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Research article

Hermite-Hadamard, Fejér and trapezoid type inequalities using Godunova-Levin Preinvex functions via Bhunia's order and with applications to quadrature formula and random variable

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Abstract: Convex and preinvex functions are two different concepts. Specifically, preinvex functions are generalizations of convex functions. We created some intriguing examples to demonstrate how these classes differ from one another. We showed that Godunova-Levin invex sets are always convex but the converse is not always true. In this note, we present a new class of preinvex functions called $(\mathtt{h}_1, \mathtt{h}_2)$-Godunova-Levin preinvex functions, which is extensions of $\mathtt{h}$-Godunova-Levin preinvex functions defined by Adem Kilicman. By using these notions, we initially developed Hermite-Hadamard and Fejér type results. Next, we used trapezoid type results to connect our inequality to the well-known numerical quadrature trapezoidal type formula for finding error bounds by limiting to standard order relations. Additionally, we use the probability density function to relate trapezoid type results for random variable error bounds. In addition to these developed results, several non-trivial examples have been provided as proofs.

Keywords: Hermite–Hadamard; Fejer; Trapezoidal formula; Godunova-Levin preinvex; mathematical operators; random variable

1. Introduction

Mathematical sciences rely heavily on convexity and it contributes to many fields such as optimization theory, economics, engineering, variational inequalities, management science and Riemannian manifolds. Convex sets and functions simplify complex problems, making them amenable to efficient computational solutions. A wide spectrum of scientific and engineering disciplines continues to benefit from concepts derived from convex analysis. Convexity is a powerful mathematical concept that can be used to simplify complicated mathematical problems and offer a theoretical framework for the creation of effective algorithms in a variety of domains. Complex systems behaviour can be deeply understood through integral inequalities that are derived from convexity concepts. These inequalities give mathematics rigour. Their ability to model, comprehend, and forecast a wide range of natural phenomena makes them indispensable instruments for engineers and physicists. Our understanding of the physical world will likely be further enhanced by the discovery of new applications and connections made possible by this field of study. To sum up, Jensen's work and later advancements in convex analysis have clarified the utility of convex functions, which is essential to understanding optimisation problems. It offers both useful techniques and theoretical underpinnings for identifying the best answers in a variety of applications. Convexity is still a major topic in mathematics, with research and applications being done in many different areas.

Approximation theory and probability distributions use generalised convexity concepts to approximate non-convex functions with convex functions. Numerous computational and numerical methods can benefit from this approximation. To summarise, integral inequalities and generalised convexity are closely related fields of study that share a mathematical framework for establishing and analysing these inequalities. The significance of comprehending the interaction between generalised convexity and integral inequalities in theoretical and practical contexts is emphasised by the applications of these ideas in a variety of fields, such as physics, functional analysis, and optimisation. Literature contains a variety of inequality types. The most crucial factor in optimization problems is Hermite-Hadamard or often called double inequality. In this context, we consider the well-known inequality owing to Hadamard and Hermite independently for convex functions; see Ref [1].

$$\mathfrak{V}\left(\frac{\mathfrak{g}_g + \mathfrak{f}_g}{2}\right) \leq \frac{1}{\mathfrak{f}_g - \mathfrak{g}_g} \int_{\mathfrak{g}_g}^{\mathfrak{f}_g} \mathfrak{V}(\nu)\, d\nu \leq \frac{\mathfrak{V}(\mathfrak{g}_g) + \mathfrak{V}(\mathfrak{f}_g)}{2}. \tag{1.1}$$

In addition to its mathematical relevance and its widespread application in a variety of domains involving different classes of generalized convexity, researchers are also investigating how to extend it to function spaces; see Refs. [2–5]. In mathematical optimization and related areas, invex functions have become important extensions of convex functions. Initially, in [6], authors introduced invex functions, that generalized classical convex mappings and discuss some of its interesting properties. In [7], Ben and Mond combined work and introduced modified form of invex sets and preinvex functions, an extension and generalization of classical convex mappings. The differentiable preinvex mappings in this class of invexity are invex, which is one of its distinguish features, but not the converse. Even though preinvex functions aren't convex, they have some lovely properties that convex functions don't.; see Ref [8]. Based on Almutairi's [9] formulation, a function $\mathfrak{V}$ is called to be $\mathfrak{h}$-Godunova-Levin (**GL**) preinvex on interval $[\mathfrak{g}_g, \mathfrak{g}_g + \varsigma(\mathfrak{f}_g, \mathfrak{g}_g)]$ iff it satisfies the following double inequality

$$\frac{h\left(\frac{1}{2}\right)}{2}\mathfrak{V}\left(\frac{2\mathfrak{g}_g+\varsigma(\mathfrak{f}_g,\mathfrak{g}_g)}{2}\right)\leq\frac{1}{\varsigma(\mathfrak{f}_g,\mathfrak{g}_g)}\int_{\mathfrak{g}_g}^{\mathfrak{g}_g+\varsigma(\mathfrak{f}_g,\mathfrak{g}_g)}\mathfrak{V}(\varrho)\,d\varrho\leq[\mathfrak{V}(\mathfrak{g}_g)+\mathfrak{V}(\mathfrak{f}_g)]\int_0^1\frac{d\eta_o}{h(\eta_o)}. \tag{1.2}$$

Based on this result, several authors extend this result in various ways, many of them using various types of preinvex functions; see Refs [10–12].

Interval-valued analysis allows one to deal with uncertainties and errors in a number of computational tasks effectively. This method ensures that results are based on uncertainties in input data by representing numerical values as intervals, making it particularly useful for applications that require accurate predictions and reliable results. By representing numerical values as intervals, it provides a realistic and conservative approach to computations. Through Moore's [13] contributions, interval analysis has developed a wide range of applications that span many fields, including math, computer science, engineering, and natural science; see Refs [14, 15]. As a result of precise results in a variety of disciplines, mathematicians are motivated to extend integral inequalities to interval-valued mappings.

Initially, authors in [16] used $\mathtt{h}$-convex mappings to link Jensen type and Hadamard type results in the setup of set-valued functions. By combining the concepts of set-valued analysis and $\mathtt{h}$-convex mappings authors in [17] developed three well-known inequalities that shed light on the characteristics and behaviour of stochastic processes within a probability space. In [18], authors used the notion of preinvex functions to create double inequality for set-valued mappings. In [19] authors utilize the notion of preinvex functions on coordinates and developed various results of the double inequality on rectangular plane. Zhou, Saleem, Nazeer, Shah [20] developed an improved form of the double inequalities by using pre-invex exponential type functions via fractional integrals in the context of set-valued mappings. Khan, Catas, Aloraini, Soliman [21] used up-down preinvex mappings in a fuzzy setup to get Fejér and Hermite-type findings. Using the concept of $(\mathtt{h}_1, \mathtt{h}_2)$-preinvex mappings, Aslam, Khalida, Saima [22] created a number of Hermite-Hadamard type results related to special functions using power mean integral inequalities. Employing the concept of harmonical $(\mathtt{h}_1, \mathtt{h}_2)$-Godunova-Levin functions through centre and radius interval order relation, the authors in [23] produced Hermite-Hadamard and Jensen type results, which expand upon a number of earlier discoveries. Using local fractional integrals, Sun [24] created a various new form of double inequalities for h-preinvex functions with applications. For generalized preinvex mappings, authors in [25] developed various novel variants of double inequalities with some interesting properties using the notion of $(\mathtt{s}, \mathtt{m}, \varphi)$ type functions. Using partial order relations, Ali et al. [26] developed different new variants of Hermite-Hadamard type results based on Godunova-Levin preinvex mappings. By combining fractional operators and generalized preinvex mappings, Tariq et al. [27] developed various new Hermite-Hadamard and Fejér type results. Sitho et al. [28] used the idea of quantum integrals to demonstrate midpoint and trapezoidal inequalities for differentiable preinvex functions. Latif, Kashuri, Hussain, Delayer [29] investigated Trapezium-type inequalities for $\mathtt{h}$-preinvex functions, as well as their applications to special means. Delavar [30] used fractional integrals to find new bounds for Hermite-Hadamard's trapezoid and midpoint type inequalities. Stojiljković et al. [31] developed some new bounds for Hermite-Hadamard type inequalities involving various types of convex functions using fractional operators. Afzal, Eldin, Nazeer, Galal [32] created several novel Hermite-Hadamard type results by employing the harmonical Godunova-levin function in a stochastic sense with centre and radius order. Tariq, Ahmad, Budak, Sahoo, Sitthiwirattham [33] conducted a thorough analysis using generalized preinvex functions of Hermite-Hadamard type inequalities. Afzal, Botmart [34] used the notion of $\mathtt{h}$-Godunova-Levin stochastic process and developed some new bounds of Hermite-Hadamard and Jensen type inclusions. Kalsoom, Latif, Idrees, Arif, Salleh [35] created Hermite-Hadamard type inequalities for generalized strongly preinvex functions using the idea of quantum calculus. Duo, Zhou [36] created some new bounds by using fractional double integral inclusion relations having exponential kernels via interval-valued coordinated convex mappings. Furthermore, comparable outcomes applying a variety of alternative fractional operators that we refer to [37–40].

This work is novel and noteworthy since it introduces a more generalized class, referred to as $(\mathtt{h}_1, \mathtt{h}_2)$-Godunova-Levin preinvex functions that unify different previously reported findings by employing different choices of bifunction ς. Since convexity and preinvexity are two different concepts, and preinvexity enjoys more nice properties than classical convex mappings, a more generalized form of inequalities is deduced with this class. Furthermore, this is the first time in literature that we have identified error bounds for quadrature type formula via this class of generalized convexity furthermore we also discuss some applications for random variables within context of error bounds that also generalize different results. The majority of literature is based on partial order or pseudo order relationships which have significant flaws in some of the inequalities results since we are not able to compare two intervals. This order relationship offers the advantage of conveniently comparing intervals and, more importantly, the endpoints of interval difference is much smaller, so a more precise result can be obtained. Recently, various authors utilized Bhunias Samanata order relation to formulate various results using different classes of convexities; see Refs. [41,42]. Stojiljković, Mirkov, Radenović [43] created a number of novel tensorial trapezoid-type inequalities for convex functions of self-ddjoint operators in Hilbert spaces. Liu, Shi, Ye, Zhao [44] employed the idea of harmonically convex functions to establish new bounds for Hermite-Hadamard type inequalities by using centre and radius orders. Regarding other recent advancements employing distinct categories of convex mappings under centre and radius order, please see [45–47].

The literature related to developed inequalities and specifically these articles; [9, 25, 41] is leading us to define a new class of preinvexity for the first time and utilizing these notions, we are developing various novel variants of the famous double and Trapezoid type inequalities and their relation to Fejér's work. The arrangement of the article is designed as: following the preliminary work in Sect. 2, we present a new class of preinvexity and talk about some of its intriguing properties in Sect. 3. The main results of this paper are presented in Sect. 4, where we developed different forms of famous double type inequalities, and in Sect. 5, where we created modified Hermite-Hadamard-Fejér type results. Section 6, focuses on error bounds of numerical integration with applications to random variable via trapezoidal type inequality. Section 7, closes with a summary of some final thoughts and suggestions for additional study.

2. Preliminaries

In this section, we discuss some current definitions and results that may provide support for the primary conclusions stated in the study. Furthermore, certain ideas are used in papers without being defined; see Ref. [9].

Definition 2.1. [9] *Suppose* $\mathtt{Q}$ *be a subset of* $\mathbf{R}$*, then it is called to be invex with respect to to the bifunction* $\varsigma(\cdot,\cdot):\mathtt{Q}\times\mathtt{Q}\to\mathbf{R}^n$*, if*

$$\mathfrak{g}_g+\eta_o\varsigma(\mathfrak{f}_g,\mathfrak{g}_g)\in\mathtt{Q}$$

for all $\mathfrak{g}_g,\mathfrak{f}_g\in\mathtt{Q}$ *and* $\eta_o\in[0,1]$.

Example 2.1. *Suppose* $\mathtt{Q}=[-4,-3]\cup[-2,3]$ *is called to be invex with respect to* $\varsigma(\cdot,\cdot)$ *and mappings is defined as:*

$$\varsigma(\rho_1,\eta_1)=\begin{cases}\rho_1-\eta_1 & \text{if } -2\le\rho_1\le 3,-1\le\eta_1\le 3\\ \rho_1-\eta_1 & \text{if } -4\le\rho_1\le -3,-4\le\eta_1\le -3;\\ -4-\eta_1 & \text{if } -2\le\rho_1\le 3,-4\le\eta_1\le -2\\ -2-\eta_1 & \text{if } -4\le\rho_1\le -3,-2\le\eta_1\le 3.\end{cases}$$

In that situation, $\mathtt{Q}$ *is definitely invex with respect to* $\varsigma(\cdot,\cdot)$*, but it is clearly not a convex set.*

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Definition 2.2. [50] *Suppose* $\mathtt{Q}$ *is a invex with respect to the* $\varsigma(\cdot,\cdot)$*. A function* $\mathfrak{V}:\mathtt{Q}\to\mathbf{R}$ *is called to be preinvex with respect to* $\varsigma(\cdot,\cdot)$ *if*

$$\mathfrak{V}\left(\mathfrak{g}_g+\eta_o\varsigma\left(\mathfrak{f}_g,\mathfrak{g}_g\right)\right)\le\eta_o\mathfrak{V}(\mathfrak{f}_g)+(1-\eta_o)\,\mathfrak{V}(\mathfrak{g}_g)$$

for all $\mathfrak{g}_g,\mathfrak{f}_g\in\mathtt{Q}$ *and* $\eta_o\in[0,1]$.

Definition 2.3. [50] *Suppose* $\mathtt{Q}$ *is a invex with respect to the* $\varsigma(\cdot,\cdot)$*. A function* $\mathfrak{V}:\mathtt{Q}\to\mathbf{R}$ *is called to be* **GL** *preinvex with respect to* ς *if*

$$\mathfrak{V}\left(\mathfrak{g}_g+\eta_o\varsigma\left(\mathfrak{f}_g,\mathfrak{g}_g\right)\right)\le\frac{\mathfrak{V}(\mathfrak{f}_g)}{\eta_o}+\frac{\mathfrak{V}(\mathfrak{g}_g)}{(1-\eta_o)}$$

for all $\mathfrak{g}_g,\mathfrak{f}_g\in\mathtt{Q}$ *and* $\eta_o\in(0,1)$.

Definition 2.4. [50] *Suppose* $\mathtt{Q}$ *is a invex with respect to the* $\varsigma(\cdot,\cdot)$*. A function* $\mathfrak{V}:\mathtt{Q}\to\mathbf{R}$ *is called to be* $\mathtt{h}$*-preinvex with respect to* ς *if*

$$\mathfrak{V}\left(\mathfrak{g}_g+\eta_o\varsigma\left(\mathfrak{f}_g,\mathfrak{g}_g\right)\right)\le\mathtt{h}(\eta_o)\mathfrak{V}(\mathfrak{f}_g)+\mathtt{h}(1-\eta_o)\,\mathfrak{V}(\mathfrak{g}_g)$$

for all $\mathfrak{g}_g,\mathfrak{f}_g\in\mathtt{Q}$ *and* $\eta_o\in(0,1)$.

