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Research article

A predator-prey fractional model with disease in the prey species

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Abstract: In this paper, we study a generalized eco-epidemiological model of fractional order for the predator-prey type in the presence of an infectious disease in the prey. The proposed model considers that the disease infects the prey, causing them to be divided into two classes, susceptible prey and infected prey, with different density-dependent predation rates between the two classes. We propose logistic growth in both the prey and predator populations, and we also propose that the predators have alternative food sources (i.e., they do not feed exclusively on these prey). The model is evaluated from the perspective of the global and local generalized derivatives by using the generalized Caputo derivative and the generalized conformable derivative. The existence, uniqueness, non-negativity, and boundedness of the solutions of fractional order systems are demonstrated for the classical Caputo derivative. In addition, we study the stability of the equilibrium points of the model and the asymptotic behavior of its solution by using the Routh-Hurwitz stability criteria and the Matignon condition. Numerical simulations of the system are presented for both approaches (the classical Caputo derivative and the conformable Khalil derivative), and the results are compared with those obtained from the model with integro-differential equations. Finally, it is shown numerically that the introduction of a predator population in a susceptible-infectious system can help to control the spread of an infectious disease in the susceptible and infected prey population.

Keywords: eco-epidemiological model; prey-predator model; susceptible-infected model; fractional order epidemiological model

1. Introduction

In recent decades, the development of new mathematical theories has introduced challenges that affect their application, particularly in efforts to find new and better solutions to problems arising in different areas related to human knowledge. This has led to the rethinking and development of new mathematical models related to biological, chemical, physical, and social processes, among others.

Particularly, in ecology, there are mathematical models that allow for the description of different types of interactions between species, such as competition, mutualism, amensalism, and antagonism, among others. One of the most studied models is the one proposed by Lotka and Volterra, which describes the dynamics between predators and prey [1, 2]. A variant of this model is presented when a change in one of the populations is incorporated, which occurs naturally in ecosystems, such as during the outbreak of an infectious disease in one of the populations. This can be studied, for example, by coupling the dynamics of population models (predator-prey type) and epidemiological models, thus giving rise to the so-called eco-epidemiological models. In this way, a susceptible-infected-susceptible (SIS) or a susceptible-infected-recovered (SIR) system can be studied, such as the one proposed by Kermack and McKendrick, which was proposed in 1927 [3], as coupled with a Lotka-Volterra-type system. The study of these eco-epidemiological models is of interest since it allows us to understand the role played by an infectious disease in species interacting in an ecosystem [4, 5]. These types of models can be as diverse as the type of diseases and relationships present in an ecosystem [6, 7]. There are studies that have focused on eco-epidemiological models with diseases in the prey [8–10], predators [11–13], or both populations [14, 15]. Another area of theoretical mathematics that has taken great importance in recent years is the so-called fractional calculus, which focuses on the study of fractional differential operators (derivatives and integrals of fractional order), which begins with the contributions of Newton, Leibniz, Lacroix, Euler, Riemann, Liouville, Caputo, Grünwald, and Letnikov, among others. In that sense, there are two approaches to fractional calculus, known as the global approach and the local approach. In the global approach, the derivatives are defined by means of an integral, which depends on the values taken by the function in an integration interval; therefore, these derivatives totally or partially conserve the behavior of the function. For this approach, there are two well-known schemes: the Riemann-Lioville scheme and the Caputo scheme [16, 17]. Bosch et al. [18] proposed an approach that generalizes the Caputo derivative through the use of a general kernel function, i.e., it is $F(\chi; q)$-admissible. On the other hand, local derivatives are defined similarly to the definition of the classical derivative of integer order as the limit of a quotient. This type of derivative can preserve some properties that are present in the classical derivative; what is more, they can converge to the classical derivatives (when the order of the derivative converges to a natural number), and when this happens, they are called conformable. Among the best known proposals is the conformable Khalil derivative [19], which is obtained by using the kernel t^{1-q}. This scheme coincides with the classical derivative when the order of the derivative is a positive integer. Other schemes for nonconformable derivatives have also been proposed, such as that developed Guzman et al. [20]. Recently, Fleitas et al. [21] proposed a scheme that generalizes the nonconformable and conformable derivatives. This new definition is based on introducing a perturbation through the use of a positive function (kernel), $T(t, q)$, in the classical definition of the n-th derivative of a function at a point t.

The importance that fractional calculus has acquired in recent years is partly due to the fact that, in the fields of engineering, physics, economics, and, in general, other applied sciences, applications have appeared in which, in the solution of inverse problems for the adjustment of data, better results have been obtained than those obtained through the use of classical calculus [22–29]. This is due to the fact that, by proposing mathematical models with fractional order derivatives, a new parameter is introduced to the models (the order of the derivative), and, in this way, we can achieve a better fit to a set of real data and thus better predict the evolution of the modeled system, such as models for the spread of a disease [22, 26]. In addition, the use of generalized derivatives (e.g., generalized Caputo or generalized conformable) can contribute in that direction by giving more degrees of freedom through the choice of an admissible function and its parameters when solving an inverse problem; furthermore, the selection of the kernel function within the admissible functions could, in itself, improve the fit. Inverse problems involve inferring or estimating the parameters of a model of some system under study. Some of these parameters can be easily obtained from previous studies, while others require calibration or estimation processes. This problem can be approached from a statistical point of view by using Bayesian inference. The problem of parameter estimation for integer order systems and fractional order systems has been of great interest in recent years, and the use of Bayesian inversion has proven to be efficient and suitable as a tool to solve inverse problems; see [23–25]; for more information on fractional models, see [10, 30–37]. Other phenomena that naturally involve the use of fractional order derivatives are chaos models; see [38] and the tautochrone problem proposed by Abel in [39,40]. One of the questions that arise when studying systems modeled with fractional order equations is as follows: are the properties of these systems preserved with integer order differential equations (such as equilibrium points, stability, solution curves, etc.)? In this paper, we present the analysis of a predator-prey-type eco-epidemiological system with disease in the prey population from three different perspectives, proposing the model with integer order derivatives by using a global fractional derivative (Caputo fractional derivative) and a local fractional derivative (conformable Khalil derivative). The paper is organized as follows. The next section presents some preliminary results for the generalized Caputo fractional derivative and the generalized conformable fractional derivative. In Section 3, we provide a detailed explanation of the proposed eco-epidemiological model. In Section 4, we prove the existence and uniqueness, as well as the non-negativity and boundedness of the solutions, for the Caputo fractional system. In addition, the conditions for the existence of equilibrium points and their stability are shown. Numerical simulations of our theoretical results are presented in Section 5. Finally, conclusions are presented in Section 6.

2. Mathematical preliminaries

In this section, we will introduce some definitions and some useful lemmas for the generalized Caputo derivative [18] and the generalized conformable derivative [21].

Definition 1. *F is an admissible kernel for the interval $[a, b]$ if $F : [0, b-a] \times (0, 1) \to [0, \infty)$ is a non-negative continuous function such that*

$$\int_0^{b-a} \frac{d\tau}{F(\tau, q)} < \infty \tag{2.1}$$

for each $q \in (0, 1)$. Moreover, F is an admissible kernel for $[a, \infty)$ if it is admissible for $[a, b]$ for every $b > a$.

Definition 2. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable function for the interval (a, b), F be an admissible kernel for $[a, b]$, and let $t \in [a, b]$ and $q \in (0, 1)$. The generalized Caputo derivative of order q of the function f at the point t is given by*

$${}^{C}D^{q}_{F,a} f(t) = \int_a^t \frac{f^{'}(\tau)}{F(t-\tau, q)} d\tau. \tag{2.2}$$

Remark 1. *Note that, if* $F(\chi, q) = \Gamma(1-q)\chi^q$*, then we obtain the classical Caputo derivative:*

$$ {}^{C}D^{q}_{F,a}f(t) = \int_a^t \frac{f'(\tau)}{F(t-\tau,q)}d\tau \approx \int_a^t \frac{f'(\tau)}{\Gamma(1-q)(t-\tau)^q}d\tau = \frac{1}{\Gamma(1-q)}\int_a^t \frac{f'(\tau)}{(t-\tau)^q}d\tau. \qquad (2.3) $$

Similarly, if $F(\chi, q) = \frac{(1-q)e^{\frac{q\chi}{1-q}}}{M(q)}$ *or* $F(\chi, q) = \frac{(1-q)}{M(q)E_q\left(-\frac{q\chi^q}{1-q}\right)}$*, we can obtain the Caputo-Fabrizio [41] and Atangana-Baleanu [42] extensions, respectively.*

The following integral operator is associated with the generalized Caputo derivative.

Definition 3. *Let* $f : [a, b] \to \mathbb{R}$ *be a differentiable function,* F *be an admissible kernel on* $[a, b]$*,* $t \in [a, b]$*, and* $q \in (0, 1)$*; then, the generalized Caputo integral operator of order* q *of the function* f *at the point* t *is defined as*

$$ {}^{C}J^{q}_{F,a}f(t) = \int_a^t \frac{f(\tau)}{F(t-\tau,q)}d\tau. \qquad (2.4) $$

Hence, ${}^{C}D^{q}_{F,a}f(t) = {}^{C}J^{q}_{F,a}f'(t)$.

Definition 4. *Let* F *be an admissible kernel on* $[a, b]$*,* $n \in \mathbb{Z}^+$*,* $q \in (n-1, n)$*, and* $t \in [a, b]$*. For an* n*-times differentiable function* $f : [a, b] \to \mathbb{R}$*, the generalized Caputo derivative of* f *of order* q *at the point* t *is given by*

$$ {}^{C}D^{q}_{F,a}f(t) = \int_a^t \frac{f^{(n)}(\tau)}{F(t-\tau,q+1-n)}d\tau. \qquad (2.5) $$

Proposition 1. *Let* F *be an admissible kernel on* $[a, b]$*,* $n \in \mathbb{Z}^+$*, and* $q \in (0, 1)$*. If* f *is an* $(n+1)$*-differentiable function on* $[a, b]$*, then*

$$ {}^{C}D^{q+n}_{F,a}f(t) = {}^{C}D^{q}_{F,a}f^{(n)}(t). \qquad (2.6) $$

Remark 2. *Note that the above proposition is important since we write* ${}^{C}D^{q+n}_{F,a}$ *as a composition of a local operator and a non-local operator.*

For more information about the generalized Caputo derivative, see [18]. In addition to these definitions, a new definition called the conformable fractional derivative was introduced by Khalil et al. in 2014 [19]; this definition introduces a perturbation through the use of a function (kernel) for the definition of the classical derivative. Following the same idea, in [21], a new definition of the generalized conformable derivative is proposed through the use of a general kernel $T(t, q)$ in the classical definition of the n-th derivative of a function at a point t, which is defined as follows.

