Mathematical Biosciences and Engineering

http://www.aimspress.com/journal/mbe

MBE, 21(2): 2233–2253.
DOI: 10.3934/mbe.2024098
Received: 29 November 2023
Revised: 02 January 2024
Accepted: 04 January 2024
Published: 12 January 2024

Research article

Input-to-state stability of stochastic nonlinear system with delayed impulses

Linni Li and Jin-E Zhang*

School of Mathematics and Statistics, Hubei Normal University, Huangshi 435002, China

* **Correspondence:** Email: zhang86021205@163.com; Tel: +867146571069; Fax: +867146571069.

Abstract: Stochastic input-to-state stability (SISS) of the stochastic nonlinear system has received extensive research. This paper aimed to investigate SISS of the stochastic nonlinear system with delayed impulses. First, when all subsystems were stable, using the average impulsive interval method and Lyapunov approach, some theoretical conditions ensuring SISS of the considered system were established. The SISS characteristic of the argumented system with both stable and unstable subsystems was also discussed, then the stochastic nonlinear system with multiple delayed impulse jumps was considered and SISS property was explored. Additionally, it should be noted that the Lyapunov rate coefficient considered in this paper is positively time-varying. Finally, several numerical examples confirmed validity of theoretical results.

Keywords: stochastic nonlinear system; input-to-state stability; delayed impulses; multiple jumps

1. Introduction

In the operating environment of actual control systems, system models will be disturbed by random factors such as perturbations of internal parameters, control inputs, external environment, random errors of state measurements, etc, and it will be very difficult to govern the real system by the deterministic model. In order to describe the real system more accurately, It$\hat{o}$ stochastic differential equation with stochastic perturbation was proposed [1], which promoted the development of the random system. In the subsequent analysis of the random system, researchers have used the It$\hat{o}$ formula as the most basic mathematical tool and the Lyapunov functional and differential inequality as the most commonly used analytical method to further explore random systems with different characteristics, such as linear random systems [2,3], nonlinear random systems [4,5], event-triggered random systems [6,7], and random multi-agent systems [8,9].

Since perturbations inevitably exist in practical engineering, it makes theoretical analysis of the system more complicated. Therefore, stability analysis of nonlinear systems under the environment of external perturbations has become an important issue in the field of control research. In 1989, input-to-state stability (ISS) was first proposed by Sontag [10]. This concept well depicts the influence of external inputs on the system, which has become one of the most important topics in the study of stability. Since then, many researchers have been studied ISS properties of diverse dynamical systems, such as stochastic systems [11,12], switch systems [13,14], discrete-time systems [15,16], and hybrid systems [17,18]. Among them, since an actual control system is more or less subject to the interference of external input, ISS of the stochastic system also has captured widespread attention, for instance, reference [11] investigated stochastic input-to-state stability (SISS) of the stochastic switched system. ISS was considered for stochastic delayed systems with Markov-switching in [12].

Implusive systems, characterized by a sudden change of state at some point, are also spread across the fields of complex networks, neural networks, etc (see [19–21]). In [21], hypothetical conditions for the ISS of a system with impulse effects were presented for the first time. On the other hand, in reality, due to the presence of stochastic perturbations, it's meaningful to study the ISS of the stochastic system with impulse effects (see [22–24]). Some ISS results for impulsive stochastic nonlinear systems were obtained based on the hypothesis [21], where the Lyapunov rate coefficient is constant, indicating that all subsystems are stable or unstable. Lately, references [25,26] discuss systems with both stable and unstable subsystems; nevertheless, here, impulse is not taken into account. Thus, it is necessary to consider impulsive stochastic systems with both stable and unstable subsystems. Meanwhile, in practical applications, time delay is unavoidable to occur during the transmission of impulses, which may lead to system instability, oscillation, and poor performance, so some interesting studies on impulses with time delay (called delayed impulse) have attracted great attention. Delayed impulse describes the phenomenon that the transient of impulse depends not only on the current state of the system but also on the historical state of the system. It can be found in many applications, for example, communication security systems where there are delayed impulses containing sampling delays; the design of impulse controllers for fishing strategies in fishing models; planar localization in submarine positioning systems, etc, where the impulses that occur are subject to a lag phenomenon. Delayed impulse phenomenon is widespread, so the study of delayed impulse is also of great practical significance. Until now, many works have been discovered on ISS for systems with delayed impulses. For example, reference [27] explored integral input-to-state stability (iISS) and ISS of deterministic systems with delayed impulses. Reference [28] investigated ISS of systems with delayed-dependent impulses. Reference [29] developed ISS and iISS of systems with distributed-delayed impulses. When the continuous part of the system was stable and unstable, ISS of the stochastic system with delayed impulses was discussed in [30]. As far as we know, there are few works to explore ISS of stochastic systems with delayed impulses, so we will focus on this issue.

In practical applications, different types of impulse jumps affect ISS characteristic: Some impulse jumps can promote stability, while others can disrupt stability. This requires a more flexible tool that calculates the effect caused by different impulse jumps in a precise way. For example, assume that the quantity of barreled water in a shop is directly proportional to the total quantity, decreasing continuously with a certain rate coefficient. Whereas, every odd day, trucks take away double the bottled water, and every even day, trucks carry away 60 percent of the bottled water. Reference [31] can well characterize the evolution of this process as an impulse system with multiple jump graphs, and two constants called rate coefficients are used to depict the behavior of the ISS-Lyapunov function along the trajectory of impulse systems during the flow and impulse jump. Meanwhile, a positive value of the rate coefficient corresponds to the case where the flow/jump has a positive effect on the

ISS characteristics and vice versa. Inspired by [31,32], to popularize deterministic results into the stochastic system, reference [33] developed SISS for stochastic systems with both multiple impulses and switch jumps, but time delay was ignored. Therefore, it should be devoted to analyzing stochastic nonlinear systems with multiple delayed impulse jumps, which is highly vital.

However, some ISS results for impulsive stochastic nonlinear systems were obtained based on the hypothesis [21], where the Lyapunov rate coefficient is constant, but such an assumption does not always hold in practice because subjecting to fluctuations in the system performance may cause the Lyapunov rate coefficient to be time-varying, i.e., they are time-dependent functions. Thus, the study of ISS for a time-varying system is more relevant than that of a constant system. Recently, ISS for a time-varying nonlinear system is considered in [34], but derivation contains several restrictive assumptions and the system is independent of random disturbances and impulses, making the results conservative. References [27,33,35] consider ISS for an impulsive stochastic nonlinear system, but their Lyapunov rate coefficients are positively constant, which is also a more stringent condition. ISS for an impulsive stochastic nonlinear time-varying system is considered in [36], but time delay is neglected, limiting the application of the obtained results. Thus, it is challenging to find out how to extend positive constant of Lyapunov rate coefficients to the the positive time-varying case, as well as considering the effect of delayed impulse, which will undoubtedly extend application of the related studies.

Enlightened by preceding discussions, this paper will investigate SISS for nonlinear stochastic systems with delayed impulses. The following three points can be considered as the main contributions of this paper: (i) In [27], Li et al. developed some ISS property for nonlinear systems with delayed impulses. In this paper, we further generalize it to a stochastic system and also consider a stochastic nonlinear system with both stable and unstable subsystems. (ii) Compared with [17,23,36], we add a time lag to the impulse term, i.e., delayed impulse, making the results more flexible. Meanwhile, according to [31,32], some criteria are explored ensuring SISS for deterministic systems with multiple impulse jumps. The generalization of deterministic results to stochastic systems is necessary. (iii) Conditions in these cases derived in this paper relax restrictions in [27,33,35], that is, the Lyapunov rate coefficient is positively time-varying instead of constant. Thereby, the derived discriminant rule is less conservative.

