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Research article

Precise tracking control via iterative learning for one-sided Lipschitz Caputo fractional-order systems

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Abstract: This paper investigates iterative learning control for Caputo fractional-order systems with one-sided Lipschitz nonlinearity. Both open- and closed-loop P-type learning algorithms are proposed to achieve perfect tracking for the desired trajectory, and their convergence conditions are established. It is shown that the algorithms can make the output tracking error converge to zero along the iteration axis. A simulation example illustrates the application of the theoretical findings, and shows the effectiveness of the proposed approach.

Keywords: fractional-order systems; one-sided Lipschitz nonlinearity; P-type learning algorithm; iterative learning control

1. Introduction

Recently, fractional calculus emerged as a crucial tool for describing the dynamics of real-world problems [1–3]. Indeed, many fractional-order systems (FOSs) have been reported in the literature, focusing on stability, fault tolerant control, and sliding mode control, among other issues (see [4–6] and references therein). Iterative learning control (ILC) is an interesting approach to obtain trajectory tracking of repetitive systems operated over finite-time [7]. In recent years, FOSs and ILC have been merged with the goal of increasing tracking performance. In [8], a D^{α}-type ILC scheme was designed and its convergence was addressed. In [9–11], both the P- and D-type learning schemes were adopted in FOSs with Lipschitz nonlinearities. In [12], fractional-order PID learning control was proposed for linear FOSs, and output convergence was analyzed using the Lebesgue-p norm. In [13], the ILC framework was adopted for FOSs with randomly varying trial lengths. In [14,15], ILC problems of multi-agent systems with fractional-order models were investigated. Despite many relevant contributions, it should be pointed out that the above mentioned works mainly address linear and Lipschitz FOSs. Moreover, a variety of control strategies have been proposed for nonlinear systems (NSs) to achieve the desired performance. In [16,17], the convergence analysis for locally Lipschitz NSs was addessed based on the contraction mapping approach. In [18], adaptive optimal control was investigated for NSs based on the policy iteration algorithm. In [19], zero-sum control for tidal turbine systems was studied though a reinforcement learning method.

Compared with classical Lipschitz nonlinearity, one-sided Lipschitz (OSL) nonlinearity possesses less conservatism. Therefore, in recent years, is has often been used in control systems. Moreover, in many practical problems, the OSL constant is much smaller than the Lipschitz one, which simplifies the estimation of the influence of nonlinearities. OSL systems are a wide class of NSs, which contain Lipschitz systems as particular cases. Practical examples are Chua's circuits, Lorenz systems, and electromechanical systems [20–22]. In [23–25], observer design issues for OSL NSs were investigated. In [26], the classical OSL was considered, and an observer was designed by introducing the quadratically inner-bounded (QIB) constraint. In recent years, observer design and control of OSL NSs has attracted considerable attention. In [27], full- and reduced-order observers were derived via the Riccati equation. In [28], exponential observer design was investigated. In [29], tracking control for OSL nonlinear differential inclusions was considered. In [30], H_∞ attenuation control was considered for OSL NSs in the finite frequency domain. In [31], event-triggered sliding mode control was studied for OSL NSs with uncertainties. In [32,33], consensus control was discussed for OSL nonlinear multi-agent systems. Other meaningful results on ILC of OSL NSs have also been reported [34–36]. In particular, the QIB constraint was employed to reach perfect trajectory tracking [34,35]. Note that the above-mentioned results are about classical integer-order systems. To the best of the authors' knowledge, for FOSs with OSL nonlinearity, the problem of how to achieve exact trajectory tracking through appropriate ILC design has not yet been investigated, which motivates the present study.

This paper deals with the ILC of a family of Caputo FOSs, where the fractional derivative is in the interval 0 and 1. The considered nonlinearity satisfies the OSL condition, which encompasses the classical Lipschitz condition. Open- and closed-loop P-type learning control algorithms are adopted. The convergence of the tracking error is guaranteed with the generalized Gronwall inequality. The novelty of this paper is summarized in the next two points.

- Unlike the control methods in references [18,19,29–33], the ILC method proposed in this paper can lead OSL nonlinear Caputo FOSs to exhibit perfect tracking capability;
- In contrast to the works of [34–36], the ILC theory is extended from integer-order OSL NSs to fractional-order OSL NSs.

This paper is divided into 5 sections. Section 2 establishes some elemental assumptions and formulates the ILC problem of fractional OSL NSs. Section 3 constructs the open- and closed-loop P-type control algorithms, and presents the corresponding convergence results. Section 4 includes a numerical example to show the suitability of the algorithms. Finally, Section 5 summarizes the conclusions.

2. Preliminaries and problem description

Some relevant lemmas and definitions are introduced. Afterwards, the problem to be tackled is formulated.

Definition 1 [37]. The Riemann-Liouville integral of order $\alpha > 0$ of a function $x(t)$ is

$$I_{0,t}^{\alpha} x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\xi)^{\alpha-1} x(\xi) \mathrm{d}\xi, t \in [0, \infty),$$

where $\Gamma(\alpha)$ stands for the Gamma function.