Definition 5. *Let* $I \subseteq \mathbb{R}$ *be an interval, a positive continuous function* $T(t, q)$ *on the interval* I*,* $f : I \to \mathbb{R}$*, and* $q \in \mathbb{R}^+$*; then, the derivative* $G^q_T f$ *of* f *of order* q *at the point* $t \in I$ *is defined by*

$$ G^q_T f(t) = \lim_{h\to 0}\frac{1}{h^{\lceil q\rceil}}\sum_{k=0}^{\lceil q\rceil}(-1)^k\binom{\lceil q\rceil}{k}f(t-khT(t,q)). \qquad (2.7) $$

If $b = max\{t \in I\}$ (respectively, $a = min\{t \in I\}$), then $G^q_T f(b)$ (respectively, $G^q_T f(a)$) is defined by using $h \to 0^+$ (respectively, $h \to 0^-$) instead of $h \to 0$ in the limit.

Remark 3. *If there exists a neighborhood of the point t for which f is defined and there exists $D^n f(t)$, then*

$$D^n f(t) = \lim_{h\to 0} \frac{1}{h^n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} f(t - kh). \tag{2.8}$$

Consequently, if f is smooth enough and $q = n \in \mathbb{N}$, then Definition 5 coincides with the classical definition of the n-th derivative of f.

Remark 4. *If $T(t, q) = t^{1-q}$, then we obtain the conformable Khalil derivative defined in [19].*

Some basic properties of the derivative $G_T^q f(t)$ can be found in [19, 21]. The following results will be used later.

Theorem 1. *Let $I \subseteq \mathbb{R}$ be an interval, $f : I \to \mathbb{R}$, and $q \in \mathbb{R}^+$.*

i. If there exists $D^{\lceil q\rceil} f$ at the point $t \in I$, then f is G_T^q-differentiable at t and $G_T^q f(t) = T(t, q)^{\lceil q\rceil} D^{\lceil q\rceil} f(t)$.
ii. If $q \in (0, 1]$, then f is G_T^q-differentiable at $t \in I$ if and only if f is differentiable at t; in this case, we have that $G_T^q f(t) = T(t, q) f'(t)$.

Theorem 2. *(Chain rule). Let $q \in (0, 1]$, g by a G_T^q-differentiable function at t, and f be a differentiable function at g(t). Then, $f \circ g$ is G_T^q-differentiable in t and $G_T^q (f \circ g)(t) = f'(g(t)) G_T^q g(t)$.*

3. Model description

The eco-epidemiological model proposed describes the interaction between a prey population and a predator population. The prey population is divided into two disjoint classes, i.e., the susceptible prey population, denoted by $S(t)$, and the infected prey population, denoted by $I(t)$. In this way, at time t, the total prey population is $N(t) = S(t) + I(t)$. On the other hand, the predator population is denoted by $Y(t)$, and this class is associated with an increase in its mortality due to the consumption of infected prey. This model is based on the following assumptions:

A1) We consider that the transmission of disease occurs through the contact between susceptible prey and infected prey, according to the mass action law.
A2) We do not consider recovery of the infected prey population because this population dies so fast due to disease that its reproduction is not possible.
A3) In the absence of predators and infected prey ($Y(t) = 0$ and $I(t) = 0$), the susceptible population exhibits grows logistically.
A4) Both susceptible and infected populations are prey for predators. However, infected prey are easier to hunt, as the disease weakens them and exposes them more to predators.
A5) The term $\frac{p_1 S^2 Y}{1+S+\alpha I}$ can be written as follows: $(SY)\left(\frac{p_1 S}{1+S+\alpha I}\right)$, i.e., the first factor, where (SY) represents the number of possible encounters between the susceptible prey and predators; and, the second one, i.e., $\frac{p_1 S}{1+S+\alpha I}$, can be considered as a density-dependent consumption rate of S. Note that, if $S \to 0$, then $\frac{p_1 S}{1+S+\alpha I} \to 0$; on the other hand, if $S \to k_1$, then $\frac{p_1 S}{1+S+\alpha I} \to p_1$; therefore, $0 < \frac{p_1 S}{1+S+\alpha I} < p_1$. Thus, p_1 is the maximum consumption rate for the predators. A similar interpretation can be given for the term $\frac{p_2 I^2 Y}{1+S+\alpha I}$. Another interpretation of this term can be found in [9, 43–45].

A6) The predator population has an alternative food source, i.e., the predators do not feed exclusively on this type of prey.

With the above assumptions, we propose the following ordinary differential equation system to model the aforementioned phenomenon:

$$
\begin{aligned}
\dot{S} &= r_1 S\left(1 - \frac{S+I}{k_1}\right) - \lambda S I - \frac{p_1 S^2 Y}{1+S+\alpha I},\\
\dot{I} &= \lambda S I - \frac{p_2 I^2 Y}{1+S+\alpha I} - \gamma I,\\
\dot{Y} &= r_2 Y\left(1 - \frac{Y}{k_2+S+mI}\right) + \delta_1\left(\frac{p_1 S^2 Y}{1+S+\alpha I}\right) - \delta_2\left(\frac{p_2 I^2 Y}{1+S+\alpha I}\right),
\end{aligned}
\tag{3.1}
$$

with the initial conditions $S(0) \geq 0$, $I(0) \geq 0$, and $Y(0) \geq 0$. The detailed biological meanings of the parameters are given in Table 1.

Table 1. Biological meanings of parameters.

Parameter	Description
r_1	Per capita growth rate of prey in the susceptible subpopulation.
r_2	Per capita growth rate of predators.
k_1	Carrying capacity of the total prey population.
k_2	Carrying capacity of the predator population.
λ	Disease transmission rate between susceptible and infected prey.
γ	Per capita mortality rate of infected prey due to the disease.
p_1, p_2	Maximum rates of consumption of susceptible and infected prey, respectively.
α	Predator's rate of preference for the infected subpopulation over the susceptible subpopulation.
m	Rate of reduction in the carrying capacity of the predator population through the consumption of infected prey.
δ_1, δ_2	Predator benefit (damage) rates due to the consumption of susceptible (infected) prey.

Thus, the variable (Ω) and parameter (Υ) spaces of interest are respectively given by

$$
\begin{aligned}
\Omega &= \left\{(S, I, Y) \in \mathbb{R}^3_+ : S \geq 0, I \geq 0, Y \geq 0\right\} \quad \textit{and}\\
\Upsilon &= \left\{(r_1, r_2, p_1, p_2, k_1, k_2, \delta_1, \delta_2, \alpha, \lambda, \gamma, m) \in \mathbb{R}^{12} : r_1, r_2, p_1, p_2, k_1, k_2, \lambda, \gamma > 0\ ;\ m, \alpha, \delta_1, \delta_2 \geq 0\right\}.
\end{aligned}
\tag{3.2}
$$

The idea of fractional calculus is a generalization of the notion of differentiation and integration from integer order to arbitrary order. In this way, the ordinary differential equation system (3.1) can be written in terms of the generalized Caputo derivative, as follows:

$$
\begin{aligned}
{}^{C}D^{q}_{F,a}S(t) &= r_1 S\left(1-\frac{S+I}{k_1}\right)-\lambda SI-\frac{p_1S^2Y}{1+S+\alpha I},\\
{}^{C}D^{q}_{F,a}I(t) &= \lambda SI-\frac{p_2I^2Y}{1+S+\alpha I}-\gamma I,\\
{}^{C}D^{q}_{F,a}Y(t) &= r_2 Y\left(1-\frac{Y}{k_2+S+mI}\right)+\delta_1\left(\frac{p_1S^2Y}{1+S+\alpha I}\right)-\delta_2\left(\frac{p_2I^2Y}{1+S+\alpha I}\right),
\end{aligned}
\tag{3.3}
$$

with $q \in (0,1)$.
Moreover, if $F(\chi, q) = \Gamma(1-q)\chi^q$, and following the method proposed in [27], the system (3.3) for the Caputo fractional derivative is given by

$$
\begin{aligned}
{}^{C}D^{q}_{t}S(t) &= r_1^q S\left(1-\frac{S+I}{k_1^q}\right)-\lambda^q SI-\frac{p_1^qS^2Y}{1+S+\alpha^q I},\\
{}^{C}D^{q}_{t}I(t) &= \lambda^q SI-\frac{p_2^qI^2Y}{1+S+\alpha^q I}-\gamma^q I,\\
{}^{C}D^{q}_{t}Y(t) &= r_2^q Y\left(1-\frac{Y}{k_2^q+S+m^qI}\right)+\delta_1^q\left(\frac{p_1^qS^2Y}{1+S+\alpha^q I}\right)-\delta_2^q\left(\frac{p_2^qI^2Y}{1+S+\alpha^q I}\right).
\end{aligned}
\tag{3.4}
$$

On the other hand, the model described by the ordinary differential equation system (3.1) in terms of the generalized conformable derivative has the following form:

$$
\begin{aligned}
G^q_T S(t) &= r_1 S\left(1-\frac{S+I}{k_1}\right)-\lambda SI-\frac{p_1S^2Y}{1+S+\alpha I},\\
G^q_T I(t) &= \lambda SI-\frac{p_2I^2Y}{1+S+\alpha I}-\gamma I,\\
G^q_T Y(t) &= r_2 Y\left(1-\frac{Y}{k_2+S+mI}\right)+\delta_1\left(\frac{p_1S^2Y}{1+S+\alpha I}\right)-\delta_2\left(\frac{p_2I^2Y}{1+S+\alpha I}\right),
\end{aligned}
\tag{3.5}
$$

with $q \in (0,1)$.
Note that taking $T(t,q) = t^{1-q}$, the generalized conformable system (3.5) becomes the conformable Khalil derivative [19]; then, applying Theorem 1(ii), such a system can be written as follow:

$$
\begin{aligned}
t^{1-q}\dot{S} &= r_1 S\left(1-\frac{S+I}{k_1}\right)-\lambda SI-\frac{p_1S^2Y}{1+S+\alpha I},\\
t^{1-q}\dot{I} &= \lambda SI-\frac{p_2I^2Y}{1+S+\alpha I}-\gamma I,\\
t^{1-q}\dot{Y} &= r_2 Y\left(1-\frac{Y}{k_2+S+mI}\right)+\delta_1\left(\frac{p_1S^2Y}{1+S+\alpha I}\right)-\delta_2\left(\frac{p_2I^2Y}{1+S+\alpha I}\right).
\end{aligned}
\tag{3.6}
$$

or, equivalently,

$$\begin{aligned}\dot{S} &= t^{q-1}\left(r_1 S\left(1-\frac{S+I}{k_1}\right)-\lambda S I-\frac{p_1 S^2 Y}{1+S+\alpha I}\right),\\ \dot{I} &= t^{q-1}\left(\lambda S I-\frac{p_2 I^2 Y}{1+S+\alpha I}-\gamma I\right),\\ \dot{Y} &= t^{q-1}\left(r_2 Y\left(1-\frac{Y}{k_2+S+mI}\right)+\delta_1\left(\frac{p_1 S^2 Y}{1+S+\alpha I}\right)-\delta_2\left(\frac{p_2 I^2 Y}{1+S+\alpha I}\right)\right).\end{aligned} \tag{3.7}$$

We can see that the fractional order systems (3.4) and (3.7) are reduced to the ordinary differential equation system (3.1) if $q \to 1$.