2. Problem formulation and preliminaries

2.1. Model

Consider stochastic nonlinear system with delayed impulses:

$$\begin{aligned} dz(t) &= \daleth(t, z(t), v(t))dt + \eth(t, z(t), v(t))d\mathfrak{w}(t),\ t \neq t_r,\ t \geq t_0, \\ z(t_r) &= \S(z(t_r^- - \tau), v(t_r^-)),\ r \in N^+, \\ z(s - t_0) &= \psi_s,\ s \in [t_0 - \tau, t_0] \end{aligned} \tag{1}$$

where ψ_s is the initial state; $N^+ = \{1, 2, \cdots\}$; $z(t) \in R^m$ is the system state; R^m is the m-dimensional space; $v(t) \in \mathbb{L}_\infty^>$ is the input; $\mathbb{L}_\infty^>$ represents all locally essentially bounded sets with norm $\|v(t)\|_{[t_0,t]} = sup_{s\in[t_0,t]}\|v(s)\|$; and $\tau > 0$ is constant delay. Let $(\Omega, \hbar, P)$ represent a complete probability space with a filtration $\{\hbar_t\}_{t\geq 0}$ satisfying usual conditions, and $\mathfrak{w}(t) = (\mathfrak{w}_1(t), \mathfrak{w}_2(t), \cdots, \mathfrak{w}_m(t))^T$ represents an m-dimensional Brownian motion in this space. E is relative to the expectation operator of probability measure P. Functions $\beth : [t_0, +\infty] \times R^n \times R^m \longrightarrow R^n$, $\eth : [t_0, +\infty] \times R^n \times R^m \longrightarrow R^{n\times m}$ and $\S$: $[t_0, +\infty] \times R^n \times R^m \longrightarrow R^n$ are Borel measurable and $R^{m\times n}$ is a $m \times n$-dimensional real matrix space. Impulsive sequence $\{t_r\}$ satisfies $0 = t_0 < t_1 < \cdots < t_r < \cdots$ and $\lim_{r\to+\infty} t_r = +\infty$.

If functions $\beth$, $\eth$, and k satisfy the Lipschitz condition [37], let $\varsigma = \psi_s$. System (1) has a unique solution $z(t, t_0, \varsigma)$. If $\beth(t, 0, 0) \equiv 0$, $k(t, 0, 0) \equiv 0$, $\eth(t, 0, 0) \equiv 0$, $\forall t \geq t_0$, then, system (1) has a trivial solution $z(t) \equiv 0$.

Denote by $\mathfrak{C}^{1,2}$ the set of all nonnegative functions $F : [t_0, +\infty) \times R^n$, which are continuously once differentiable in t and twice in x. If $F \in \mathfrak{C}^{1,2}$, define $\mathcal{L}F : [t_0, +\infty) \times R^n$ [23]:

$$\mathcal{L}F(t,z) = \frac{\partial F(t,z)}{\partial t} + \frac{\partial F(t,z)}{\partial z}\beth(t,z,v(t)) + \frac{1}{2}tr[\eth^T(t,z,v(t))\frac{\partial^2 F(t,z)}{\partial z^2}\eth(t,z,v(t))]$$

2.2. Definitions

Definition 1. *[38] A function $\Im : R^+ \to R^+$ is said to be of class κ if $\Im$ is continuous strictly increasing with $\Im(0) = 0$. κ_∞ is a radially unbounded subset of κ. $\upsilon\kappa_\infty$ is called a convex subset of κ_∞. A function $\aleph : R^+ \times R^+ \to R^+$ is of class $\kappa\iota$ if $\aleph(\cdot, u) \in \kappa$ for every fixed $u \geq 0$, and $\aleph(e, u)$ decreases to 0 as $u \to +\infty$ for every fixed $e \geq 0$.*

Definition 2. *[35] System* (1) *is said to be SISS if $\forall\Lambda > 0$ and functions $\xi \in \kappa\iota$ and $\eta \in \kappa_\infty$ satisfy:*

$$P\left\{|z(t)| < \xi(\|\psi\|_\tau, t - t_0) + \eta(\|v\|_{[t_0,t]})\right\} \geq 1 - \Lambda,\ t \geq t_0$$

where $\|\psi\|_\tau = sup_{[t_0-\tau,t_0]}|\psi|$.

Definition 3. *[27] There exists constants θ, Δ, $d \in R^+$, and function $\wp_\theta^{\Delta,d} = \{\wp \in C(R^+, R)\}$ satisfying:*
(i) $\wp(b_1) + \wp(b_2) \leq \wp(b_1 + b_2) + \Delta$, for $\forall b_1, b_2 \in R^+$;
(ii) $2\Delta - d < \wp(\tau)$;
(iii) $\wp(b) + \theta b < d$ and tends towards $-\infty$ as $b \to +\infty$.

Definition 4. *[21] For impulsive time $\{t_r\}_{r\in N^+}$, $\mathcal{N}(t, g)$ is the number of impulses in $(g, t]$. If*

$$\frac{t-g}{\mathcal{T}_a} - \mathcal{N}_0 \leq \mathcal{N}(t,g) \leq \frac{t-g}{\mathcal{T}_a} + \mathcal{N}_0$$

where $\mathcal{T}_a > 0$, $\mathcal{N}_0 > 0$, $\mathcal{T}_a$ is said to be the average impulsive interval and $\mathcal{N}_0$ is called the elasticity number.

3. Main results

In this section, using the Lyapunov approach and average impulsive interval method, some conditions ensuring SISS of system (1) are established. Thus, we will discuss three situations: (A) All subsystems are ISS, (B) some subsystems are ISS, and (C) stochastic systems with multiple delayed impulse jumps.

3.1. All subsystems are ISS for the considered system

Theorem 1. *If there are functions $\Theta \in \kappa_\infty$, $\wp \in \wp_\theta^{\Delta,d}$, $\beta_1 \in \nu\kappa_\infty$, $\beta_2 \in \kappa_\infty$ and a continuous function $\varphi(t) : [t_0, +\infty] \to R^+$, such that*
(I_1) $\beta_1(|z(t)|) \le F(t, z(t)) \le \beta_2(|z(t)|)$
(I_2) If $F(t, z(t)) \in \mathfrak{C}^{1,2}$ and constant $d > 0$ satisfy:

$$EF(t, z(t)) \ge \Theta(\|v(t)\|) \Rightarrow \begin{cases} E\mathcal{L}F(t, z(t)) \le -\varphi(t)EF(t, z(t)), & t \ge t_0,\ t \ne t_r, \\ EF(t_r, z(t_r)) \le e^{-d}EF(t_r^- - \tau, z(t_r^- - \tau)), & r \in N^+ \end{cases} \tag{2}$$

(I_3) Let impulsive sequence $\{t_r\}_{r\in N^+}$ satisfy:

$$-d\mathcal{N}(t, g) + \int_t^g \varphi(u)du \le \wp(t - g),\ \forall t \ge g \ge t_0 \tag{3}$$

then, system (1) is SISS.

Proof. For $\beta_1 \in \nu\kappa_\infty$, using Jensen's inequality and (I_1), we have

$$\beta_1(E|z(t)|) \le E\beta_1(|z(t)|) \le E(F(t, z(t))) \tag{4}$$

and

$$F^* \le \beta_2(E|\psi_s|) \tag{5}$$

where $F^* = sup_{n\in[t_0-\tau,t_0]}EF(n, z(n))$.