Definition 2.5. [50] *Suppose* $\mathtt{Q}$ *is a invex with respect to the* $\varsigma(\cdot,\cdot)$*. A Function* $\mathfrak{V}:\mathtt{Q}\to\mathbf{R}$ *is called to be* $\mathtt{h}$-**GL** *preinvex with respect to* ς *if*

$$\mathfrak{V}\left(\mathfrak{g}_g+\eta_o\varsigma\left(\mathfrak{f}_g,\mathfrak{g}_g\right)\right)\le\frac{\mathfrak{V}(\mathfrak{f}_g)}{\mathtt{h}(\eta_o)}+\frac{\mathfrak{V}(\mathfrak{g}_g)}{\mathtt{h}(1-\eta_o)}$$

for all $\mathfrak{g}_g,\mathfrak{f}_g\in\mathtt{Q}$ *and* $\eta_o\in(0,1)$.

Definition 2.6. [50] *Suppose* $\mathtt{Q}$ *is a invex with respect to* $\varsigma(\cdot,\cdot)$*. If for all* $\mathfrak{g}_g,\mathfrak{f}_g\in\mathtt{Q}$ *and* $\eta_o\in[0,1]$,

$$\varsigma\left(\mathfrak{f}_g,\mathfrak{f}_g+\eta_o\,\varsigma\left(\mathfrak{g}_g,\mathfrak{f}_g\right)\right)=-\eta_o\,\varsigma\left(\mathfrak{g}_g,\mathfrak{f}_g\right)\tag{2.1}$$

and

$$\varsigma\left(\mathfrak{g}_g,\mathfrak{f}_g+\eta_o\,\varsigma\left(\mathfrak{g}_g,\mathfrak{f}_g\right)\right)=(1-\eta_o)\,\varsigma\left(\mathfrak{g}_g,\mathfrak{f}_g\right).\tag{2.2}$$

for all $\mathfrak{g}_g,\mathfrak{f}_g\in\mathtt{Q}$ *and* $\eta_{o1},\eta_{o2}\in[0,1]$*, and this is said to be Condition C, if one has*

$$\varsigma\left(\mathfrak{f}_g+\eta_{o2}\,\varsigma\left(\mathfrak{g}_g,\mathfrak{f}_g\right),\mathfrak{f}_g+\eta_{o1}\varsigma\left(\mathfrak{g}_g,\mathfrak{f}_g\right)\right)=(\eta_{o2}-\eta_{o1})\,\varsigma\left(\mathfrak{g}_g,\mathfrak{f}_g\right).$$

2.1. Some Basic Notions of Set-Valued Functions

As we proceed through the article, we will cover a few basic information regarding interval analysis.

$$[\diamond] = [\underline{\diamond}, \overline{\diamond}] \quad (\underline{\diamond} \leq \nu \leq \overline{\diamond};\ \nu \in \mathbf{R}),$$
$$[\varsigma] = [\underline{\varsigma}, \overline{\varsigma}] \quad (\underline{\varsigma} \leq \nu \leq \overline{\varsigma};\ \nu \in \mathbf{R}),$$
$$[\diamond] + [\varsigma] = [\underline{\diamond}, \overline{\diamond}] + [\underline{\varsigma}, \overline{\varsigma}] = [\underline{\diamond} + \underline{\varsigma}, \overline{\diamond} + \overline{\varsigma}]$$

and

$$\Lambda\diamond = \Lambda[\underline{\diamond}, \overline{\diamond}] = \begin{cases} [\Lambda\underline{\diamond}, \Lambda\overline{\diamond}], & \text{if } \Lambda > 0; \\ \{0\}, & \text{if } \Lambda = 0; \\ [\Lambda\overline{\diamond}, \Lambda\underline{\diamond}], & \text{if } \Lambda < 0, \end{cases}$$

where $\Lambda \in \mathbf{R}$.

Let $\mathbf{R}_I$ be the pack of all intervals and $\mathbf{R}_I^+$ be the collection of all positive intervals of set of real number $\mathbf{R}$. As a next step, we define how we calculate the relation we use throughout the article. It is called midpoint and radii of interval order relation.

More precisely $\diamond$ can be represented as follows:

$$\diamond = \langle \diamond_c, \diamond_r \rangle = \left\langle \frac{\overline{\diamond} + \underline{\diamond}}{2}, \frac{\overline{\diamond} - \underline{\diamond}}{2} \right\rangle.$$

Accordingly, we can describe the $C\mathcal{R}$ order relation for intervals in this manner:

Definition 2.7. [45] *The Bhunia and Samanta interval order relation for* $\diamond = [\underline{\diamond}, \overline{\diamond}] = \langle \diamond_c, \diamond_r \rangle$ *and* $\varsigma = [\underline{\varsigma}, \overline{\varsigma}] = \langle \varsigma_c, \varsigma_r \rangle \in \mathbf{R}_I$ *is defined as:*

$$\diamond \preceq_{cr} \varsigma \Longleftrightarrow \begin{cases} \diamond_c < \varsigma_c, & \text{if } \diamond_c \neq \varsigma_c; \\ \diamond_r \leq \varsigma_r, & \text{if } \diamond_c = \varsigma_c. \end{cases}$$

For the intervals $\diamond, \varsigma \in \mathbf{R}_I$, then this relation hold $\diamond \preceq_{cr} \varsigma$ or $\varsigma \preceq_{cr} \diamond$.

Definition 2.8. [46] *Let* $\mathfrak{V} : [\mathfrak{g}_g, \mathfrak{f}_g]$ *be an* **I.V.F** *where* $\mathfrak{V} = [\underline{\mathfrak{V}}, \overline{\mathfrak{V}}]$*, then* $\mathfrak{V}$ *is Riemann integrable* (**IR**) *on* $[\mathfrak{g}_g, \mathfrak{f}_g]$ *iff* $\underline{\mathfrak{V}}$ *and* $\overline{\mathfrak{V}}$ *are* (**IR**) *on* $[\mathfrak{g}_g, \mathfrak{f}_g]$*, that is,*

$$(\mathbf{IR}) \int_{\mathfrak{g}_g}^{\mathfrak{f}_g} \mathfrak{V}(\varrho)\, d\varrho = \left[(\mathbf{R}) \int_{\mathfrak{g}_g}^{\mathfrak{f}_g} \underline{\mathfrak{V}}(\varrho)\, d\varrho, (\mathbf{R}) \int_{\mathfrak{g}_g}^{\mathfrak{f}_g} \overline{\mathfrak{V}}(\varrho)\, d\varrho \right].$$

The pack of all **I.V.F.S** *for Riemann integrable on* $[\mathfrak{g}_g, \mathfrak{f}_g]$ *is denoted by* $\mathbf{IR}_{([\mathfrak{g}_g, \mathfrak{f}_g])}$.

Theorem 2.1. [47] *Let* $\mathfrak{V}, \eta_o : [\mathfrak{g}_g, \mathfrak{f}_g]$ *be an* **I.V.F.S** *defined as* $\mathfrak{V} = [\underline{\mathfrak{V}}, \overline{\mathfrak{V}}]$ *and* $\eta_o = [\underline{\eta_o}, \overline{\eta_o}]$*. If* $\mathfrak{V}(\varrho) \preceq_{cr} \eta_o(\varrho)\ \forall\ \varrho \in [\mathfrak{g}_g, \mathfrak{f}_g]$*, then*

$$\int_{\mathfrak{g}_g}^{\mathfrak{f}_g} \mathfrak{V}(\varrho)\, d\varrho \preceq_{C\mathcal{R}} \int_{\mathfrak{g}_g}^{\mathfrak{f}_g} \eta_o(\varrho)\, d\varrho.$$

With the help of an example, we show that the preceding Theorem holds true.

Example 2.2. *Let $\mathfrak{V} = [\varrho, 2\varrho]$ and $\eta_o = [\varrho^2, \varrho^2+2]$. Then, for $\varrho \in [0,1]$, $\mathfrak{V}_C = \frac{3\varrho}{2}$, $\mathfrak{V}_{\mathcal{R}} = \frac{\varrho}{2}$, $\eta_{oC} = \varrho^2+1$ and $\eta_{o\mathcal{R}} = 1$. As a result, by utilizing the Definition 2.7, one has $\mathfrak{V}(\varrho) \preceq_{C\mathcal{R}} \eta_o(\varrho)$ for $\varrho \in [0,1]$. Since,*

$$\int_0^1 [\varrho, 2\varrho]\, d\varrho = \left[\frac{1}{2}, 1\right]$$

and

$$\int_0^1 [\varrho^2, \varrho^2+2]\, d\varrho = \left[\frac{1}{3}, \frac{7}{3}\right].$$

From Theorem 2.1, one has

$$\int_0^1 \mathfrak{V}(\varrho)\, d\varrho \preceq_{C\mathcal{R}} \int_0^1 \eta_o(\varrho)\, d\varrho.$$

3. Some Novel Definitions and its Special Cases

The purpose of this section is to introduce a new type of preinvexity called Godunova-Levin preinvex functions of the $(\mathrm{h}_1, \mathrm{h}_2)$ type, based on total order relations, that generalizes several existing definitions.

Definition 3.1. *Suppose $\mathfrak{V} : [\mathfrak{g}_g, \mathfrak{f}_g]$ be an set-valued function given by $\mathfrak{V} = [\underline{\mathfrak{V}}, \overline{\mathfrak{V}}]$. Let $\mathrm{h}_1, \mathrm{h}_2 : (0,1) \to (0,\infty)$ where $\mathrm{h}_1, \mathrm{h}_2 \neq 0$, then $\mathfrak{V}$ is called to be $C\mathcal{R}$-$(\mathrm{h}_1, \mathrm{h}_2)$-**GL**-preinvex with respect to ς if*

$$\mathfrak{V}(\mathfrak{g}_g + \eta_o\varsigma\left(\mathfrak{f}_g, \mathfrak{g}_g\right)) \preceq_{C\mathcal{R}} \frac{\mathfrak{V}(\mathfrak{f}_g)}{\mathrm{H}(\eta_o, 1-\eta_o)} + \frac{\mathfrak{V}(\mathfrak{g}_g)}{\mathrm{H}(1-\eta_o, \eta_o)},$$

for all $\mathfrak{g}_g, \mathfrak{f}_g \in \mathrm{Q}$ and $\eta_o \in (0,1)$.

Remark 3.1. *Choosing $\mathrm{h}_1(\eta_o) = \frac{1}{\eta_o}, \mathrm{h}_2(\eta_o) = 1$, in Definition 3.1, the $C\mathcal{R}$-$(\mathrm{h}_1, \mathrm{h}_2)$-**GL**-preinvex function reduces to the $C\mathcal{R}$-preinvex function.*

$$\mathfrak{V}\left(\mathfrak{g}_g + \eta_o\varsigma\left(\mathfrak{f}_g, \mathfrak{g}_g\right)\right) \preceq_{C\mathcal{R}} \eta_o\mathfrak{V}(\mathfrak{f}_g) + (1-\eta_o)\,\mathfrak{V}(\mathfrak{g}_g).$$

Remark 3.2. *Choosing $\mathrm{h}_1(\eta_o) = \frac{1}{\eta_o^s}, \mathrm{h}_2(\eta_o) = 1$, in Definition 3.1, the $C\mathcal{R}$-$(\mathrm{h}_1, \mathrm{h}_2)$-**GL**-preinvex function reduces to the $C\mathcal{R}$-s-preinvex function.*

$$\mathfrak{V}\left(\mathfrak{g}_g + \eta_o\varsigma\left(\mathfrak{f}_g, \mathfrak{g}_g\right)\right) \preceq_{C\mathcal{R}} \eta_o^s\mathfrak{V}(\mathfrak{f}_g) + (1-\eta_o)^s\,\mathfrak{V}(\mathfrak{g}_g).$$

Remark 3.3. *Choosing $\mathrm{h}_1(\eta_o) = \eta_o, \mathrm{h}_2(\eta_o) = 1$, in Definition 3.1, the $C\mathcal{R}$-$(\mathrm{h}_1, \mathrm{h}_2)$-**GL**-preinvex function reduces to the $C\mathcal{R}$-**GL**-preinvex function.*

$$\mathfrak{V}\left(\mathfrak{g}_g + \eta_o\varsigma\left(\mathfrak{f}_g, \mathfrak{g}_g\right)\right) \preceq_{C\mathcal{R}} \frac{\mathfrak{V}(\mathfrak{f}_g)}{\eta_o} + \frac{\mathfrak{V}(\mathfrak{g}_g)}{(1-\eta_o)}.$$

Remark 3.4. *Choosing $\mathrm{h}_1(\eta_o) = \frac{1}{\eta_o(1-\eta_o)}, \mathrm{h}_2(\eta_o) = 1$, in Definition 3.1, the $C\mathcal{R}$-$(\mathrm{h}_1, \mathrm{h}_2)$-**GL**-preinvex function reduces to the tgs $C\mathcal{R}$ preinvex function [52].*

$$\mathfrak{V}\left(\mathfrak{g}_g + \eta_o\varsigma\left(\mathfrak{f}_g, \mathfrak{g}_g\right)\right) \preceq_{C\mathcal{R}} \eta_o(1-\eta_o)[\mathfrak{V}(\mathfrak{f}_g) + \mathfrak{V}(\mathfrak{g}_g)].$$

Remark 3.5. *Choosing* $\varsigma(\mathfrak{f}_g, \mathfrak{g}_g) = \mathfrak{f}_g - \mathfrak{g}_g$ *and* $\mathrm{h}_1(\eta_o) = \frac{1}{\eta_o}, \mathrm{h}_2(\eta_o) = 1$*, in Definition 3.1, the* $C\mathcal{R}$*-*$(\mathrm{h}_1, \mathrm{h}_2)$*-***GL***-preinvex function reduces to the* $C\mathcal{R}$*-convex function [52].*

$$\mathfrak{V}\left(\eta_o \mathfrak{f}_g + (1-\eta_o)\mathfrak{g}_g\right) \preceq_{C\mathcal{R}} \eta_o \mathfrak{V}(\mathfrak{f}_g) + (1-\eta_o)\,\mathfrak{V}(\mathfrak{g}_g).$$

Remark 3.6. *Choosing* $\underline{\mathfrak{V}} = \overline{\mathfrak{V}}$*, in Definition 3.1, the* $C\mathcal{R}$*-*$(\mathrm{h}_1, \mathrm{h}_2)$*-***GL***-preinvex function reduces to the* h-**GL***-preinvex function [48].*

$$\mathfrak{V}\left(\mathfrak{g}_g + \eta_o \varsigma\left(\mathfrak{f}_g, \mathfrak{g}_g\right)\right) \leq \frac{\mathfrak{V}(\mathfrak{f}_g)}{\mathrm{h}(\eta_o)} + \frac{\mathfrak{V}(\mathfrak{g}_g)}{\mathrm{h}(1-\eta_o)}.$$