4. Mathematical analysis

4.1. Existence and uniqueness

The existence and uniqueness of the solutions of the Caputo fractional differential system (3.4) are studied in the region $\Omega \times [t_0, T]$, where $\Omega = \{(S, I, Y) \in \mathbb{R}^3 : max(|S|, |I|, |Y|) \leq \psi\}$.

Theorem 3. *For each $X_0 = (S_0, I_0, Y_0) \in \Omega$, there exists a unique solution $X(t) \in \Omega$ of the Caputo fractional differential system (3.4) with the initial condition X_0, which is defined for all $t \geq 0$.*

Proof. Let $X = (S, I, P)$ and $\bar{X} = (\bar{S}, \bar{I}, \bar{P}) \in \Omega$. Consider a mapping $H : \mathbb{R}^3 \to \mathbb{R}^3$ that is defined by $H(X) = (H_1(X), H_2(X), H_3(X))$, with

$$\begin{aligned}H_1(X) &= r_1^q S\left(1-\frac{S+I}{k_1^q}\right)-\lambda^q S I-\frac{p_1^q S^2 Y}{1+S+\alpha^q I},\\ H_2(X) &= \lambda^q S I-\frac{p_2^q I^2 Y}{1+S+\alpha^q I}-\gamma^q I,\\ H_3(X) &= r_2^q Y\left(1-\frac{Y}{k_2^q+S+m^q I}\right)+\delta_1^q\left(\frac{p_1^q S^2 Y}{1+S+\alpha^q I}\right)-\delta_2^q\left(\frac{p_2^q I^2 Y}{1+S+\alpha^q I}\right).\end{aligned} \tag{4.1}$$

For any $X, \bar{X} \in \Omega$, it follows from (4.1) that

$$\| H(X) - H(\bar{X}) \| = | H_1(X) - H_1(\bar{X}) | + | H_2(X) - H_2(\bar{X}) | + | H_3(X) - H_3(\bar{X}) |,$$

with

$$\begin{aligned}| H_1(X) - H_1(\bar{X}) | &= | r_1^q S - \frac{r_1^q S(S+I)}{k_1^q} - \lambda^q S I - \frac{p_1^q S^2 Y}{1+S+\alpha^q I} - r_1^q \bar{S} + \frac{r_1^q \bar{S}(\bar{S}+\bar{I})}{k_1^q}\\ &\quad + \lambda^q \bar{S}\bar{I} + \frac{p_1^q \bar{S}^2 \bar{Y}}{1+\bar{S}+\alpha^q \bar{I}} |\\ &= | r_1^q(S-\bar{S}) - \lambda^q(SI-\bar{S}\bar{I}) - \frac{r_1^q}{k_1^q}\left(S(S+I)-\bar{S}(\bar{S}+\bar{I})\right)\end{aligned}$$

$$
\begin{aligned}
&- \quad p_1^q\left(\frac{S^2Y}{1+S+\alpha^q I}-\frac{\bar{S^2}\bar{Y}}{1+\bar{S}+\alpha^q\bar{I}}\right)|\\
&= \quad |\, r_1^q(S-\bar{S})-\lambda^q(SI-\bar{S}\bar{I})-\frac{r_1^q}{k_1^q}\Big((S^2-\bar{S^2})+(SI-\bar{S}\bar{I})\Big)\\
&- \quad p_1^q\left(\frac{S^2Y}{1+S+\alpha^q I}-\frac{\bar{S^2}\bar{Y}}{1+\bar{S}+\alpha^q\bar{I}}\right)|\\
&\leq \quad r_1^q\,|\,S-\bar{S}\,|+\lambda^q\,|\,SI-\bar{S}\bar{I}\,|+\frac{r_1^q}{k_1^q}\,|\,S^2-\bar{S^2}\,|+\frac{r_1^q}{k_1^q}\,|\,SI-\bar{S}\bar{I}\,|\\
&+ \quad p_1^q\,|\,\frac{S^2Y(1+\bar{S}+\alpha^q\bar{I})-\bar{S^2}\bar{Y}(1+S+\alpha^q I)}{(1+S+\alpha^q I)(1+\bar{S}+\alpha^q\bar{I})}\,|\,.
\end{aligned}
$$

For this last inequality, note that

$$
\begin{aligned}
|S^2-\bar{S^2}| &= |(S+\bar{S})(S-\bar{S})|\\
&\leq |S|\,|\,S-\bar{S}\,|+|\,\bar{S}\,|\,|S-\bar{S}|\\
&\leq \psi\,|\,S-\bar{S}\,|+\psi|S-\bar{S}|\\
&\leq 2\psi\,|\,S-\bar{S}\,|,
\end{aligned}
$$

and

$$
\begin{aligned}
|SI-\bar{S}\bar{I}| &= |SI-\bar{S}I+\bar{S}I-\bar{S}\bar{I}|\\
&\leq |I||S-\bar{S}|+|\bar{S}||I-\bar{I}|\\
&\leq \psi|S-\bar{S}|+\psi|I-\bar{I}|.
\end{aligned}
$$

Also,

$$
\begin{aligned}
|S^2Y(1+\bar{S}+\alpha^q\bar{I})-\bar{S^2}\bar{Y}(1+S+\alpha^q I)| &= |S^2Y+\bar{S}S^2Y+\alpha^q\bar{I}S^2Y-\bar{S^2}\bar{Y}-S\bar{S^2}\bar{Y}-\alpha^q I\bar{S^2}\bar{Y}|\\
&\leq |S^2Y-\bar{S^2}\bar{Y}|+|\bar{S}S^2Y-S\bar{S^2}\bar{Y}|+\alpha^q|\bar{I}S^2Y-I\bar{S^2}\bar{Y}|\\
&\leq 2\psi^2|S-\bar{S}|+\psi^2|Y-\bar{Y}|+\psi^3|S-\bar{S}|+\psi^3|Y-\bar{Y}|\\
&+ \psi^3\alpha^q|S-\bar{S}|+\psi^3\alpha^q|Y-\bar{Y}|+\psi^3\alpha^q|I-\bar{I}|.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
|\,H_1(X)-H_1(\bar{X})\,| \quad &\leq \quad r_1^q\,|\,S-\bar{S}\,|+\lambda^q\,|\,SI-\bar{S}\bar{I}\,|+\frac{r_1^q}{k_1^q}\,|\,S^2-\bar{S^2}\,|+\frac{r_1^q}{k_1^q}\,|\,SI-\bar{S}\bar{I}\,|\\
&+ \quad p_1^q\,|\,\frac{S^2Y(1+\bar{S}+\alpha^q\bar{I})-\bar{S^2}\bar{Y}(1+S+\alpha^q I)}{(1+S+\alpha^q I)(1+\bar{S}+\alpha^q\bar{I})}\,|\\
&\leq \quad r_1^q\,|\,S-\bar{S}\,|+\lambda^q\psi|S-\bar{S}|+\lambda^q\psi|I-\bar{I}|+\frac{2\psi r_1^q}{k_1^q}|S-\bar{S}|\\
&+ \quad \frac{r_1^q\psi}{k_1^q}|S-\bar{S}|+\frac{r_1^q\psi}{k_1^q}|I-\bar{I}|+2p_1^q\psi^2|S-\bar{S}|+p_1^q\psi^2|Y-\bar{Y}|
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad p_1^q\psi^3|S-\bar{S}| + p_1^q\psi^3|Y-\bar{Y}| + 2p_1^q\psi^3\alpha^q|S-\bar{S}|\\
&+ \quad p_1^q\psi^3\alpha^q|Y-\bar{Y}| + p_1^q\psi^3\alpha^q|I-\bar{I}|\\
&= \quad \left(r_1^q + \lambda^q\psi + \frac{3\psi r_1^q}{k_1^q} + 2p_1^q\psi^2 + p_1^q\psi^3 + 2p_1^q\alpha^q\psi^3\right)|S-\bar{S}|\\
&+ \quad \left(\lambda^q\psi + \frac{r_1^q\psi}{k_1^q} + p_1^q\alpha^q\psi^3\right)|I-\bar{I}| + \left(p_1^q\psi^2 + p_1^q\psi^3 + p_1^q\alpha^q\psi^3\right)|Y-\bar{Y}|.
\end{aligned}
$$

Similarly, for $|\, H_2(X) - H_2(\bar{X})\,|$ and $|\, H_3(X) - H_3(\bar{X})\,|$, we have

$$
\begin{aligned}
|\, H_2(X) - H_2(\bar{X})\,| &= |\,\lambda^q S I - \frac{p_2^q I^2 Y}{1+S+\alpha^q I} - \gamma^q I - \lambda^q\bar{S}\bar{I} + \frac{p_2^q\bar{I}^2\bar{Y}}{1+\bar{S}+\alpha^q\bar{I}} + \gamma^q\bar{I}\,|\\
&\le (\psi\lambda^q + \psi^3 p_2^q)|S-\bar{S}| + (\psi\lambda^q + \gamma^q + 2\psi^2 p_2^q + 2\psi^3 p_2^q + 3\psi^3 p_2^q\alpha^q)|I-\bar{I}|\\
&+ (\psi^2 p_2^q + \psi^3 p_2^q + \psi^3 p_2^q\alpha^q)|Y-\bar{Y}|,
\end{aligned}
$$