Using Itô differential formula [1], then

$$dF(t, z(t)) = \mathcal{L}F(t, z(t))dt + F_z(t, z(t))\eth(t, z(t), v(t))d\mathfrak{w}(t),\ t \in [t_r, t_{r+1}],\ r \in N^+$$

Let $\triangle t$ stand for sufficiently small, satisfying $t+\triangle t \in (t_r, t_{r+1})$. For each integer $w \ge 1$, the stopping time is defined: $T_w = inf\{t \in (t_r, t_{r+1}) : |z(t)| \ge w\}$, and it implies that $\lim_{w\to+\infty} T_w = t_{r+1}$. Using Lemma 3.2 in [39] and Fubini's theorem:

$$EF(t \wedge T_w, z(t \wedge T_w)) - EF(t_r, z(t_r)) = \int_{t_r}^{t\wedge T_w} E\mathcal{L}F(n, z(n))dn$$

Set $n \to +\infty$, then

$$EF(t, z(t)) - EF(t_r, z(t_r) = \int_{t_r}^{t} E\mathcal{L}F(n, z(n))dn$$

which yields

$$EF(t + \triangle t, z(t + \triangle t)) - EF(t, z(t)) = \int_t^{t+\triangle t} E\mathcal{L}F(n, z(n))dn$$

where $t,\ t + \triangle t \in (t_r, t_{r+1})$. Due to the continuity of $\mathcal{L}F(t, z(t))$,

$$\begin{aligned} \frac{dEF(t, z(t))}{dt} &= \lim_{\triangle t\to 0} \frac{EF(t + \triangle t, z(t + \triangle t)) - EF(t, z(t))}{\triangle t} \\ &= \lim_{\triangle t\to 0} \frac{\int_t^{t+\triangle t} E\mathcal{L}F(n, z(n))dn}{\triangle t} \\ &= E\mathcal{L}F(t, z(t)),\ t \in [t_r, t_{r+1}) \end{aligned} \tag{6}$$

From (2) and (6), we obtain

$$\frac{dEF(t,z(t))}{dt} \le -\varphi(t)EF(t,z(t)),$$

$$\Rightarrow EF(t,z(t)) \le exp(\int_{t_r}^{t} -\varphi(t)dt)EF(t_r,z(t_r)),\ t \in [t_r,t_{r+1}) \tag{7}$$

Since $EF(t,z(t))$ is right continuous, there is a time series $t_0 = \bar{t}_0 < \bar{t}_1 < \bar{t}_2 < \bar{t}_3 < \bar{t}_4 < \cdots$ satisfying:

$$\begin{aligned} &\bar{t}_{2j+1} = inf\left\{t \ge \bar{t}_{2j} : EF(t,z(t)) \le \Theta(\|v\|_{[t_0,t]})\right\} \le +\infty; \\ &\bar{t}_{2j} = inf\left\{t \ge \bar{t}_{2j+1} : EF(t,z(t)) \ge \Theta(\|v\|_{[t_0,t]})\right\} \le +\infty;\ j = 0,1,2,\cdots \end{aligned} \tag{8}$$

The time series decomposes $[t_0,+\infty]$ into several disjoint subintervals. If the decomposition is finite, the last subinterval is infinite, or if the decomposition is infinite, then all subintervals are finite.

The proof will be split into two scenarios, i.e., $-d+\int_{t_r-\tau}^{t_r}\varphi(t)dt > 0$, $r \in N^+$ and $-d+\int_{t_r-\tau}^{t_r}\varphi(t)dt \le 0$, $r \in N^+$, respectively.

Case I: $-d+\int_{t_r-\tau}^{t_r}\varphi(t)dt > 0$, $r \in N^+$, then, we will discuss two possible cases about t_r, i.e., $t_r \le \bar{t}_1$, $r \in N^+$, and $t_r > \bar{t}_1$, $r \in N^+$.

For $t_r \le \bar{t}_1$, $r \in N^+$, it can be seen from (7) that $EF(t,z(t)) \le exp(\int_{t_0}^{t} -\varphi(t)dt)F^*$, $t \in [t_0, t_1 \wedge \bar{t}_1)$. From (2) and $\varphi(t) > 0$,

$$\begin{aligned} EF(t_1,z(t_1)) &\le exp(-d)EF(t_1^- - \tau, z(t_1^- - \tau)) \\ &\le \begin{cases} exp(-d - \int_{t_0}^{t_1-\tau}\varphi(t)dt)F^*, & t_1-\tau \ge t_0, \\ exp(-d)F^*, & t_1 - \tau < t_0, \end{cases} \\ &\le exp(-d - \int_{t_0}^{t_1-\tau}\varphi(t)dt)F^* \end{aligned}$$

and

$$\begin{aligned} EF(t,z(t)) \le& exp(\int_{t_1}^{t} -\varphi(t)dt)EF(t_1,z(t_1)) \\ &\le exp(-d - \int_{t_0}^{t_1-\tau}\varphi(t)dt - \int_{t_1}^{t}\varphi(t)dt)F^*,\ t \in [t_1, t_2 \wedge \bar{t}_1) \end{aligned}$$

Similarly,

$$\begin{aligned} EF(t_2,z(t_2)) &\le exp(-d)EF(t_2^- - \tau, z(t_2^- - \tau)) \\ &\le \begin{cases} exp(-2d - \int_{t_0}^{t_1-\tau}\varphi(t)dt - \int_{t_1}^{t_2-\tau}\varphi(t)dt)F^*, & t_2-\tau \ge t_1, \\ exp(-d - \int_{t_0}^{t_2-\tau}\varphi(t)dt)F^*, & t_1 > t_2 - \tau \ge t_0, \\ exp(-d)F^*, & t_2-\tau < t_0, \end{cases} \\ &\le exp(-2d - \int_{t_0}^{t_1-\tau}\varphi(t)dt - \int_{t_1}^{t_2-\tau}\varphi(t)dt)F^* \end{aligned}$$

Based on (7), we gain

$$\begin{aligned}EF(t,z(t)) \leq & exp(\int_{t_2}^{t} -\varphi(t)dt)EF(t_2,z(t_2))\\ \leq & exp(-2d-\int_{t_0}^{t_1-\tau}\varphi(t)dt-\int_{t_1}^{t_2-\tau}\varphi(t)dt-\int_{t_2}^{t}\varphi(t)dt)F^*,\ t\in[t_2,t_3\wedge\bar{t}_1)\end{aligned}$$

Thus, we get for any $t \in [t_0, \bar{t}_1)$,

$$\begin{aligned}EF(t,z(t)) &\leq exp(-\mathcal{N}(t,t_0)d-\int_{t_0}^{t_1-\tau}\varphi(t)dt-\int_{t_1}^{t_2-\tau}\varphi(t)dt-\cdots-\int_{t_{N(t,t_0)}}^{t}\varphi(t)dt)F^*\\ &\leq exp(-\mathcal{N}(t,t_0)d+\int_{t_0}^{t}\varphi(t)dt)F^*\end{aligned} \tag{9}$$

Together with (3) and (9),

$$EF(t,z(t)) \leq exp(\wp(t-t_0))F^*,\ t\in[t_0,\bar{t}_1) \tag{10}$$

If $\bar{t}_1 = +\infty$, it's not difficult to know system (1) is ISS. Otherwise, in $[t_0, \bar{t}_1)$, taking into account the definition of $\left\{\bar{t}_j\right\}_{j\geq 0}$, we get

$$EF(t,z(t)) \leq \Theta(\|v\|_{[t_0,t]}),\ t\geq t_0 \tag{11}$$

If $\bar{t}_2 = +\infty$, combining with (10) and (11) and for any $t \geq t_0$,

$$EF(t,z(t)) \leq exp(\wp(t-t_0))F^* + \Theta(\|v\|_{[t_0,t]})$$

If $\bar{t}_2 < +\infty$, $\bar{t}_2$ may or may not be the impulse instant.
Supposing that $\bar{t}_2$ represents an impulsive instant, then

$$\begin{aligned}EF(\bar{t}_2,z(\bar{t}_2)) &\leq exp(-d)EF(\bar{t}_2^- -\tau, z(\bar{t}_2^- -\tau))\\ &\leq \begin{cases} exp(-d)\Theta(\|v\|_{[t_0,\bar{t}_2-\tau]}), & \bar{t}_2-\tau\geq\bar{t}_1,\\ exp(-d+\wp(\bar{t}_2-\tau-t_0))F^*, & \bar{t}_1>\bar{t}_2-\tau\geq t_0,\\ exp(-d)F^*, & \bar{t}_2-\tau<t_0,\end{cases}\\ &\leq exp(-d+\wp(\bar{t}_2-\tau-t_0))F^* + exp(-d)\Theta(\|v\|_{[t_0,\bar{t}_2-\tau]})\end{aligned} \tag{12}$$

