbances, *IEEE Trans. Neural Networks*, **6** (2018), 52570–52581. https://doi.org/10.1109/TNN.2010.2101081

24. X. Li, X. Zhang, S. Song, Effect of delayed impulses on input-to-state stability of nonlinear systems, *Automatica*, **76** (2017), 378–382. https://doi.org/10.1016/j.automatica.2016.08.009

25. X. Lv, J. Cao, X. Li, M. Abdel-Aty, U. A. Al-Juboori, Synchronization analysis for complex dynamical networks with coupling delay via event-triggered delayed impulsive control, *IEEE Trans. Cybern.*, **51** (2020), 5269–5278. https://doi.org/10.1109/TCYB.2020.2974315

26. N. Wang, X. Li, J. Lu, M. F. E. Alsaadi, Unified synchronization criteria in an array of coupled neural networks with hybrid impulses, *Neural Networks*, **101** (2018), 25–32. https://doi.org/10.1016/j.neunet.2018.01.017

27. X. Li, N. Wang, J. Lou, J. Lu, Global μ-synchronization of impulsive pantograph neural networks, *Neural Networks*, **131** (2020), 78–92. https://doi.org/10.1016/j.neunet.2020.07.004

28. Z. Guan , Z. Liu , G. Feng, Y. Wang, Synchronization of complex dynamical networks with time-varying delays via impulsive distributed control, *IEEE Trans. Circuits Syst. I Regul. Pap.*, **57** (2010), 2182–2195. https://doi.org/10.1109/TCSI.2009.2037848

29. W. Chen, W. Zheng, Input-to-state stability and integral input-to-state stability of nonlinear impulsive systems with delays, *Automatica*, **45** (2009), 1481–1488. https://doi.org/10.1016/j.automatica.2009.02.005

30. W. Chen, J. Chen, X. Lu, Effects of impulse delays on Lp-stability of a class of nonlinear time-delay systems, *J. Franklin. Inst.*, **357** (2020), 7983–8007. https://doi.org/10.1016/j.jfranklin.2020.05.028

31. W. Chen, Z. Ruan, W. X. Zheng, Stability and L_2-gain analysis for linear time-delay systems with delayed impulses: An augmentation-based switching impulse approach, *IEEE Trans. Autom. Control*, **64** (2019), 4209–4216. https://doi.org/10.1109/TAC.2019.2893149

32. W. Chen, J. Wen, X. Lu, S. Niu, New stability criteria for linear impulsive systems with interval impulse-delay, *J. Franklin. Inst.*, **358** (2021), 6775–6797. https://doi.org/10.1016/j.jfranklin.2021.07.011

AIMS Press