and

$$
\begin{aligned}
|\, H_3(X) - H_3(\bar{X})\,| &= |\, r_2^q Y - \frac{r_2^q Y^2}{k_2^q + S + m^q I} + \frac{\delta_1^q p_1^q S^2 Y}{1+S+\alpha^q I} - \frac{\delta_2^q p_2^q I^2 Y}{1+S+\alpha^q I}\\
&- r_2^q\bar{Y} + \frac{r_2^q\bar{Y}^2}{k_2^q + \bar{S} + m^q\bar{I}} - \frac{\delta_1^q p_1^q\bar{S}^2\bar{Y}}{1+\bar{S}+\alpha^q\bar{I}} + \frac{\delta_2^q p_2^q\bar{I}^2\bar{Y}}{1+\bar{S}+\alpha^q\bar{I}}\,|\\
&\le \left(2\psi^2\delta_1^q p_1^q + \psi^3\delta_1^q p_1^q + 2\psi^3\alpha^q\delta_1^q p_1^q + \psi^3\delta_2^q p_2^q\right)|\, S-\bar{S}\,|\\
&+ \left(\psi^3\alpha^q\delta_1^q p_1^q + 2\psi^2\delta_2^q p_2^q + 2\psi^3\delta_2^q p_2^q + 3\psi^3\alpha^q\delta_2^q p_2^q\right)|\, I-\bar{I}\,|\\
&+ \left(r_2^q + \frac{2r_2^q\psi}{k_2^q} + \psi^2\delta_1^q p_1^q + \psi^3\delta_1^q p_1^q + \psi^3\alpha^q\delta_1^q p_1^q\right)|\, Y-\bar{Y}\,|\\
&+ \left(\psi^2\delta_2^q p_2^q + \psi^3\delta_2^q p_2^q + \psi^3\alpha^q\delta_2^q p_2^q\right)|\, Y-\bar{Y}\,|\,.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\|\, H(X) - H(\bar{X})\,\| &= |\, H_1(X) - H_1(\bar{X})\,| + |\, H_2(X) - H_2(\bar{X})\,| + |\, H_3(X) - H_3(\bar{X})\,|\\
&\le \left(r_1^q + 2\psi\lambda^q + \frac{3\psi r_1^q}{k_1^q} + (2\psi^2 p_1^q + \psi^3(1+2\alpha^q))(1+\delta_1^q)\right)\left|S-\bar{S}\right|\\
&+ \left(\psi^3 p_2^q(1+\delta_2^q)\right)\left|S-\bar{S}\right| + \left(2\psi\lambda^q + \frac{r_1^q\psi}{k_1^q} + \gamma^q\right)\left|I-\bar{I}\right|\\
&+ \left(\psi^3 p_1^q\alpha^q(1+\delta_1^q) + (2\psi^2 p_2^q + \psi^3 p_2^q(2+3\alpha^q))(1+\delta_2^q)\right)\left|I-\bar{I}\right|\\
&+ \left(r_2^q + \frac{2r_2^q\psi}{k_2^q} + (\psi^3 p_1^q(1+\delta_1^q) + \psi^3 p_2^q(1+\delta_2^q))(1+\alpha^q)\right)\left|Y-\bar{Y}\right|\\
&+ \left(\psi^2 p_1^q(1+\delta_1^q) + \psi^2 p_2^q(1+\delta_2^q)\right)\left|Y-\bar{Y}\right|\\
&\le L\|X-\bar{X}\|,
\end{aligned}
$$

with

$$L = max\{r_1^q + 2\psi\lambda^q + \frac{3\psi r_1^q}{k_1^q} + (2\psi^2 p_1^q + \psi^3(1+2\alpha^q))(1+\delta_1^q) + \psi^3 p_2^q(1+\delta_2^q),$$
$$2\psi\lambda^q + \frac{r_1^q\psi}{k_1^q} + \gamma^q + \psi^3 p_1^q\alpha^q(1+\delta_1^q) + (2\psi^2 p_2^q + \psi^3 p_2^q(2+3\alpha^q))(1+\delta_2^q), \tag{4.2}$$
$$r_2^q + \frac{2r_2^q\psi}{k_2^q} + (\psi^3 p_1^q(1+\delta_1^q) + \psi^3 p_2^q(1+\delta_2^q))(1+\alpha^q) + \psi^2 p_1^q(1+\delta_1^q) + \psi^2 p_2^q(1+\delta_2^q)\}.$$

Thus, $H(X)$ satisfies the Lipschitz condition with respect to X. In accordance with Theorem 3.4 in [17], it follows that there exists a unique solution $X(t)$ of the Caputo fractional differential system (3.4) with the initial condition $X_0 = (S_0, I_0, Y_0)$.

□

4.2. Non-negativity and boundedness

The following results show the non-negativity of the solutions of the Caputo fractional differential system (3.4).

Theorem 4. *$\mathbb{R}_+^3$ is a positively invariant domain of the Caputo fractional differential system (3.4).*

Proof. We need to show that the domain $\mathbb{R}_+^3$ is positively invariant. For the Caputo fractional differential system (3.4), it holds that

$$\begin{aligned} {}^C D_t^q S(t)|_{S=0} &= 0, \\ {}^C D_t^q I(t)|_{I=0} &= 0, \\ {}^C D_t^q Y(t)|_{Y=0} &= 0. \end{aligned} \tag{4.3}$$

In accordance with Theorem 1 in [46] and Lemma 6 in [47], we have that the solutions of the Caputo fractional differential system (3.4) are non-negative. Consequently, for each hyperplane bounding the non-negative orthant, the vector field points into $\mathbb{R}_+^3$. Therefore, the domain $\mathbb{R}_+^3$ is a positively invariant region. □

Now, the boundedness of the solutions of the Caputo fractional differential system (3.4) is investigated in the following theorem:

Theorem 5. *All solutions of the Caputo fractional differential system (3.4) starting in $\mathbb{R}_+^3$ are uniformly bounded.*

Proof. Let $(s(t), i(t), y(t))$ be a solution of the system with non-negative initial conditions, and considering the function $V(t) = S(t) + I(t) + \frac{1}{\delta_1^q}Y(t)$, then, for each $\eta > 0$,

$$\begin{aligned} {}^C D_t^q V(t) + \eta V(t) &= r_1^q S\left(1 - \frac{S+I}{k_1^q}\right) - \frac{p_2^q I^2 Y}{1+S+\alpha^q I} - \gamma^q I + \frac{r_2^q Y}{\delta_1^q}\left(1 - \frac{Y}{k_2^q + S + m^q I}\right) \\ &- \frac{\delta_2^q}{\delta_1^q}\left(\frac{p_2^q I^2 Y}{1+S+\alpha^q I}\right) + \eta S + \eta I + \frac{\eta}{\delta_1^q} Y \end{aligned}$$

$$
\begin{aligned}
&= (\eta + r_1^q)S - (\gamma^q - \eta)I + \frac{1}{\delta_1^q}(\eta + r_2^q)Y - \frac{p_2^q I^2 Y}{1 + S + \alpha^q I}\left(1 + \frac{\delta_2^q}{\delta_1^q}\right) \\
&- \frac{r_2^q Y^2}{\delta_1^q(k_2^q + S + m^q I)} - \frac{r_1^q S(S + I)}{k_1^q}.
\end{aligned}
$$

Thus, if we choose $\eta < \gamma^q$, then

$$
\begin{aligned}
{}^C D_t^q V(t) + \eta V(t) &\leq (\eta + r_1^q)S + \frac{1}{\delta_1^q}(\eta + r_2^q)Y \\
&\leq (\eta + r_1^q)k_1^q + \frac{1}{\delta_1^q}(\eta + r_2^q)(k_1^q + k_2^q) \\
&= \frac{k_1^q}{\delta_1^q}\left(\eta + r_2^q + \delta_1^q(\eta + r_1^q)\right) + \frac{k_2^q}{\delta_1^q}(\eta + r_2^q) = M.
\end{aligned}
$$

Consequently, we can find a positive number M such that

$$
{}^C D_t^q V(t) + \eta V(t) \leq M. \tag{4.4}
$$

In accordance with Lemma 9 in [48], it follows that

$$
V(t) \leq \left(V(t_0) - \frac{M}{\eta}\right) E_q\left[\eta(t - t_0)^q\right] + \frac{M}{\eta}, \tag{4.5}
$$

with E_q as the Mittag-Leffler function, defined as $E_q(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(jq + 1)}$ for $q > 0$, and Γ is the gamma function. Since $E_q\left[\eta(t - t_0)^q\right] \to 0$ as $t \to \infty$ (Lemma 5 and Corollary 6 in [48]), we have

$$
V(t) \leq \frac{M}{\eta}, \quad t \to \infty. \tag{4.6}
$$

Therefore, all solutions of the Caputo fractional differential system (3.4) starting in $\mathbb{R}_+^3$ are uniformly bounded in the region Λ, with

$$
\Lambda = \left\{(S, I, Y) \in \mathbb{R}_+^3 : V(t) \leq \frac{M}{\eta} + \epsilon, \epsilon > 0\right\}. \tag{4.7}
$$

□

Remark 5. *A similar analysis is used to prove the existence, uniqueness, non-negativity, and boundedness of the solutions for the Khalil fractional system (3.7).*

4.3. Equilibrium points and stability

It is easy to see that the equilibrium points of the integer order model, the Caputo derivative model, and the Khalil derivative model are the same since the parameter q in the Caputo fractional differential system (3.4) only affects the units of the parameters but not their numerical value. So, we will focus on the study of the equilibrium points of the Caputo fractional differential system.
Local stability for the Caputo fractional differential system (3.4) around the biologically feasible equilibrium points has been investigated as follows. In order to do this, we chose to calculate the basic reproduction number corresponding to this model, which we will call the fractional basic reproduction number ($\mathcal{R}_0^f$), defined by $\mathcal{R}_0^f = \frac{\lambda^q k_1^q}{\gamma^q}$, to determine the conditions for the existence and stability of the equilibrium points of the system.

Remark 6. *Note that the conditions for the fractional basic reproduction number, $\mathcal{R}_0^f = \frac{\lambda^q k_1^q}{\gamma^q}$, to be greater than one are preserved with respect to the basic reproduction number for the ordinary differential equation system (3.1), $\mathcal{R}_0$. Furthermore, observe that $\mathcal{R}_0^f$ can be written in terms of $\mathcal{R}_0$, i.e., $\mathcal{R}_0^f = (\mathcal{R}_0)^q$ with $\mathcal{R}_0 = \frac{\lambda k_1}{\gamma}$, and both coincide if $q = 1$.*

The Caputo fractional differential system (3.4) has the following equilibrium points:

i) Trivial equilibrium point $E_0 = (0, 0, 0)$.
ii) Axial equilibrium point $E_1 = (k_1^q, 0, 0)$, where only the susceptible prey population exists.
iii) Axial equilibrium point $E_2 = (0, 0, k_2^q)$, where only the predator population exists.
iv) Predator-free equilibrium point $E_3 = \left(\frac{\gamma^q}{\lambda^q}, \frac{r_1^q \gamma^q (\mathcal{R}_0^f - 1)}{\lambda^q \left(\gamma^q \mathcal{R}_0^f + r_1^q\right)}, 0\right)$, which belongs to region Ω if $\mathcal{R}_0^f > 1$.
v) Disease-free equilibrium point $E_4 = (\tilde{S}, 0, \tilde{Y})$, where $\tilde{Y} = \dfrac{r_1^q (k_1^q - S)(S + 1)}{p_1^q k_1^q S}$ and $\tilde{S}$ is a positive root of the quintic polynomial equation

$$(S - k_1^q)(AS^4 + BS^3 + CS^2 + DS + E) = 0 \tag{4.8}$$

with

$$\begin{aligned}
A &= \delta_1^q p_1^{2q} k_1^q, \\
B &= \delta_1^q p^{2q} k_1^q k_2^q + r_2^q p_1^q k_1^q + r_1^q r_2^q, \\
C &= r_2^q p_1^q k_1^q k_2^q + r_2^q p_1^q k_1^q - r_1^q r_2^q k_1^q + 2 r_1^q r_2^q, \\
D &= r_2^q p_1^q k_1^q k_2^q - 2 r_1^q r_2^q k_1^q + r_1^q r_2^q, \\
E &= -r_1^q r_2^q k_1^q.
\end{aligned}$$

Note that, if $S = k_1^q$, then the equilibrium point E_4 coincides with the equilibrium point E_1. Let $p_1(S) = (AS^4 + BS^3 + CS^2 + DS + E)$; using Descartes' sign rule, we can prove that $p_1(S)$ has at least one positive root in $(0, k_1^q)$. In this way, the disease-free equilibrium point E_4 belongs to the region Ω if and only if $0 < S < k_1^q$.
vi) Coexistence equilibrium point $E_5 = (\hat{S}, \hat{I}, \hat{Y})$ (its existence will be described numerically).