Supposing that $\bar{t}_2$ does not represent an impulsive instant, from (8) we know that there exists a bound in $[\bar{t}_1, \bar{t}_2)$ due to continuity of $EF(t, z(t))$, then

$$EF(\bar{t}_2,z(\bar{t}_2)) = \Theta(\|v\|_{[t_0,\bar{t}_2]}) \tag{13}$$

Together with (12) and (13), we find

$$EF(\bar{t}_2,z(\bar{t}_2)) \leq exp(-d+\wp(\bar{t}_2-\tau-t_0))F^* + \Theta(\|v\|_{[t_0,\bar{t}_2]}) \tag{14}$$

Moreover, using the analogous argument of (10) and Definition 3, replacing t_0 with $\bar{t}_2$ yields that in $[\bar{t}_2, \bar{t}_3)$:

$$\begin{aligned} EF(t,z(t)) &\le exp(\wp(t-\bar{t}_2))EF(\bar{t}_2, z(\bar{t}_2)) \\ &\le exp(-d + \wp(t-\bar{t}_2) + \wp(\bar{t}_2 - \tau - t_0))F^* + exp(\wp(t-\bar{t}_2))\Theta(\|v\|_{[t_0,t]}) \\ &\le exp(\wp(t-t_0) + 2\Delta - d - \wp(\tau))F^* + exp(\wp(t-\bar{t}_2))\Theta(\|v\|_{[t_0,t]}) \\ &\le exp(\wp(t-t_0))F^* + \Theta(\|v\|_{[t_0,t]}) \end{aligned} \tag{15}$$

where $\wp(+\infty) = -\infty$. Using the same method, it follows that for any $t \ge t_0$,

$$EF(t,z(t)) \le exp(\wp(t-t_0))F^* + \Theta(\|v\|_{[t_0,t]}) \tag{16}$$

For $t_r > \bar{t}_1$, $r \in N^+$; hence, $t_r \in [\bar{t}_{2j-1}, \bar{t}_{2j})$, $r,\ j \in N^+$, or $t_r \in [\bar{t}_{2j}, \bar{t}_{2j+1})$, $r,\ j \in N^+$.

If $t_r \in [\bar{t}_{2j-1}, \bar{t}_{2j})$, $r,\ j \in N^+$, from (7) and (8) we derive $EF(t,z(t)) \le exp(\int_{t_0}^{t} -\varphi(t)dt)F^*$, $t \in [t_0, t_1 \wedge \bar{t}_1)$, and $EF(t,z(t)) \le \Theta(\|v\|_{[t_0,t]})$. Thus, $EF(t,z(t)) \le \Theta(\|v\|_{[t_0,t]})$, $t \in [\bar{t}_{2j-1}, \bar{t}_{2j})$. Finally, we gain for any $t \ge t_0$:

$$EF(t,z(t)) \le exp(\int_{t_0}^{t} -\varphi(t)dt)F^* + \Theta(\|v\|_{[t_0,t]}) \tag{17}$$

If $t_r \in [\bar{t}_{2j}, \bar{t}_{2j+1})$, $r,\ j \in N^+$, it can be referred to as Case I, and (16) still holds.

Together with (4), (16), and (17), for any $t \ge t_0$,

$$\begin{aligned} \beta_1(E|z(t)|) &\le E(F(t,z(t)) \\ &\le exp(\wp(t-t_0))\beta_2(\|\psi\|_\tau) + \Theta(\|v\|_{[t_0,t]}) \end{aligned}$$

which indicates

$$\begin{aligned} E|z(t)| &\le \beta_1^{-1}\left[exp(\wp(t-t_0))\beta_2(\|\psi\|_\tau) + \Theta(\|v\|_{[t_0,t]})\right] \\ &\le \xi_1\left(\|\psi\|_\tau, t-t_0\right) + \eta_1\left(\|v\|_{[t_0,t]}\right) \end{aligned} \tag{18}$$

where $\xi_1\left(\|\psi\|_\tau, t-t_0\right) = \beta_1^{-1}\left(2exp(\wp(t-t_0))\beta_2(\|\psi\|_\tau)\right)$ and $\eta_1\left(\|v\|_{[t_0,t]}\right) = \beta_1^{-1}\left(2\Theta(\|v\|_{[t_0,t]})\right)$. For $\forall\Lambda > 0$, set $\bar{\xi}\left(\|\psi\|_\tau, t-t_0\right) = 1/\Lambda\beta_1^{-1}\left(2exp(\wp(t-t_0))\beta_2(\|\psi\|_\tau)\right)$ and $\bar{\eta}\left(\|v\|_{[t_0,t]}\right) = 1/\Lambda\beta_1^{-1}\left(2\Theta(\|v\|_{[t_0,t]})\right)$. According to (18) and Chebyshev's inequality, for any $t \ge t_0$,

$$\begin{aligned} &P\left\{|z(t)| \ge \bar{\xi}(\|\psi\|_\tau, t-t_0) + \bar{\eta}(\|v\|_{[t_0,t]})\right\} \\ &\le \frac{E|z(t)|}{\bar{\xi}(\|\psi\|_\tau, t-t_0) + \bar{\eta}(\|v\|_{[t_0,t]})} \le \Lambda \end{aligned}$$

which yields

$$P\left\{|z(t)| < \bar{\xi}(\|\psi\|_\tau, t-t_0) + \bar{\eta}(\|v\|_{[t_0,t]})\right\} \ge 1 - \Lambda$$

Thus, system (1) is SISS.

Case II: $-d + \int_{t_r-\tau}^{t_r} \varphi(t)dt \le 0$, $r \in N^+$. For $t_r \le \bar{t}_1$, $r \in N^+$, it can be seen from (7) that $EF(t,z(t)) \le exp(\int_{t_0}^{t} -\varphi(t)dt)F^*$, $t \in [t_0, \bar{t}_1)$. When $\bar{t}_1 = +\infty$, we know system (1) is ISS. When $\bar{t}_1 < +\infty$, then take into account $t \in [\bar{t}_1, \bar{t}_2)$. When $\bar{t}_2 = +\infty$, the estimate of ISS is obtained as $EF(t,z(t)) \le exp(\int_{t_0}^{t} -\varphi(t)dt)F^* + \Theta(\|v\|_{[t_0,t]})$, $t \ge t_0$. Otherwise, $\bar{t}_2 < +\infty$, and we need discuss whether $\bar{t}_2$ is the impulsive time.

Supposing that $\bar{t}_2$ represents an impulsive instant, then

$$\begin{aligned} EF(\bar{t}_2, z(\bar{t}_2)) &\leq exp(-d)EF(\bar{t}_2^- - \tau, z(\bar{t}_2^- - \tau)) \\ &\leq exp(-d - \int_{t_0}^{\bar{t}_2-\tau} \varphi(t)dt)F^* + exp(-d)\Theta(\|v\|_{[t_0,\bar{t}_2]}) \\ &\leq exp(-d + \int_{\bar{t}_2-\tau}^{\bar{t}_2} \varphi(t)dt - \int_{t_0}^{\bar{t}_2} \varphi(t)dt)F^* + \Theta(\|v\|_{[t_0,\bar{t}_2]}) \\ &\leq exp(-\int_{t_0}^{\bar{t}_2} \varphi(t)dt)F^* + \Theta(\|v\|_{[t_0,\bar{t}_2]}) \end{aligned}$$

Supposing that $\bar{t}_2$ does not represent an impulsive instant, hence, $EF(\bar{t}_2, z(\bar{t}_2)) = \Theta(\|v\|_{[t_0,\bar{t}_2]})$, similarly with Case I and proceeding as before, the case of $t_r > \bar{t}_1$, $r \in N^+$ is omitted here. As well, to use Chebyshev's inequality, the SISS characteristic is obtained.

