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Research article

Dynamics of a modified Leslie-Gower predator-prey model with double Allee effects

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Abstract: In this paper, we investigate the dynamic behavior of a modified Leslie-Gower predator-prey model with the Allee effect on both prey and predator. It is shown that the model has at most two positive equilibria, where one is always a hyperbolic saddle and the other is a weak focus with multiplicity of at least three by concrete example. In addition, we analyze the bifurcations of the system, including saddle-node bifurcation, Hopf bifurcation and Bogdanov-Takens bifurcation. The results show that the model has a cusp of codimension three and undergoes a Bogdanov-Takens bifurcation of codimension two. The system undergoes a degenerate Hopf bifurcation and has two limit cycles (the inner one is stable and the outer one is unstable). These enrich the dynamics of the modified Leslie-Gower predator-prey model with the double Allee effects.

Keywords: Leslie-Gower; Allee effect; limit cycle; Hopf bifurcation; Bogdanov-Takens bifurcation

1. Introduction

Leslie and Gower proposed a predator-prey model [1, 2], as follows:

$$\begin{aligned} \dot{x} &= rx\left(1-\frac{x}{K}\right)-mxy, \\ \dot{y} &= sy\left(1-\frac{y}{nx}\right), \end{aligned} \tag{1.1}$$

where x and y denote the average population densities of the prey and predator at time t, respectively; K, r, s, m and n are all positive; K represents the environmental carrying capacity; r and s are the intrinsic growth rates of the prey and predator, respectively; m is the maximum per capita predation rate; n is a measure of the quality of the prey as food for the predator. In model (1.1), the environmental capacity of predators is directly proportional to the number of prey (i.e., nx), and $\frac{y}{nx}$ is called the Leslie-Gower term.

Korobeinikov [3] and Hsu and Huang [4] showed that system (1.1) has a globally asymptotically stable positive equilibrium under certain conditions. Lindstrom [5] studied the nonexistence and existence of limit cycles of system (1.1).

In system (1.1), predators are thought to feed on a single prey species. In the world, several predators pursue a wide range of prey. That is, the predator will seek alternative food sources if the predator's favorite food is not in sufficient supply. This type of predator is called a generalist predator, and it includes foxes, common buzzards, cats, etc. [6]. In order to study the situation in which the predator is a generalist predator, Aziz-Alaoui and Okiye [7] replaced nx with $nx + c$ and considered a predator-prey model with other food sources and a Holling type II functional response as follows:

$$\begin{aligned} \dot{x} &= rx\left(1 - \frac{x}{K}\right) - \frac{mxy}{a + x}, \\ \dot{y} &= sy\left(1 - \frac{y}{nx + c}\right), \end{aligned} \tag{1.2}$$

where $\frac{y}{nx+c}$ is called the modified Leslie-Gower term; c can be seen as other food sources for the predator. The authors [7] studied the boundedness of the solution of system (1.2) and proved that the interior positive equilibrium is globally asymptotically stable under certain parameter conditions by constructing a suitable Lyapunov function. Nindjin et al. [8] discussed the effect of time delay on the stability of the positive equilibrium. Xiang et al. [9] rigorously analyzed the high codimension bifurcation of system (1.2), such as the Hopf bifurcation of codimension 2 and degenerate Bogdanov-Takens bifurcation of codimension 3. Under the condition that the coefficients of system (1.2) are periodic, Zhu and Wang [10] proved the existence of positive periodic solutions and obtained some sufficient conditions for the global attractivity of positive periodic solutions. For more interesting results on the modified Leslie-Gower term, please refer to [11–13].

Allee [14] pointed out that clustering is conducive to the growth and survival of the population, but excessive sparsity and overcrowding can prevent the growth of the population; he also found that each population has its optimal population density. A population is endangered when its density falls below a certain threshold; that is, it has a minimum density to sustain the population. When the population density is too low, it will be difficult for individuals to find mates or resist natural enemies, which will lead to a decrease in the birth rate and an increase in the death rate of the population. This phenomenon is called the Allee effect, which leads to more complex dynamic behavior. Lots of biological phenomena can cause the Allee effect, such as mating difficulty, anti predator defense and genetic drift. One of the Allee effect's forms expression is the multiplicative Allee effect, which can be written as follows for a single species:

$$\dot{x} = rx\left(1 - \frac{x}{K}\right)(x - a),$$

where a is the Allee threshold. When $0 < a < K$, that is, the population density is small, the per capita growth rate of the population is negative, which is called a strong Allee effect.

Arancibia-Ibarra [15] proposed the following model with the multiplicative Allee effect and a generalist predator:

$$\begin{aligned} \dot{x} &= rx\left(1 - \frac{x}{K}\right)(x - m) - qxy, \\ \dot{y} &= sy\left(1 - \frac{y}{nx + c}\right). \end{aligned} \tag{1.3}$$

The author demonstrated the existence of separatrices that separate basins of attraction in the phase plane, which is associated with the oscillation, coexistence and extinction of predator-prey populations.