Definition 2 [37]. The Caputo derivative of order $0 < \alpha < 1$ of a function $x(t)$ is

$${}_C\mathcal{D}_{0,t}^{\alpha} x(t) = I_{0,t}^{1-\alpha} \frac{\mathrm{d}}{\mathrm{d}t} x(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\xi)^{-\alpha} \dot{x}(\xi) \mathrm{d}\xi, t \in [0, \infty).$$

Lemma 1 [38]. Consider the differentiable vector $x(t) \in R^n$. It follows that, for any time instant $t \geq 0$, we have

$${}_C\mathcal{D}_{0,t}^{\alpha}(x^{\mathrm{T}}(t)x(t)) \leq 2x^{\mathrm{T}}(t)\,{}_C\mathcal{D}_{0,t}^{\alpha} x(t), \ \forall \alpha \in (0, 1),$$

where the superscript T denotes the vector (or matrix) transpose.

Lemma 2. (Generalized Gronwall Inequality) [9] Consider that the function $u(t)$ is continuous on the interval $t \in [0, T]$, and let $v(t-\xi)$ be nonnegative and continuous on $0 \leq \xi \leq t \leq T$. Additionally, consider that the function $w(t)$ is positive continuous and nondecreasing on $t \in [0, T]$. If

$$u(t) \leq w(t) + \int_0^t v(t-\xi)u(\xi)\mathrm{d}\xi, \ t \in [0, T],$$

then we have

$$u(t) \leq w(t)\mathrm{e}^{\int_0^t v(t-\xi)\mathrm{d}\xi}, \ t \in [0, T].$$

To simplify the notation, in the following, we use $\mathcal{D}^{\alpha}$ to refer to the Caputo derivative ${}_C\mathcal{D}_{0,t}^{\alpha}$.

Let us consider the nonlinear FOS

$$\begin{cases} \mathcal{D}^{\alpha} x_k(t) = Ax_k(t) + Bu_k(t) + f(x_k(t)), \\ y_k(t) = Cx_k(t) + Du_k(t), \end{cases} \tag{2.1}$$

where $\alpha \in (0, 1)$, $t \in [0, T]$, and $k = 0, 1, 2, \cdots$ is the repetition. Moreover, $x_k(t) \in R^n$, $u_k(t) \in R^m$, and $y_k(t) \in R^p$ represent the state, control, and output of (2.1), respectively; $f(x_k(t)) \in R^n$ stands for a continuous nonlinear function; and A, B, C, and D are constant coefficients matrices.

Assumption 1. The nonlinear function $f(\cdot)$ is OSL, meaning that, for $\forall x(t), \hat{x}(t) \in R^n$,

$$\langle f(x(t)) - f(\hat{x}(t)), x(t) - \hat{x}(t) \rangle \leq \sigma \|x(t) - \hat{x}(t)\|^2,$$

where $\|\cdot\|$ denotes the Euclidean norm, $\langle \cdot, \cdot \rangle$ represents the inner product, and $\sigma \in R$ is the OSL constant.

Remark 1. Note that the above constant σ can assume any real value, while the Lipschitz constant is positive. From [26], a Lipschitz function is OSL ($\sigma > 0$), but the converse may not hold.

Assumption 2. The desired trajectory $y_d(t)$ is possible, meaning that a control $u_d(t)$ exists, guaranteeing

$$\begin{cases} \mathcal{D}^{\alpha} x_d(t) = A x_d(t) + B u_d(t) + f(x_d(t)), \\ y_d(t) = C x_d(t) + D u_d(t), \end{cases}$$

with $x_d(t)$ being the desired state.
Assumption 3. The system defined by expression (2.1) meets the initial condition

$$x_k(0) = x_d(0), \ k = 0, 1, 2, \cdots ,$$

where $x_d(0)$ represents the desired initial state.
Remark 2. Assumption 2 is a representative condition for OSL Caputo FOSs in control law design. Assumption 3 is the identical initialization condition, which has been widely used in ILC design to obtain perfect tracking [7].

The main objective herein is to design a control sequence $u_k(t)$ so that the output $y_k(t)$ of (1) can track the specified trajectory $y_d(t)$, with $t \in [0, T]$, as $k \to \infty$.

