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Research article

On the date of the epidemic peak

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Abstract: Epidemiologists have used the timing of the peak of an epidemic to guide public health interventions. By determining the expected peak time, they can allocate resources effectively and implement measures such as quarantine, vaccination, and treatment at the right time to mitigate the spread of the disease. The peak time also provides valuable information for those modeling the spread of the epidemic and making predictions about its future trajectory. In this study, we analyze the time needed for an epidemic to reach its peak by presenting a straightforward analytical expression. Utilizing two epidemiological models, the first is a generalized $SEIR$ model with two classes of latent individuals, while the second incorporates a continuous age structure for latent infections. We confirm the conjecture that the peak occurs at approximately $T \sim (\ln N)/\lambda$, where N is the population size and λ is the largest eigenvalue of the linearized system in the first model or the unique positive root of the characteristic equation in the second model. Our analytical results are compared to numerical solutions and shown to be in good agreement.

Keywords: SEIR model; disease-age structure; final size

1. Introduction

Epidemic modeling has become a current issue with the 2020 pandemic caused by a coronavirus. Technical concepts, such as the parameter R_0, have emerged in the discourse of policymakers. This type of modeling is necessary to understand the transmission dynamics of infectious diseases, forecast the future trajectory of outbreaks, and assess strategies for epidemic control [1–4]. Its significance has been emphasized by a series of viral infection epidemics, including HIV since the 1980s [5, 6], the SARS epidemic in 2002–2003 [7–9], H5N1 influenza in 2005 [10, 11], H1N1 in 2009 [12, 13], Ebola in 2014 [14, 15], and the recent COVID-19 pandemic [16–19].

The most important features to assess the severity of an epidemic are its final size and timescale. The final size of the epidemic, typically measured by the number of people ultimately affected by the disease, serves as a crucial indicator for the assessment of the extent of the disease's impact on the population. Grasping this dimension allows authorities to effectively plan for necessary resources, such as hospital beds, medical equipment, and healthcare personnel. This, in turn, helps to prevent healthcare systems from becoming overwhelmed and ensures an appropriate response to the increasing demand during the epidemic's peak. Nevertheless, effective resource management during an epidemic can be optimized with precise knowledge of the peak time [20]. Researchers use various data sources and mathematical models to determine peak date of the epidemic and gain insights into the dynamics of the disease spread. In [21], the authors present an analytical method for determining the peak time of an epidemic outbreak by utilizing the SIR (susceptible-infected-recovered) model. The formula takes into account variables such as the fraction of susceptible individuals, the infectious ratio, and the initial count of infected and susceptible individuals. Empirical testing has confirmed the accuracy and utility of the formula in terms of predicting the peak time of different epidemic diseases, including COVID-19. More recently, in [22], the author proposes a simple yet accurate formula, relying on Pade approximations, to estimate the peak date of an epidemic in an SIR-type epidemic model. The author compares the results of their estimation with those of other existing formulas, highlighting that the current estimation is the most accurate with a negligible error margin across the entire parameter regime of the SIR model. In [23], the $SEIR$ model is being studied to understand the spread of infectious diseases like COVID-19, and to provide analytical expressions for the peak and and their timing, as well as the long-term behavior of the affected populations. In [20, Chapter 1], the author examines the asymptotic behavior of the time T required for an epidemic, as modeled by SIR differential system, to reach its peak when the population size N is large. The formula derived by the author for T is $T \sim \frac{1}{a-b}\left\{\log\frac{N}{I_0} + \log\left[\left(1-\frac{b}{a}\right)\log\frac{a}{b}\right] + \int_0^{\log\frac{a}{b}} \frac{-1+e^{-u}+u}{u\left(1-e^{-u}-\frac{b}{a}u\right)} du\right\}$, where I_0 is the initial number of infected individuals, a is the effective contact rate and b is the recovery rate, with the assumption that $a > b$. In [20, Chapter 2], the authors evaluated the $SEIR$ epidemic model with the aim of finding the date of the peak. They determined a lower bound for the date of the epidemic peak and conjectured that it occurs at time $T \sim (\ln N)/\lambda$, where λ is the largest eigenvalue of the linearized system.

Based on these works, we aim to determine the lower bound for the date of the epidemic peak and the final number of individuals who contract the disease in two epidemiological models. Our first model is a non-structured $SEIR$ epidemic model that includes two latent classes (SE_1E_2IR), while the second model incorporates a continuous age structure for latent infections. The results confirm the conjecture that the peak occurs at approximately $T \sim (\ln N)/\lambda$ where N is the population size and λ is the largest eigenvalue of the linearized system in the first model or the unique positive root of the characteristic equation in the second model. The findings from this study will contribute to the understanding of the dynamics of disease spread and inform public health interventions and planning for future outbreaks.

2. A simple differential equation model

Consider a population of size N subjected to a contagious disease outbreaks. The population is divided into four classes: susceptible $S(t)$, exposed $E(t)$, infectious $I(t)$ or recovered from infection $R(t)$. The following model describes the evolution of this epidemic in time [24, 25]:

$$S'(t) = -aS(t)\frac{I(t)}{N} \tag{2.1}$$

$$E_1'(t) = aS(t)\frac{I(t)}{N} - b_1E_1(t) \tag{2.2}$$

$$E_2'(t) = b_1E_1(t) - b_2E_2(t) \tag{2.3}$$

$$I'(t) = b_2E_2(t) - cI(t) \tag{2.4}$$

$$R'(t) = cI(t) \tag{2.5}$$

In this model, there are two latent (exposed) periods. Individuals remain in the exposed class for a certain latent period, passing through two stages $E_1(t)$ and $E_2(t)$ before becoming infective [24]. Recovered individuals are assumed to be immune and retain their immunity without a time limit. The parameter a ($a > 0$) represents the average number of contacts per unit time that a susceptible individual has with an infectious individual. b_1 ($b_1 > 0$) is the rate of progression from the first exposed class to the second exposed class. b_2 ($b_2 > 0$) is the rate at which the exposed individuals in the second exposed class become infectious and c ($c > 0$) is the rate of recovery. The figure below displays the transfer diagram for the SE_1E_2IR system. Consider the initial conditions

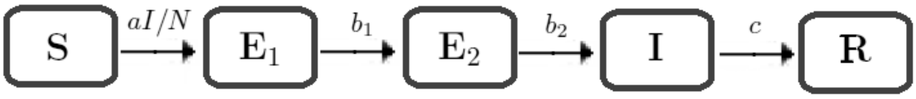

Figure 1. Flow diagram representing the structure of the model described by Eqs (2.1)–(2.5).

$$S(0) = N-(n_{E_1}+n_{E_2}+n_I) > 0,\ E_1(0) = n_{E_1} \geq 0,\ E_2(0) = n_{E_2} \geq 0,\ I(0) = n_I \geq 0,\ R(0) = 0, \tag{2.6}$$

with $n_{E_1} + n_{E_2} + n_I > 0$.