Through numerical simulations, they showed that system (1.3) undergoes Hopf bifurcation and Bogdanov-Takens bifurcation. Considering system (1.3) with Holling's type II functional response, Arancibia-Ibarra and Flores [16] investigated the different bifurcation and showed that the system exhibits the multi-stability phenomenon. Yin et al. [17] studied a predator-prey model with the Allee effect and prey refuge, and they showed that the system has two limit cycles and a Bogdanov-Takens bifurcation of codimension 3. Some scholars [18–22] have studied the impact of the other functional response functions and Allee effect on the dynamical behaviors of Leslie-Gower systems.

Most scholars have considered the impace of the Allee effect on prey. In fact, the Allee effect on predator populations can also affect the dynamic behavior of the system. For predators, when the population density is low, the success rate of cooperative hunting will decrease; the probability of finding a mate will also decrease, which will lead to a decrease in the population birth rate. Therefore, there is also the Allee effect in predator populations. Recently, the study of predator-prey models with the Allee effect on the predator has attracted the interest of a number of scholars. Fox example, Feng and Kang [23] investigated a predator-prey model with the Allee effect on both prey and predator; they showed that the double Allee effect greatly altered the survival of these two species. Alves and Hilker [24] studied the relationship between hunting cooperation and the Allee effect in the predator population, as well as its impact on the predator-prey system.

Therefore, inspired by [15, 23], we consider a modified Leslie-Gower predator-prey system with double Allee effects on predator and prey, as follows:

$$\begin{aligned} \dot{x} &= rx\left(1 - \frac{x}{K}\right)(x-a) - mxy, \\ \dot{y} &= sy\left(\frac{y}{y+b} - \frac{y}{nx+c}\right), \end{aligned} \tag{1.4}$$

where all parameters are positive and $0 < a < K$. Here, $\frac{y}{y+b}$ represents the Allee effect, and the per capita growth rate of the predator changes from s to $\frac{sy}{y+b}$ with the influence of the Allee effect [25]. Obviously, as b increases, the Allee effect becomes stronger and the per capita growth rate of the predator is slower.

For simplicity, making a dimensionless transformation by using

$$\bar{x} = \frac{x}{K}, \quad \bar{y} = \frac{y}{nK}, \quad \bar{t} = rKt, \quad \bar{a} = \frac{a}{K},$$
$$\bar{m} = \frac{mn}{r}, \quad \bar{s} = \frac{s}{rK}, \quad p = \frac{b}{nK}, \quad \bar{c} = \frac{c}{nK},$$

and dropping the bar, system (1.4) becomes

$$\begin{aligned} \dot{x} &= x(1-x)(x-a) - mxy, \\ \dot{y} &= sy\left(\tfrac{y}{y+p} - \tfrac{y}{x+c}\right), \end{aligned} \tag{1.5}$$

where $0 < a < 1$ and the other parameters are positive. From the biological background, assume that the following initial conditions:

$$(x(0), y(0)) \in \Omega \triangleq \{(x, y) \in \mathbb{R}^2 | x \geq 0, y \geq 0\}.$$

The rest of this paper is organized as follows. We respectively discuss the existence and stability of the equilibria in Sections 2 and 3. In Section 4, we investigate the existence of various bifurcations, such as saddle-node bifurcation, Hopf bifurcation and Bogdanov-Takens bifurcation. In Section 5, we give numerical simulations to show the influence of the Allee effect in the predator population on the dynamical behavior of the system. The paper ends with a brief conclusion.

2. Existence of equilibria

First, we show that all solutions of system (1.5) are positive and bounded if the initial condition is given by $(x(0), y(0)) \in \Omega_+ \triangleq \{(x, y) \in \mathbb{R}^2 | x > 0, y > 0\}$.

Lemma 2.1. *All solutions of system* (1.5) *are positive and bounded for the initial condition* $(x(0), y(0)) \in \Omega_+$.

Proof. System (1.5) has the solution $(x(t), y(t))$, as follows:

$$\begin{aligned} x(t) &= x(0) \exp\left\{\int_0^t \left[(1 - x(\tau))(x(\tau) - a) - my(\tau)\right] d\tau\right\}, \\ y(t) &= y(0) \exp\left\{\int_0^t \left[s\left(\frac{y(\tau)}{y(\tau)+p} - \frac{y(\tau)}{x(\tau)+c}\right)\right] d\tau\right\}. \end{aligned}$$

Since the initial values $x(0) > 0$ and $y(0) > 0$, we get that $x(t) > 0$ and $y(t) > 0$ for $t > 0$. Therefore, solutions of system (1.5) are positive.