Proposition 3.1. *Let* $\mathfrak{V} : [\mathfrak{g}_g, \mathfrak{f}_g] \to \mathbf{R}_I$ *be an set-valued function given by* $\mathfrak{V} = [\underline{\mathfrak{V}}, \overline{\mathfrak{V}}] = \langle \mathfrak{V}_C, \mathfrak{V}_\mathcal{R} \rangle$*. If* $\mathfrak{V}_C$ *and* $\mathfrak{V}_\mathcal{R}$ *are* $(\mathrm{h}_1, \mathrm{h}_2)$*-***GL***-preinvex functions, then* $\mathfrak{V}$ *is a* $C\mathcal{R}$*-*$(\mathrm{h}_1, \mathrm{h}_2)$*-***GL***-preinvex mapping.*

Proof. Since $\mathfrak{V}_C$ and $\mathfrak{V}_\mathcal{R}$ are $(\mathrm{h}_1, \mathrm{h}_2)$-**GL**-preinvex functions, and $\forall\ \eta_o \in (0,1)$, one has

$$\mathfrak{V}_C\left(\mathfrak{g}_g + \eta_o \varsigma\left(\mathfrak{f}_g, \mathfrak{g}_g\right)\right) \leq \frac{\mathfrak{V}_C(\mathfrak{f}_g)}{\mathrm{H}(\eta_o, 1-\eta_o)} + \frac{\mathfrak{V}_C(\mathfrak{g}_g)}{\mathrm{H}(1-\eta_o, \eta_o)}$$

and

$$\mathfrak{V}_\mathcal{R}\left(\mathfrak{g}_g + \eta_o \varsigma\left(\mathfrak{f}_g, \mathfrak{g}_g\right)\right) \leq \frac{\mathfrak{V}_\mathcal{R}(\mathfrak{f}_g)}{\mathrm{H}(\eta_o, 1-\eta_o)} + \frac{\mathfrak{V}_\mathcal{R}(\mathfrak{g}_g)}{\mathrm{H}(1-\eta_o, \eta_o)}.$$

If $\mathfrak{V}_C\left(\mathfrak{g}_g + \eta_o \varsigma\left(\mathfrak{f}_g, \mathfrak{g}_g\right)\right) \neq \frac{\mathfrak{V}_C(\mathfrak{f}_g)}{\mathrm{H}(\eta_o, 1-\eta_o)} + \frac{\mathfrak{V}_C(\mathfrak{g}_g)}{\mathrm{H}(1-\eta_o, \eta_o)}$, then

$$\mathfrak{V}_C\left(\mathfrak{g}_g + \eta_o \varsigma\left(\mathfrak{f}_g, \mathfrak{g}_g\right)\right) < \frac{\mathfrak{V}_C(\mathfrak{f}_g)}{\mathrm{H}(\eta_o, 1-\eta_o)} + \frac{\mathfrak{V}_C(\mathfrak{g}_g)}{\mathrm{H}(1-\eta_o, \eta_o)}.$$

This implies

$$\mathfrak{V}_C\left(\mathfrak{g}_g + \eta_o \varsigma\left(\mathfrak{f}_g, \mathfrak{g}_g\right)\right) \preceq_{C\mathcal{R}} \frac{\mathfrak{V}_C(\mathfrak{f}_g)}{\mathrm{H}(\eta_o, 1-\eta_o)} + \frac{\mathfrak{V}_C(\mathfrak{g}_g)}{\mathrm{H}(1-\eta_o, \eta_o)}.$$

Otherwise, $\mathfrak{V}_\mathcal{R}\left(\mathfrak{g}_g + \eta_o \varsigma\left(\mathfrak{f}_g, \mathfrak{g}_g\right)\right) \leq \frac{\mathfrak{V}_\mathcal{R}(\mathfrak{f}_g)}{\mathrm{H}(\eta_o, 1-\eta_o)} + \frac{\mathfrak{V}_\mathcal{R}(\mathfrak{g}_g)}{\mathrm{H}(1-\eta_o, \eta_o)}$ this implies

$$\mathfrak{V}_\mathcal{R}\left(\mathfrak{g}_g + \eta_o \varsigma\left(\mathfrak{f}_g, \mathfrak{g}_g\right)\right) \preceq_{C\mathcal{R}} \frac{\mathfrak{V}_\mathcal{R}(\mathfrak{f}_g)}{\mathrm{H}(\eta_o, 1-\eta_o)} + \frac{\mathfrak{V}_\mathcal{R}(\mathfrak{g}_g)}{\mathrm{H}(1-\eta_o, \eta_o)}.$$

From Definition 3.1, we have

$$\mathfrak{V}(\mathfrak{g}_g + \eta_o \varsigma\left(\mathfrak{f}_g, \mathfrak{g}_g\right)) \preceq_{C\mathcal{R}} \frac{\mathfrak{V}(\mathfrak{f}_g)}{\mathrm{H}(\eta_o, 1-\eta_o)} + \frac{\mathfrak{V}(\mathfrak{g}_g)}{\mathrm{H}(1-\eta_o, \eta_o)}$$

This demonstrate that, if $\mathfrak{V}_C$ and $\mathfrak{V}_\mathcal{R}$ are $(\mathrm{h}_1, \mathrm{h}_2)$-**GL**-preinvex functions, then $\mathfrak{V}$ is a $C\mathcal{R}$-$(\mathrm{h}_1, \mathrm{h}_2)$-**GL**-preinvex function. □

4. Hermite-Hadamard Type Inequalities for $\mathcal{CR}$-$(\mathtt{h}_1, \mathtt{h}_2)$-Godunova-Levin preinvex functions

As part of this section, we present several new Hermite-Hadamard and Fejér type inequalities for Godunova-Levin-preinvex functions of the $(\mathtt{h}_1, \mathtt{h}_2)$ type.

Theorem 4.1. *Let* $\mathfrak{V} : [\mathfrak{g}_g, \mathfrak{g}_g + \varsigma(\mathfrak{f}_g, \mathfrak{g}_g)] \to \mathbf{R}_I$ *be an set-valued function defined as* $\mathfrak{V}(\varrho) = \left[\underline{\mathfrak{V}}(\varrho), \overline{\mathfrak{V}}(\varrho)\right]$. *If* $\mathfrak{V} : [\mathfrak{g}_g, \mathfrak{g}_g + \varsigma(\mathfrak{f}_g, \mathfrak{g}_g)] \to \mathbf{R}$ *is a* $\mathcal{CR}$-$(\mathtt{h}_1, \mathtt{h}_2)$-**GL**-*preinvex mapping and satisfies the* ***Condition C****, then the following relation holds:*

$$\frac{\left[\mathtt{H}\left(\frac{1}{2}, \frac{1}{2}\right)\right]}{2}\mathfrak{V}\left(\frac{2\mathfrak{g}_g + \varsigma(\mathfrak{f}_g, \mathfrak{g}_g)}{2}\right) \preceq_{\mathcal{CR}} \frac{1}{\varsigma(\mathfrak{f}_g, \mathfrak{g}_g)} \int_{\mathfrak{g}_g}^{\mathfrak{g}_g+\varsigma(\mathfrak{f}_g,\mathfrak{g}_g)} \mathfrak{V}(\varrho)\, d\varrho$$
$$\preceq_{\mathcal{CR}} [\mathfrak{V}(\mathfrak{g}_g) + \mathfrak{V}(\mathfrak{f}_g)] \int_0^1 \frac{d\eta_o}{\mathtt{H}(\eta_o, 1-\eta_o)}.$$

Proof. By definition of $\mathcal{CR}$-$(\mathtt{h}_1, \mathtt{h}_2)$-**GL**-preinvex function, one has

$$\mathfrak{V}\left(\frac{2\nu_1 + \varsigma(\nu_2, \nu_1)}{2}\right) \preceq_{\mathcal{CR}} \frac{1}{\left[\mathtt{H}\left(\frac{1}{2}, \frac{1}{2}\right)\right]}[\mathfrak{V}(\nu_1) + \mathfrak{V}(\nu_2)].$$

Choosing $\nu_1 = \mathfrak{g}_g + \eta_o\varsigma(\mathfrak{f}_g, \mathfrak{g}_g)$ and $\nu_2 = \mathfrak{g}_g + (1-\eta_o)\varsigma(\mathfrak{f}_g, \mathfrak{g}_g)$, we have

$$\mathfrak{V}\left(\mathfrak{g}_g + \eta_o\varsigma(\mathfrak{f}_g, \mathfrak{g}_g) + \frac{1}{2}\varsigma\left(\mathfrak{g}_g + (1-\eta_o)\varsigma(\mathfrak{f}_g, \mathfrak{g}_g), \mathfrak{g}_g + \eta_o\varsigma(\mathfrak{f}_g, \mathfrak{g}_g)\right)\right)$$
$$\preceq_{\mathcal{CR}} \frac{1}{\left[\mathtt{H}\left(\frac{1}{2}, \frac{1}{2}\right)\right]}[\mathfrak{V}\left(\mathfrak{g}_g + \eta_o\varsigma(\mathfrak{f}_g, \mathfrak{g}_g)\right) + \mathfrak{V}\left(\mathfrak{g}_g + (1-\eta_o)\varsigma(\mathfrak{f}_g, \mathfrak{g}_g)\right)].$$

This implies

$$\left[\mathtt{H}\left(\frac{1}{2}, \frac{1}{2}\right)\right]\mathfrak{V}\left(\frac{2\mathfrak{g}_g + \varsigma(\mathfrak{f}_g, \mathfrak{g}_g)}{2}\right) \preceq_{\mathcal{CR}} [\mathfrak{V}\left(\mathfrak{g}_g + \eta_o\varsigma(\mathfrak{f}_g, \mathfrak{g}_g)\right) + \mathfrak{V}\left(\mathfrak{g}_g + (1-\eta_o)\varsigma(\mathfrak{f}_g, \mathfrak{g}_g)\right)]. \tag{4.1}$$

Integrating aforementioned inequality (4.1), we obtain

$$\left[\mathtt{H}\left(\frac{1}{2}, \frac{1}{2}\right)\right]\mathfrak{V}\left(\frac{2\mathfrak{g}_g + \varsigma(\mathfrak{f}_g, \mathfrak{g}_g)}{2}\right) \preceq_{\mathcal{CR}} \left[\int_0^1 \mathfrak{V}\left(\mathfrak{g}_g + \eta_o\varsigma(\mathfrak{f}_g, \mathfrak{g}_g)\right) d\eta_o + \int_0^1 \mathfrak{V}\left(\mathfrak{g}_g + (1-\eta_o)\varsigma(\mathfrak{f}_g, \mathfrak{g}_g)\right) d\eta_o\right]$$
$$= \int_0^1 \left(\underline{\mathfrak{V}}\left(\mathfrak{g}_g + \eta_o\varsigma(\mathfrak{f}_g, \mathfrak{g}_g)\right) + \underline{\mathfrak{V}}\left(\mathfrak{g}_g + (1-\eta_o)\varsigma(\mathfrak{f}_g, \mathfrak{g}_g)\right)\right) d\eta_o,$$
$$\int_0^1 \left(\overline{\mathfrak{V}}\left(\mathfrak{g}_g + \eta_o\varsigma(\mathfrak{f}_g, \mathfrak{g}_g)\right) + \overline{\mathfrak{V}}\left(\mathfrak{g}_g + (1-\eta_o)\varsigma(\mathfrak{f}_g, \mathfrak{g}_g)\right)\right) d\eta_o$$
$$= \frac{2}{\varsigma(\mathfrak{f}_g, \mathfrak{g}_g)} \int_{\mathfrak{g}_g}^{\mathfrak{g}_g+\varsigma(\mathfrak{f}_g,\mathfrak{g}_g)} \underline{\mathfrak{V}}(\varrho)\, d\varrho, \frac{2}{\varsigma(\mathfrak{f}_g, \mathfrak{g}_g)} \int_{\mathfrak{g}_g}^{\mathfrak{g}_g+\varsigma(\mathfrak{f}_g,\mathfrak{g}_g)} \overline{\mathfrak{V}}(\varrho)\, d\varrho$$
$$= \frac{2}{\varsigma(\mathfrak{f}_g, \mathfrak{g}_g)} \int_{\mathfrak{g}_g}^{\mathfrak{g}_g+\varsigma(\mathfrak{f}_g,\mathfrak{g}_g)} \mathfrak{V}(\varrho)\, d\varrho.$$

From the previous developments, we can infer that

$$\frac{\left[\mathtt{H}\left(\frac{1}{2},\frac{1}{2}\right)\right]}{2}\mathfrak{V}\left(\frac{2\mathfrak{g}_g+\varsigma(\mathfrak{f}_g,\mathfrak{g}_g)}{2}\right)\preceq_{CR}\frac{1}{\varsigma(\mathfrak{f}_g,\mathfrak{g}_g)}\int_{\mathfrak{g}_g}^{\mathfrak{g}_g+\varsigma(\mathfrak{f}_g,\mathfrak{g}_g)}\mathfrak{V}(\varrho)\,d\varrho. \tag{4.2}$$

From Definition 3.1, we have

$$\mathfrak{V}\left(\mathfrak{g}_g+\eta_o\varsigma\left(\mathfrak{f}_g,\mathfrak{g}_g\right)\right)\preceq_{CR}\frac{\mathfrak{V}(\mathfrak{f}_g)}{\mathtt{H}(\eta_o,1-\eta_o)}+\frac{\mathfrak{V}(\mathfrak{g}_g)}{\mathtt{H}(1-\eta_o,\eta_o)}.$$

Integrating the above result, we get

$$\int_0^1\mathfrak{V}(\mathfrak{g}_g+\eta_o\varsigma\left(\mathfrak{f}_g,\mathfrak{g}_g\right))\,d\eta_o\preceq_{CR}\mathfrak{V}(\mathfrak{f}_g)\int_0^1\frac{d\eta_o}{\mathtt{H}(\eta_o,1-\eta_o)}+\mathfrak{V}(\mathfrak{g}_g)\int_0^1\frac{d\eta_o}{\mathtt{H}(1-\eta_o,\eta_o)}.$$

This implies

$$\frac{1}{\varsigma(\mathfrak{f}_g,\mathfrak{g}_g)}\int_{\mathfrak{g}_g}^{\mathfrak{g}_g+\varsigma(\mathfrak{f}_g,\mathfrak{g}_g)}\mathfrak{V}(\varrho)\,d\varrho\preceq_{CR}[\mathfrak{V}(\mathfrak{g}_g)+\mathfrak{V}(\mathfrak{f}_g)]\int_0^1\frac{d\eta_o}{\mathtt{H}(\eta_o,1-\eta_o)}. \tag{4.3}$$

By combining (4.2) and (4.3), we get required result.