One may argue that an infective individual in a totally susceptible population causes a new infections per unit time, and the mean time spent in the infective compartment is $1/c$. Thus, at the beginning of the epidemic, an infected individual will infect on average $\mathcal{R}_0 = a/c$ secondary cases before entering the compartment R, despite undergoing the latent phase. The parameter $\mathcal{R}_0$ is called the basic reproduction number. It is one of the most important parameters in epidemiology, and it is defined as the average number of secondary cases generated by a single case in a completely susceptible population [26]. The basic reproduction number is an important measure in the spread of infectious diseases, as it helps to predict the potential magnitude and speed of an outbreak. The value of $\mathcal{R}_0$ can vary depending on several factors, including the contagiousness of the disease, the susceptibility of the population, and the effectiveness of public health interventions. If $\mathcal{R}_0 > 1$, then the disease may emerge in the population. However, if $\mathcal{R}_0 < 1$, the disease disappears [27]. We assume throughout this paper that $a > c$, which means that $\mathcal{R}_0 > 1$.

Let us commence by recalling the following proposition, proven via the same method as in the classical $SEIR$ model.

Proposition 1. *The system defined by Eqs* (2.1)–(2.5) *has a unique solution which is defined for all* $t > 0$. *Furthermore,* $S(t) > 0, E_1(t) > 0, E_2(t) > 0, I(t) > 0, R(t) > 0$ *and* $S(t) + E_1(t) + E_2(t) + I(t) + R(t) = N$ *for all* $t > 0$.

Proof

The Cauchy-Lipschitz theorem [28, p. 74] ensures the existence and uniqueness of a solution to the system described by Eqs (2.1)–(2.5) over a maximal interval $[0; T_{\max}]$. Additionally, for all $t \in]0, T_{\max}[$, we have

$$S(t) = S(0)\exp\Big(-\frac{a}{N}\int_0^t I(u)du\Big) > 0$$

since $S(0) > 0$. Let

$$X(t) = \begin{pmatrix} E_1(t) \\ E_2(t) \\ I(t) \end{pmatrix}, \quad F(t) = \begin{pmatrix} -b_1 & 0 & aS(t)/N \\ b_1 & -b_2 & 0 \\ 0 & b_2 & -c \end{pmatrix} \tag{2.7}$$

We then have

$$\frac{dX}{dt} = F(t)X(t) \tag{2.8}$$

We observe that $X(0) \geq 0$ and $X(0) \neq 0$ because $n_{E_1} + n_{E_2} + n_I > 0$, where the inequality $\geq$ between vectors signifies an inequality for all respective components. Additionally, because the off-diagonal entries of the matrix $F(t)$ are non-negative for all $t \in [0; T_{\max})$ and $F(0)$ is irreducible, the system (2.8) is consequently cooperative and irreducible; then, it is strongly monotone [20, 29, 30]. Therefore, for all $t \in]0; T_{\max}[$, $E_1(t) > 0$, $E_2(t) > 0$, and $I(t) > 0$. Since

$$R(t) = c\int_0^t I(u)du$$

we also deduce that $R(t) > 0$ for all $t \in]0; T_{\max}[$. Note that

$$\frac{d}{dt}(S + E_1 + E_2 + I + R) = 0$$

we derive

$$S(t) + E_1(t) + E_2(t) + I(t) + R(t) = S(0) + E_1(0) + E_2(0) + I(0) + R(0) = N$$

and

$$0 < S(t) < N, \ \ 0 < E_1(t) < N, \ \ 0 < E_2(t) < N, 0 < I(t) < N, 0 < R(t) < N$$

for all $0 < t < T_{\max}$. Since the solutions remain bounded on the maximal interval $]0; T_{\max}[$, it follows that $T_{\max} = +\infty$ [31, Corollary A.5]. Therefore, the system described by (2.1)–(2.5) has a unique solution defined for all $t > 0$.

2.1. Final size of the epidemic

The final size of an epidemic, represents the ultimate number of infected individuals in a population during the course of an outbreak. It is a mathematical measure used to evaluate the spread and impact of a disease.

The function $S(t)$ is a non-negative, smooth, and decreasing function which converges to a limit $S_\infty \geq 0$ as $t \to +\infty$. The function $R(t)$ is a non-negative, smooth, and increasing function which converges to a limit $R_\infty \leq N$ as $t \to +\infty$. The sum of Eqs (2.4) and (2.5) is given by

$$(I + R)'(t) = b_2 E_2(t) \geq 0 \tag{2.9}$$

Thus, $I + R$ is a non-negative, smooth, and increasing function, and it converges to a limit as $t \to +\infty$. As $R(t)$ converges, $I(t)$ also converges to a limit $I_\infty \geq 0$. The sum of Eqs (2.3)–(2.5) is given by

$$(E_2 + I + R)'(t) = b_1 E_1(t) \geq 0 \tag{2.10}$$

Therefore, $E_2 + I + R$ is a non-negative and smooth increasing function that is bounded by N; hence it converges to a limit as $t \to +\infty$. As both $R(t)$ and $I(t)$ converge, $E_2(t)$ also converges to a limit $E_{2,\infty} \geq 0$. By utilizing Eqs (2.2)–(2.5), it can be proven that $E_1 + E_2 + I + R$ is a non-negative, smooth, increasing, and bounded function that also converges to a limit as $t \to +\infty$. Consequently, as $E_2(t)$, $R(t)$, and $I(t)$ converge, $E_1(t)$ converges to a limit $E_{1,\infty} \geq 0$.