3. The ILC design

For the nonlinear FOS (2.1), we design an open-loop P-type learning control algorithm

$$u_{k+1}(t) = u_k(t) + \Psi e_k(t), \tag{3.1}$$

where the output tracking error at the kth iteration is defined as $e_k(t) = y_d(t) - y_k(t)$ and the learning gain matrix is $\Psi \in R^{m \times p}$.
Theorem 1. Let us assume that Assumptions 1–3 hold for the FOS (2.1) with algorithm (3.1). If Ψ can be chosen such that

$$\rho_1 = \|I - D\Psi\| < 1, \tag{3.2}$$

then $y_k(t)$ converges to $y_d(t)$ for $t \in [0, T]$.
Proof. Let us use $\delta(\cdot)_k(t) = (\cdot)_{k+1}(t) - (\cdot)_k(t)$, where $(\cdot)$ stands for the variables x, u, and f. It follows from (2.1) and (3.1) that

$$\mathcal{D}^{\alpha}(\delta x_k(t)) = A\delta x_k(t) + B\delta u_k(t) + \delta f_k(t) = A\delta x_k(t) + \delta f_k(t) + B\Psi e_k(t). \tag{3.3}$$

If we left-multiply (3.3) by $(\delta x_k(t))^{\mathrm{T}}$ and use Assumption 1, then we have

$$\begin{aligned} (\delta x_k(t))^{\mathrm{T}} \mathcal{D}^{\alpha}(\delta x_k(t)) &= \langle A\delta x_k(t), \delta x_k(t)\rangle + \langle B\Psi e_k(t), \delta x_k(t)\rangle + \langle \delta f_k(t), \delta x_k(t)\rangle \\ &\leq (A\delta x_k(t))^{\mathrm{T}} \delta x_k(t) + (B\Psi e_k(t))^{\mathrm{T}} \delta x_k(t) + \sigma \|\delta x_k(t)\|^2 \\ &\leq (\|A\| + |\sigma|)\|\delta x_k(t)\|^2 + \|B\Psi\|\|\delta x_k(t)\|\|e_k(t)\|. \end{aligned} \tag{3.4}$$

According to Lemma 1,

$$\mathcal{D}^{\alpha}((\delta x_k(t))^{\mathrm{T}} \delta x_k(t)) \leq 2(\delta x_k(t))^{\mathrm{T}} \mathcal{D}^{\alpha}(\delta x_k(t)). \tag{3.5}$$

From (3.4) and (3.5), we get

$$\begin{aligned}\mathcal{D}^{\alpha}(\|\delta x_k(t)\|^2) &\leq 2(\|A\|+|\sigma|)\|\delta x_k(t)\|^2+2\|B\Psi\|\|\delta x_k(t)\|\|e_k(t)\| \\ &\leq (2\,\|A\|+2\,|\sigma|+1)\|\delta x_k(t)\|^2+\|B\Psi\|^2\|e_k(t)\|^2 \\ &= c_1\|\delta x_k(t)\|^2+c_2\|e_k(t)\|^2, \end{aligned} \tag{3.6}$$

where $c_1 = 2\,\|A\| + 2\,|\sigma| + 1$ and $c_2 = \|B\Psi\|^2$. Applying the α-order integral on (3.6), we get

$$I^{\alpha}_{0,t}\mathcal{D}^{\alpha}(\|\delta x_k(t)\|^2) \leq I^{\alpha}_{0,t}(c_1\|\delta x_k(t)\|^2+c_2\|e_k(t)\|^2). \tag{3.7}$$

It follows from Assumption 3 that $\|\delta x_k(0)\|^2 = 0$, and we further get

$$\begin{aligned}I^{\alpha}_{0,t}\mathcal{D}^{\alpha}(\|\delta x_k(t)\|^2) &= I^{\alpha}_{0,t}I^{1-\alpha}_{0,t}\frac{\mathrm{d}}{\mathrm{d}t}(\|\delta x_k(t)\|^2) = I^{1}_{0,t}\frac{\mathrm{d}}{\mathrm{d}t}(\|\delta x_k(t)\|^2) \\ &= \|\delta x_k(t)\|^2-\|\delta x_k(0)\|^2 = \|\delta x_k(t)\|^2,\end{aligned}$$

which, together with (3.7), leads to

$$\begin{aligned}\|\delta x_k(t)\|^2 &\leq I^{\alpha}_{0,t}(c_1\|\delta x_k(t)\|^2+c_2\|e_k(t)\|^2) \\ &= \frac{c_1}{\Gamma(\alpha)}\int_0^t (t-\xi)^{\alpha-1}\|\delta x_k(\xi)\|^2\mathrm{d}\xi+\frac{c_2}{\Gamma(\alpha)}\int_0^t (t-\xi)^{\alpha-1}\|e_k(\xi)\|^2\mathrm{d}\xi \\ &= \frac{c_1}{\Gamma(\alpha)}\int_0^t (t-\xi)^{\alpha-1}\|\delta x_k(\xi)\|^2\mathrm{d}\xi+\frac{c_2}{\Gamma(\alpha)}\int_0^t (t-\xi)^{\alpha-1}\mathrm{e}^{2\lambda\xi}\{\mathrm{e}^{-2\lambda\xi}\|e_k(\xi)\|^2\}\mathrm{d}\xi \\ &\leq \frac{c_1}{\Gamma(\alpha)}\int_0^t (t-\xi)^{\alpha-1}\|\delta x_k(\xi)\|^2\mathrm{d}\xi+\frac{c_2}{\Gamma(\alpha)}\int_0^t (t-\xi)^{\alpha-1}\mathrm{e}^{2\lambda\xi}\mathrm{d}\xi\,\|e_k\|^2_{\lambda}.\end{aligned} \tag{3.8}$$