□

Note: Based on our newly developed results, several previously published results have been unified.

Remark 4.1.

- *Choosing* $\mathtt{h}_1(\eta_o)=\mathtt{h}(\eta_o),\mathtt{h}_2(\eta_o)=1$ *and* $\varsigma(\mathfrak{f}_g,\mathfrak{g}_g)=\mathfrak{f}_g-\mathfrak{g}_g$, *then Theorem* 4.1 *generates outcomes for* $\mathcal{CR}$-h-**GL** *functions [41].*
- *Choosing* $\mathtt{h}_1(\eta_o)=\frac{1}{\mathtt{h}(\eta_o)},\mathtt{h}_2(\eta_o)=1$ *and* $\varsigma(\mathfrak{f}_g,\mathfrak{g}_g)=\mathfrak{f}_g-\mathfrak{g}_g$, *then Theorem* 4.1 *generates outcomes for* $\mathcal{CR}$-h*-convex functions [51].*
- *Choosing* $\mathtt{h}_1(\eta_o)=\frac{1}{\mathtt{h}_1(\eta_o)},\mathtt{h}_2(\eta_o)=\frac{1}{\mathtt{h}_2(\eta_o)}$ *and* $\varsigma(\mathfrak{f}_g,\mathfrak{g}_g)=\mathfrak{f}_g-\mathfrak{g}_g$, *then Theorem* 4.1 *generates outcomes for* $\mathcal{CR}$-$(\mathtt{h}_1,\mathtt{h}_2)$*-convex functions [49].*

Example 4.1. *Let* $\mathfrak{V}(\varrho)=[1-\varrho^{\frac{1}{2}},(9-3\varrho^{\frac{1}{2}})]$, $\varsigma(\mathfrak{f}_g,\mathfrak{g}_g)=\mathfrak{f}_g-\mathfrak{g}_g$, $\mathfrak{f}_g=2$ *and* $\mathfrak{g}_g=0$, *then for* $\mathtt{h}_1(\eta_o)=\frac{1}{\eta_o},\mathtt{h}_2(\eta_o)=1$, *we have*

$$\frac{\left[\mathtt{H}\left(\frac{1}{2},\frac{1}{2}\right)\right]}{2}\mathfrak{V}\left(\frac{2\mathfrak{g}_g+\varsigma(\mathfrak{f}_g,\mathfrak{g}_g)}{2}\right)\approx[0,5.999],$$

$$\frac{1}{\varsigma(\mathfrak{f}_g,\mathfrak{g}_g)}\int_{\mathfrak{g}_g}^{\mathfrak{g}_g+\varsigma(\mathfrak{f}_g,\mathfrak{g}_g)}\mathfrak{V}(\varrho)\,d\varrho\approx[0.057,6.171],$$

$$[\mathfrak{V}(\mathfrak{g}_g)+\mathfrak{V}(\mathfrak{f}_g)]\int_0^1\frac{d\eta_o}{\mathtt{H}(\eta_o,1-\eta_o)}\approx[0.585,13.757].$$

As a result, Theorem 4.1 is validated as accurate.

$$[0,5.999]\preceq_{CR}[0.057,6.171]\preceq_{CR}[0.585,13.757].$$

Theorem 4.2. *Let* $\mathfrak{V}, \mathfrak{Y} : [\mathfrak{g}_g, \mathfrak{g}_g + \varsigma(\mathfrak{f}_g, \mathfrak{g}_g)] \rightarrow \mathbf{R}_I$ *be an set-valued functions, which are given by* $\mathfrak{Y}(\varrho) = \left[\underline{\mathfrak{Y}}(\varrho), \overline{\mathfrak{Y}}(\varrho)\right]$ *and* $\mathfrak{V}(\varrho) = \left[\underline{\mathfrak{V}}(\varrho), \overline{\mathfrak{V}}(\varrho)\right]$. *If* $\mathfrak{V}, \mathfrak{Y} : [\mathfrak{g}_g, \mathfrak{g}_g + \varsigma(\mathfrak{f}_g, \mathfrak{g}_g)] \rightarrow \mathbf{R}$ *are* $C\mathcal{R}$-$(\mathrm{h}_1, \mathrm{h}_2)$-$\mathbf{GL}$*-preinvex functions, then the following double inequality applies:*

$$\frac{1}{\varsigma(\mathfrak{f}_g, \mathfrak{g}_g)} \int_{\mathfrak{g}_g}^{\mathfrak{g}_g+\varsigma(\mathfrak{f}_g,\mathfrak{g}_g)} \mathfrak{V}(\varrho)\mathfrak{Y}(\varrho)\, d\varrho \preceq_{C\mathcal{R}} \mathbf{M}(\mathfrak{g}_g, \mathfrak{f}_g) \int_0^1 \frac{d\eta_o}{\mathrm{H}^2(\eta_o, 1-\eta_o)} + \mathbf{N}(\mathfrak{g}_g, \mathfrak{f}_g) \int_0^1 \frac{d\eta_o}{\mathrm{H}(1-\eta_o, 1-\eta_o)\mathrm{H}(\eta_o, \eta_o)}. \tag{4.4}$$

where

$$\mathbf{M}(\mathfrak{g}_g, \mathfrak{f}_g) = \mathfrak{V}(\mathfrak{g}_g)\mathfrak{Y}(\mathfrak{g}_g) + \mathfrak{V}(\mathfrak{f}_g)\mathfrak{Y}(\mathfrak{f}_g)$$

and

$$\mathbf{N}(\mathfrak{g}_g, \mathfrak{f}_g) = \mathfrak{V}(\mathfrak{g}_g)\mathfrak{Y}(\mathfrak{f}_g) + \mathfrak{V}(\mathfrak{f}_g)\mathfrak{Y}(\mathfrak{g}_g).$$

Proof. Since $\mathfrak{V}, \mathfrak{Y}$ are $C\mathcal{R}$-$(\mathrm{h}_1, \mathrm{h}_2)$-$\mathbf{GL}$-preinvex functions, we have

$$\mathfrak{V}\left(\mathfrak{g}_g + \mathfrak{y}\varsigma\left(\mathfrak{f}_g, \mathfrak{g}_g\right)\right) \preceq_{C\mathcal{R}} \frac{\mathfrak{V}(\mathfrak{f}_g)}{\mathrm{H}(\eta_o, 1-\eta_o)} + \frac{\mathfrak{V}(\mathfrak{g}_g)}{\mathrm{H}(1-\eta_o, \eta_o)}$$

and

$$\mathfrak{Y}\left(\mathfrak{g}_g + \mathfrak{y}\varsigma\left(\mathfrak{f}_g, \mathfrak{g}_g\right)\right) \preceq_{C\mathcal{R}} \frac{\mathfrak{Y}(\mathfrak{f}_g)}{\mathrm{H}(\eta_o, 1-\eta_o)} + \frac{\mathfrak{Y}(\mathfrak{g}_g)}{\mathrm{H}(1-\eta_o, \eta_o)}.$$

The product of the two aforementioned results gives us

$$\begin{aligned}
&\mathfrak{V}\left(\mathfrak{g}_g + \mathfrak{y}\varsigma\left(\mathfrak{f}_g, \mathfrak{g}_g\right)\right)\mathfrak{Y}\left(\mathfrak{g}_g + \mathfrak{y}\varsigma\left(\mathfrak{f}_g, \mathfrak{g}_g\right)\right)\\
&\preceq_{C\mathcal{R}} \left[\frac{\mathfrak{V}(\mathfrak{f}_g)}{\mathrm{H}(\eta_o, 1-\eta_o)} + \frac{\mathfrak{V}(\mathfrak{g}_g)}{\mathrm{H}(1-\eta_o, \eta_o)}\right]\left[\frac{\mathfrak{Y}(\mathfrak{f}_g)}{\mathrm{H}(\eta_o, 1-\eta_o)} + \frac{\mathfrak{Y}(\mathfrak{g}_g)}{\mathrm{H}(1-\eta_o, \eta_o)}\right]\\
&= \frac{[\mathfrak{V}(\mathfrak{f}_g)\mathfrak{Y}(\mathfrak{f}_g)]}{\mathrm{H}^2(\eta_o, 1-\eta_o)} + \frac{[\mathfrak{V}(\mathfrak{g}_g)\mathfrak{Y}(\mathfrak{g}_g)]}{\mathrm{H}^2(1-\eta_o, \eta_o)} + \frac{[\mathfrak{V}(\mathfrak{f}_g)\mathfrak{Y}(\mathfrak{g}_g)] + [\mathfrak{V}(\mathfrak{g}_g)\mathfrak{Y}(\mathfrak{f}_g)]}{\mathrm{H}(1-\eta_o, 1-\eta_o)\mathrm{H}(\eta_o, \eta_o)}.
\end{aligned} \tag{4.5}$$

For integrating (4.5), we have

$$\begin{aligned}
&\int_0^1 \mathfrak{V}(\mathfrak{g}_g + \mathfrak{y}\varsigma\left(\mathfrak{f}_g, \mathfrak{g}_g\right))\mathfrak{Y}(\mathfrak{g}_g + \mathfrak{y}\varsigma\left(\mathfrak{f}_g, \mathfrak{g}_g\right))\, d\eta_o\\
&\preceq_{C\mathcal{R}} [\mathfrak{V}(\mathfrak{f}_g)\mathfrak{Y}(\mathfrak{f}_g)] \int_0^1 \frac{d\eta_o}{\mathrm{H}^2(\eta_o, 1-\eta_o)} + [\mathfrak{V}(\mathfrak{g}_g)\mathfrak{Y}(\mathfrak{g}_g)] \int_0^1 \frac{d\eta_o}{\mathrm{H}^2(1-\eta_o, \eta_o)}\\
&\quad + [\mathfrak{V}(\mathfrak{f}_g)\mathfrak{Y}(\mathfrak{g}_g) + \mathfrak{V}(\mathfrak{g}_g)\mathfrak{Y}(\mathfrak{f}_g)] \int_0^1 \frac{d\eta_o}{\mathrm{H}(1-\eta_o, 1-\eta_o)\mathrm{H}(\eta_o, \eta_o)}.
\end{aligned}$$

From Definition 2.8, we obtain

$$\begin{aligned}
&\frac{1}{\varsigma(\mathfrak{f}_g, \mathfrak{g}_g)} \int_{\mathfrak{g}_g}^{\mathfrak{g}_g+\varsigma(\mathfrak{f}_g,\mathfrak{g}_g)} \mathfrak{V}(\varrho)\mathfrak{Y}(\varrho)\, d\varrho \preceq_{C\mathcal{R}} [\mathfrak{V}(\mathfrak{g}_g)\mathfrak{Y}(\mathfrak{g}_g) + \mathfrak{V}(\mathfrak{f}_g)\mathfrak{Y}(\mathfrak{f}_g)] \int_0^1 \frac{d\eta_o}{\mathrm{H}^2(1-\eta_o, \eta_o)}\\
&+ [\mathfrak{V}(\mathfrak{g}_g)\mathfrak{Y}(\mathfrak{f}_g) + \mathfrak{V}(\mathfrak{f}_g)\mathfrak{Y}(\mathfrak{g}_g)] \int_0^1 \frac{d\eta_o}{\mathrm{H}(1-\eta_o, 1-\eta_o)\mathrm{H}(\eta_o, \eta_o)}
\end{aligned}$$

$$= \mathbf{M}(\mathfrak{g}_g, \mathfrak{f}_g) \int_0^1 \frac{d\eta_o}{\mathrm{H}^2(1-\eta_o, \eta_o)} + \mathbf{N}(\mathfrak{g}_g, \mathfrak{f}_g) \int_0^1 \frac{d\eta_o}{\mathrm{H}(1-\eta_o, 1-\eta_o)\mathrm{H}(\eta_o, \eta_o)}.$$

□

Remark 4.2. *Choosing* $\mathrm{h}_1(\eta_o) = \mathrm{h}(\eta_o), \mathrm{h}_2(\eta_o) = 1$ *and* $\varsigma(\mathfrak{f}_g, \mathfrak{g}_g) = \mathfrak{f}_g - \mathfrak{g}_g$, *then Theorem* 4.2 *generates outcomes for* $\mathcal{CR}$-h-**GL** *functions [41].*

Remark 4.3. *Choosing* $\mathrm{h}_1(\eta_o) = \frac{1}{\mathrm{h}(\eta_o)}, \mathrm{h}_2(\eta_o) = 1$ *and* $\varsigma(\mathfrak{f}_g, \mathfrak{g}_g) = \mathfrak{f}_g - \mathfrak{g}_g$, *then Theorem* 4.2 *generates outcomes for* $\mathcal{CR}$-h*-convex functions [51].*

$$\frac{1}{\mathfrak{f}_g - \mathfrak{g}_g} \int_{\mathfrak{g}_g}^{\mathfrak{f}_g} \mathfrak{V}(\varrho)\mathfrak{Y}(\varrho)\, d\varrho \preceq_{\mathcal{CR}} \mathbf{M}(\mathfrak{g}_g, \mathfrak{f}_g) \int_0^1 h(\eta_o)^2 d\eta_o$$
$$+\mathbf{N}(\mathfrak{g}_g, \mathfrak{f}_g) \int_0^1 \mathrm{h}(1-\eta_o) h(\eta_o) d\eta_o.$$

Remark 4.4. *Choosing* $\mathrm{h}_1(\eta_o) = \frac{1}{\mathrm{h}_1(\eta_o)}, \mathrm{h}_2(\eta_o) = \frac{1}{\mathrm{h}_2(\eta_o)}$ *and* $\varsigma(\mathfrak{f}_g, \mathfrak{g}_g) = \mathfrak{f}_g - \mathfrak{g}_g$, *then Theorem* 4.2 *generates outcomes for* $\mathcal{CR}$-$(\mathrm{h}_1, \mathrm{h}_2)$*-convex functions [49].*

$$\frac{1}{\mathfrak{f}_g - \mathfrak{g}_g} \int_{\mathfrak{g}_g}^{\mathfrak{f}_g} \mathfrak{V}(\varrho)\mathfrak{Y}(\varrho)\, d\varrho \preceq_{\mathcal{CR}} \mathbf{M}(\mathfrak{g}_g, \mathfrak{f}_g) \int_0^1 \mathrm{H}^2(\eta_o, 1-\eta_o) d\eta_o$$
$$+\mathbf{N}(\mathfrak{g}_g, \mathfrak{f}_g) \int_0^1 \mathrm{H}(1-\eta_o, 1-\eta_o)\mathrm{H}(\eta_o, \eta_o) d\eta_o.$$