Remark 1. *In (15), due to $\wp \in \wp_\theta^{\Delta,d}$, by using condition (ii) in Definition 3, the calculation of exponential power is simplified: $exp(-d+\wp(t-\bar{t}_2)+\wp(\bar{t}_2-\tau-t_0) \leq exp(\wp(t-t_0)+2\Delta-d-\wp(\tau)) \leq exp(\wp(t-t_0))$.*

Remark 2. *In recent years, when the coefficient d in (2) belongs to R, i.e., $d \in R$, relevant results are investigated in [36,40], which implies that impulse may be stabilizing or destabilizing. Although the coefficient in (2) is $d > 0$, due to time delay in impulse term, impulse may also be destabilizing.*

Remark 3. *The Lyapunov rate coefficient to be a constant has been studied in [35], whereas, in this paper, the Lyapunov rate coefficient is extended to be positively time-varying, which makes the derived criterion less conservative. In addition, in [36,41], stability of an impulsive stochastic system is also investigated, but time delay is not taken into account and the two cases of impulsive time t_r, i.e., $t_r \leq \bar{t}_1$, $r \in N^+$ and $t_r > \bar{t}_1$, $r \in N^+$, are also not discussed.*

Corollary 1. *Let conditions (I_1) and (I_2) in Theorem 1 hold. In addition, let ω be a positive constant, and $\mathcal{T}_a \leq \omega$, satisfying:*

$$\int_{t_0}^{+\infty} (\varphi(s) + \frac{-d}{\omega})ds = -\infty \tag{19}$$

Thus, system (1) is SISS.

Proof. From Definition 4 and $d > 0$,

$$-d\mathcal{N}(t, g) \leq \frac{-d(t-g)}{\mathcal{T}_a} + d\mathcal{N}_0 \leq \frac{-d(t-g)}{\omega} + d\mathcal{N}_0$$

Let

$$v(y) = \frac{-dy}{\omega} + \int_0^y (\varphi(s + t_0))ds + d\mathcal{N}_0, \ y \geq 0$$

In addition, $\int_{t_0}^{+\infty}(\varphi(s) + \frac{-d}{\omega})ds = -\infty$, which means that

$$\int_0^{+\infty} (\varphi(s + t_0) + \frac{-d}{\omega})ds = -\infty$$

This yields that $\wp \in C(R^+, R)$ and $\wp(+\infty) = -\infty$, then, system (1) is SISS.

3.2. Some subsystems are ISS for considered system

In this subsection, we consider this case where some subsystems are ISS and some subsystems are non-ISS. We will seek that under certain conditions, the whole system can maintain SISS property even if some subsystems are non-ISS.

Because there exists stable subsystems and unstable subsystems, we hypothesize that the whole system T is parted into T_p and T_q, where T_p is subsystems that are ISS and T_q is subsystems that are non-ISS, and $\mathsf{T} = \mathsf{T}_p \cup \mathsf{T}_q$. In T_p, let $\mathbb{T}_p(g,t)$ stand for total activation time, or in T_q, let $\mathbb{T}_q(g,t)$ represent total activation time. Additionally, let $\breve{\mathbb{T}}_q(t_r) = \left\{t_{r+1} - t_r \mid t \in \mathsf{T}_q, t \in [t_r, t_{r+1}), r \in N\right\}$ represent activation time of a single non-ISS subsystem so we can set $\mathbb{T}_n := max\left\{\breve{\mathbb{T}}_q(t_r), r \in N\right\}$.

Theorem 2. *If there are functions* $\Theta \in \kappa_\infty$, $\wp \in \wp_\theta^{\Delta,d}$, $\beta_1 \in \nu\kappa_\infty$, $\beta_2 \in \kappa_\infty$ *and two continuous functions* $\varphi_p(t) : [t_0, +\infty] \to R^+$ *and* $\varphi_q(t) : [t_0, +\infty] \to R^+$, *such that*
(I_1) $\beta_1(|z(t)|) \leq F(t, z(t)) \leq \beta_2(|z(t)|)$
(I_2) *If* $F(t, z(t)) \in \mathfrak{C}^{1,2}$ *and constant* $d > 0$ *satisfy:*

$$EF(t,z(t)) \geq \Theta(\|v(t)\|) \Rightarrow \begin{cases} \begin{cases} E\mathcal{L}F(t,z(t)) \leq -\varphi_p(t)EF(t,z(t)), & t \geq t_0,\ t \in \mathsf{T}_p, \\ E\mathcal{L}F(t,z(t)) \leq \varphi_q(t)EF(t,z(t)), & t \geq t_0,\ t \in \mathsf{T}_q, \end{cases} \\ EF(t_r, z(t_r)) \leq e^{-d}EF(t_r^- - \tau, z(t_r^- - \tau)),\ r \in N^+ \end{cases} \tag{20}$$

(I_3) *Let impulsive sequence* $\{t_r\}_{r\in N^+}$ *satisfy:*

$$[-d + \int_{\mathbb{T}_n} (\varphi_p(u) + \varphi_q(u))du]N(t,g) + \int_{\mathbb{T}_n} (\varphi_p(u) + \varphi_q(u))du + \int_g^t \varphi_p(u)du \leq \wp(t-g),\ \forall t \geq g \geq t_0 \tag{21}$$

Thus, system (1) is SISS.

Proof. Proof is similar to Theorem 1. We will omit the derivation process before (7). According to Fubini's theorem, (20) can be rewritten as:

$$E\mathcal{L}F(t,z(t)) \leq \underline{\varphi}(t)EF(t,z(t)),\ t \geq t_0,\ t \in \mathsf{T},$$
$$EF(t_r, z(t_r)) \leq e^{-d}EF(t_r^- - \tau, z(t_r^- - \tau)),\ r \in N^+$$

where $\underline{\varphi}(t) = \begin{cases} -\varphi_p(t), & t \in \mathsf{T}_p, \\ \varphi_q(t), & t \in \mathsf{T}_q. \end{cases}$

As similar with Theorem 1, a time series is defined: $t_0 = \bar{t}_0 < \bar{t}_1 < \bar{t}_2 < \bar{t}_3 < \bar{t}_4 < \cdots$.

Next, we will also prove SISS in two cases, i.e., $-d + \int_{\mathbb{T}_n}(\varphi_p(u)+\varphi_q(u))du + \int_{t_r-\tau}^{t_r} \varphi_p(u)du > 0$, $r \in N^+$ and $-d + \int_{\mathbb{T}_n}(\varphi_p(u) + \varphi_q(u))du + \int_{t_r-\tau}^{t_r} \varphi_p(u)du \leq 0$, $r \in N^+$, respectively.

Case I: $-d + \int_{\mathbb{T}_n}(\varphi_p(u) + \varphi_q(u))du + \int_{t_r-\tau}^{t_r} \varphi_p(u)du > 0$, $r \in N^+$, then we also discuss two possible cases about t_r, i.e., $t_r \leq \bar{t}_1$, $r \in N^+$ and $t_r > \bar{t}_1$, $r \in N^+$.

For $t_r \leq \bar{t}_1$, $r \in N^+$, it can be seen from (7) that $EF(t,z(t)) \leq exp(\int_{t_0}^t(-\underline{\varphi}(t)dt)F^*$, $t \in [t_0, t_1 \wedge \bar{t}_1)$, due to $\varphi_p(t) > 0$ and $\varphi_q(t) > 0$, and we gain

$$\begin{aligned}EF(t,z(t)) &\leq exp(\int_{t_0}^{t}(-\underline{\varphi}(t)dt)F^* \\ &= exp[\int_{\mathbb{T}_p} -\varphi_p(u)du + \int_{\mathbb{T}_q} \varphi_q(u)du]F^* \\ &\leq exp[\int_{t_0}^{t} -\varphi_p(u)du + \int_{\mathbb{T}_n}(\varphi_p(u)+\varphi_q(u))du]F^*,\ t\in[t_0, t_1\wedge \bar{t}_1)\end{aligned}$$