Local stability of the equilibrium points for the Caputo fractional differential system (3.4) is analyzed by using the Jacobian matrix, the Routh-Hurwitz stability criteria [49, 50], and the following theorem [51, 52].

Theorem 6. *Consider the following fractional order system:*

$$\begin{aligned} {}^{C}D^{q}x(t) &= f(t,x), \qquad 0 < q < 1 \\ x(0) &= x_0, \end{aligned} \tag{4.9}$$

where $f(t,x) : \mathbb{R}^+ \times \mathbb{R}^n \rightarrow \mathbb{R}^n$. The equilibrium points of the Caputo fractional differential system (4.9) are locally asymptotic stable if all eigenvalues μ_i of the Jacobian matrix $\frac{\partial f(t,x)}{\partial x}$ evaluated at the equilibrium points satisfy the following condition:

$$| \, arg(\mu_i) \, | > \frac{q\pi}{2}. \tag{4.10}$$

A similar theorem for the stability analysis of the conformable Khalil fractional system (3.7) can be found in [53].

Remark 7. *The trivial equilibrium point $E_0 = (0,0,0)$ is not biologically viable because the population will go extinct at E_0. Note that the trivial equilibrium point E_0 is an unstable equilibrium point.*

Theorem 7. *i) Axial equilibrium point E_1 is always unstable.*
ii) Axial equilibrium point E_2 is always unstable.
iii) Predator-free equilibrium point $E_3 \in \Omega$ is locally asymptotically stable if and only if $\mathcal{R}_0^f > 1$.

Proof. i) The Jacobian matrix of the Caputo fractional differential system (3.4) around the axial equilibrium point $E_1 = (k_1^q, 0, 0)$ is given by

$$J \,|_{E_1} = \begin{pmatrix} -r_1^q & -\lambda^q k_1^q - r^q & -\frac{p_1^q k_1^{2q}}{1+k_1^q} \\ 0 & \lambda^q k_1^q - \gamma^q & 0 \\ 0 & 0 & r_2^q + \frac{\delta_1^q p_1^q k_1^{2q}}{1+k_1^q} \end{pmatrix}. \tag{4.11}$$

The eigenvalues of matrix (4.11) are as follows: $\mu_1 = -r^q$, $\mu_2 = \lambda^q k_1^q - \gamma^q$, and $\mu_3 = r_2^q + \frac{\delta_1^q p_1^q k_1^{2q}}{1+k_1^q}$. Following Theorem 6, $|arg(\mu_1)| = \pi$, $|arg(\mu_2)| = \pi$ if $\mathcal{R}_0^f < 1$, and $|arg(\mu_3)| = 0$. Since the eigenvalues μ_2 (when $\mathcal{R}_0^f > 1$) and μ_3 do not satisfy the condition $|arg(\mu_{2,3})| > \frac{q\pi}{2}$ for all $q \in (0,1)$, the axial equilibrium point E_1 is an unstable saddle point if $\mathcal{R}_0^f < 1$.

ii) The Jacobian matrix of the Caputo fractional differential system (3.4) around the axial equilibrium point $E_2 = (0,0,k_2^q)$ is given by

$$J \,|_{E_2} = \begin{pmatrix} r_1^q & 0 & 0 \\ 0 & -\gamma^q & 0 \\ r_2^q & m^q r_2^q & -r_2^q \end{pmatrix}. \tag{4.12}$$

The eigenvalues of matrix (4.12) are $\mu_4 = r_1^q$, $\mu_5 = -\gamma^q$, and $\mu_6 = -r_2^q$. Following Theorem 6, it can be observed that $| \, arg(\mu_{5,6}) \, | = \pi$ and $| \, arg(\mu_4) \, | = 0$. Since the eigenvalue μ_4 does not satisfy that $| \, arg(\mu_4) \, | > \frac{q\pi}{2}$ for all $0 < q < 1$, the axial equilibrium point E_2 is always unstable.

iii) The Jacobian matrix of the Caputo fractional differential system (3.4) around the predator-free equilibrium point $E_3 = \left(\frac{\gamma^q}{\lambda^q}, \frac{r_1^q\gamma^q(\mathcal{R}_0^f-1)}{\lambda^q\left(\gamma^q\mathcal{R}_0^f+r_1^q\right)}, 0\right)$ is given by

$$J\mid_{E_3}=\begin{pmatrix} -\frac{r_1^q}{\mathcal{R}_0^f} & -(\frac{r_1^q}{\mathcal{R}_0^f}+\gamma^q) & -\frac{p_1^q S^{*2}}{(1+S^*+\alpha^q I^*)} \\ \frac{r_1^q\gamma^q(\mathcal{R}_0^f-1)}{\gamma^q\mathcal{R}_0^f+r_1^q} & 0 & -\frac{p_2^q I^{*2}}{(1+S^*+\alpha^q I^*)} \\ 0 & 0 & J_{3,3} \end{pmatrix}, \tag{4.13}$$

with $S^* = \frac{\gamma^q}{\lambda^q}$, $I^* = \frac{r_1^q\gamma^q(\mathcal{R}_0^f-1)}{\lambda^q(\gamma^q\mathcal{R}_0^f+r_1^q)}$, $J_{3,3} = \left(r_2^q\lambda^{2q} + r_2^q\lambda^q\gamma^q + \delta_1^q p_1^q\gamma^{2q}\right)Z^2 + r_1^q r_2^q\gamma^q\alpha^q\lambda^q\left(\mathcal{R}_0^f - 1\right)Z - \delta_2^q p_2^q r_1^q\gamma^{2q}(\mathcal{R}_0^f - 1)^2$, and $Z = \gamma^q\mathcal{R}_0^f + r_1^q$.

The characteristic polynomial associated with (4.13) is given by

$$P(\mu) = (\mu - J_{3,3})\left(\mu^2 + \frac{r_1^q}{\mathcal{R}_0^f}\mu + \frac{\gamma^q r_1^q\left(\mathcal{R}_0^f - 1\right)}{\mathcal{R}_0^f}\right). \tag{4.14}$$

Therefore, the eigenvalues associated with the characteristic polynomial are denoted by $J_{3,3}$, and the solutions of the quadratic polynomial are represented by $p_1(\mu) = \mu^2 + \frac{r_1^q}{\mathcal{R}_0^f}\mu + \frac{\gamma^q r_1^q\left(\mathcal{R}_0^f-1\right)}{\mathcal{R}_0^f}$. By Theorem 6, $\mid arg(\mu_7) \mid= \pi$ if $J_{33} < 0$. On the other hand, note that the quadratic polynomial $p_1(\mu)$ is Hurwitz if $\frac{r_1^q}{\mathcal{R}_0^f} > 0$ and $\frac{\gamma^q r_1^q\left(\mathcal{R}_0^f-1\right)}{\mathcal{R}_0^f} > 0$. This ensures that the eigenvalues of the polynomial $p_1(\mu)$ have a negative real part; thus, by Theorem 6, $\mid arg(\mu_{8,9}) \mid= \pi$ if and only if $\mathcal{R}_0^f > 1$. Therefore, the predator-free equilibrium point E_3 of the Caputo fractional differential system (3.4) is locally asymptotically stable.

□

The following results present conditions for the existence of equilibrium points E_4 and E_5.

Theorem 8. *The disease-free equilibrium point $E_4 = (\tilde{S}, 0, \tilde{Y})$ of the Caputo fractional differential system (3.4) is locally asymptotically stable if the characteristic polynomial associated with the Jacobian matrix evaluated on E_4 is Hurwitz.*

Proof. Assume that the Jacobian matrix evaluated at point $E_4 = (\tilde{S}, 0, \tilde{Y})$ is given by

$$J\mid_{E_4}=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \tag{4.15}$$

with

$$
\begin{aligned}
a_{11} &= r_1^q\left(1-\frac{\tilde{S}}{k_1^q}\right)-\frac{r_1^q\tilde{S}}{k_1^q}-\frac{2p_1^q\tilde{S}\tilde{Y}}{1+\tilde{S}}+\frac{p_1^q\tilde{S}^2\tilde{Y}}{(1+\tilde{S})^2},\\
a_{12} &= -\frac{r_1^q\tilde{S}}{k_1^q}-\lambda^q\tilde{S}+\frac{p_1^q\alpha^q\tilde{S}^2\tilde{Y}}{(1+\tilde{S})^2},\\
a_{13} &= -\frac{p_1^q\tilde{S}^2}{1+\tilde{S}},\\
a_{22} &= \lambda^q\tilde{S}-\gamma^q,\\
a_{31} &= \frac{r_2^q\tilde{Y}^2}{(\tilde{S}+k_2^q)^2}+\frac{2p_1^q\delta_1^q\tilde{S}\tilde{Y}}{1+\tilde{S}}-\frac{p_1^q\delta_1^q\tilde{S}^2\tilde{Y}}{(1+\tilde{S})^2},\\
a_{32} &= \frac{r_2^q\alpha^q\tilde{Y}^2}{(\tilde{S}+k_2^q)^2}-\frac{\delta_1^q p_1^q\alpha^q\tilde{S}^2\tilde{Y}}{(1+\tilde{S})^2},\\
a_{33} &= r_2^q\left(1-\frac{\tilde{Y}}{\tilde{S}+k_2^q}\right)-\frac{r_2^q\tilde{Y}}{\tilde{S}+k_2^q}+\frac{\delta_1^q p_1^q\tilde{S}^2}{1+\tilde{S}}.
\end{aligned}
$$

The characteristic polynomial associated with (4.15) is given by

$$\mu^3+A_1\mu^2+A_2\mu+A_3=0, \tag{4.16}$$

with $A_1 = -(a_{33}+a_{22}+a_{11})$, $A_2 = -(-a_{22}a_{11}-a_{33}a_{11}+a_{31}a_{13}-a_{33}a_{22})$, and $A_3 = a_{31}a_{13}a_{22}-a_{33}a_{22}a_{11}$. By the Routh-Hurwitz stability criteria, the disease-free equilibrium point E_4 is locally asymptotically stable if and only if $A_1 > 0$, $A_3 > 0$, and $A_1A_2 > A_3$. Note that all three of the above conditions can be satisfied by appropriately selecting the parameter values. □