Remark 4.5. *Choosing* $\mathrm{h}_1(\eta_o) = \frac{1}{\mathfrak{y}}, h_2(\eta_o) = 1$, *then Theorem 4.2 generates outcomes for* $\mathcal{CR}$*-preinvex functions, i.e.,*

$$\frac{1}{\varsigma(\mathfrak{f}_g, \mathfrak{g}_g)} \int_{\mathfrak{g}_g}^{\mathfrak{g}_g + \varsigma(\mathfrak{f}_g, \mathfrak{g}_g)} \mathfrak{V}(\varrho)\mathfrak{Y}(\varrho)\, d\varrho \preceq_{\mathcal{CR}} \frac{\mathbf{M}(\mathfrak{g}_g, \mathfrak{f}_g)}{3} + \frac{\mathbf{N}(\mathfrak{g}_g, \mathfrak{f}_g)}{6}.$$

Remark 4.6. *Choosing* $\mathrm{h}_1(\eta_o) = \frac{1}{\mathfrak{y}}, h_2(\eta_o) = 1$ *and* $\varsigma(\mathfrak{f}_g, \mathfrak{g}_g) = \mathfrak{f}_g - \mathfrak{g}_g$, *then Theorem 4.2 generates outcomes for* $\mathcal{CR}$*-convex functions, i.e.,*

$$\frac{1}{\mathfrak{f}_g - \mathfrak{g}_g} \int_{\mathfrak{g}_g}^{\mathfrak{f}_g} \mathfrak{V}(\varrho)\mathfrak{Y}(\varrho)\, d\varrho \preceq_{\mathcal{CR}} \frac{\mathbf{M}(\mathfrak{g}_g, \mathfrak{f}_g)}{3} + \frac{\mathbf{N}(\mathfrak{g}_g, \mathfrak{f}_g)}{6}.$$

Example 4.2. *Let* $\mathfrak{V}(\varrho) = \left[2 - \varrho^{\frac{1}{2}}, (6 - 3\varrho^{\frac{1}{2}})\right], \mathfrak{Y}(\varrho) = [e^{\varrho} - \varrho, e^{\varrho} + \varrho]$, $\varsigma(\mathfrak{f}_g, \mathfrak{g}_g) = \mathfrak{f}_g - \mathfrak{g}_g, \mathfrak{g}_g = 0$ *and* $\mathfrak{f}_g = 2$. *Then, for* $\mathrm{h}_1(\eta_o) = \frac{1}{\mathfrak{y}}, h_2(\eta_o) = 1$, *we have*

$$\frac{1}{\varsigma(\mathfrak{f}_g, \mathfrak{g}_g)} \int_{\mathfrak{g}_g}^{\mathfrak{g}_g + \varsigma(\mathfrak{f}_g, \mathfrak{g}_g)} \mathfrak{V}(\varrho)\mathfrak{Y}(\varrho)\, d\varrho \approx [1.95, 10.9]$$

and

$$\mathbf{M}(\mathfrak{g}_g, \mathfrak{f}_g) \int_0^1 \frac{d\eta_o}{\mathrm{H}^2(\eta_o, 1-\eta_o)} + \mathbf{N}(\mathfrak{g}_g, \mathfrak{f}_g) \int_0^1 \frac{d\eta_o}{\mathrm{H}(1-\eta_o, 1-\eta_o)\mathrm{H}(\eta_o, \eta_o)} d\eta_o \approx [4.32, 15.96].$$

Thus, we have

$$[1.95, 10.9] \preceq_{\mathcal{CR}} [4.32, 15.96].$$

Theorem's 4.2 validity is therefore confirmed.

Theorem 4.3. *Following the same hypothesis as Theorem 4.2, the following relationship holds:*

$$\frac{\left[\mathtt{H}\left(\frac{1}{2},\frac{1}{2}\right)\right]^2}{2}\mathfrak{V}\left(\frac{2\mathfrak{g}_g+\varsigma(\mathfrak{f}_g,\mathfrak{g}_g)}{2}\right)\mathfrak{Y}\left(\frac{2\mathfrak{g}_g+\varsigma(\mathfrak{f}_g,\mathfrak{g}_g)}{2}\right)$$
$$\preceq_{C\mathcal{R}} \frac{1}{\varsigma(\mathfrak{f}_g,\mathfrak{g}_g)}\int_{\mathfrak{g}_g}^{\mathfrak{g}_g+\varsigma(\mathfrak{f}_g,\mathfrak{g}_g)}\mathfrak{V}(\varrho)\mathfrak{Y}(\varrho)\,d\varrho$$
$$+\mathbf{M}(\mathfrak{g}_g,\mathfrak{f}_g)\int_0^1\frac{d\eta_o}{\mathtt{H}(1-\eta_o,1-\eta_o)\mathtt{H}(\eta_o,\eta_o)}+\mathbf{N}(\mathfrak{g}_g,\mathfrak{f}_g)\int_0^1\frac{d\eta_o}{\mathtt{H}^2(\eta_o,1-\eta_o)}.$$

Proof. The proof is completed by taking into account Definition 3.1 and using the same technique as [An et al. [53], Theorem 5].

□

Remark 4.7. *Choosing* $\mathtt{h}_1(\eta_o)=\frac{1}{\mathtt{h}(\eta_o)},\mathtt{h}_2(\eta_o)=1$ *and* $\varsigma(\mathfrak{f}_g,\mathfrak{g}_g)=\mathfrak{f}_g-\mathfrak{g}_g$, *then Theorem* 4.3 *generates outcomes for* $C\mathcal{R}$-h-*convex functions [51].*

Remark 4.8. *Choosing* $\mathtt{h}_1(\eta_o)=\frac{1}{\mathtt{h}_1(\eta_o)},\mathtt{h}_2(\eta_o)=\frac{1}{\mathtt{h}_2(\eta_o)}$ *and* $\varsigma(\mathfrak{f}_g,\mathfrak{g}_g)=\mathfrak{f}_g-\mathfrak{g}_g$, *then Theorem* 4.3 *generates outcomes for* $C\mathcal{R}$-$(\mathtt{h}_1,\mathtt{h}_2)$-*convex functions [49].*

Remark 4.9. *Choosing* $\mathtt{h}_1(\eta_o)=\mathtt{h}(\eta_o),\mathtt{h}_2(\eta_o)=1$ *and* $\varsigma(\mathfrak{f}_g,\mathfrak{g}_g)=\mathfrak{f}_g-\mathfrak{g}_g$, *then Theorem* 4.3 *generates outcomes for* $C\mathcal{R}$-h-**GL** *functions [41].*

Example 4.3. *Suppose* $\mathfrak{V}(\varrho)=\left[-\varrho^2,2\varrho^2+1\right],\mathfrak{Y}(\varrho)=[-\varrho,\varrho],\varsigma(\mathfrak{f}_g,\mathfrak{g}_g)=\mathfrak{f}_g-\mathfrak{g}_g,\mathfrak{g}_g=1$ *and* $\mathfrak{f}_g=3$. *Then, for* $\mathsf{h}_1(\eta_o)=\frac{1}{\mathfrak{y}},\mathsf{h}_2(\eta_o)=\frac{1}{4}$, *we have*

$$\frac{\left[\mathtt{H}\left(\frac{1}{2},\frac{1}{2}\right)\right]^2}{2}\mathfrak{V}\left(\mathfrak{g}_g+\frac{1}{2}\varsigma(\mathfrak{f}_g,\mathfrak{g}_g)\right)\mathfrak{Y}\left(\mathfrak{g}_g+\frac{1}{2}\varsigma(\mathfrak{f}_g,\mathfrak{g}_g)\right)\approx[-1.031,1.031]$$

and

$$\frac{1}{\varsigma(\mathfrak{f}_g,\mathfrak{g}_g)}\int_{\mathfrak{g}_g}^{\mathfrak{g}_g+\varsigma(\mathfrak{f}_g,\mathfrak{g}_g)}\mathfrak{V}(\varrho)\mathfrak{Y}(\varrho)\,d\varrho+\mathbf{M}(\mathfrak{g}_g,\mathfrak{f}_g)\int_0^1\frac{d\eta_o}{\mathtt{H}(\eta_o,\eta_o)\mathtt{H}(1-\eta_o,1-\eta_o)}$$
$$+\mathbf{N}(\mathfrak{g}_g,\mathfrak{f}_g)\int_0^1\frac{d\eta_o}{\mathtt{H}^2(\eta_o,1-\eta_o)}\approx[-132.25,45].$$

Thus, we have

$$[-1.031,1.031]\preceq_{C\mathcal{R}}[-132.25,45].$$

Theorem's 4.3 validity is therefore confirmed.

5. Hermite-Hadamard-Fejér Type Inequality For $C\mathcal{R}$-$(\mathtt{h}_1,\mathtt{h}_2)$-GL-Preinvex Functions

Theorem 5.1. *Let* $\mathfrak{V}:[\mathfrak{g}_g,\mathfrak{g}_g+\varsigma(\mathfrak{f}_g,\mathfrak{g}_g)]\to\mathbf{R}_I$ *be an set-valued function is defined as* $\mathfrak{V}(\varrho)=\left[\underline{\mathfrak{V}}(\varrho),\overline{\mathfrak{V}}(\varrho)\right]$ *for all* $\varrho\in\left[\mathfrak{g}_g,\mathfrak{f}_g\right]$. *If* $\mathfrak{V}:[\mathfrak{g}_g,\mathfrak{g}_g+\varsigma(\mathfrak{f}_g,\mathfrak{g}_g)]\to\mathbf{R}$ *is an* $C\mathcal{R}$-$(\mathtt{h}_1,\mathtt{h}_2)$-**GL**-*preinvex and* $\mathtt{W}:[\mathfrak{g}_g,\mathfrak{g}_g+\varsigma(\mathfrak{f}_g,\mathfrak{g}_g)]\to\mathbf{R}$ *is symmetric with respect to* $\mathfrak{g}_g+\frac{1}{2}\varsigma(\mathfrak{f}_g,\mathfrak{g}_g)$, *then the following outcome holds:*

$$\frac{1}{\varsigma(\mathfrak{f}_g,\mathfrak{g}_g)}\int_{\mathfrak{g}_g}^{\mathfrak{g}_g+\varsigma(\mathfrak{f}_g,\mathfrak{g}_g)}\mathfrak{V}(\varrho)\mathtt{W}(\varrho)\,d\varrho\preceq_{C\mathcal{R}}[\mathfrak{V}(\mathfrak{g}_g)+\mathfrak{V}(\mathfrak{f}_g)]\int_0^1\frac{\mathtt{W}(\mathfrak{g}_g+\mathfrak{y}\varsigma(\mathfrak{f}_g,\mathfrak{g}_g))\,d\eta_o}{\mathtt{H}(\eta_o,1-\eta_o)}.$$

Proof. As $\mathfrak{V}$ is an $\mathcal{CR}$-($\mathrm{h}_1, \mathrm{h}_2$)-**GL**-preinvex function and $\mathtt{W}$ is symmetric function, we have

$$\mathfrak{V}\left(\mathfrak{g}_g + \eta\varsigma\left(\mathfrak{f}_g, \mathfrak{g}_g\right)\right)\mathtt{W}\left(\mathfrak{g}_g + \eta\varsigma\left(\mathfrak{f}_g, \mathfrak{g}_g\right)\right)$$
$$\preceq_{\mathcal{CR}} \left[\frac{\mathfrak{V}(\mathfrak{f}_g)}{\mathrm{H}(\eta_o, 1-\eta_o)} + \frac{\mathfrak{V}(\mathfrak{g}_g)}{\mathrm{H}(1-\eta_o, \eta_o)}\right]\mathtt{W}\left(\mathfrak{g}_g + \eta\varsigma\left(\mathfrak{f}_g, \mathfrak{g}_g\right)\right)$$

and

$$\mathfrak{V}\left(\mathfrak{g}_g + (1-\eta_o)\varsigma\left(\mathfrak{f}_g, \mathfrak{g}_g\right)\right)\mathtt{W}\left(\mathfrak{g}_g + (1-\eta_o)\varsigma\left(\mathfrak{f}_g, \mathfrak{g}_g\right)\right)$$
$$\preceq_{\mathcal{CR}} \left[\frac{\mathfrak{V}(\mathfrak{g}_g)}{\mathrm{H}(\eta_o, 1-\eta_o)} + \frac{\mathfrak{V}(\mathfrak{f}_g)}{\mathrm{H}(1-\eta_o, \eta_o)}\right]\mathtt{W}\left(\mathfrak{g}_g + (1-\eta_o)\varsigma\left(\mathfrak{f}_g, \mathfrak{g}_g\right)\right).$$

Including the two aforementioned results and then integrating, we have

$$\int_0^1 \mathfrak{V}\left(\mathfrak{g}_g + \eta\varsigma\left(\mathfrak{f}_g, \mathfrak{g}_g\right)\right)\mathtt{W}\left(\mathfrak{g}_g + \eta\varsigma\left(\mathfrak{f}_g, \mathfrak{g}_g\right)\right)\, d\eta_o \tag{5.1}$$
$$+ \int_0^1 \mathfrak{V}\left(\mathfrak{g}_g + (1-\eta_o)\varsigma\left(\mathfrak{f}_g, \mathfrak{g}_g\right)\right)\mathtt{W}\left(\mathfrak{g}_g + (1-\eta_o)\varsigma\left(\mathfrak{f}_g, \mathfrak{g}_g\right)\right)\, d\eta_o$$
$$\preceq_{\mathcal{CR}} \int_0^1 \left[\mathfrak{V}(\mathfrak{g}_g)\left(\frac{(\mathtt{W}\left(\mathfrak{g}_g + \eta\varsigma\left(\mathfrak{f}_g, \mathfrak{g}_g\right)\right)}{\mathrm{H}(1-\eta_o, \eta_o)} + \frac{\mathtt{W}\left(\mathfrak{g}_g + (1-\eta_o)\varsigma\left(\mathfrak{f}_g, \mathfrak{g}_g\right)\right)}{\mathrm{H}(\eta_o, 1-\eta_o)}\right)\right.$$
$$\left.+\mathfrak{V}(\mathfrak{f}_g)\left(\frac{(\mathtt{W}\left(\mathfrak{g}_g + \eta\varsigma\left(\mathfrak{f}_g, \mathfrak{g}_g\right)\right)}{\mathrm{H}(\eta_o, 1-\eta_o)} + \frac{\mathtt{W}\left(\mathfrak{g}_g + (1-\eta_o)\varsigma\left(\mathfrak{f}_g, \mathfrak{g}_g\right)\right))}{\mathrm{H}(1-\eta_o, \eta_o)}\right)\right]\, d\eta_o$$
$$= 2\mathfrak{V}(\mathfrak{g}_g)\int_0^1 \frac{\mathtt{W}\left(\mathfrak{g}_g + (1-\eta_o)\varsigma\left(\mathfrak{f}_g, \mathfrak{g}_g\right)\right)}{\mathrm{H}(\eta_o, 1-\eta_o)}\, d\eta_o + 2\mathfrak{V}(\mathfrak{f}_g)\int_0^1 \frac{\mathtt{W}\left(\mathfrak{g}_g + \eta\varsigma\left(\mathfrak{f}_g, \mathfrak{g}_g\right)\right)}{\mathrm{H}(\eta_o, 1-\eta_o)}\, d\eta_o$$
$$= 2[\mathfrak{V}(\mathfrak{g}_g) + \mathfrak{V}(\mathfrak{f}_g)]\int_0^1 \frac{\mathtt{W}\left(\mathfrak{g}_g + \eta\varsigma\left(\mathfrak{f}_g, \mathfrak{g}_g\right)\right)}{\mathrm{H}(\eta_o, 1-\eta_o)}\, d\eta_o. \tag{5.2}$$