From (20), we derive

$$\begin{aligned}EF(t_1,z(t_1)) &\leq e^{-d}EF(t_1^- - \tau, z(t_1^- - \tau)) \\ &\leq \begin{cases} exp[-d+\int_{\mathbb{T}_n}(\varphi_p(u)+\varphi_q(u))du - \int_{t_0}^{t_1-\tau}\varphi_p(u)du]F^*, & t_1-\tau > t_0, \\ exp[-d+\int_{\mathbb{T}_n}(\varphi_p(u)+\varphi_q(u))du]F^*, & t_1-\tau\leq t_0, \end{cases} \\ &\leq exp[-d+\int_{\mathbb{T}_n}(\varphi_p(u)+\varphi_q(u))du - \int_{t_0}^{t_1-\tau}\varphi_p(u)du]F^*\end{aligned}$$

as well as

$$\begin{aligned}EF(t,z(t)) &\leq exp(\int_{t_1}^{t}(\underline{\varphi}(t)dt)EF(t_1,z(t_1)) \\ &\leq exp[-d+2\int_{\mathbb{T}_n}(\varphi_p(u)+\varphi_q(u))du - \int_{t_0}^{t_1-\tau}\varphi_p(u)du - \int_{t_1}^{t}\varphi_p(u)du]F^*,\ t\in[t_1, t_2\wedge\bar{t}_1)\end{aligned}$$

Similarly,

$$\begin{aligned}EF(t_2,z(t_2)) &\leq e^{-d}EF(t_2^- - \tau, z(t_2^- - \tau)) \\ &\leq \begin{cases} exp[-2d+2\int_{\mathbb{T}_n}(\varphi_p(u)+\varphi_q(u))du - \int_{t_0}^{t_1-\tau}\varphi_p(u)du - \int_{t_1}^{t_2-\tau}\varphi_p(u)du]F^*, & t_2-\tau > t_1, \\ exp[-d+\int_{\mathbb{T}_n}(\varphi_p(u)+\varphi_q(u))du - \int_{t_0}^{t_2-\tau}\varphi_p(u)du]F^*, & t_1 \geq t_2-\tau > t_0, \\ exp[-d+\int_{\mathbb{T}_n}(\varphi_p(u)+\varphi_q(u))du]F^*, & t_2-\tau\leq t_0, \end{cases} \\ &\leq exp[-2d+2\int_{\mathbb{T}_n}(\varphi_p(u)+\varphi_q(u))du - \int_{t_0}^{t_1-\tau}\varphi_p(u)du - \int_{t_1}^{t_2-\tau}\varphi_p(u)du]F^*\end{aligned}$$

and

$$\begin{aligned}EF(t,z(t)) &\leq exp(\int_{t_2}^{t}(\underline{\varphi}(t)dt)EF(t_2,z(t_2)) \\ &\leq exp[-2d+2\int_{\mathbb{T}_n}(\varphi_p(u)+\varphi_q(u))du - \int_{t_0}^{t_1-\tau}\varphi_p(u)du - \int_{t_1}^{t_2-\tau}\varphi_p(u)du \\ &\quad - \int_{t_2}^{t}\varphi_p(u)du + \int_{\mathbb{T}_n}(\varphi_p(u)+\varphi_q(u))du]F^*,\ t\in[t_2, t_3\wedge\bar{t}_1)\end{aligned}$$

Thus, we obtain

$$\begin{aligned}EF(t,z(t)) &\leq exp[(-d+\int_{\mathbb{T}_n}(\varphi_p(u)+\varphi_q(u))du)\mathcal{N}(t,s)-\int_{t_0}^{t_1-\tau}\varphi_p(u)du-\int_{t_1}^{t_2-\tau}\varphi_p(u)du-\cdots\\&\quad-\int_{t_{N(t,g)}}^{t}\varphi_p(u)du+\int_{\mathbb{T}_n}(\varphi_p(u)+\varphi_q(u))du]F^*\\&\leq exp[(-d+\int_{\mathbb{T}_n}(\varphi_p(u)+\varphi_q(u))du)N(t,g)+\int_g^t\varphi_p(u)du+\int_{\mathbb{T}_n}(\varphi_p(u)+\varphi_q(u))du]F^*,\ t\in[t_0,\bar{t}_1)\end{aligned} \tag{22}$$

Together with (21) and (22),

$$EF(t,z(t)) \leq exp(\wp(t-t_0))F^*,\ t\in[t_0,\bar{t}_1) \tag{23}$$

If $\bar{t}_1 = +\infty$, it's not difficult to know system (1) is ISS. Otherwise, in $[t_0,\bar{t}_1)$, with the definition of $\left\{\bar{t}_j\right\}_{j\geq 0}$, we gain

$$EF(t,z(t)) \leq \Theta(\|v\|_{[t_0,t]}),\ t\geq t_0 \tag{24}$$

The remaining part of the reasoning is analogous to Theorem 1 and is omitted in this section. Finally, SISS of system (1) is proved.

Corollary 2. *Let conditions (I_1) and (I_2) in Theorem 2 hold. In addition, let ω be a positive constant. If $h=-d+\int_{\mathbb{T}_n}(\varphi_p(u)+\varphi_q(u))du<0$ and $\mathcal{T}_a\geq\omega$, it satisfies:*

$$\int_{t_0}^{+\infty}(\varphi_p(s)+\frac{h}{\omega})ds=-\infty \tag{25}$$

Therefore, system (1) is SISS.

Proof. From Definition 4 and $h<0$, we get

$$h\mathcal{N}(t,g)\leq\frac{h(t-g)}{\mathcal{T}_a}+(-h)\mathcal{N}_0\leq\frac{h(t-g)}{\omega}+(-h)\mathcal{N}_0$$

Let

$$v(y)=\frac{hy}{\omega}+\int_0^y(\varphi(s+t_0))ds+(-h)\mathcal{N}_0,\ y\geq 0$$

Additionally, $\int_{t_0}^{+\infty}(\varphi_p(s)+\frac{h}{\omega})ds=-\infty$, which implies that

$$\int_0^{+\infty}(\varphi(s+t_0)+\frac{h}{\omega})ds=-\infty$$

This yields that $\wp\in C(R^+,R)$ and $\wp(+\infty)=-\infty$, then, system (1) is SISS.

4. ISS for considered system with multiple jumps

Lemma 1. *[36] For $l = 1, \ldots, n$, $n \in N^+$, let $T_l = \left\{t_1^l, t_2^l, \ldots\right\}$ represent an impulsive time series, be strictly increasing in $(t_0, +\infty)$ with no finite accumulation point, and*

$$T = \underset{l=1,\ldots,n}{\cup} T_l, \quad T_l \cap T_z = \emptyset, \quad l,\ z \in 1, \ldots, n,\ l \neq z$$

where t_0 is initial time and T is the set of impulse moments in increasing order.

Consider the stochastic nonlinear system with multiple delayed impulse jumps:

$$\begin{aligned} dz(t) &= \beth(t, z(t), v(t))dt + \eth(t, z(t), v(t))d\mathfrak{w}(t),\ t \notin T,\ t \geq t_0, \\ z(t_r) &= \S_i(z(t_r^- - \tau), v(t_r^-)),\ t \in T_l,\ l = 1, \ldots, p \end{aligned} \tag{26}$$

where function $\S_i : [t_0, +\infty] \times R^n \times R^m \longrightarrow R^n$ is Borel measurable. A time series $\{t_r\}_{r\in N^+}$ is given. Let times $t,\ g$ satisfy $t > g \geq t_0$ and $N_l(t, g)$ stand for the number of impulses $t_r^l \in T_l$ in $(g, t]$.

Because of effects of multiple jumps, Definition 3 is reformulated as follows:

Definition 5. *[32] There exists constants $\theta,\ \Delta,\ d_l \in R^+ (l = 1, \ldots, n)$ and function $\mathfrak{v}_\theta^{\Delta,d} = \{\mathfrak{v} \in C(R^+, R)\}$ satisfying:*
(i) $\mathfrak{v}(b_1) + \mathfrak{v}(b_2) \leq \mathfrak{v}(b_1 + b_2) + \theta$, for $\forall b_1,\ b_2 \in R^+$;
(ii) $2\Delta - d_l < \mathfrak{v}(\tau)$;
(iii) $\mathfrak{v}(b) + \sigma b < d_l$ and tends towards $-\infty$ as $b \to +\infty$.