Let us prove that $I_\infty = E_{2,\infty} = E_{1,\infty} = 0$. Integration of Eq (2.5) gives

$$R(t) = c \int_0^t I(u)\, du \tag{2.11}$$

Because $R(t)$ is bounded by N, $I(t)$ cannot converge to a strictly positive limit; then $I_\infty = 0$.

Similarly, using the sum of the equations for I and R, gives

$$I(t) + R(t) - I(0) = b_2 \int_0^t E_2(u)\, du \tag{2.12}$$

from which we conclude that $E_{2,\infty} = 0$.

By performing a similar calculation using the sum of the equations for E_2, I, and R, we obtain

$$E_2(t) + I(t) + R(t) - E_2(0) - I(0) = b_1 \int_0^t E_1(u)\, du \tag{2.13}$$

it follows that $E_{1,\infty} = 0$.

By utilizing the equations for S and R, Eq (2.5) can be written as

$$R'(t) = \frac{-cN}{aS(t)} S'(t) \tag{2.14}$$

Then

$$R(t) = -\frac{cN}{a} \ln \frac{S(t)}{S(0)} \tag{2.15}$$

we obtain the following at the limit

$$R_\infty = -\frac{cN}{a} \ln \frac{S_\infty}{S(0)} \tag{2.16}$$

this shows that $S_\infty > 0$, and it leads to the equation

$$S_\infty = N - R_\infty = N + \frac{cN}{a} \ln \frac{S_\infty}{S(0)} \tag{2.17}$$

Denote by $x_\infty = \dfrac{S_\infty}{N}$, $x_0 = \dfrac{S(0)}{N}$ and define the function

$$f(x) = 1 - x + \frac{c}{a} \ln \frac{x}{x_0} \tag{2.18}$$

Clearly $f(x_\infty) = 0$ and $0 < x_\infty \leq 1$. In addition

$$\begin{aligned} f'(x) &= \frac{c}{ax} - 1 > 0 \quad \text{if } 0 < x < \frac{c}{a} \\ f'(x) &< 0 \qquad\qquad\quad \text{if } \frac{c}{a} < x < 1 \end{aligned} \tag{2.19}$$

On the other hand

$$\begin{aligned} f(1) &= \frac{c}{a} \ln \frac{1}{x_0} > 0 \\ f(x) &\to -\infty \quad \text{if} \quad x \to 0^+ \\ f\left(\frac{c}{a}\right) &= 1 - \frac{c}{a} + \frac{c}{a} \ln \frac{\frac{c}{a}}{x_0} > 1 - \frac{c}{a} + \frac{c}{a} \ln \frac{c}{a} > 0 \end{aligned} \tag{2.20}$$

The last inequality stems from the assumption that $c/a < 1$ and the property of $g(u) = 1 - u + u \ln u$ being a monotonically decreasing function in the interval $]0, 1[$ with $g(1) = 0$ and $g(u) > 0$ for $u \in]0, 1[$.

As a result, the function f increases from $-\infty$ to $f(\frac{c}{a}) > 0$ on the interval $]0, \frac{c}{a}[$ and decreases from $f(\frac{c}{a}) > 0$ to $f(1) > 0$ on the interval $]\frac{c}{a}, 1[$. Therefore, the equation $f(x) = 0$ has a unique solution in the interval $]0, 1[$, and this solution lies within the interval $]0, \frac{c}{a}[$. We conclude that

$$0 < \frac{S_\infty}{N} < \frac{c}{a} \tag{2.21}$$

2.2. Epidemic peak

To start, clarify the definition of the epidemic peak; we have

$$(E_1 + E_2 + I)'(t) = \left(\frac{aS(t)}{N} - c\right) I(t) \tag{2.22}$$

Suppose that

$$\frac{S(0)}{N} = 1 - \frac{(n_{E_1} + n_{E_2} + n_I)}{N} > \frac{c}{a} \tag{2.23}$$

Under the assumption that $a > c$, this inequality holds as long as N is sufficiently large. Because $I(t) > 0$ for all $t > 0$, the function $S(t)$ is monotonically decreasing from $S(0) > cN/a$ to $S_\infty < cN/a$ on the interval $[0, +\infty[$ also, there exists a unique time $T > 0$ such that $S(T) = cN/a$. According to Eq (2.22), the function $E_1 + E_2 + I$ is monotonically increasing on the interval $[0, T]$

and monotonically decreasing on the interval $[T, +\infty[$. We refer to T as the time of the epidemic peak. It is worth noting that in general, this time does not correspond to the maximum of I or the maximum of E_i ($i = 1, 2$). Using Eq (2.15), we get

$$E_1(T) + E_2(T) + I(T) = N - S(T) - R(T) = N - \frac{cN}{a} + \frac{cN}{a}\ln\frac{\frac{c}{a}}{\frac{S(0)}{N}} \tag{2.24}$$

2.3. Lower bound

Consider the following linear system

$$\begin{aligned} e_1'(t) &= -b_1 e_1(t) + a i(t) \\ e_2'(t) &= b_1 e_1(t) - b_2 e_2(t) \\ i'(t) &= b_2 e_2(t) - c i(t) \end{aligned} \tag{2.25}$$

Define

$$A = \begin{pmatrix} -b_1 & 0 & a \\ b_1 & -b_2 & 0 \\ 0 & b_2 & -c \end{pmatrix} \tag{2.26}$$

Note that the determinant of matrix A is given by $\det(A) = b_1 b_2(a - c) > 0$. Whether A has three real eigenvalues or one real eigenvalue and two complex conjugates, it always has at least one eigenvalue that is strictly positive. Define

$$\sigma(A) = \{\lambda \in \mathbb{C} \mid \det(A - \lambda I) = 0\} \text{ the spectrum of } A$$

and

$$\lambda_+ = \max\{\mathrm{Re}(\lambda) \mid \lambda \in \sigma(A)\} \text{ the stability modulus of matrix } A$$