Since

$$\int_0^1 \mathfrak{V}\left(\mathfrak{g}_g + \eta\varsigma\left(\mathfrak{f}_g, \mathfrak{g}_g\right)\right)\mathtt{W}\left(\mathfrak{g}_g + \eta\varsigma\left(\mathfrak{f}_g, \mathfrak{g}_g\right)\right)\, d\eta_o \tag{5.3}$$
$$+ \int_0^1 \mathfrak{V}\left(\mathfrak{g}_g + (1-\eta_o)\varsigma\left(\mathfrak{f}_g, \mathfrak{g}_g\right)\right)\mathtt{W}\left(\mathfrak{g}_g + (1-\eta_o)\varsigma\left(\mathfrak{f}_g, \mathfrak{g}_g\right)\right)\, d\eta_o$$
$$= \frac{2}{\varsigma(\mathfrak{f}_g, \mathfrak{g}_g)}\int_{\mathfrak{g}_g}^{\mathfrak{g}_g+\varsigma(\mathfrak{f}_g,\mathfrak{g}_g)} \mathfrak{V}(\varrho)\mathtt{W}(\varrho)\, d\varrho, \tag{5.4}$$

We achieve the desired outcome by accounting results (5.1) and (5.3). □

Remark 5.1. *If* $\mathrm{h}_1(\eta_o) = \frac{1}{\mathrm{h}(\eta_o)}, \mathrm{h}_2(\eta_o) = 1$ *with* $\underline{\mathfrak{V}} = \overline{\mathfrak{V}}$*, then Theorem* 5.1 *generates outcomes for* h-**GL**-*preinvex functions, i.e.,*

$$\frac{1}{\varsigma(\mathfrak{f}_g, \mathfrak{g}_g)}\int_{\mathfrak{g}_g}^{\mathfrak{g}_g+\varsigma(\mathfrak{f}_g,\mathfrak{g}_g)} \mathfrak{V}(\varrho)\mathtt{W}(\varrho)\, d\varrho \leq [\mathfrak{V}(\mathfrak{g}_g) + \mathfrak{V}(\mathfrak{f}_g)]\int_0^1 \frac{\mathtt{W}(\mathfrak{g}_g + \eta\varsigma(\mathfrak{f}_g, \mathfrak{g}_g))}{\mathrm{h}(\eta_o)}\, d\eta_o.$$

Remark 5.2. *If* $\mathtt{h}_1(\eta_o) = \frac{1}{\mathfrak{y}}, \mathtt{h}_2(\eta_o) = 1$*, then Theorem* 5.1 *generates outcomes for* $\mathcal{CR}$*-preinvex functions, i.e.,*

$$\frac{1}{\varsigma(\mathfrak{f}_g, \mathfrak{g}_g)} \int_{\mathfrak{g}_g}^{\mathfrak{g}_g+\varsigma(\mathfrak{f}_g,\mathfrak{g}_g)} \mathfrak{V}(\varrho)\mathtt{W}(\varrho)\, d\varrho \preceq_{\mathcal{CR}} [\mathfrak{V}(\mathfrak{g}_g) + \mathfrak{V}(\mathfrak{f}_g)] \int_0^1 \mathfrak{y}\mathtt{W}(\mathfrak{g}_g + \mathfrak{y}\varsigma(\mathfrak{f}_g, \mathfrak{g}_g))\, d\eta_o.$$

Remark 5.3. *If* $\mathtt{h}_1(\eta_o) = \frac{1}{\mathtt{h}(\eta_o)}, \mathtt{h}_2(\eta_o) = 1$ *and* $\varsigma(\mathfrak{f}_g, \mathfrak{g}_g) = \mathfrak{f}_g - \mathfrak{g}_g$*, then Theorem* 5.1 *generates outcomes for* $\mathcal{CR}$-h-**GL** *functions, i.e.,*

$$\frac{1}{\mathfrak{f}_g - \mathfrak{g}_g} \int_{\mathfrak{g}_g}^{\mathfrak{f}_g} \mathfrak{V}(\varrho)\mathtt{W}(\varrho)\, d\varrho \preceq_{\mathcal{CR}} [\mathfrak{V}(\mathfrak{g}_g) + \mathfrak{V}(\mathfrak{f}_g)] \int_0^1 \frac{\mathtt{W}((1-\eta_o)\mathfrak{g}_g + \mathfrak{y}\mathfrak{f}_g)}{\mathtt{h}(\eta_o)}\, d\eta_o.$$

Example 5.1. *Suppose* $\mathfrak{V}(\varrho) = \left[3 - \sqrt{\varrho}, 8 - \sqrt{\varrho}\right], \varsigma(\mathfrak{f}_g, \mathfrak{g}_g) = \mathfrak{f}_g - \mathfrak{g}_g, \mathfrak{g}_g = 0$ *and* $\mathfrak{f}_g = 2$*. Then, for* $\mathtt{h}_1(\eta_o) = \frac{1}{\mathfrak{y}}, \mathtt{h}_2(\eta_o) = 1$, $\mathtt{W}(\varrho) = \varrho$ *for* $\varrho \in [0,1]$ *and* $\mathtt{W}(\varrho) = -\varrho + 3$ *for* $\varrho \in [1,2]$*, one has*

$$\begin{aligned}
&\frac{1}{\varsigma(\mathfrak{f}_g, \mathfrak{g}_g)} \int_{\mathfrak{g}_g}^{\mathfrak{g}_g+\varsigma(\mathfrak{f}_g,\mathfrak{g}_g)} \mathfrak{V}(\varrho)\mathtt{W}(\varrho)\, d\varrho \\
&= \frac{1}{2}\int_0^2 \mathfrak{V}(\varrho)\mathtt{W}(\varrho)\, d\varrho \\
&= \frac{1}{2}\int_0^1 \left[\left(3 - \varrho^{\frac{1}{2}}\right)\varrho, \varrho\left(8 - \varrho^{\frac{1}{2}}\right)\right] d\varrho \\
&\quad + \frac{1}{2}\int_1^2 \left[\left(3 - \varrho^{\frac{1}{2}}\right)(-\varrho+3), (-\varrho+3)\left(8 - 4\varrho^{\frac{1}{2}}\right)\right] d\varrho \\
&\approx [1.9029, 3.6117]
\end{aligned}$$

and

$$\begin{aligned}
&[\mathfrak{V}(\mathfrak{g}_g) + \mathfrak{V}(\mathfrak{f}_g)] \int_0^1 \frac{\mathtt{W}(\mathfrak{g}_g + \mathfrak{y}\varsigma(\mathfrak{f}_g, \mathfrak{g}_g))}{\mathtt{H}(\eta_o, 1-\eta_o)}\, d\eta_o \\
&= \left([3,8] + \left[3 - 2^{\frac{1}{2}}, \left(8 - 4\sqrt{2}\right)\right]\right) \int_0^1 \mathfrak{y}\mathtt{W}(2\mathfrak{y})\, d\eta_o \\
&= \left[6 - 2^{\frac{1}{2}}, \left(16 - 4\sqrt{2}\right)\right]\left(\int_0^{\frac{1}{2}} 2\mathfrak{y}^2 dt + \int_{\frac{1}{2}}^1 \mathfrak{y}(-2\mathfrak{y}+3)\, d\eta_o\right) \\
&\approx [2.8661, 6.4646].
\end{aligned}$$

Thus, we have

$$[1.9029, 3.6117] \preceq_{\mathcal{CR}} [2.8661, 6.4646].$$

Theorem's 5.1 validity is therefore confirmed.

Theorem 5.2. *Following the same hypothesis as Theorem 5.1, the following relationship holds true*

$$\mathfrak{V}\left(\frac{2\mathfrak{g}_g + \varsigma(\mathfrak{f}_g, \mathfrak{g}_g)}{2}\right) \preceq_{\mathcal{CR}} \frac{2}{\left[\mathtt{H}\left(\frac{1}{2}, \frac{1}{2}\right)\right] \int_{\mathfrak{g}_g}^{\mathfrak{g}_g+\varsigma(\mathfrak{f}_g,\mathfrak{g}_g)} \mathtt{W}(\varrho)\, d\varrho} \int_{\mathfrak{g}_g}^{\mathfrak{g}_g+\varsigma(\mathfrak{f}_g,\mathfrak{g}_g)} \mathfrak{V}(\varrho)\mathtt{W}(\varrho)\, d\varrho.$$

Proof. As $\mathfrak{V}$ is an $\mathcal{CR}$-$(\mathrm{h}_1, \mathrm{h}_2)$-**GL**-preinvex function, one has

$$\mathfrak{V}\left(\frac{2\mathfrak{g}_g + \varsigma(\mathfrak{f}_g, \mathfrak{g}_g)}{2}\right) \preceq_{CR} \frac{1}{\left[\mathtt{H}\left(\frac{1}{2}, \frac{1}{2}\right)\right]} \left[\mathfrak{V}\left(\mathfrak{g}_g + \mathfrak{y}\varsigma(\mathfrak{f}_g, \mathfrak{g}_g)\right) + \mathfrak{V}\left(\mathfrak{g}_g + (1-\eta_o)\varsigma(\mathfrak{f}_g, \mathfrak{g}_g)\right)\right].$$

Multiplying aforementioned inequality by $\mathtt{W}\left(\mathfrak{g}_g + \mathfrak{y}\varsigma\left(\mathfrak{f}_g, \mathfrak{g}_g\right)\right) = \mathtt{W}\left(\mathfrak{g}_g + (1-\eta_o)\varsigma\left(\mathfrak{f}_g, \mathfrak{g}_g\right)\right)$ and integrating, we have

$$\begin{aligned} &\mathfrak{V}\left(\frac{2\mathfrak{g}_g + \varsigma(\mathfrak{f}_g, \mathfrak{g}_g)}{2}\right) \int_0^1 \mathtt{W}\left(\mathfrak{g}_g + \mathfrak{y}\varsigma\left(\mathfrak{f}_g, \mathfrak{g}_g\right)\right)\, d\eta_o \\ &\preceq_{CR} \frac{1}{\left[\mathtt{H}\left(\frac{1}{2}, \frac{1}{2}\right)\right]} \left[\int_0^1 \mathfrak{V}\left(\mathfrak{g}_g + \mathfrak{y}\varsigma\left(\mathfrak{f}_g, \mathfrak{g}_g\right)\right) \mathtt{W}\left(\mathfrak{g}_g + \mathfrak{y}\varsigma\left(\mathfrak{f}_g, \mathfrak{g}_g\right)\right)\, d\eta_o \right. \\ &\quad \left. + \int_0^1 \mathfrak{V}\left(\mathfrak{g}_g + (1-\eta_o)\varsigma\left(\mathfrak{f}_g, \mathfrak{g}_g\right)\right) \mathtt{W}\left(\mathfrak{g}_g + (1-\eta_o)\varsigma\left(\mathfrak{f}_g, \mathfrak{g}_g\right)\right)\, d\eta_o\right]. \end{aligned} \tag{5.5}$$

Since

$$\begin{aligned} &\int_0^1 \mathfrak{V}\left(\mathfrak{g}_g + \mathfrak{y}\varsigma\left(\mathfrak{f}_g, \mathfrak{g}_g\right)\right) \mathtt{W}\left(\mathfrak{g}_g + \mathfrak{y}\varsigma\left(\mathfrak{f}_g, \mathfrak{g}_g\right)\right)\, d\eta_o \\ &= \int_0^1 \mathfrak{V}\left(\mathfrak{g}_g + (1-\eta_o)\varsigma\left(\mathfrak{f}_g, \mathfrak{g}_g\right)\right) \mathtt{W}\left(\mathfrak{g}_g + (1-\eta_o)\varsigma\left(\mathfrak{f}_g, \mathfrak{g}_g\right)\right)\, d\eta_o \\ &= \frac{1}{\varsigma(\mathfrak{f}_g, \mathfrak{g}_g)} \int_{\mathfrak{g}_g}^{\mathfrak{g}_g + \varsigma(\mathfrak{f}_g, \mathfrak{g}_g)} \mathfrak{V}(\varrho)\mathtt{W}(\varrho)\, d\varrho \end{aligned} \tag{5.6}$$

and

$$\int_0^1 \mathtt{W}\left(\mathfrak{g}_g + \mathfrak{y}\varsigma\left(\mathfrak{f}_g, \mathfrak{g}_g\right)\right) d\eta_o = \frac{1}{\varsigma(\mathfrak{f}_g, \mathfrak{g}_g)} \int_{\mathfrak{g}_g}^{\mathfrak{g}_g + \varsigma(\mathfrak{f}_g, \mathfrak{g}_g)} \mathtt{W}(\varrho)\;\; d\varrho. \tag{5.7}$$

Using (5.6) and (5.7) in (5.5), we have

$$\mathfrak{V}\left(\frac{2\mathfrak{g}_g + \varsigma(\mathfrak{f}_g, \mathfrak{g}_g)}{2}\right) \preceq_{CR} \frac{2}{\left[\mathtt{H}\left(\frac{1}{2}, \frac{1}{2}\right)\right] \int_{\mathfrak{g}_g}^{\mathfrak{g}_g + \varsigma(\mathfrak{f}_g, \mathfrak{g}_g)} \mathtt{W}(\varrho)\, d\varrho} \int_{\mathfrak{g}_g}^{\mathfrak{g}_g + \varsigma(\mathfrak{f}_g, \mathfrak{g}_g)} \mathfrak{V}(\varrho)\mathtt{W}(\varrho)\, d\varrho.$$

□

Remark 5.4. *If* $\underline{\mathfrak{V}} = \overline{\mathfrak{V}}$*, then Theorem* 5.2 *generates outcomes for* $(\mathtt{h}_1, \mathtt{h}_2)$-**GL**-*preinvex function, i.e.,*

$$\mathfrak{V}\left(\frac{2\mathfrak{g}_g + \varsigma(\mathfrak{f}_g, \mathfrak{g}_g)}{2}\right) \leq \frac{2}{\left[\mathtt{H}\left(\frac{1}{2}, \frac{1}{2}\right)\right] \int_{\mathfrak{g}_g}^{\mathfrak{g}_g + \varsigma(\mathfrak{f}_g, \mathfrak{g}_g)} \mathtt{W}(\varrho)\, d\varrho} \int_{\mathfrak{g}_g}^{\mathfrak{g}_g + \varsigma(\mathfrak{f}_g, \mathfrak{g}_g)} \mathfrak{V}(\varrho)\mathtt{W}(\varrho)\, d\varrho.$$