We can see that

$$\begin{aligned}\int_0^t (t-\xi)^{\alpha-1}\mathrm{e}^{2\lambda\xi}\mathrm{d}\xi &\xrightarrow{t-\xi=\tau} \int_0^t \tau^{\alpha-1}\mathrm{e}^{2\lambda(t-\tau)}\mathrm{d}\tau \\ &= \mathrm{e}^{2\lambda t}\int_0^t \tau^{\alpha-1}\mathrm{e}^{-2\lambda\tau}\mathrm{d}\tau \\ &\xrightarrow{2\lambda\tau=\xi} \frac{\mathrm{e}^{2\lambda t}}{(2\lambda)^{\alpha}}\int_0^{2\lambda t} \xi^{\alpha-1}\mathrm{e}^{-\xi}\mathrm{d}\xi \\ &< \frac{\mathrm{e}^{2\lambda t}}{(2\lambda)^{\alpha}}\int_0^{+\infty} \xi^{\alpha-1}\mathrm{e}^{-\xi}\mathrm{d}\xi \\ &= \frac{\mathrm{e}^{2\lambda t}}{(2\lambda)^{\alpha}}\Gamma(\alpha).\end{aligned} \tag{3.9}$$

From (3.8) and (3.9), we have

$$\|\delta x_k(t)\|^2 \leq \frac{c_1}{\Gamma(\alpha)}\int_0^t (t-\xi)^{\alpha-1}\|\delta x_k(\xi)\|^2\mathrm{d}\xi+\frac{c_2\mathrm{e}^{2\lambda t}}{(2\lambda)^{\alpha}}\,\|e_k\|^2_{\lambda}.$$

Setting

$$v(t-\xi) = \frac{c_1}{\Gamma(\alpha)}(t-\xi)^{\alpha-1},$$

$$w(t) = \frac{c_2 e^{2\lambda t}}{(2\lambda)^\alpha} \|e_k\|_\lambda^2 ,$$

and using Lemma 2, we get

$$\begin{aligned}
\|\delta x_k(t)\|^2 &\le \frac{c_2 e^{2\lambda t}}{(2\lambda)^\alpha} e^{\frac{c_1}{\Gamma(\alpha)} \int_0^t (t-\xi)^{\alpha-1} d\xi} \|e_k\|_\lambda^2 \\
&= \frac{c_2 e^{2\lambda t}}{(2\lambda)^\alpha} e^{\frac{c_1}{\Gamma(\alpha)} \frac{t^\alpha}{\alpha}} \|e_k\|_\lambda^2 \\
&\le \frac{c_2 e^{2\lambda t}}{(2\lambda)^\alpha} e^{\frac{c_1 T^\alpha}{\Gamma(\alpha+1)}} \|e_k\|_\lambda^2 .
\end{aligned}$$

Multiplying the above inequality by $e^{-2\lambda t}$, and using the λ-norm $\|\cdot\|_\lambda$, we have

$$\|\delta x_k\|_\lambda^2 \le \frac{c_2 e^{\frac{c_1 T^\alpha}{\Gamma(\alpha+1)}}}{(2\lambda)^\alpha} \|e_k\|_\lambda^2 ,$$

where $\|\cdot\|_\lambda = \sup_{t\in[0,T]}\{e^{-\lambda t}\|\cdot\|\}$.

Therefore, we get

$$\|\delta x_k\|_\lambda \le \frac{c_3}{\sqrt{\lambda^\alpha}} \|e_k\|_\lambda , \tag{3.10}$$

where

$$c_3 = \sqrt{\frac{c_2 e^{\frac{c_1 T^\alpha}{\Gamma(\alpha+1)}}}{2^\alpha}}.$$

It is obvious that

$$e_{k+1}(t) = e_k(t) - C\delta x_k(t) - D\delta u_k(t) = (I - D\Psi)e_k(t) - C\delta x_k(t). \tag{3.11}$$

It follows from (3.2), (3.10), and (3.11) that

$$\begin{aligned}
\|e_{k+1}\|_\lambda &\le \|I - D\Psi\|\|e_k\|_\lambda + \|C\|\|\delta x_k\|_\lambda \\
&\le \rho_1\|e_k\|_\lambda + \|C\|\|\delta x_k\|_\lambda \\
&\le \rho_1\|e_k\|_\lambda + \frac{c_3\|C\|}{\sqrt{\lambda^\alpha}} \|e_k\|_\lambda \\
&= \hat{\rho}_1\|e_k\|_\lambda, \qquad (3.12)
\end{aligned}$$

where

$$\hat{\rho}_1 = \rho_1 + \frac{c_3\|C\|}{\sqrt{\lambda^\alpha}}.$$

As $0 \le \rho_1 < 1$ by (3.2), we can select λ as large as needed so that $\hat{\rho}_1 < 1$. Thus, we obtain

$$\lim_{k\to\infty} \|e_k\|_\lambda = 0.$$

Note that $\|e_k\|_s \le e^{\lambda T}\|e_k\|_\lambda$, with $\|\cdot\|_s = \sup_{t\in[0,T]} \|\cdot\|$ denoting the supremum norm. Therefore, $\lim_{k\to\infty} \|e_k\|_s = 0$, meaning that

$$\lim_{k\to\infty} y_k(t) = y_d(t),\ t \in [0, T].$$

This ends the proof.