From Eq (2.2), we have

$$E_1'(t) \leq -b_1 E_1(t) + a I(t) \tag{2.27}$$

Due to the non-negative off-diagonal coefficients of matrix A, we can apply a comparison theorem for cooperative systems [20, Corollary 2.3.1]; we obtain

$$\begin{pmatrix} E_1(t) \\ E_2(t) \\ I(t) \end{pmatrix} \leq e^{tA} \begin{pmatrix} n_{E_1} \\ n_{E_2} \\ n_I \end{pmatrix} \tag{2.28}$$

for all $t \geq 0$, where the inequality between vectors is component by component. Thus

$$\begin{aligned} E_1(T) + E_2(T) + I(T) &\leq (1\ 1\ 1)\, e^{TA} \begin{pmatrix} n_{E_1} \\ n_{E_2} \\ n_I \end{pmatrix} \\ &\leq (n_{E_1} + n_{E_2} + n_I) e^{TA} \end{aligned} \tag{2.29}$$

Because A is a Metzler matrix (the off-diagonal coefficients are non-negative [32]) and

$$A^2 = \begin{pmatrix} b_1^2 & ab_2 & -a(b_1+c) \\ -b_1(b_1+b_2) & b_2^2 & ab_1 \\ b_1b_2 & -b_2(b_2+c) & c^2 \end{pmatrix} \neq 0$$

A is irreducible [33]; thus, λ_+ is an eigenvalue of multiplicity equal to 1 [29, Corollary 4.3.2]. Therefore there exists an invertible matrix X [34] such that

$$A = XJX^{-1} \tag{2.30}$$

We distinguish two cases:

- Case 1: A is diagonalizable. Thus

$$J = \begin{pmatrix} \lambda_+ & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix} \tag{2.31}$$

  and

$$e^{JT} = \begin{pmatrix} e^{\lambda_+ T} & 0 & 0 \\ 0 & e^{\lambda_1 T} & 0 \\ 0 & 0 & e^{\lambda_2 T} \end{pmatrix} \tag{2.32}$$

  Consequently, there exists a constant $k_1 > 0$, independent of T and N, such that

$$\|e^{AT}\| \leq k_1\, e^{\lambda_+ T} \tag{2.33}$$

- Case 2: A is not diagonalizable. Thus

$$J = \begin{pmatrix} \lambda_+ & 0 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{pmatrix} \tag{2.34}$$

  and

$$e^{JT} = \begin{pmatrix} e^{\lambda_+ T} & 0 & 0 \\ 0 & e^{\lambda_1 T} & Te^{\lambda_1 T} \\ 0 & 0 & e^{\lambda_1 T} \end{pmatrix} \tag{2.35}$$

Let us define the function $g(T) = Te^{(\lambda_1 - \lambda_+)T}$; this function increases on the interval $[0, 1/(\lambda_+ - \lambda_1)]$ and decreases on the interval $[1/(\lambda_+ - \lambda_1), +\infty]$. Moreover $g(0) = 0$ and $\lim_{T\to+\infty} g(T) = 0$. Hence

$$g(T) \leq g\Big(\frac{1}{\lambda_+ - \lambda_1}\Big) = \frac{1}{\lambda_+ - \lambda_1} e^{-1} \tag{2.36}$$

We get

$$Te^{\lambda_1 T} \leq \frac{1}{\lambda_+ - \lambda_1} e^{-1+\lambda_+ T} \tag{2.37}$$

Consequently, there exists a constant $k_2 > 0$, independent of T and N, such that

$$\|e^{AT}\| \leq k_2 e^{\lambda_+ T} \tag{2.38}$$

It then follows from Eq (2.29) that

$$E_1(T) + E_2(T) + I(T) \leq (n_{E_1} + n_{E_2} + n_I)\|e^{AT}\| \leq k e^{\lambda_+ T} \tag{2.39}$$

where $k > 0$ is a constant which depends only on a, b_1, b_2, c, n_{E_1}, n_{E_2} and n_I, but not on N. Because $S(0)/N < 1$, it follows that $-\ln(S(0)/N) > 0$. Hence, we obtain from Eq (2.24) the following inequality

$$N\left(1 - \frac{c}{a} + \frac{c}{a}\ln\frac{c}{a}\right) \leq E_1(T) + E_2(T) + I(T) \tag{2.40}$$

In conclusion, based on Eqs (2.39) and (2.40), the following inequalities are derived

$$N(1 - \frac{c}{a} + \frac{c}{a}\ln\frac{c}{a}) \leq E_1(T) + E_2(T) + I(T) \leq k e^{\lambda_+ T} \tag{2.41}$$

Finally, there exists another constant $K \in \mathbb{R}$ that only depends on the parameters a, b_1, b_2, c, n_{E_1}, n_{E_2} and n_I, but not on N, such that

$$T \geq \frac{\ln N}{\lambda_+} + K \tag{2.42}$$

This lower bound supports the following conjecture

$$T \sim \frac{\ln N}{\lambda_+}, \quad N \to \infty \tag{2.43}$$

3. A model with age of infection

In the previous section, a model was presented by using ordinary differential equations to describe the progression of an infection. This section extends that model by stratifying the latency phase based on the time since infection. The new model tracks the time since infection, represented by the variable x, and considers individuals who have been in the exposed class for a duration of x. These individuals move to the infected class at a rate of $b(x)$, which is a non-negative function of x. The model is described by the following system, with notation from the previous section

$$\begin{cases} S'(t) = -aS(t)\dfrac{I(t)}{N} \\ \dfrac{\partial E}{\partial t} + \dfrac{\partial E}{\partial x} = -b(x)E(t,x) \\ E(t,0) = aS(t)\dfrac{I(t)}{N} \\ I'(t) = \int_0^\infty b(x)E(t,x)\,dx - cI(t) \\ R'(t) = cI(t) \end{cases} \tag{3.1}$$

The following assumptions are made regarding the parameters of the system:

(H1) $a, c > 0$.

(H2) $b \in L^{\infty}(\mathbb{R}^{+})$, with the essential upper bound $\bar{b}$.

(H3) We suppose that there exist $B > 0$ and $X > 0$ such that for all $x > X$, $b(x) > B$.