Remark 5.5. *If* $\mathrm{h}_1(\eta_o) = \frac{1}{\mathfrak{y}}, \mathrm{h}_2(\eta_o) = 1$*, then Theorem* 5.2 *generates outcomes for* $\mathcal{CR}$-*preinvex functions, i.e.,*

$$\mathfrak{V}\left(\frac{2\mathfrak{g}_g + \varsigma(\mathfrak{f}_g, \mathfrak{g}_g)}{2}\right) \preceq_{CR} \frac{1}{\int_{\mathfrak{g}_g}^{\mathfrak{g}_g + \varsigma(\mathfrak{f}_g, \mathfrak{g}_g)} \mathtt{W}(\varrho)\, d\varrho} \int_{\mathfrak{g}_g}^{\mathfrak{g}_g + \varsigma(\mathfrak{f}_g, \mathfrak{g}_g)} \mathfrak{V}(\varrho)\mathtt{W}(\varrho)\, d\varrho.$$

Remark 5.6. *If $\varsigma(\mathfrak{f}_g, \mathfrak{g}_g) = \mathfrak{f}_g - \mathfrak{g}_g$, then Theorem 5.2 generates outcomes for $\mathcal{CR}$-$(\mathtt{h}_1, \mathtt{h}_2)$-**GL** function, i.e.,*

$$\mathfrak{V}\left(\frac{\mathfrak{g}_g + \mathfrak{f}_g}{2}\right) \preceq_{CR} \frac{2}{\left[\mathtt{H}\left(\frac{1}{2}, \frac{1}{2}\right)\right] \int_{\mathfrak{g}_g}^{\mathfrak{f}_g} \mathtt{W}(\varrho)\, d\varrho} \int_{\mathfrak{g}_g}^{\mathfrak{f}_g} \mathfrak{V}(\varrho)\mathtt{W}(\varrho)\, d\varrho.$$

Remark 5.7. *If $\mathtt{h}_1(\eta_o) = \frac{1}{\mathfrak{y}}, \mathtt{h}_2(\eta_o) = 1$ and $\varsigma(\mathfrak{f}_g, \mathfrak{g}_g) = \mathfrak{f}_g - \mathfrak{g}_g$, then Theorem 5.2 generates outcomes for $\mathcal{CR}$-convex functions, i.e.,*

$$\mathfrak{V}\left(\frac{\mathfrak{g}_g + \mathfrak{f}_g}{2}\right) \preceq_{CR} \frac{1}{\int_{\mathfrak{g}_g}^{\mathfrak{f}_g} \mathtt{W}(\varrho)\, d\varrho} \int_{\mathfrak{g}_g}^{\mathfrak{f}_g} \mathfrak{V}(\varrho)\mathtt{W}(\varrho)\, d\varrho.$$

Example 5.2. *Following the same hypothesis as Example 5.1, we have*

$$\mathfrak{V}\left(\mathfrak{g}_g + \frac{1}{2}\varsigma(\mathfrak{f}_g, \mathfrak{g}_g)\right) = \mathfrak{V}(1) = [2, 7]$$

and

$$\begin{aligned}
&\frac{2}{\mathtt{H}\left(\frac{1}{2}, \frac{1}{2}\right) \int_{\mathfrak{g}_g}^{\mathfrak{g}_g+\varsigma(\mathfrak{f}_g,\mathfrak{g}_g)} \mathtt{W}(\varrho)\, d\varrho} \int_{\mathfrak{g}_g}^{\mathfrak{g}_g+\varsigma(\mathfrak{f}_g,\mathfrak{g}_g)} \mathfrak{V}(\varrho)\mathtt{W}(\varrho)\, d\varrho \\
&= \frac{1}{\int_0^2 \mathtt{W}(\varrho)\, d\varrho} \int_0^2 \left[3 - \sqrt{\varrho}, (8 - 4\sqrt{\varrho})\right] \mathtt{W}(\varrho)\, d\varrho \\
&\approx [3.80588, 7.22354].
\end{aligned}$$

Thus, we have

$$[2, 7] \preceq_{CR} [3.80588, 7.22354].$$

Theorem's 5.2 validity is therefore confirmed.

6. Applications of the Numerical Quadrature Formula based on Generalized Convexity

This section aims to develop several applications of the numerical quadrature rule, specifically the trapezoid type rule, using the standard order relation ($\leq$) via generalised convexity defined in [53].

Theorem 6.1. *Consider $\mathfrak{Y} : \mathcal{I} \subseteq \mathbf{R} \to \mathbf{R}$ be a differentiable function on $\mathcal{I}^\circ$, $\mathfrak{g}_g, \mathfrak{f}_g \in \mathcal{I}^\circ$ with $\mathfrak{g}_g < \mathfrak{f}_g$ and $\mathfrak{V} : [\mathfrak{g}_g, \mathfrak{f}_g] \to \mathbf{R}^+$ be a differentiable function symmetric to $\frac{\mathfrak{g}_g+\mathfrak{f}_g}{2}$. If $|\mathfrak{Y}'|$ is an $(\mathtt{h}_1, \mathtt{h}_2)$-convex function on $[\mathfrak{g}_g, \mathfrak{f}_g]$, then*

$$\begin{aligned}
&\left|\frac{\mathfrak{Y}(\mathfrak{g}_g) + \mathfrak{Y}(\mathfrak{f}_g)}{2} \int_{\mathfrak{g}_g}^{\mathfrak{f}_g} \mathfrak{V}(\nu)d\nu - \int_{\mathfrak{g}_g}^{\mathfrak{f}_g} \mathfrak{Y}(\nu)\mathfrak{V}(\nu)d\nu\right| \\
&\leq (\mathfrak{f}_g - \mathfrak{g}_g)\left(\left|\mathfrak{Y}'(\mathfrak{g}_g)\right| + \left|\mathfrak{Y}'(\mathfrak{f}_g)\right|\right) \int_{\mathfrak{g}_g}^{\frac{\mathfrak{f}_g+\mathfrak{g}_g}{2}} \int_0^1 \mathfrak{V}(\nu) \left[\mathtt{H}(\eta_o, 1 - \eta_o) + \mathtt{H}(1 - \eta_o, \eta_o)\right] d\eta_o d\nu.
\end{aligned}$$

where

$$\mathcal{Z}_{\mathfrak{V}}(\eta_o) = \begin{cases} 2\int_{\mathfrak{y}}^{\frac{1}{2}} \mathfrak{V}(\mathfrak{s}\mathfrak{g}_g + (1-\mathfrak{s})\mathfrak{f}_g)d\mathfrak{s} & 0 \le \mathfrak{y} \le \frac{1}{2} \\ -2\int_{\frac{1}{2}}^{\mathfrak{y}} \mathfrak{V}(\mathfrak{s}\mathfrak{g}_g + (1-\mathfrak{s})\mathfrak{f}_g)d\mathfrak{s} & \frac{1}{2} \le \mathfrak{y} \le 1. \end{cases}$$

Proof. From the definition of $\mathcal{Z}_{\mathfrak{V}}(\eta_o)$ and $(\mathrm{h}_1, \mathrm{h}_2)$- convexity of $|\mathfrak{Y}'|$ we have

$$\begin{aligned}
&\left|\frac{\mathfrak{Y}(\mathfrak{g}_g)+\mathfrak{Y}(\mathfrak{f}_g)}{2}\int_{\mathfrak{g}_g}^{\mathfrak{f}_g}\mathfrak{V}(\nu)d\nu - \int_{\mathfrak{g}_g}^{\mathfrak{f}_g}\mathfrak{Y}(\nu)\mathfrak{V}(\nu)d\nu\right| = \frac{(\mathfrak{f}_g-\mathfrak{g}_g)^2}{2}\left|\int_0^1 \mathcal{Z}_{\mathfrak{V}}(\eta_o)\mathfrak{Y}'(\mathfrak{y}\mathfrak{g}_g+(1-\eta_o)\mathfrak{f}_g)d\mathfrak{y}\right| \\
&\le \frac{(\mathfrak{f}_g-\mathfrak{g}_g)^2}{2}\left\{\int_0^{\frac{1}{2}} |\mathcal{Z}_{\mathfrak{V}}(\eta_o)|\left|\mathfrak{Y}'(\mathfrak{y}\mathfrak{g}_g+(1-\eta_o)\mathfrak{f}_g)\right|d\mathfrak{y} + \int_{\frac{1}{2}}^1 |\mathcal{Z}_{\mathfrak{V}}(\eta_o)|\left|\mathfrak{Y}'(\mathfrak{y}\mathfrak{g}_g+(1-\eta_o)\mathfrak{f}_g)\right|d\mathfrak{y}\right\} \\
&= \frac{(\mathfrak{f}_g-\mathfrak{g}_g)^2}{2}\left\{\int_0^{\frac{1}{2}} \mathcal{Z}_{\mathfrak{V}}(\eta_o)\left|\mathfrak{Y}'(\mathfrak{y}\mathfrak{g}_g+(1-\eta_o)\mathfrak{f}_g)\right|d\mathfrak{y} - \int_{\frac{1}{2}}^1 \mathcal{Z}_{\mathfrak{V}}(\eta_o)\left|\mathfrak{Y}'(\mathfrak{y}\mathfrak{g}_g+(1-\eta_o)\mathfrak{f}_g)\right|d\mathfrak{y}\right\} \\
&\le \frac{(\mathfrak{f}_g-\mathfrak{g}_g)^2}{2}\left\{2\int_0^{\frac{1}{2}}\int_{\mathfrak{y}}^{\frac{1}{2}} \mathfrak{V}(\mathfrak{s}\mathfrak{g}_g+(1-\mathfrak{s})\mathfrak{f}_g)\left(\mathtt{H}(\eta_o,1-\eta_o)\left|\mathfrak{Y}'(\mathfrak{g}_g)\right| + \mathrm{h}_1(1-\eta_o)\mathrm{h}_2(\eta_o)\left|\mathfrak{Y}'(\mathfrak{f}_g)\right|\right)d\mathfrak{s}d\eta_o\right. \\
&\left. +2\int_{\frac{1}{2}}^1\int_{\frac{1}{2}}^{\mathfrak{y}} \mathfrak{V}(\mathfrak{s}\mathfrak{g}_g+(1-\mathfrak{s})\mathfrak{f}_g)\left(\mathtt{H}(\eta_o,1-\eta_o)\left|\mathfrak{Y}'(\mathfrak{g}_g)\right| + \mathrm{h}_1(1-\eta_o)\mathrm{h}_2(\eta_o)\left|\mathfrak{Y}'(\mathfrak{f}_g)\right|\right)d\mathfrak{s}d\eta_o\right\}.
\end{aligned}$$

Modify the integration order,

$$\begin{aligned}
&\left|\frac{\mathfrak{Y}(\mathfrak{g}_g)+\mathfrak{Y}(\mathfrak{f}_g)}{2}\int_{\mathfrak{g}_g}^{\mathfrak{f}_g}\mathfrak{V}(\nu)d\nu - \int_{\mathfrak{g}_g}^{\mathfrak{f}_g}\mathfrak{Y}(\nu)\mathfrak{V}(\nu)d\nu\right| \\
&\le (\mathfrak{f}_g-\mathfrak{g}_g)^2\left\{\int_0^{\frac{1}{2}}\int_0^{s} \mathfrak{V}(\mathfrak{s}\mathfrak{g}_g+(1-\mathfrak{s})\mathfrak{f}_g)\left(\mathtt{H}(\eta_o,1-\eta_o)\left|\mathfrak{Y}'(\mathfrak{g}_g)\right| + \mathtt{H}(1-\eta_o,\eta_o)\left|\mathfrak{Y}'(\mathfrak{f}_g)\right|\right)d\eta_o d\mathfrak{s}\right. \\
&\left. +\int_{\frac{1}{2}}^1\int_{s}^{1} \mathfrak{V}(\mathfrak{s}\mathfrak{g}_g+(1-\mathfrak{s})\mathfrak{f}_g)\left(\mathtt{H}(\eta_o,1-\eta_o)\left|\mathfrak{Y}'(\mathfrak{g}_g)\right| + \mathtt{H}(1-\eta_o,\eta_o)\left|\mathfrak{Y}'(\mathfrak{f}_g)\right|\right)d\eta_o d\mathfrak{s}\right\}.
\end{aligned}$$

Using the variable change $\nu = \mathfrak{s}\mathfrak{g}_g + (1-\mathfrak{s})\mathfrak{f}_g$, one has

$$\begin{aligned}
&\left|\frac{\mathfrak{Y}(\mathfrak{g}_g)+\mathfrak{Y}(\mathfrak{f}_g)}{2}\int_{\mathfrak{g}_g}^{\mathfrak{f}_g}\mathfrak{V}(\nu)d\nu - \int_{\mathfrak{g}_g}^{\mathfrak{f}_g}\mathfrak{Y}(\nu)\mathfrak{V}(\nu)d\nu\right| \\
&\le (\mathfrak{f}_g-\mathfrak{g}_g)\left\{\int_{\frac{\mathfrak{f}_g+\mathfrak{g}_g}{2}}^{\mathfrak{f}_g}\int_0^{\frac{\mathfrak{f}_g-\nu}{\mathfrak{f}_g-\mathfrak{g}_g}} \mathfrak{V}(\nu)\left(\mathtt{H}(\eta_o,1-\eta_o)\left|\mathfrak{Y}'(\mathfrak{g}_g)\right| + \mathrm{h}_1(1-\eta_o)\mathrm{h}_2(\eta_o)\left|\mathfrak{Y}'(\mathfrak{f}_g)\right|\right)d\eta_o d\nu\right. \\
&\left. +\int_{\mathfrak{g}_g}^{\frac{\mathfrak{f}_g+\mathfrak{g}_g}{2}}\int_{\frac{\mathfrak{f}_g-\nu}{\mathfrak{f}_g-\mathfrak{g}_g}}^{1} \mathfrak{V}(\nu)\left(\mathtt{H}(\eta_o,1-\eta_o)\left|\mathfrak{Y}'(\mathfrak{g}_g)\right| + \mathtt{H}(1-\eta_o,\eta_o)\left|\mathfrak{Y}'(\mathfrak{f}_g)\right|\right)d\eta_o d\nu\right\}. \qquad (6.1)
\end{aligned}$$