Now, we design a closed-loop P-type learning control algorithm, such that

$$u_{k+1}(t) = u_k(t) + \Phi e_{k+1}(t), \tag{3.13}$$

where the learning gain is $\Phi \in R^{m\times p}$.

Theorem 2. Consider that Assumptions 1–3 hold for the FOS (2.1) with the learning algorithm (3.13). If the gain Φ can be chosen such that

$$\rho_2 = \|(I + D\Phi)^{-1}\| < 1, \tag{3.14}$$

then $y_k(t)$ converges to $y_d(t)$ for $t \in [0, T]$.

Proof. From (2.1) and (3.13), we get

$$\mathcal{D}^{\alpha}(\delta x_k(t)) = A\delta x_k(t) + B\delta u_k(t) + \delta f_k(t) = A\delta x_k(t) + \delta f_k(t) + B\Phi e_{k+1}(t). \tag{3.15}$$

Left multiplying (3.15) by $(\delta x_k(t))^{\mathrm{T}}$ and considering Assumption 1, we obtain

$$\begin{aligned}(\delta x_k(t))^{\mathrm{T}}\mathcal{D}^{\alpha}(\delta x_k(t)) &= \langle A\delta x_k(t), \delta x_k(t)\rangle + \langle B\Phi e_{k+1}(t), \delta x_k(t)\rangle + \langle \delta f_k(t), \delta x_k(t)\rangle \\ &\le (A\delta x_k(t))^{\mathrm{T}}\delta x_k(t) + (B\Phi e_{k+1}(t))^{\mathrm{T}}\delta x_k(t) + \sigma\|\delta x_k(t)\|^2 \\ &\le (\|A\| + |\sigma|)\|\delta x_k(t)\|^2 + \|B\Phi\|\|\delta x_k(t)\|\|e_{k+1}(t)\|. \end{aligned}\tag{3.16}$$

Obviously, (3.16) together with (3.5) implies

$$\begin{aligned}\mathcal{D}^{\alpha}(\|\delta x_k(t)\|^2) &\le 2(\|A\| + |\sigma|)\|\delta x_k(t)\|^2 + 2\|B\Phi\|\|\delta x_k(t)\|\|e_{k+1}(t)\| \\ &\le (2\,\|A\| + 2\,|\sigma| + 1)\|\delta x_k(t)\|^2 + \|B\Psi\|^2\|e_{k+1}(t)\|^2 \\ &= c_1\|\delta x_k(t)\|^2 + c_4\|e_{k+1}(t)\|^2, \end{aligned}\tag{3.17}$$

where $c_4 = \|B\Phi\|^2$. Similarly to the procedure adopted in Theorem 1, we get

$$\|\delta x_k\|_{\lambda} \le \frac{c_5}{\sqrt{\lambda^{\alpha}}}\,\|e_{k+1}\|_{\lambda}\,, \tag{3.18}$$

where

$$c_5 = \sqrt{\frac{c_4 \mathrm{e}^{\frac{c_1 T^{\alpha}}{\Gamma(\alpha+1)}}}{2^{\alpha}}}.$$

From expressions (2.1) and (3.13), we have

$$e_{k+1}(t) = e_k(t) - C\delta x_k(t) - D\delta u_k(t) = e_k(t) - C\delta x_k(t) - D\Phi e_{k+1}(t),$$

that is

$$(I + D\Phi)e_{k+1}(t) = e_k(t) - C\delta x_k(t),$$

where the symbol I stands for the identity matrix. Since $I + D\Phi$ is nonsingular, we get

$$e_{k+1}(t) = (I + D\Phi)^{-1}(e_k(t) - C\delta x_k(t)).$$

Furthermore, we derive

$$\begin{aligned}\|e_{k+1}\|_\lambda &\le \|(I + D\Phi)^{-1}\|\|e_k\|_\lambda + \|(I + D\Phi)^{-1}C\|\|\delta x_k\|_\lambda \\ &\le \rho_2\|e_k\|_\lambda + \|(I + D\Phi)^{-1}C\|\|\delta x_k\|_\lambda. \end{aligned} \tag{3.19}$$