The phase space for the system is defined as $Y = \mathbb{R}^{+} \times L^{1}_{+} \times \mathbb{R}^{+} \times \mathbb{R}^{+}$, where L^{1}_{+} represents the space of non-negative and integrable functions on $(0, \infty)$, with a norm defined by

$$\|(y_1, y_2, y_3, y_4)\|_Y = |y_1| + \int_0^{\infty} |y_2(x)| dx + |y_3| + |y_4|$$

Biologically, this norm gives the total population size N. In our case, $N = S + \int_0^{\infty} E(t, x) dx + I + R$ which is constant. The initial condition for Eq (3.1) is

$$(S(0), E(0, .), I(0), R(0)) \in Y$$

where

$$S(0) = N - n_E - n_I > 0, \quad \int_0^{\infty} E(0, x)\, dx = n_E \geq 0, \quad I(0) = n_I \geq 0, \quad R(0) = 0 \tag{3.2}$$

with $n_E + n_I > 0$.
Let $\pi(x)$ denote the probability of survival in the exposed class until age x, where for $x \geq 0$, $\pi(x)$ is defined as

$$\pi(x) = e^{-\int_0^x b(s) ds} \tag{3.3}$$

3.1. Final size of the epidemic

It can be proved, as done in Subsection 2.1, that

$$S_{\infty} = \lim_{t \to \infty} S(t) > 0 \text{ and } R_{\infty} = \lim_{t \to \infty} R(t) < N \tag{3.4}$$

The sum of the equations for I and R gives

$$(I + R)'(t) = \int_0^{\infty} b(x) E(t, x)\, dx \geq 0 \tag{3.5}$$

Hence $I + R$ is an increasing function that is bounded by N; also, it follows that it has a limit. Because $R(t)$ converges, $I(t)$ also converges to a limit of $I_{\infty} \geq 0$.
Therefore, because $R(t) = c \int_0^t I(s)\, ds$ is bounded by N, $I(t)$ cannot converge to a positive limit and instead reaches $I_{\infty} = 0$. Furthermore,

$$\Big(\int_0^{\infty} E(t, x)\, dx + I(t) + R(t) \Big)' = aS(t) \frac{I(t)}{N} \geq 0 \tag{3.6}$$

Therefore, because $\int_0^{\infty} E(., x)\, dx + I + R$ is a increasing and bounded function, it converges to a limit. Because I and R also converge, it can be concluded that $\int_0^{\infty} E(t, x)\, dx$ converges to a limit as well.

Next, we will prove that $\lim\limits_{t\to\infty}\int_0^\infty E(t,x)\,dx = 0$.
Integrating the second equation of Eq (3.1) with the boundary equation and initial condition, yields

$$E(t,x) = \begin{cases} E(t-x,0)\pi(x) & t > x \\ E(0,x-t)\dfrac{\pi(x)}{\pi(x-t)} & t \le x \end{cases} \tag{3.7}$$

It follows that

$$\int_0^\infty E(t,x)\,dx = \int_0^t E(t,x)\,dx + \int_t^\infty E(t,x)\,dx = \int_0^t E(t-x,0)\pi(x)\,dx + \int_t^\infty E(0,x-t)\frac{\pi(x)}{\pi(x-t)}\,dx$$
$$= G_1(t) + G_2(t) \tag{3.8}$$

where $G_1(t) = \int_0^t E(t-x,0)\pi(x)\,dx$ and $G_2(t) = \int_t^\infty E(0,x-t)\dfrac{\pi(x)}{\pi(x-t)}\,dx$
It follows that

$$\lim_{t\to\infty}\int_0^\infty E(t,x)\,dx = \lim_{t\to\infty} G_1(t) + \lim_{t\to\infty} G_2(t) \tag{3.9}$$

We have the following inequalities

$$0 \le G_1(t) \le \int_0^\infty E(t-x,0)\pi(x)\,dx \tag{3.10}$$

then

$$0 \le \lim_{t\to\infty} G_1(t) \le \lim_{t\to\infty}\int_0^\infty E(t-x,0)\pi(x)\,dx \tag{3.11}$$

Bacause $\lim\limits_{t\to\infty} I(t) = 0$, it follows that

1) $\lim\limits_{t\to\infty} E(t-x,0)\pi(x) = \lim\limits_{t\to\infty} a\frac{S(t-x)}{N} I(t-x)\pi(x) = 0$

2) $E(t-x,0)\pi(x) = a\frac{S(t-x)}{N} I(t-x)\pi(x) \le aN\pi(x)$; then, $|E(t-x,0)\pi(x)| \le aN\pi(x)$

Therefore, by the dominated convergence theorem [35], we obtain

$$\lim_{t\to\infty}\int_0^\infty E(t-x,0)\pi(x)\,dx = \int_0^\infty \lim_{t\to\infty} E(t-x,0)\pi(x)\,dx = 0 \tag{3.12}$$

On the other hand, we have

$$G_2(t) = \int_t^\infty E(0,x-t)\frac{\pi(x)}{\pi(x-t)}\,dx = \int_0^\infty E(0,x)\frac{\pi(x+t)}{\pi(x)}\,dx \tag{3.13}$$

Using (H3), we obtain

$$G_2(t) = \int_0^\infty E(0,x)e^{-\int_x^{x+t} b(s)\,ds}\,dx \le \int_0^\infty E(0,x)\,dx e^{-Bt} \tag{3.14}$$

It can be deduced that

$$\lim_{t\to\infty} G_2(t) = 0 \tag{3.15}$$

To summarize, we have established that

$$\lim_{t\to\infty}\int_0^\infty E(t,x)\,dx = 0 \tag{3.16}$$

Bacause $N = S + \int_0^\infty E(t,x)\,dx + I + R$, we obtain, at the limit, that $S_\infty + R_\infty = N$.
By utilizing arguments similar to those in Subsection 2.1, it can be demonstrated that

$$0 < \frac{S_\infty}{N} < \frac{c}{a} \tag{3.17}$$

3.2. Epidemic peak

First, clarify the definition of the epidemic peak; we have

$$\Big(\int_0^\infty E(t,x)\,dx + I(t)\Big)' = \Big(\frac{aS(t)}{N} - c\Big)I(t) \tag{3.18}$$