As $\mathfrak{V}$ is symmetric to $\frac{\mathfrak{f}_g+\mathfrak{g}_g}{2}$, then one has

$$\int_{\frac{\mathfrak{f}_g+\mathfrak{g}_g}{2}}^{\mathfrak{f}_g}\int_0^{\frac{\mathfrak{f}_g-\nu}{\mathfrak{f}_g-\mathfrak{g}_g}} \mathfrak{V}(\nu)\left(\mathtt{H}(\eta_o,1-\eta_o)\left|\mathfrak{Y}'(\mathfrak{g}_g)\right| + \mathtt{H}(1-\eta_o,\eta_o)\left|\mathfrak{Y}'(\mathfrak{f}_g)\right|\right) d\eta_o d\nu$$
$$= \int_{\mathfrak{g}_g}^{\frac{\mathfrak{f}_g+\mathfrak{g}_g}{2}}\int_0^{\frac{\nu-\mathfrak{g}_g}{\mathfrak{f}_g-\mathfrak{g}_g}} \mathfrak{V}(\nu)\left(\mathtt{H}(\eta_o,1-\eta_o)\left|\mathfrak{Y}'(\mathfrak{g}_g)\right| + \mathtt{H}(1-\eta_o,\eta_o)\left|\mathfrak{Y}'(\mathfrak{f}_g)\right|\right) d\eta_o d\nu. \tag{6.2}$$

Replacing (6.2) in (6.1) it follows that

$$\left|\frac{\mathfrak{Y}(\mathfrak{g}_g)+\mathfrak{Y}(\mathfrak{f}_g)}{2}\int_{\mathfrak{g}_g}^{\mathfrak{f}_g}\mathfrak{V}(\nu)d\nu - \int_{\mathfrak{g}_g}^{\mathfrak{f}_g}\mathfrak{Y}(\nu)\mathfrak{V}(\nu)d\nu\right|$$
$$\leq (\mathfrak{f}_g-\mathfrak{g}_g)\left(\left|\mathfrak{Y}'(\mathfrak{g}_g)\right| + \left|\mathfrak{Y}'(\mathfrak{f}_g)\right|\right)\int_{\mathfrak{g}_g}^{\frac{\mathfrak{f}_g+\mathfrak{g}_g}{2}}\int_0^1 \mathfrak{V}(\nu)\left[\mathtt{H}(\eta_o,1-\eta_o)+\mathtt{H}(1-\eta_o,\eta_o)\right] d\eta_o d\nu. \tag{6.3}$$

□

6.1. *Quadrature formula*

Consider $\mathfrak{p}$ be a partition of $\left[\mathfrak{g}_g,\mathfrak{f}_g\right]$, i.e., $\mathfrak{p}:\mathfrak{g}_g=\mathfrak{v}_0<\mathfrak{v}_1<\cdots<\mathfrak{v}_{n-1}<\mathfrak{v}_n=\mathfrak{f}_g$, of this quadrature formula

$$\int_{\mathfrak{g}_g}^{\mathfrak{f}_g}\mathfrak{Y}(\nu)\mathfrak{V}(\nu)d\nu = \mathcal{T}(\mathfrak{Y},\mathfrak{V},\mathfrak{p}) + S(\mathfrak{Y},\mathfrak{V},\mathfrak{p}),$$

where

$$\mathcal{T}(\mathfrak{Y},\mathfrak{V},\mathfrak{p}) = \sum_{i=0}^{n-1}\frac{\mathfrak{Y}(\mathfrak{e}_i)+\mathfrak{Y}(\mathfrak{e}_{i+1})}{2}\int_{\mathfrak{e}_i}^{\mathfrak{e}_{i+1}}\mathfrak{V}(\nu)d\nu,$$

is called to be trapezoidal formula. Consider a subinterval $[\mathfrak{e}_i,\mathfrak{e}_{i+1}]$ while using Theorem 6.1. This gives the following as:

$$\left|\frac{\mathfrak{Y}(\mathfrak{e}_i)+\mathfrak{Y}(\mathfrak{e}_{i+1})}{2}\int_{\mathfrak{e}_i}^{\mathfrak{e}_{i+1}}\mathfrak{V}(\nu)d\nu - \int_{\mathfrak{e}_i}^{\mathfrak{e}_{i+1}}\mathfrak{Y}(\nu)\mathfrak{V}(\nu)d\nu\right|$$
$$\leq (\mathfrak{e}_{i+1}-\mathfrak{e}_i)\left[|\mathfrak{Y}'(\mathfrak{e}_i)|+|\mathfrak{Y}'(\mathfrak{e}_{i+1})|\right]\int_{\frac{\mathfrak{e}_i+\mathfrak{e}_{i+1}}{2}}^{\mathfrak{e}_{i+1}}\int_0^{\frac{\mathfrak{e}_{i+1}-\nu}{\mathfrak{e}_{i+1}-\mathfrak{e}_i}}\mathfrak{V}(\nu)\left[\mathtt{H}(\eta_o,1-\eta_o)+\mathtt{H}(1-\eta_o,\eta_o)\right]d\eta_o d\nu, \tag{6.4}$$

Using the inequality (6.4) and the triangular inequality, we obtain

$$\left|\,\mathcal{T}(\mathfrak{Y},\mathfrak{V},\mathfrak{p}) - \int_{\mathfrak{g}_g}^{\mathfrak{f}_g}\mathfrak{Y}(\nu)\mathfrak{V}(\nu)d\nu\,\right|$$
$$= \left|\sum_{i=0}^{n-1}\left[\frac{\mathfrak{Y}(\mathfrak{e}_i)+\mathfrak{Y}(\mathfrak{e}_{i+1})}{2}\int_{\mathfrak{e}_i}^{\mathfrak{e}_{i+1}}\mathfrak{V}(\nu)d\nu - \int_{\mathfrak{e}_i}^{\mathfrak{e}_{i+1}}\mathfrak{Y}(\nu)\mathfrak{V}(\nu)d\nu\right]\right|$$

$$\leq \sum_{i=0}^{n-1} \left| \frac{\mathfrak{Y}(\mathfrak{e}_i) + \mathfrak{Y}(\mathfrak{e}_{i+1})}{2} \int_{\mathfrak{e}_i}^{\mathfrak{e}_{i+1}} \mathfrak{V}(\nu) d\nu - \int_{\mathfrak{e}_i}^{\mathfrak{e}_{i+1}} \mathfrak{Y}(\nu)\mathfrak{V}(\nu) d\nu \right|$$

$$\leq \sum_{i=0}^{n-1} (\mathfrak{e}_{i+1} - \mathfrak{e}_i) \left[|\mathfrak{Y}'(\mathfrak{e}_i)| + |\mathfrak{Y}'(\mathfrak{e}_{i+1})| \right] \int_{\frac{\mathfrak{e}_i+\mathfrak{e}_{i+1}}{2}}^{\mathfrak{e}_{i+1}} \int_0^{\frac{\mathfrak{e}_{i+1}-\nu}{\mathfrak{e}_{i+1}-\mathfrak{e}_i}} \mathfrak{V}(\nu)$$

$$\times \left[\mathtt{H}(\eta_o, 1-\eta_o) + \mathtt{H}(1-\eta_o, \eta_o) \right] d\eta_o d\nu.$$

This provides us with the error bound:

$$|S(\mathfrak{Y}, \mathfrak{V}, \mathfrak{p})| \leq \sum_{i=0}^{n-1} (\mathfrak{e}_{i+1} - \mathfrak{e}_i) \left[|\mathfrak{Y}'(\mathfrak{e}_i)| + |\mathfrak{Y}'(\mathfrak{e}_{i+1})| \right]$$

$$\times \int_{\frac{\mathfrak{e}_i+\mathfrak{e}_{i+1}}{2}}^{\mathfrak{e}_{i+1}} \int_0^{\frac{\mathfrak{e}_{i+1}-\nu}{\mathfrak{e}_{i+1}-\mathfrak{e}_i}} \mathfrak{V}(\nu) \left[\mathtt{H}(\eta_o, 1-\eta_o) + \mathtt{H}(1-\eta_o, \eta_o) \right] d\eta_o d\nu.$$

Remark 6.1. *If* $\mathtt{h}_1(\eta_o) = \mathfrak{y}^k, \mathtt{h}_2(\eta_o) = 1$ *with* $\mathtt{k} = 1$ *in (6.4), then we reiterate the disparity revealed in [54].*

$$|S(\mathfrak{Y}, \mathfrak{p})| \leq \frac{1}{8} \sum_{\mathtt{i}=0}^{\mathtt{n}-1} \left[|\mathfrak{Y}'(\mathfrak{e}_i)| + |\mathfrak{Y}'(\mathfrak{e}_{i+1})| \right] (\mathfrak{e}_{i+1} - \mathfrak{e}_i)^2.$$

Applications to Random Variable

Consider a probability density function. $\mathfrak{V} : [\mathfrak{g}_g, \mathfrak{f}_g] \to \mathbf{R}^+$ with $0 < \mathfrak{g}_g < \mathfrak{f}_g$, then

$$\int_{\mathfrak{g}_g}^{\mathfrak{f}_g} \mathfrak{V}(\nu) d\nu = 1,$$

which is symmetric to $\frac{\mathfrak{f}_g+\mathfrak{g}_g}{2}$ and let $\mathtt{u}$ be a moment where $\mathtt{u} \in \mathbf{R}$ then, we have

$$\mathcal{E}_{\mathtt{u}}(X) = \int_{\mathfrak{g}_g}^{\mathfrak{f}_g} \nu^{\mathtt{u}} \mathfrak{V}(\nu) d\nu,$$

is finite. From Theorem (6.1) and the fact that for any $\mathfrak{g}_g \leq \nu \leq \frac{\mathfrak{f}_g+\mathfrak{g}_g}{2}$ we have $0 \leq \frac{\nu-\mathfrak{g}_g}{\mathfrak{f}_g-\mathfrak{g}_g} \leq \frac{1}{2}$, the following result holds.

$$\left| \frac{\mathfrak{Y}(\mathfrak{g}_g) + \mathfrak{Y}(\mathfrak{f}_g)}{2} \int_{\mathfrak{g}_g}^{\mathfrak{f}_g} \mathfrak{V}(\nu) d\nu - \int_{\mathfrak{g}_g}^{\mathfrak{f}_g} \mathfrak{Y}(\nu)\mathfrak{V}(\nu) d\nu \right| \leq (\mathfrak{f}_g - \mathfrak{g}_g) \left(\left| \mathfrak{Y}'(\mathfrak{g}_g) \right| + \left| \mathfrak{Y}'(\mathfrak{f}_g) \right| \right)$$

$$\times \int_{\mathfrak{g}_g}^{\frac{\mathfrak{f}_g+\mathfrak{g}_g}{2}} \int_0^{\frac{1}{2}} \mathfrak{V}(\nu) \left[\mathtt{H}(\eta_o, 1-\eta_o) + \mathtt{H}(1-\eta_o, \eta_o) \right] d\eta_o d\nu = \frac{(\mathfrak{f}_g - \mathfrak{g}_g)}{2} \left(\left| \mathfrak{Y}'(\mathfrak{g}_g) \right| + \left| \mathfrak{Y}'(\mathfrak{f}_g) \right| \right)$$

$$\times \int_0^{\frac{1}{2}} \left[\mathtt{H}(\eta_o, 1-\eta_o) + \mathtt{H}(1-\eta_o, \eta_o) \right] d\mathfrak{y},$$

since $\mathfrak{V}$ is symmetric and $\int_{\mathfrak{g}_g}^{\mathfrak{f}_g} \mathfrak{V}(\nu) d\nu = 1$, we have $\int_{\mathfrak{g}_g}^{\frac{\mathfrak{f}_g+\mathfrak{g}_g}{2}} \mathfrak{V}(\nu) d\nu = \frac{1}{2}$.

Example 6.1. *If we take into account*

$$\begin{cases} \mathfrak{Y}(\nu) = \frac{1}{\mathrm{u}}\nu^{\mathrm{u}}, & \nu > 0, \mathrm{u} \in (-\infty, 0) \cup (0, 2] \cup [3, +\infty); \\ \mathrm{h}_1(\eta_o) = \mathfrak{y}^{\mathrm{k}}, \mathrm{h}_2(\eta_o) = \frac{1}{4} & \mathrm{k} \in (-\infty, -1) \cup (-1, 1]; \\ \mathfrak{B}(\nu) = 1. \end{cases}$$

Since $|\mathfrak{Y}'|$ *is* $(\mathrm{h}_1, \mathrm{h}_2)$*-convex and so from Theorem 6.1 we have*

$$\begin{aligned} \left| \frac{\mathfrak{g}_g{}^{\mathrm{u}} + \mathfrak{f}_g{}^{\mathrm{u}}}{2\mathrm{u}} - \mathcal{E}_{\mathrm{u}}(X) \right| &\leq \frac{\mathrm{u}(\mathfrak{f}_g - \mathfrak{g}_g)}{2} \left(\mathfrak{g}_g{}^{\mathrm{u}-1} + \mathfrak{f}_g{}^{\mathrm{u}-1} \right) \int_0^{\frac{1}{2}} \left[\frac{\mathfrak{y}^k}{4} + \frac{(1-\eta_o)^k}{4} \right] d\mathfrak{y} \\ &= \frac{\mathrm{u}(\mathfrak{f}_g - \mathfrak{g}_g)}{4(k+1)} \left(\mathfrak{g}_g{}^{\mathrm{u}-1} + \mathfrak{f}_g{}^{\mathrm{u}-1} \right). \end{aligned}$$

As a result, the required bound is

$$\left| \frac{\mathfrak{g}_g{}^{\mathrm{u}} + \mathfrak{f}_g{}^{\mathrm{u}}}{2\mathrm{u}} - \mathcal{E}_{\mathrm{u}}(X) \right| \leq \frac{\mathrm{u}(\mathfrak{f}_g - \mathfrak{g}_g)}{4(k+1)} \left(\mathfrak{g}_g{}^{\mathrm{u}-1} + \mathfrak{f}_g{}^{\mathrm{u}-1} \right),$$

Remark 6.2. *If* $\mathrm{u} = 1, \mathrm{h}_2(\eta_o) = 1, \mathrm{k} = 1$*, then we can get the following known bound as follows:*

$$\left| \frac{\mathfrak{f}_g + \mathfrak{g}_g}{2} - \mathcal{E}(X) \right| \leq \frac{\mathfrak{f}_g - \mathfrak{g}_g}{4}.$$

7. Conclusion

In this work, Godunova-Levin type mappings via set-valued functions are used to study a variety of inequalities associated with a new class of preinvexity. To start, we define the Godunova-Levin preinvex mappings under the full-order relation and examine some of its induced properties. We generalize many previously reported results and build novel forms by using arbitrary non-negative functions and related bifunctions of Hermite, Hadamard, and Fejér-type inequalities. We also discuss some special cases of these inequalities. To further illustrate the accuracy of the obtained results, a few numerical examples are given. In the subsequent, we concentrate on numerical integration error bounds and their applications to random variables through trapezoidal type inequality, utilising standard order via generalized convexity. Further research into other kinds of convex inequalities is feasible using the idea and concepts established in this work, with potential applications to issues like differential equations with convex shapes attached and optimisation problems.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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