Theorem 3. *If there are functions $\Theta \in \kappa_\infty$, $\bar{\mathfrak{v}} \in \mathfrak{v}_\theta^{\Delta,d}$, $\beta_1 \in \nu\kappa_\infty$, $\beta_2 \in \kappa_\infty$ and a continuous function $\varphi(t) : [t_0, +\infty] \to R^+$, such that*
(I_1) $\beta_1(|z(t)|) \leq F(t, z(t)) \leq \beta_2(|z(t)|)$
(I_2) If $F(t, z(t)) \in \mathfrak{C}^{1,2}$ and constant $d_l > 0$ satisfy:

$$EF(t, z(t)) \geq \Theta(\|v(t)\|) \Rightarrow \begin{cases} E\mathcal{L}F(t, z(t)) \leq -\varphi(t)EF(t, z(t)), & t \geq t_0,\ t \neq t_r, \\ EF(t_r, z(t_r)) \leq e^{-d_l}EF(t_r^- - \tau, z(t_r^- - \tau)), & r \in N^+ \end{cases} \tag{27}$$

(I_3) Let impulsive sequence $\{t_r\}_{r\in N^+}$ satisfy:

$$-\sum_{l=1}^{n} d_l \mathcal{N}(t, g) + \int_t^s \varphi(u)du \leq \bar{\mathfrak{v}}(t - g),\ \forall t \geq g \geq t_0 \tag{28}$$

Thus, system (26) is SISS.

Proof. The approach to proof is similar to Theorem 1, except that d is replaced by d_l and $d\mathcal{N}(t, g)$ is replaced by $\sum_{l=1}^n d_l \mathcal{N}(t, g)$.

Remark 4. *The multiple jumps in this paper imply that the combination of l different impulsive time series constitutes impulse of the whole system. When $l = 1$, system (26) will degenerate into system (1). When $l \neq 1$, due to the existence of l different jumps, the impulse coefficient in (27) has been changed from d to d_l.*

5. Numerical examples

Example 1. *Consider the stochastic nonlinear system with delayed impulses:*

$$\begin{aligned} dz(t) &= [(-\frac{3}{2}sin(t) - 1)z(t) + \frac{1}{4}v(t)]dt + \frac{1}{2}z(t)d\mathfrak{w}(t),\ t \neq t_r, \\ z(t_r) &= \frac{1}{4}z(t_r^- - \tau) + \frac{1}{4}v(t_r^-),\ r \in N^+ \end{aligned} \tag{29}$$

where $t \geq 0$.

To choose $F(t, z) = z^2$ and $\Theta(z) = z^2$, then

$$\begin{aligned} E\mathcal{L}F(t, z(t)) &= (-3sin(t) - 2)Ez^2(t) + \frac{1}{2}Ez(t)v(t) + \frac{1}{4}Ez^2(t) \\ &\leq (-3sin(t) - \frac{3}{2})Ez^2(t) + \frac{1}{4}\|v(t)\|^2 \\ &\leq (-3sin(t) - \frac{5}{4})Ez^2(t) \end{aligned}$$

and

$$\begin{aligned} EF(t_r, z(t_r)) &\leq \frac{1}{8}Ex^2(t_r^- - \tau) + \frac{1}{8}\|v(t)\|^2 \\ &\leq exp(-ln4)Ez^2(t_r^- - \tau) \end{aligned}$$

If all subsystems are stable, let $\varphi(t) = 3sin(t) + \frac{5}{4}$, $d = ln4$ and $\omega = 1$. If $\mathcal{T}_a \leq \omega$, by computing $\int_0^{+\infty}(\varphi(s) + \frac{-d}{\omega})ds = -\infty$ according to Corollary 1, system (29) is SISS.

To choose $\tau = 1$. Figure 1 depicts state trajectory with $v(t) = sin(t)$ and Figure 2 depicts state trajectory with $v(t) = z(t)$.

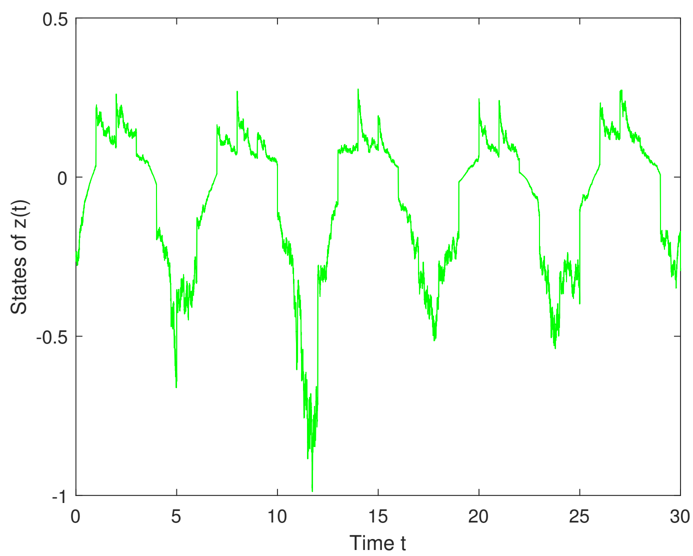

Figure 1. State trajectory with $v(t) = sin(t)$.

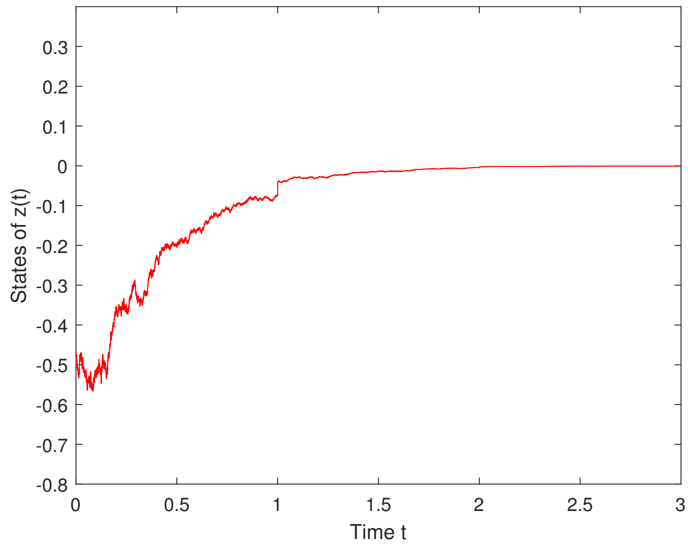

Figure 2. State trajectory with $v(t) = z(t)$.

If there exists both stable and unstable subsystems, let $\varphi_p(u) = 3sin(u) + \frac{5}{4}$, $\varphi_q(u) = sin(u)$, $d = ln4$, $\mathbb{T}_n = 0.3$, and $\omega = 0.05$. If $\mathcal{T}_a \geq \omega$, by computing we can get $-d + \int_{\mathbb{T}_n}(\varphi_p(u) + \varphi_q(u))du < 0$, $\int_0^{+\infty}(\varphi_p(s) + \frac{-d+\int_{\mathbb{T}_n}(\varphi_p(u)+\varphi_q(u))du}{\omega})ds = -\infty$ according to Corollary 2, and system (29) is SISS.

To choose $\tau = 1$. Figure 3 depicts state trajectory with $v(t) = sin(t)$ and Figure 4 depicts state trajectory with $v(t) = z(t)$.

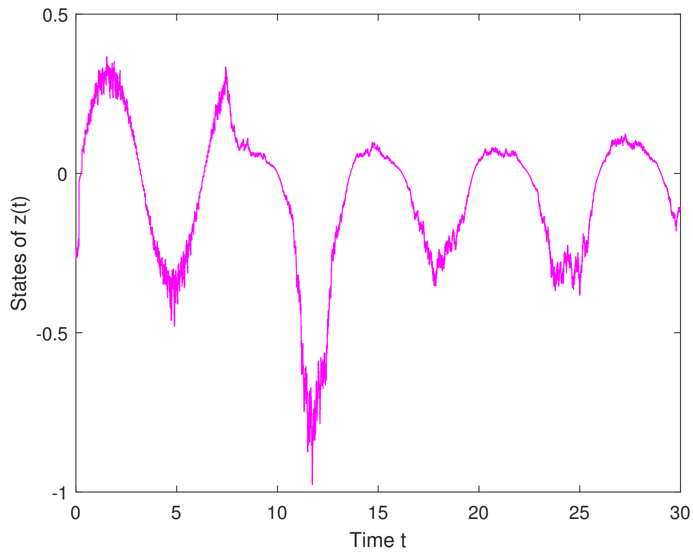

Figure 3. State trajectory with $v(t) = sin(t)$.