Theorem 9. *The coexistence equilibrium point $E_5 = (\hat{S}, \hat{I}, \hat{Y})$ of the Caputo fractional differential system (3.4) is locally asymptotically stable if the characteristic polynomial associated with the Jacobian matrix evaluated on E_5 is Hurwitz.*

Proof. Assume that the Jacobian matrix evaluated at point $E_5 = (\hat{S}, \hat{I}, \hat{Y})$ is given by

$$J\mid_{E_5}=\begin{pmatrix} b_{11} & b_{12} & b_{13}\\ b_{21} & b_{22} & b_{23}\\ b_{31} & b_{32} & b_{33}\end{pmatrix}, \tag{4.17}$$

with

$$
\begin{aligned}
b_{11} &= r_1^q\left(1-\frac{\hat{S}+\hat{I}}{k_1^q}\right)-\frac{r_1^q\hat{S}}{k_1^q}-\lambda^q\hat{I}-\frac{2p_1^q\hat{S}\hat{Y}}{1+\hat{S}+\alpha^q\hat{I}}+\frac{p_1^q\hat{S}^2\hat{Y}}{(1+\hat{S}+\alpha^q\hat{I})^2},\\
b_{12} &= -\frac{r_1^q\hat{S}}{k_1^q}-\lambda^q\hat{S}+\frac{\alpha^q p_1^q\hat{S}^2\hat{Y}}{(1+\hat{S}+\alpha^q\hat{I})^2},\\
b_{13} &= -\frac{p_1^q\hat{S}^2}{1+\hat{S}+\alpha^q\hat{I}},\\
b_{21} &= \lambda^q\hat{I}+\frac{p_2^q\hat{I}^2\hat{Y}}{(1+\hat{S}+\alpha^q\hat{I})^2},\\
b_{23} &= -\frac{p_2^q\hat{I}^2}{1+\hat{S}+\alpha^q\hat{I}},\\
b_{22} &= \lambda^q\hat{S}-\frac{2p_2^q\hat{I}\hat{Y}}{1+\hat{S}+\alpha^q\hat{I}}+\frac{p_2^q\alpha^q\hat{I}^2\hat{Y}}{(1+\hat{S}+\alpha^q\hat{I})^2}-\gamma^q,\\
b_{31} &= \frac{r_2^q\hat{Y}^2}{(k_2^q+m^q\hat{I}+\hat{S})^2}+\frac{2\delta_1^q p_1^q\hat{S}\hat{Y}}{1+\alpha^q\hat{I}+\hat{S}}-\frac{p_1^q\delta_1^q\hat{S}^2\hat{Y}}{(1+\alpha^q\hat{I}+\hat{S})^2}+\frac{\delta_2^q p_2^q\hat{I}^2\hat{Y}}{(1+\alpha^q\hat{I}+\hat{S})^2},\\
b_{32} &= \frac{r_2^q m^q\hat{Y}^2}{(k_2^q+m^q\hat{I}+\hat{S})^2}-\frac{\delta_1^q p_1^q\alpha^q\hat{S}^2\hat{Y}}{(1+\alpha^q\hat{I}+\hat{S})^2}-\frac{2\delta_2^q p_2^q\hat{I}\hat{Y}}{1+\alpha^q\hat{I}+\hat{S}}+\frac{\delta_2^q p_2^q\alpha^q\hat{I}^2\hat{Y}}{(1+\alpha^q\hat{I}+\hat{S})^2},\\
b_{33} &= r_2^q\left(1-\frac{\hat{Y}}{k_2^q+m^q\hat{I}+\hat{S}}\right)-\frac{r_2^q\hat{Y}}{k_2^q+m^q\hat{I}+\hat{S}}+\frac{p_1^q\delta_1^q\hat{S}^2}{1+\alpha^q\hat{I}+\hat{S}}-\frac{p_2^q\delta_2^q\hat{I}^2}{1+\alpha^q\hat{I}+\hat{S}}.
\end{aligned}
$$

The characteristic polynomial associated with (4.17) is given by

$$\mu^3+B_1\mu^2+B_2\mu+B_3=0, \tag{4.18}$$

with $B_1 = -(b_{33}+b_{22}+b_{11})$, $B_2 = b_{22}b_{11}+b_{33}b_{11}-b_{21}b_{12}-b_{31}b_{13}+b_{33}b_{22}-b_{32}b_{23}$, and $B_3 = -b_{11}b_{22}b_{33}+b_{11}b_{23}b_{32}+a_{12}b_{21}b_{33}-b_{12}b_{23}b_{31}-b_{13}b_{21}b_{32}+b_{13}b_{22}b_{31}$. According to the Routh-Hurwitz stability criteria, the coexistence equilibrium point E_5 is locally asymptotically stable if and only if $B_1 > 0$, $B_3 > 0$, and $B_1B_2 > B_3$. The stability of this equilibrium point will be demonstrated numerically. □

Remark 8. *A study of global stability in the subspaces S, SI, Y, and SY, as well as the space SIY, can be conducted by constructing Lyapunov functions [54], for example. On the other hand, although an analysis of bifurcations has not been performed for this model, conditions can be given for the existence of different bifurcations. For more information on bifurcations in fractional order models, see [55–59], and, for ordinary differential equation models, see [60–63].*

5. Numerical simulations

In this section, we show, through numerical simulations, some of the analytical results that were obtained in the previous sections. For the numerical simulations of the analyzed system, we chose to use the standard Euler method [64] (the conformable Khalil derivative) and the fractional forward Euler method (the classical Caputo derivative); also, the last method introduced by Tomášek [65] transforms the fractional differential equation into a fractional integral equation with a posterior discretization of the fractional integral. The simulations were developed in an R environment (version 4.1.1). Numerical results for the system (direct problem) are presented and have been compared from the perspective of fractional order (Caputo and Khalil) and integer order differential equations. In addition, we show the different scenarios of coexistence between the three populations (i.e., susceptible prey, infected prey, and predators), as well as the coexistence between susceptible and infected prey or the coexistence between susceptible prey and predators. For this numerical analysis, we chose the following set of parameters based on the eco-epidemiological study of tilapia and pelicans in the Salton Sea, as provided by Chattopadhyay et al. [66,67] and Greenhalgh et al. [8,9], with $r_2 = 0.0015$ and $m = 0.25$ (see Table 2).

Table 2. Set of parameter values used for integer order and fractional order models.

p_1	α	r_1	p_2	k_1	δ_1	δ_2	γ	λ	r_2	k_2	m
0.05	6	1.8	0.05	50	0.35	0.18	0.24	0.06	0.0015	20	0.25

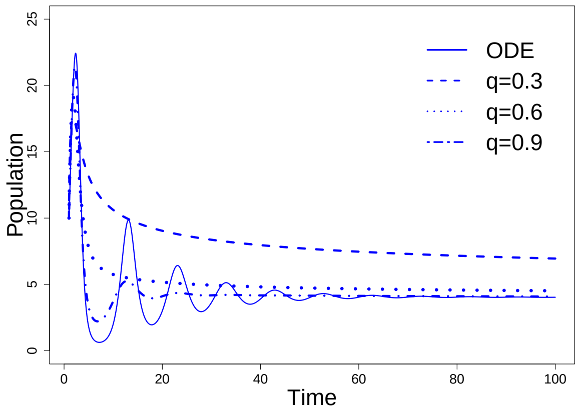
For this set of parameter values, the equilibrium points of the models were set as $E_0 = (0, 0, 0)$, $E_1 = (50, 0, 0)$, $E_2 = (0, 0, 20)$, $E_3 = (4, 17.25, 0)$, $E_4 = (0.68, 0, 87.51)$, and $E_5 = (9.33, 12.82, 43.55)$, and it also holds that $\mathcal{R}_0^f = 12.5 > 1$. The following Table 3 shows the stability of the equilibrium points.

Table 3. Equilibrium points and their stability, corresponding to the parameters in Table 2.

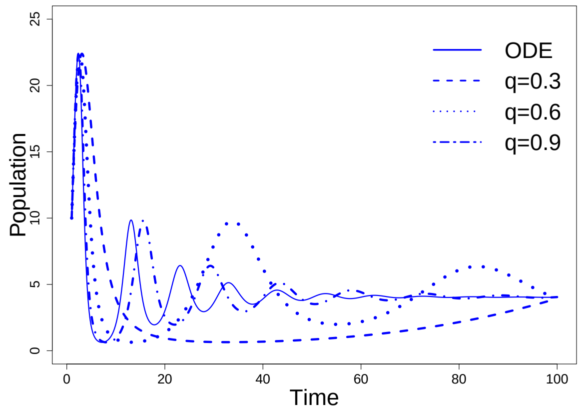
Equilibrium points	Eigenvalues	Stability
$E_0 = (0, 0, 0)$	$\mu_1 = 1.8, \mu_2 = -0.24, \mu_3 = 0.0015$	Unstable
$E_1 = (50, 0, 0)$	$\mu_1 = -1.8, \mu_2 = 2.76, \mu_3 = 0.85$	Unstable
$E_2 = (0, 0, 20)$	$\mu_1 = 1.80, \mu_2 = -0.24, \mu_3 = -0.0015,$	Unstable
$E_3 = (4, 17.25, 0)$	$\mu_1 = -0.07 + 0.62i, \mu_2 = -0.07 - 0.62i, \mu_3 = -0.02$	Locally asymptotic stable
$E_4 = (0.68, 0, 87.51)$	$\mu_1 = -1.06, \mu_2 = -0.01, \mu_3 = -0.19$	Locally asymptotic stable
$E_5 = (9.33, 12.82, 43.55)$	$\mu_1 = -0.30 + 0.74i, \mu_2 = -0.30 - 0.74i, \mu_3 = 0.03$	Unstable

Remark 9. *Note that, for this set of parameter values, we have two locally asymptotically stable equilibrium points, the predator-free equilibrium point E_3, and the disease-free equilibrium point E_4. Also, we have that $\mathcal{R}_0^f > 1$. This result is apparently a contradiction of Theorem 2 of [68] since, despite the setting of $\mathcal{R}_0^f > 1$, the disease-free equilibrium point E_4 is asymptotically stable. However, this theorem does not take into account the presence of a predator population. This shows that the presence of predators can help to control the spread of a disease by decreasing the population of infected prey through the consumption of predators and, thus, decreasing the interaction between susceptible and infected prey.*

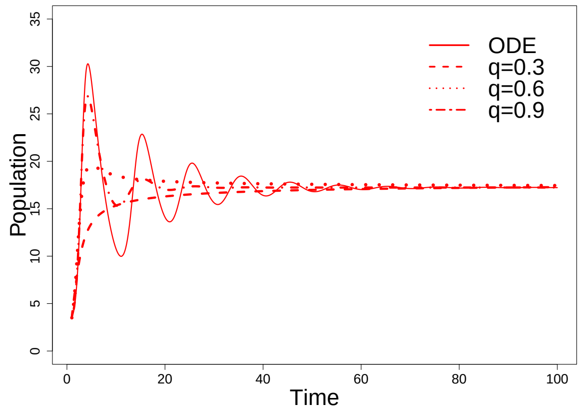
In the following simulations, the blue and red curves represent the subpopulations of susceptible and infected prey, respectively, and the green curves represent the population of predators.