Assume that $\frac{S(0)}{N} = 1 - \frac{(n_E + n_I)}{N} > \frac{c}{a} < 1$, which is true for sufficiently large N; we have that $I(t) > 0$ for all $t > 0$. The function $S(t)$ decreases monotonically from $S(0) > \frac{Nc}{a}$ to $S_\infty < \frac{Nc}{a}$ on the interval $[0, +\infty[$. Therefore, there exists a unique time $T > 0$ such that $S(T) = \frac{cN}{a}$. As in Subsection 3.2, the time of the epidemic peak is referred to as T.
We have

$$\int_0^\infty E(T,x)\,dx + I(T) = N - S(T) - R(T) = N - \frac{cN}{a} + \frac{cN}{a}\ln\frac{\frac{c}{a}}{\frac{S(0)}{N}} \tag{3.19}$$

3.3. Lower bound

Let $(S(t), E(t,x), I(t), R(t))$ be a solution of Eq (3.1) with the initial condition given by Eq (3.2). The fact that $S/N \le 1$ implies that $(E(t,x), I(t))$ satisfies the following conditions:

$$\begin{cases} \dfrac{\partial E}{\partial t} &= -\dfrac{\partial E}{\partial x} - b(x)E(t,x) \\ E(t,0) &\le aI(t), \\ I'(t) &= \int_0^\infty b(x)E(t,x)\,dx - cI(t) \end{cases} \tag{3.20}$$

Let $(\bar{E}(t,x), \bar{I}(t))$ be the solution of the following auxiliary system

$$\begin{cases} \dfrac{\partial \bar{E}}{\partial t} &= -\dfrac{\partial \bar{E}}{\partial x} - b(x)\bar{E}(t,x) \\ \bar{E}(t,0) &= a\bar{I}(t) \\ \bar{I}'(t) &= \int_0^\infty b(x)\bar{E}(t,x)\,dx - c\bar{I}(t) \end{cases} \tag{3.21}$$

with $(\bar{E}(0,x), \bar{I}(0)) = (E(0,x), I(0))$.
Let $F(t,x) = \bar{E}(t,x) - E(t,x)$, $J(t) = \bar{I}(t) - I(t)$; then, $(F(t,x), J(t))$ satisfies

$$\begin{cases} \dfrac{\partial F}{\partial t} & = -\dfrac{\partial F}{\partial x} - b(x)F(t,x) \\ F(t,0) & = aJ(t) + W(t) \\ J'(t) & = \int_0^\infty b(x)F(t,x)\,dx - cJ(t) \\ F(0,x) & = J(0) = 0 \end{cases} \tag{3.22}$$

where $W(t) = aI(t)(1 - \frac{S(t)}{N}) \geq 0$.
Next, we need to show that $(F, J) \geq (0, 0)$.

The integration of the first equation of Eq (3.22) with the specified boundary and initial conditions yields

$$F(t,x) = F(t-x, 0)\pi(x) \tag{3.23}$$

The integration of the equation for J in Eq (3.22), combined with Eq (3.23), yields

$$J(t) = \int_0^t e^{-c(t-s)} \int_0^t b(x)F(s-x,0)\pi(x)\,dx\,ds \tag{3.24}$$

Define $B(t) = F(t,0)$; then,

$$B(t) = \int_0^t \int_0^t \Psi(x)B(s-x)\,dx e^{-c(t-s)}\,ds + W(t) \tag{3.25}$$

where $\Psi(x) = ab(x)\pi(x)$.
Let us consider the following sequence

$$\begin{cases} B_{n+1}(t) = \int_0^t \int_0^t e^{-c(t-s)}\Psi(x)B_n(s-x)\,dx\,ds + W(t) \quad n \in \mathbb{N} \\ B_0(t) = W(t) \geq 0 \end{cases} \tag{3.26}$$

Because B_0, Ψ and W are non-negative and continuous functions, by the induction method, we can prove that it is the same for B_n, for all $n \geq 0$.

Next, we prove the convergence of $\{B_n(t)\}_n$ to $B(t)$ for all $t \in [0, \tilde{T}]$ by using the contraction mapping principle. For this purpose, we introduce a variable

$$\bar{B}_n(t) = e^{-\bar{\lambda}t}B_n(t) \text{ for some } \bar{\lambda} > 0 \tag{3.27}$$

Multiplying both sides of Eq (3.26) by $e^{-\bar{\lambda}t}$ yields

$$\bar{B}_{n+1}(t) = \int_0^t \int_0^t \Psi(x)\bar{B}_n(s-x)e^{\bar{\lambda}(s-x)}dx e^{-c(t-s)-\bar{\lambda}t}\,ds + \bar{W}(t) \tag{3.28}$$

where $\bar{W}(t) = e^{-\bar{\lambda}t}W(t)$.

It can be deduced that if convergence of $\bar{B}_n(t)$ to $\bar{B}(t)$ for any $t \in [0, \tilde{T}]$ is established, then $B_n(t)$ will converge to $B(t)$.
For any $n \in \mathbb{N}$, we have

$$\begin{aligned} \|\bar{B}_{n+1}(t) - \bar{B}_n(t)\|_\infty &\leq \int_0^t \int_0^t \Psi(x)|\bar{B}_n(s-x) - \bar{B}_{n-1}(s-x)|e^{\bar{\lambda}(s-x)}\,dx e^{-c(t-s)-\bar{\lambda}t}\,ds \\ &\leq \int_0^t \int_0^t \Psi(x)e^{\bar{\lambda}(s-x)}\,dx e^{-c(t-s)-\bar{\lambda}t}\,ds \|\bar{B}_n(s-x) - \bar{B}_{n-1}(s-x)\|_\infty \end{aligned} \tag{3.29}$$

or $\int_0^t \int_0^t \Psi(x)e^{\bar{\lambda}(s-x)}\,dx e^{-c(t-s)-\bar{\lambda}t} ds \leq \dfrac{a\bar{b}}{\bar{\lambda}^2} := M_{\bar{\lambda}}$ for $\bar{\lambda} > 0$, where $\bar{b}$ is defined in (H2).
Substituting this estimate into Eq (3.29) yields

$$\|\bar{B}_{n+1}(t) - \bar{B}_n(t)\|_\infty \leq M_{\bar{\lambda}}\|\bar{B}_n(t) - \bar{B}_{n-1}(t)\|_\infty \leq M_{\bar{\lambda}}^n\|\bar{B}_1(t) - \bar{B}_0(t)\|_\infty \tag{3.30}$$