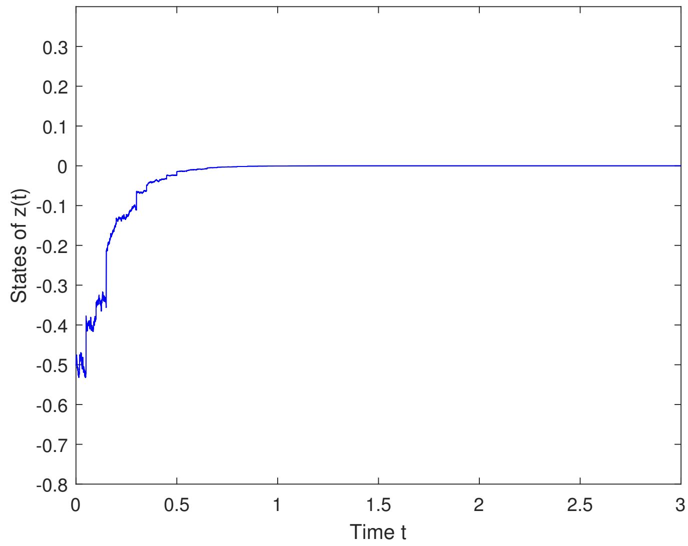

Figure 4. State trajectory with $v(t) = z(t)$.

Example 2. *Consider the stochastic nonlinear system with multiple delayed impulse jumps:*

$$
\begin{aligned}
dz(t) &= [(-\frac{3}{2}sin(t) - 1)z(t) + \frac{1}{4}v(t)]dt + \frac{1}{2}z(t)d\mathfrak{w}(t),\ t \neq T, \\
z(t_r) &= \begin{cases} \frac{1}{5}z(t_r^- - \tau) + \frac{1}{10}v(t_r^-),\ t \in T_1, \\ \frac{1}{3}z(t_r^- - \tau) + \frac{1}{8}v(t_r^-),\ \ t \in T_2 \end{cases}
\end{aligned} \tag{30}
$$

where $t \geq 0$.

To choose $F(t, z) = z^2$ and $\Theta(z) = z^2$, then

$$
\begin{aligned}
E\mathcal{L}F(t, z(t)) &= (-3sin(t) - 2)Ez^2(t) + \frac{1}{2}Ez(t)v(t) + \frac{1}{4}Ez^2(t) \\
&\leq (-3sin(t) - \frac{3}{2})Ez^2(t) + \frac{1}{4}||v(t)||^2 \\
&\leq (-3sin(t) - \frac{5}{4})Ez^2(t)
\end{aligned}
$$

Set $\varphi(t) = 3sin(t) + \frac{5}{4}$.
When $t \in T_1$,

$$
\begin{aligned}
EF(t_r, z(t_r)) &\leq \frac{2}{25}Ez^2(t_r^- - \tau) + \frac{1}{50}||v(t)||^2 \\
&\leq exp(-ln10)Ez^2(t_r^- - \tau)
\end{aligned}
$$

and we get $d_1 = ln10$ and $\omega_1 = 1$.
When $t \in T_2$,

$$
\begin{aligned}
EF(t_r, z(t_r)) &\leq \frac{2}{9}Ez^2(t_r^- - \tau) + \frac{1}{32}||v(t)||^2 \\
&\leq exp(-ln(\frac{288}{73}))Ez^2(t_r^- - \tau)
\end{aligned}
$$

and we obtain $d_2 = ln(\frac{288}{73})$ and $\omega_2 = 0.5$.

If $\mathcal{T}_{a1} \leq \omega_1, \mathcal{T}_{a2} \leq \omega_2$, by computing we obtain $\int_0^{+\infty}(\varphi(s) + \frac{-d}{\omega_1})ds = -\infty$, $\int_0^{+\infty}(\varphi(s) + \frac{-d}{\omega_2})ds = -\infty$, respectively, according to Corollary 1, and system (30) is SISS.

To choose $\tau = 1$. Figure 5 depicts state trajectory with $v(t) = sin(t)$ and Figure 6 depicts state trajectory with $v(t) = z(t)$.

Remark 5. *In Example 2, $l = 2$, which implies that there are two different delayed impulses, so Figure 5 and Figure 6 display two lines representing different jumps, respectively.*

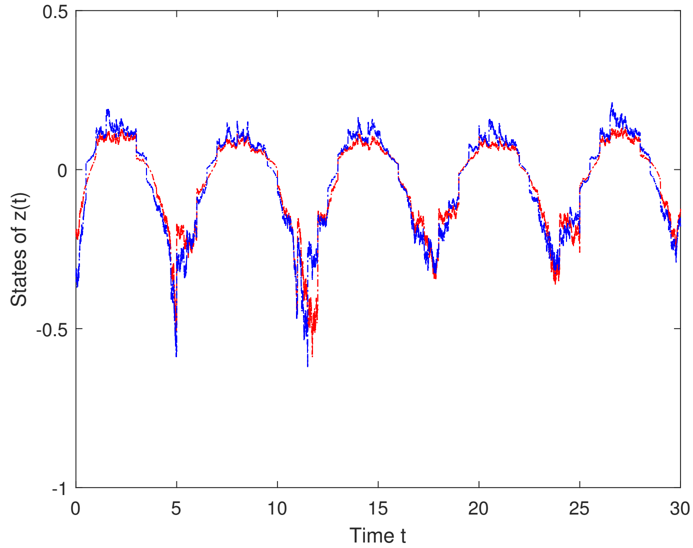

Figure 5. State trajectory with $v(t) = sin(t)$.

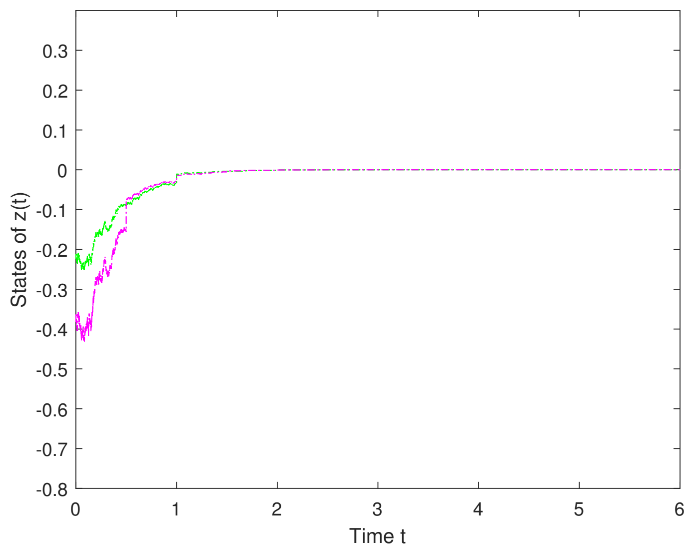

Figure 6. State trajectory with $v(t) = z(t)$.

6. Conclusions and outlook

This paper was committed to SISS of stochastic nonlinear systems with delayed impulses. Using the average impulsive interval method, Chebyshev's inequality, and Lyapunov approach, we not only derived SISS criteria when all subsystems were stable, but also established conditions that ensured SISS of the considered systems with both stable and unstable subsystems simultaneously. Next, when there existed multiple delayed impulse jumps in the stochastic system, SISS characteristic of the argumented system was also obtained. In particular, the conditions considered in this paper were based on the fact that the Lyapunov rate coefficient was positively time-varying instead of constant. Finally, two examples manifested the validity of the above results. Since this paper discussed the time delay of impulses, systems with time delay in both states and impulses can be considered in the future. The time delay is not necessarily constant delay, but can be time-varying delay, unbounded delay, etc, so that the results are more general.

Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this paper.

Conflict of interest

The authors declare there is no conflict of interest.

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