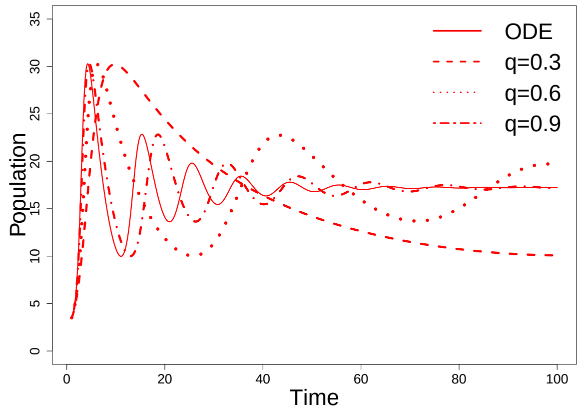
(a) Numerical solution of the Caputo approach.



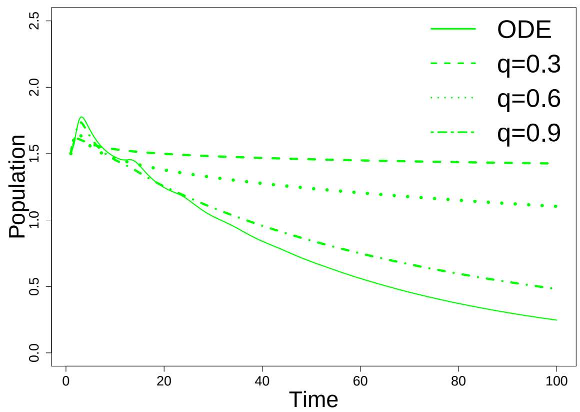
(b) Numerical solution of the conformable Khalil approach.



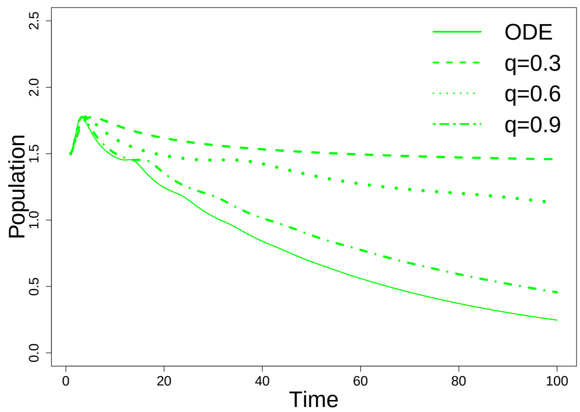
(c) Numerical solution of the Caputo approach.



(d) Numerical solution of the conformable Khalil approach.



(e) Numerical solution of the Caputo approach.



(f) Numerical solution of the conformable Khalil approach.

Figure 1. Stable dynamical behavior of the systems (3.1), (3.4), and (3.7) for the parameter values given in Table 2, with the initial condition $(S, I, Y) = (10, 3.5, 1.5)$.

In the Figure 1, we show the family of solutions associated with systems of differential equations of integer order (3.1) and fractional order (the classical Caputo derivative (3.4) and the conformable Khalil derivative (3.7)). For the first values of the parameters of Table 2, it is observed that all trajectories of the fractional and ordinary systems approach the predator-free equilibrium point, E_3. Note that the eigenvalues of the Jacobian matrix of the system at E_3 were found to be $\mu_{1,2} = -0.07 \pm 0.62i$ and $\mu_3 = -0.02$, which satisfy that $| \, arg(\mu_{1,2,3}) \, | = \pi > \frac{q\pi}{2}$; thus, the predator-free equilibrium point is locally asymptotically stable, as proved in Theorem 7. On the other hand, in order to verify the Routh–Hurwitz criteria, note that $A_1 = 1.2850 > 0$, $A_3 = 0.0041 > 0$, and $A_1A_2 - A3 = 0.3005 > 0$, which ensures the stability of the disease-free equilibrium E_4, as stated in Theorem 8.

Given that there are two locally asymptotically stable equilibrium points for the parameter values given in Table 2, the models (3.1), (3.4) and (3.7) show a bistability phenomenon in Figure 2(a); it is observed that the systems converge to two different equilibrium points for the same parameter values based on the variation of the initial conditions. For the phase diagrams, we applied two different initial values, i.e., $(10, 3.5, 1.5)$ and $(12.5, 1.35, 30.5)$, for which the systems converge to the different equilibrium points $E_3 = (4, 17.25, 0)$ and $E_4 = (0.68, 0, 87.51)$, respectively.

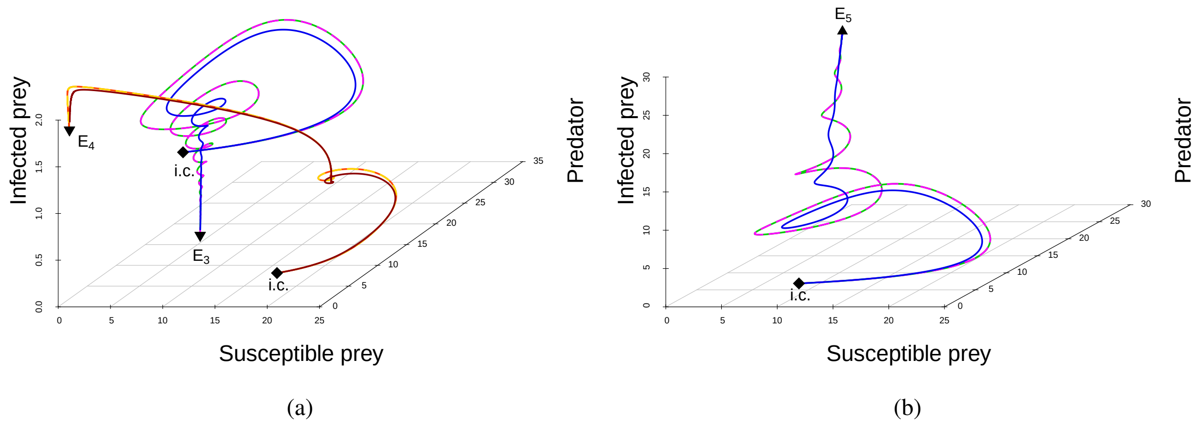
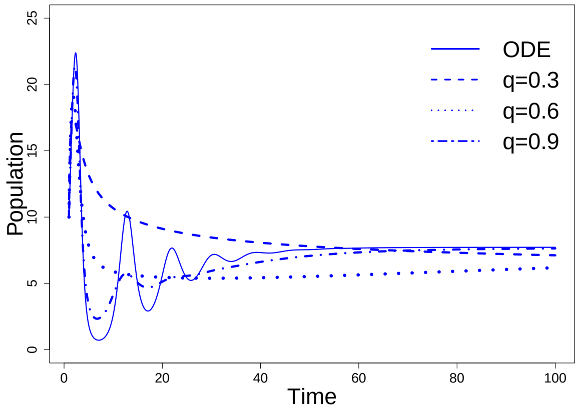
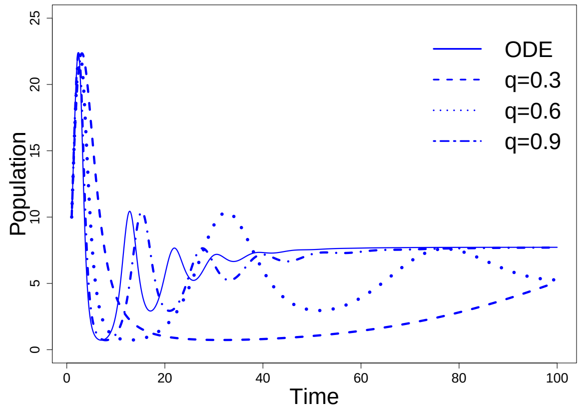


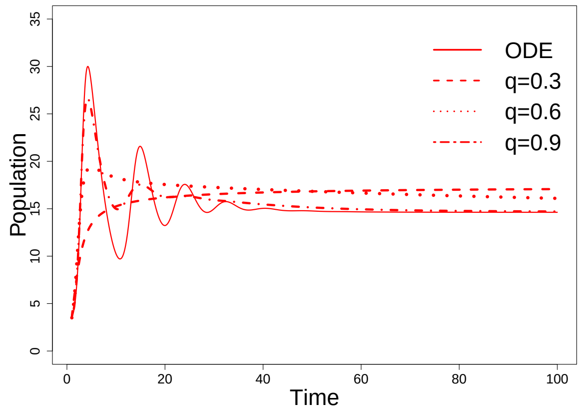
Figure 2. Phase portraits of the systems (3.1), (3.4) and (3.7) for the parameter values given in Table 2. (a) Bistability phenomenon in Scenario 1: Caputo fractional derivative (blue), Khalil fractional derivative (pink), integer order derivative (green), and Scenario 2: Caputo fractional derivative (brown), Khalil fractional derivative (yellow), integer order derivative (orange); b) Phase portrait of the coexistence equilibrium point $E_5 = (9.33, 12.82, 43.55)$.



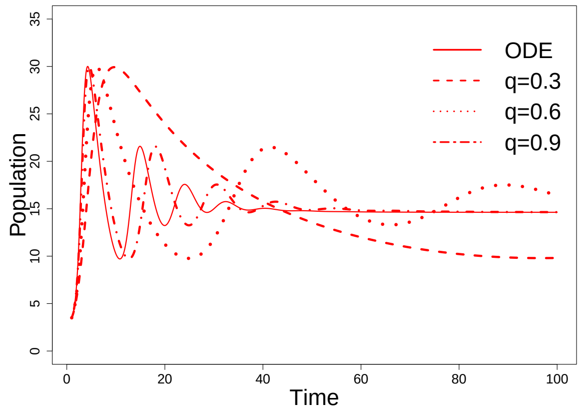
(a) Numerical solution of the Caputo approach.



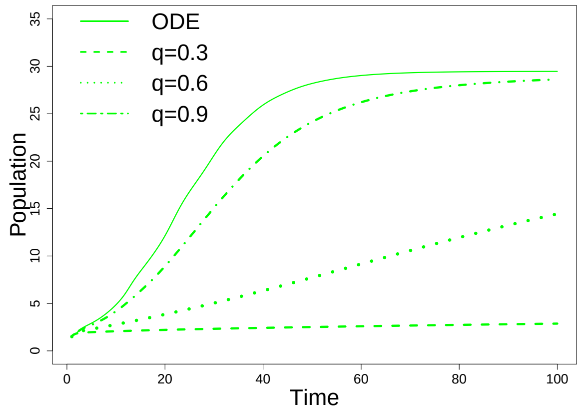
(b) Numerical solution of the conformable Khalil approach.