As a result, for any $m, n \in \mathbb{N}$ $(m > n)$ and $\bar{\lambda}$ that is sufficiently large,

$$\begin{aligned} \|\bar{B}_m(t) - \bar{B}_n(t)\|_\infty &\leq \|\bar{B}_m(t) - \bar{B}_{m-1}(t)\|_\infty + ... + \|\bar{B}_{n+1}(t) - \bar{B}_n(t)\|_\infty \\ &\leq (M_{\bar{\lambda}}^{m-1} + ... + M_{\bar{\lambda}}^n)\|\bar{B}_1(t) - \bar{B}_0(t)\|_\infty \\ &\leq \frac{M_{\bar{\lambda}}^{m-n}}{1 - M_{\bar{\lambda}}}\|\bar{B}_1(t) - \bar{B}_0(t)\|_\infty \end{aligned} \tag{3.31}$$

We choose $\bar{\lambda}$ large enough such that $M_{\bar{\lambda}} < 1$.
Therefore, as m and n approach infinity, $\|\bar{B}_m(t) \to \bar{B}_n(t)\|_\infty$ tends to zero, which means that $\{\bar{B}_n(t)\}_n$ is a Cauchy sequence. It follows that $\bar{B}_n(t)$ converges to $\bar{B}(t)$; thus, $B_n(t)$ also converges to $B(t)$ as n approaches infinity.
Let

$$\bar{N}(t) = \int_0^\infty F(t, x)\,dx + J(t) \tag{3.32}$$

for $t \in [0, \tilde{T})$; we have

$$\bar{N}'(t) = (a - c)J(t) + W(t) \leq (a - c)\bar{N}(t) + aN \tag{3.33}$$

for $t \in [0, \tilde{T})$. By using the standard comparison lemma [36], we can conclude that

$$\bar{N}(t) \leq \frac{aN}{a - c}\left(e^{(a-c)t} - 1\right) \tag{3.34}$$

which implies that $\lim_{t \to \tilde{T}^-} \sup \bar{N}(t) < \infty$. Thus, the maximal interval of existence for the solution of Eq (3.22) is $\mathbb{R}_+$.
To summarize, we have shown that the solution of Eq (3.22) is non-negative, which implies that

$$E(t, x) \leq \bar{E}(t, x), \quad I(t) \leq \bar{I}(t) \quad \forall\, t \geq 0,\ x \geq 0 \tag{3.35}$$

When looking for solutions of Eq (3.21) of the form $\bar{E}(t,x) = u(x)e^{\lambda t}$ and $\bar{I}(t) = ve^{\lambda t}$ where λ is constant, by substituting in Eq (3.21), we obtain the following:

$$\begin{cases} \lambda u &= -\dfrac{\partial u}{\partial x} - b(x)u(x) \\ u(0) &= av \\ \lambda v &= \int_0^\infty b(x)u(x)\,dx - cv \end{cases} \tag{3.36}$$

Solving the first equation of Eq (3.36) yields

$$u(x) = ave^{-\lambda x - \int_0^x b(s)ds} \tag{3.37}$$

By inserting Eq (3.37) into the last equation of Eq (3.36), we obtain the following characteristic equation,

$$a\int_0^\infty b(x)e^{-\lambda x - \int_0^x b(s)\,ds}\,dx = \lambda + c \tag{3.38}$$

The characteristic equation can be rewritten in the following form:

$$\mathcal{R}(\lambda) = 1 \tag{3.39}$$

where

$$\mathcal{R}(\lambda) = \frac{a\int_0^\infty b(x)e^{-\lambda x - \int_0^x b(s)\,ds}\,dx}{\lambda + c} \tag{3.40}$$

We define the basic reproduction number by

$$\mathcal{R}_0 = \mathcal{R}(0) = \frac{a}{c}(1 - e^{-\int_0^\infty b(x)\,dx}) \tag{3.41}$$

For $\lambda < -c$, $\mathcal{R}(\lambda)$ is negative and the characteristic equation Eq (3.39) does not have a solution.

For $\lambda > -c$, $\mathcal{R}(\lambda)$ is a decreasing function of λ which approaches ∞ as $\lambda \to -c$ and zero as $\lambda \to \infty$. As a result, the characteristic equation given by Eq (3.39) has a unique real solution λ^*. In addition, if $\mathcal{R}_0 > 1$, this real solution satisfies that $\lambda^* > 0$.
Therefore, the solution of Eq (3.21) is given by:

$$\bar{E}(t,x) = ave^{\lambda^*(t-x) - \int_0^x b(s)ds},\ \bar{I}(t) = ve^{\lambda^* t} \tag{3.42}$$

We denote $u(x) = ave^{-\lambda^* x - \int_0^x b(s)\,ds}$. Using Eq (3.35), we get

$$\begin{aligned} \int_0^\infty E(t,x)\,dx + I(t) &\le \int_0^\infty u(x)\,dxe^{\lambda^* t} + ve^{\lambda^* t} \\ &\le \Big(\int_0^\infty u(x)\,dx + v\Big)e^{\lambda^* t} \\ &\le ke^{\lambda^* T} \end{aligned} \tag{3.43}$$

where k is a positive constant that is independent of both N and T.
Because $S(0)/N < 1$, we have that $-\ln(S(0)/N) > 0$. We can deduce the following inequality from Eq (3.19):

$$N(1 - \frac{c}{a} + \frac{c}{a}\ln(\frac{c}{a})) \leq \int_0^\infty E(T,x)\,dx + I(T) \tag{3.44}$$

Therefore, from Eqs (3.43) and (3.44), we arrive at the following inequalities:

$$N\Big(1 - \frac{c}{a} + \frac{c}{a}\ln(\frac{c}{a})\Big) \leq \int_0^\infty E(T,x)\,dx + I(T) \leq ke^{\lambda^* T} \tag{3.45}$$