Substituting (3.18) into (3.19) yields

$$\|e_{k+1}\|_\lambda \le \rho_2\|e_k\|_\lambda + \frac{c_5}{\sqrt{\lambda^\alpha}}\|(I + D\Phi)^{-1}C\|\|e_{k+1}\|_\lambda.$$

Taking λ such that

$$\frac{c_5}{\sqrt{\lambda^\alpha}}\|(I + D\Phi)^{-1}C\| < 1,$$

then we have

$$\|e_{k+1}\|_\lambda \le \hat{\rho}_2\|e_k\|_\lambda, \tag{3.20}$$

where

$$\hat{\rho}_2 = \frac{\rho_2}{1 - \frac{c_5}{\sqrt{\lambda^\alpha}}\|(I + D\Phi)^{-1}C\|}.$$

As $0 \le \rho_2 < 1$, we can choose λ as large as needed so that $\hat{\rho}_2 < 1$. From expression (3.20), we can obtain

$$\lim_{k\to\infty} \|e_k\|_\lambda = 0.$$

As $\|e_k\|_s \le e^{\lambda T}\|e_k\|_\lambda$, we know that $\lim_{k\to\infty} \|e_k\|_s = 0$, and it follows that

$$\lim_{k\to\infty} y_k(t) = y_d(t), \ t \in [0, T].$$

This completes the proof.

4. An illustrative example

We illustrate the applicability of the P-type learning algorithms by means of a practical example.

Let us choose the following nonlinear FOS, which can be used to describe the motion of a moving object in Cartesian coordinates [39]

$$\begin{cases} \mathcal{D}^{0.5}x_k(t) = Ax_k(t) + Bu_k(t) + f(x_k(t)), \\ y_k(t) = Cx_k(t) + Du_k(t), \end{cases}$$

where $x_k(t) = [x_{1k}(t) \;\; x_{2k}(t)]^{\mathrm{T}}$, with $t \in [0, 1]$,

$$A = \begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix}, \ B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \ C = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \ D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$f(x_k(t)) = \begin{bmatrix} -x_{1k}(t)(x_{1k}^2(t) + x_{2k}^2(t)) \\ -x_{2k}(t)(x_{1k}^2(t) + x_{2k}^2(t)) \end{bmatrix}.$$

We know from [26] that the nonlinear function $f(\cdot)$ is globally OSL with $\sigma = 0$ in R^2. Let us use

$$y_d(t) = \begin{bmatrix} \sin(3\pi t) \\ te^{-0.1t} \end{bmatrix},$$

and consider

$$x_k(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \ u_0(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

i) Open-loop algorithm (3.1).
Using the gain matrix

$$\Psi = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix},$$

we then have

$$\rho_1 = \|I - D\Psi\| = 0.5 < 1.$$

Figures 1 and 2 depict the desired trajectories $y_d^{(1)}(t)$ and $y_d^{(2)}(t)$, and the outputs $y_k^{(1)}(t)$ and $y_k^{(2)}(t)$, respectively, at the 3rd, 5th, and 7th iterations, obtained with the learning algorithm (3.1). Figure 3 represents the errors, showing that perfect tracking is reached as the number of iterations increases.

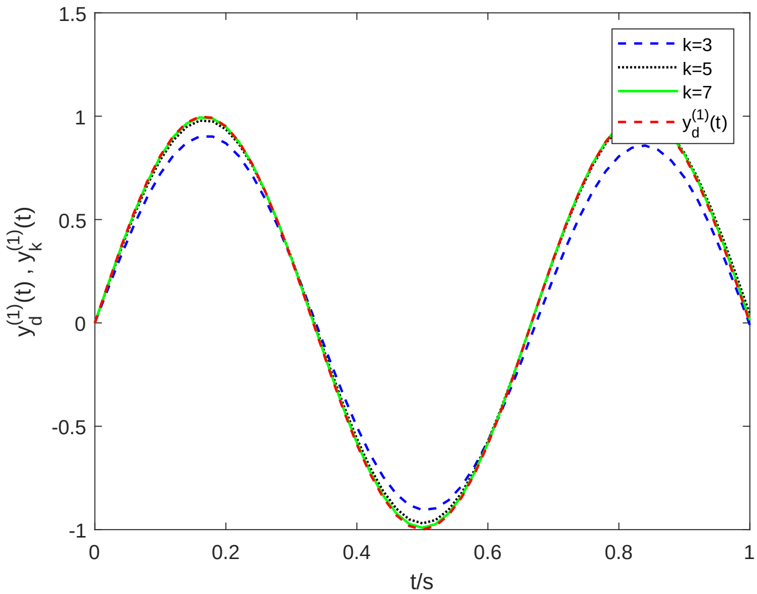

Figure 1. Desired trajectory and system output, $y_d^{(1)}(t)$ and $y_k^{(1)}(t)$, at the 3rd, 5th, and 7th iterations, when using the learning algorithm (3.1).