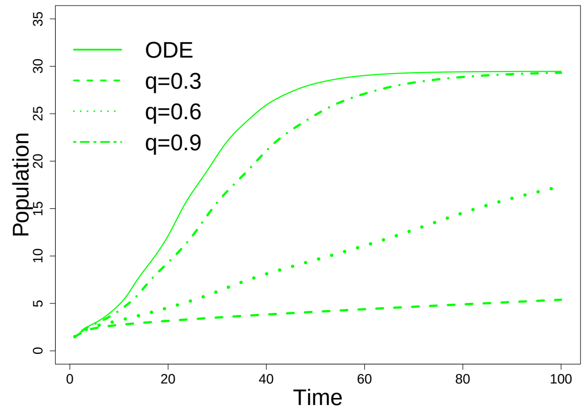
(c) Numerical solution of the Caputo approach.



(d) Numerical solution of the conformable Khalil approach.



(e) Numerical solution of the Caputo approach.



(f) Numerical solution of the conformable Khalil approach.

Figure 3. Stable dynamical behavior of the systems (3.1), (3.4), and (3.7) for the parameter values given in Table 2 (except r_2), with the initial condition $(S, I, Y) = (10, 3.5, 1.5)$.

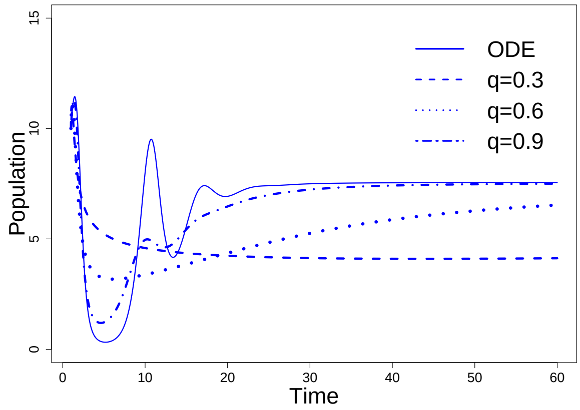
In the Figure 3, we chose to use the same values from Table 2, except for the parameter r_2, which was applied with the value of $r_2 = 0.15$. With this set of parameter values, the equilibrium points E_0, E_1, E_2, and E_3 were found, which coincide with the equilibrium points of the previous simulation, as they do not depend on this parameter; in addition, the $E_4 = (5.26, 0, 38.32)$ and $E_5 = (7.72, 14.62, 29.46)$ were found. In this case, it is observed that all trajectories of the fractional and ordinary systems approach the coexistence equilibrium point E_5 with the initial condition $(S_0, I_0, Y_0) = (10, 3.5, 1.5)$ (see Figure 2(b) and Figure 3). Moreover, for the parameter values were $B_1 = 0.5475 > 0$, $B_3 = 0.0704 > 0$, and $B_1B_2 - B_3 = 0.3086 > 0$, which satisfy the Routh-Hurwitz stability criteria. Therefore, the coexistence equilibrium point E_5 is locally asymptotically stable, as indicated in Theorem 9.

For the next set of simulations, we took the following hypothetical values for the parameters (see Table 4).

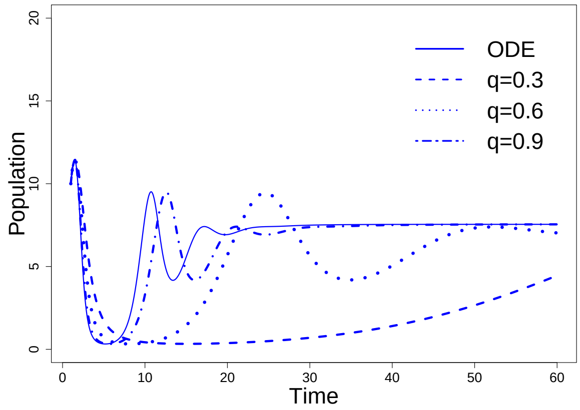
Table 4. Set of parameter values used for integer order and fractional order models.

p_1	α	r_1	p_2	k_1	δ_1	δ_2	γ	λ	r_2	k_2	m
0.05	3.5	1.5	0.15	50	0.35	0.20	0.30	0.15	0.35	20	0.25

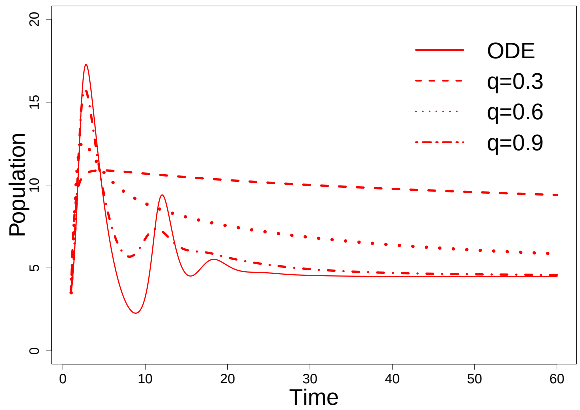
In the Figure 4, we considered that the maximum rate of consumption of infected prey is greater than that of susceptible prey, i.e., $p_2 > p_1$. From this set of values, we obtained the following equilibrium points: $E_0 = (0, 0, 0)$, $E_1 = (50, 0, 0)$, $E_2 = (0, 0, 20)$, $E_3 = (2, 8, 0)$, $E_4 = (5.52, 0, 31.50)$, and $E_5 = (7.54, 4.47, 30)$. In this case, only E_5 is locally asymptotically stable; also, it is observed that all trajectories of the fractional and ordinary systems approach the coexistence equilibrium point E_5 (see Figure 4), and that the susceptible prey population is bigger than the infected prey population, which evidences that the predator population can help to control the spread of the disease in the prey population. Note that, for these parameter values, we have that $B_1 = 1.2081 > 0$, $B_3 = 0.2158 > 0$, and $B_1B_2 - B_3 = 1.2496253 > 0$, which satisfies the Routh-Hurwitz stability criteria, as stated in Theorem 9.



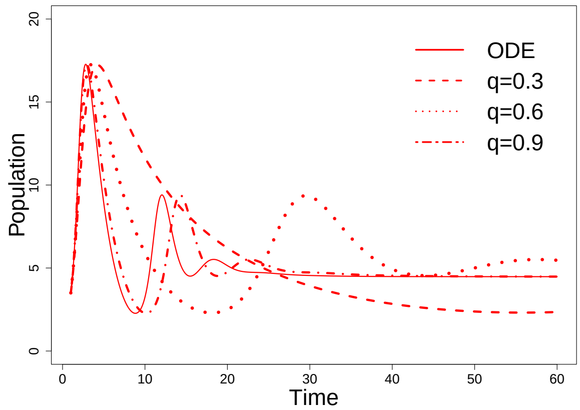
(a) Numerical solution of the Caputo approach.



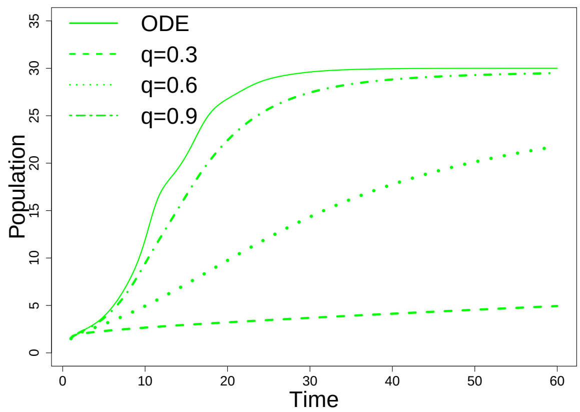
(b) Numerical solution of the conformable Khalil approach.



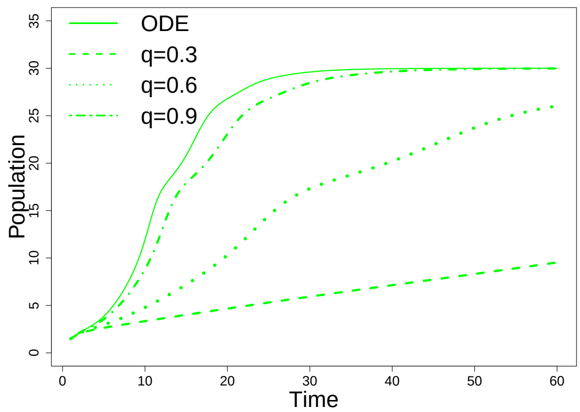
(c) Numerical solution of the Caputo approach.



(d) Numerical solution of the conformable Khalil approach.



(e) Numerical solution of the Caputo approach.



(f) Numerical solution of the conformable Khalil approach.

Figure 4. Stable dynamical behavior of the systems (3.1), (3.4), and (3.5) for the parameter values given in Table 3, with the initial condition $(S, I, Y) = (10, 3.5, 1.5)$.

6. Conclusions

In this paper, we have proposed an epidemiological model, first with the generalized Caputo fractional derivative and the generalized conformable fractional derivative to analyze a predator-prey model in the presence of an infectious disease in the prey and density-dependent predation rates. We have demonstrated the existence and uniqueness of the solution of the Caputo fractional derivative system (3.4), and that the solution remains positive and bounded whenever it starts with a positive initial value, showing that the model is well-posed. We found that there are five biologically meaningful equilibrium points, for which we have determined the existence and stability conditions. In particular, we show that, if $\mathcal{R}_0^f > 1$, the predator-free equilibrium point, E_3, is in Ω and is locally asymptotically stable. We have also shown that the equilibrium points E_4 and E_5 are locally asymptotically stable if the Routh-Hurwitz criterion is satisfied. Numerical simulations have been performed, and the results show that the solutions of the Caputo fractional derivative and Khalil fractional derivative converge to the solution of the integer order system when $q \to 1$. Similarly, we have shown that both fractional order models preserve the form of the solution of the integer order system, i.e., they preserve the oscillations of the susceptible and infected prey population and the solution tends to the equilibrium points of the integer order system.
From a biological point of view, determining the conditions of existence and stability for the coexistence equilibrium point, E_5, is important, as it guarantees the interaction between all species of the system. Particularly, in Table 3, we can see that, apparently, Theorem 2 of [68] is contradicted since the disease-free equilibrium point, E_4, is locally asymptotically stable despite the fact that $\mathcal{R}_0^f > 1$; however, this theorem does not take into account that the predation of infected prey decreases the population of the infectious class, so it can be seen that the predator population can aid in the control of infectious diseases among a population of susceptible and infected prey.

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Use of AI tools declaration

The authors declare that they have not used artificial intelligence tools in the creation of this article.

Conflict of interest

The authors declare that there is no conflict of interest.

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