Hence, there exists another constant $K \in \mathbb{R}$ that is independent of both N and T such that

$$T \geq \frac{\ln N}{\lambda^*} + K \tag{3.46}$$

This lower bound suggests the following conjecture

$$T \sim \frac{\ln N}{\lambda^*}, \quad N \to \infty \tag{3.47}$$

4. Numerical simulations

4.1. Example 1

We have chosen the parameters $a = 9/10$, $b_1 = 3/4$, and $b_2 = 2/5$, with the initial conditions $n_{E_1} = n_{E_2} = 0$ and $n_I(0) = 1$. We have also chosen three values of the recovery rate c such that $\mathcal{R}_0 = a/c \in \{2, 2.5, 4.5\}$. We have considered various values of the population N ranging from 10^3 to 10^6. We have solved the system of equations for SE_1E_2IR and determined the peak time T, which is the time at which the number of cases reaches its maximum, represented by the sum of E_1, E_2, and I. Figure 2 depicts the variation of T as a function of $\ln N$ (represented by solid lines), as well as $(\ln N)/\lambda_+$ (represented by small circles). The curves appear to align, which is consistent with the conjecture. Additionally, Figure 2 suggests that the next term in the asymptotic expansion of T is a constant, which is negative when $\mathcal{R}_0$ is close to 1 and becomes positive as $\mathcal{R}_0$ increases.

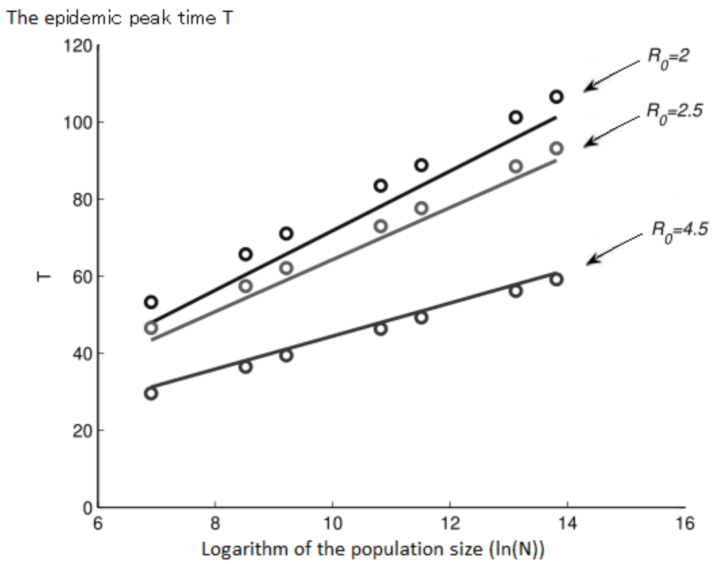

Figure 2. Epidemic peak time in the SE_1E_2IR model as a function of $\ln N$ with solid lines, and $(\ln N)/\lambda_+$ with small circles.

4.2. Example 2

For the model given by Eq (3.1), we have applied the following:

$$b(x) = \begin{cases} 0 & \text{if } x \leq 1 \\ 0.75 & \text{if } x > 1 \end{cases}$$

The value of a is set to $9/10$, n_E is zero, and $n_I(0) = 1$. We have also considered three different recovery rates c to obtain $\mathcal{R}_0$ values of $\{2.5, 3, 4.5\}$. The population N is varied between 10^3 and 10^8. We have solved Eq (3.1) and determined the peak time T, which is the time at which the maximum of $\int_0^\infty E(t,x)dx + I(t)$ is reached. Figure 3 displays the relationship between T and $\ln N$ with continuous lines, and $(\ln N)/\lambda^*$ with small circles. The data appear to indicate that the two sets of values coincide, in accordance with the conjecture.

5. Conclusions

The calculation of the peak epidemic is important because it gives information about the timing and severity of an outbreak. Assuming that the contact coefficient already incorporates certain health restrictions, having an idea of the peak date may allow one to at least predict the order of magnitude of the duration of these restrictions (weeks or months).

In this work, we have established a lower bound for the peak date of two epidemic models. The first model is an SE_1E_2IR epidemic model with two latent categories. These categories encompass two chains of different lengths and are placed between the susceptible and infectious compartments. The second model includes a continuous age structure for latently infected individuals. We have demonstrated that the formula for the lower bound of the peak date is given by $T \sim (\ln N)/\lambda$, where N is the population size and λ represents the largest eigenvalue of the

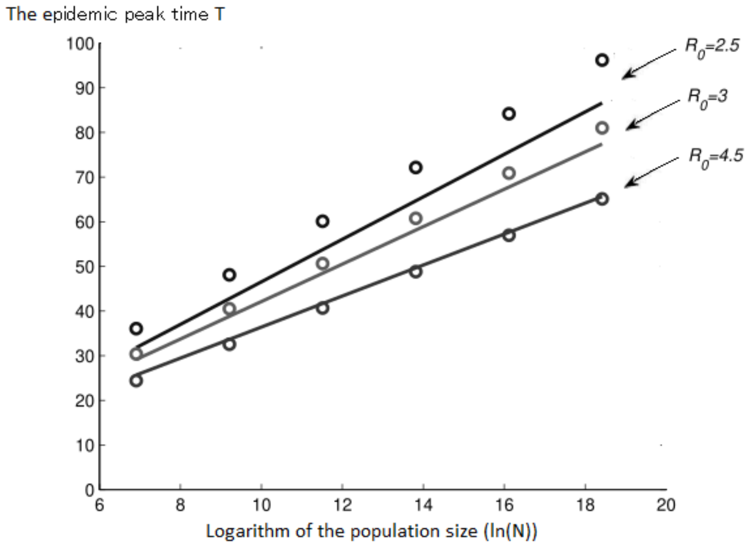

Figure 3. Epidemic peak time in the age-structured SEIR model as a function of $\ln N$ with solid lines and $(\ln N)/\lambda^*$ with small circles.

linearized system for the first model, and the unique positive root of the characteristic equation given by Eq (3.39) for the second model.

Although we were unable to prove that the epidemic peak is reached at $T \sim (\ln N)/\lambda$, our numerical findings support this conjecture. Looking ahead, a future avenue for this research involves endeavors to formally demonstrate this equality and extend the technique employed in this work to determine the date of the epidemic peak for other models that are more general, complex, and realistic.

Use of AI tools declaration

The authors declare that they have not used artificial intelligence tools in the creation of this article.

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Conflict of interest

The authors declare that there is no conflict of interest.

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