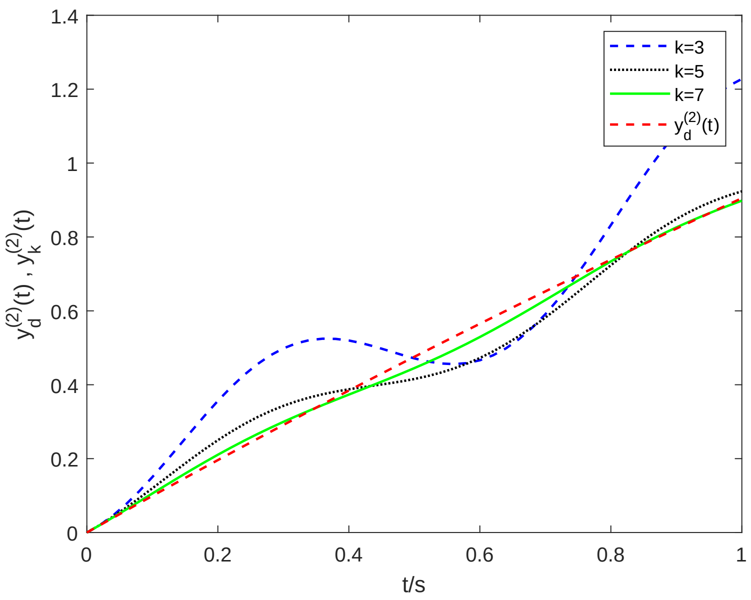

Figure 2. Desired trajectory and system outputs, $y_d^{(2)}(t)$ and $y_k^{(2)}(t)$, at the 3rd, 5th, and 7th iterations, obtained with the learning algorithm (3.1).

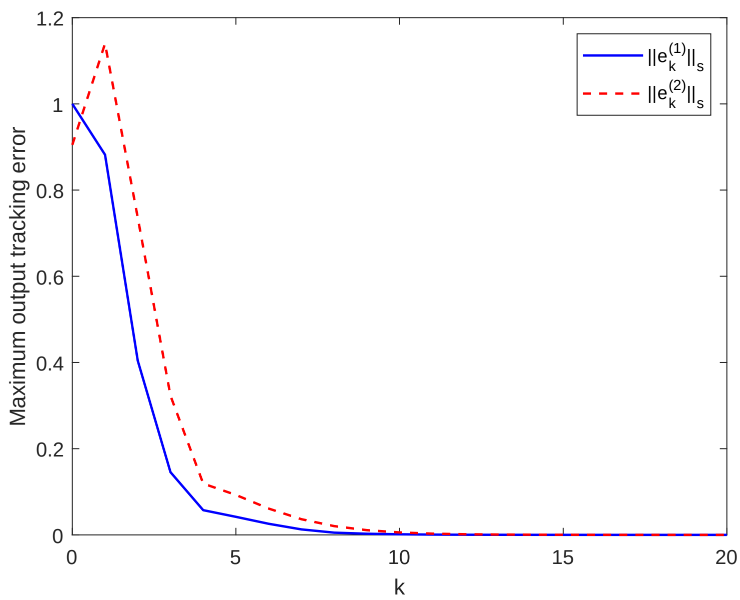

Figure 3. The output tracking errors versus the number of iterations, obtained with the learning algorithm (3.1).

ii) Closed-loop algorithm (3.13).

Using the gain matrix

$$\Phi = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

then we have

$$\rho_2 = \|(I + D\Phi)^{-1}\| = 0.5 < 1,$$

meaning that the convergence is verified. Figures 4 and 5 illustrate that $y_k^{(1)}(t)$ and $y_k^{(2)}(t)$ follow the desired trajectories from the 6th iteration. Figure 6 shows that the error converges under algorithm (3.13).

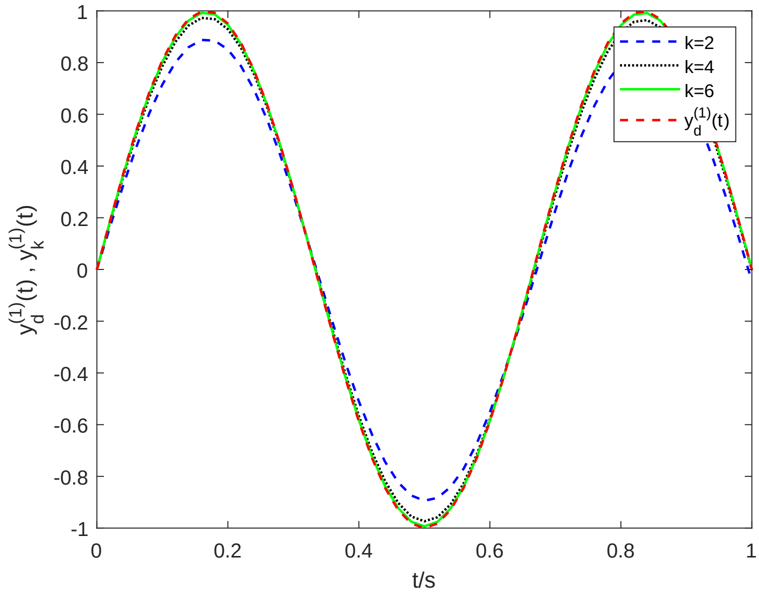

Figure 4. Desired trajectory and system output, $y_d^{(1)}(t)$ and $y_k^{(1)}(t)$, at the 2nd, 4th, and 6th iterations, when using the learning algorithm (3.13).

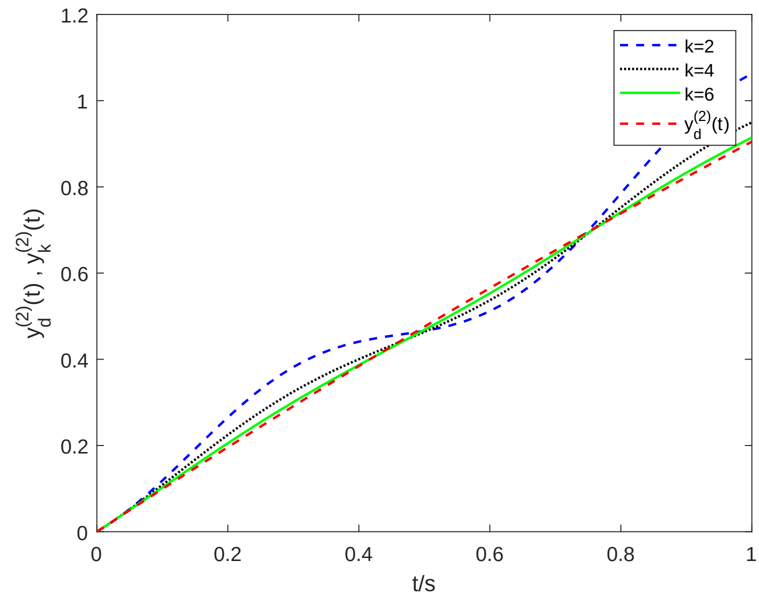

Figure 5. Desired trajectory and system output, $y_d^{(2)}(t)$ and $y_k^{(2)}(t)$, at the 2nd, 4th, and 6th iterations, when using the learning algorithm (3.13).

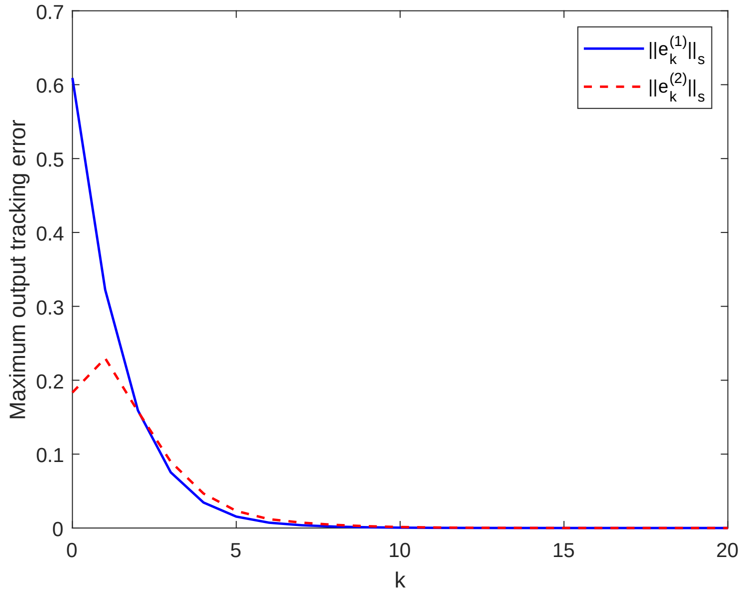

Figure 6. The output tracking errors versus the number of iterations, obtained with the learning algorithm (3.13).

We also verify that, at the 10th iteration, $\|e_k^{(i)}\|_s$ $(i = 1, 2)$ achieves $[1.4 \times 10^{-3},\ 5.7 \times 10^{-3}]$ and $[0.5 \times 10^{-3},\ 1.4 \times 10^{-3}]$ when using algorithms (3.1) and (3.13), respectively. This confirms the results observed in Figures 3 and 6, meaning that algorithm (3.13) performs better than (3.1) in terms of convergence speed.

5. Conclusions

The ILC for a class of Caputo FOSs with OSL nonlinearity was investigated. Open- and closed-loop P-type learning algorithms were designed to guarantee perfect tracking of a desired trajectory, and their convergence was verified using the generalized Gronwall inequality. An example was provided to verify the theoretical results. It should be noted that the QIB constraint was used in the framework of ILC for OSL NSs with irregular dynamics [34,35]. To some extent, this limits the applicability of NSs due to their need to simultaneously satisfy the OSL and QIB constraints. Therefore, the ILC for irregular OSL nonlinear FOSs needs to be further investigated by relaxing or removing the QIB constraint.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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