If $x \geq 1$, from the prey equation of system (1.5), we obtain that $\dot{x} < x(1 - x)(x - a) \leq 0$, which implies that $\limsup_{t\to\infty} x(t) \leq 1$. According to $\limsup_{t\to\infty} x(t) \leq 1$, we have that $x(t) < 1$ for large values of time t. In addition, from the predator equation of system (1.5), we have that $\dot{y} < sy\left(1 - \frac{y}{1+c}\right)$ for large values of time t. Then, we have $\limsup_{t\to\infty} y(t) < 1 + c$. Hence, solutions of system (1.5) are bounded. The proof is completed.

Next, we study the existence of equilibria of system (1.5). Obviously, system (1.5) has a trivial equilibrium $E_0(0, 0)$ and two predator-free equilibria $E_1(1, 0)$ and $E_2(a, 0)$. When $c > p$, system (1.5) has a prey-free equilibrium $E_3(0, c - p)$. Note that the positive equilibria of system (1.5) satisfy the following equations:

$$\begin{cases} (1 - x)(x - a) - my = 0, \\ \dfrac{1}{y + p} - \dfrac{1}{x + c} = 0. \end{cases} \tag{2.1}$$

From the first equation of (2.1), we can obtain the prey isocline of system (1.5):

$$y_1 = \frac{1}{m}(1 - x)(x - a).$$

From the second equation of (2.1), we get the predator isocline of system (1.5):

$$y_2 = x + c - p.$$

The prey isocline y_1 is a parabola passing through points E_1, E_2 with vertex at $\left(\frac{a+1}{2}, \frac{(a-1)^2}{4m}\right)$. The predator isocline y_2 is a monotonically increasing straight line passing through E_3 with a slope of 1. Also, we calculate that the slope of curve y_1 at E_2 is $\frac{1-a}{m}$.

From (2.1), we obtain

$$f(x) = x^2 + (m - 1 - a)x + cm - pm + a = 0. \tag{2.2}$$

The discriminant of Eq (2.2) is

$$\Delta = (m-1-a)^2 - 4cm + 4pm - 4a.$$

Define

$$x_1^* = \frac{a+1-m-\sqrt{\Delta}}{2}, \quad x_2^* = \frac{a+1-m+\sqrt{\Delta}}{2}, \quad x_* = \frac{a+1-m}{2}.$$

Obviously, the number of intersection points of curves y_1 and y_2 depends on the sign of Δ. If $\Delta > 0$, Eq (2.2) has two real roots x_1^* and x_2^*, which means that the number of intersection points of curves y_1 and y_2 is 2. If $\Delta = 0$, Eq (2.2) has a unique real root x_*; that is, the curve y_1 has only one intersection point with y_2.

Note that the positive equilibria are determined by the intersection of the curves y_1 and y_2 in the first quadrant. Based on the positional relationships of E_1, E_2, E_3 and the slope of curve y_1 at E_2, we consider four cases:

(1) When $p - c < a$,

(1.a) if $\frac{1-a}{m} > 1$, the number of intersections of y_1 and y_2 may be 0, 1 or 2, depending on the sign of Δ. For example, y_1 and y_2 have two intersections (see Figure 1(a)) or only one intersection (see Figure 1(b)) if $\Delta > 0$ or $\Delta = 0$, respectively;

(1.b) if $0 < \frac{1-a}{m} \le 1$, the predator isocline is above the prey isocline in the first quadrant, as shown in Figure 1(c). So, there is no intersection of y_1 and y_2, i.e., system (1.5) has no positive equilibrium.

(2) When $p - c = a$,

(2.a) if $\frac{1-a}{m} > 1$, y_1 and y_2 have only one intersection in the first quadrant; see Figure 1(d). Hence, system (1.5) has only one positive equilibrium;

(2.b) if $0 < \frac{1-a}{m} \le 1$, the predator isocline and the prey isocline have only one intersection at E_2, as shown in Figure 1(e). Hence, there is no intersection in the first quadrant, which implies that system (1.5) has no positive equilibrium.

(3) When $a < p - c < 1$, there is one intersection of y_1 and y_2 in the first quadrant; see Figure 1(f). Hence, system (1.5) has a positive equilibrium.

(4) When $p - c \ge 1$, in the first quadrant, the prey isocline is to the left of the predator isocline, as shown in Figure 1(g) and (h). Therefore, system (1.5) has no positive equilibrium.

Let

$$p^{**} = a + c, \quad p^* = c - c_1^* \text{ and } c_1^* = \frac{m^2 - 2(a+1)m + (a-1)^2}{4m};$$

then,

$$\Delta = 4m(p - p^*).$$

Notice that $p^{**} \ge p^*$. Clearly, when $c_1^* \le 0$, or when $c_1^* > 0$ and $c > c_1^*$, we can get that $p^* > 0$. By a simple calculation, we get that $c_1^* > 0$ if $0 < m < (1-\sqrt{a})^2$ or $m > (1+\sqrt{a})^2$, and $c_1^* \le 0$ if $(1-\sqrt{a})^2 \le m \le (1+\sqrt{a})^2$. Based on the above discussion, we derive the following theorem about the existence of the positive equilibria of system (1.5).

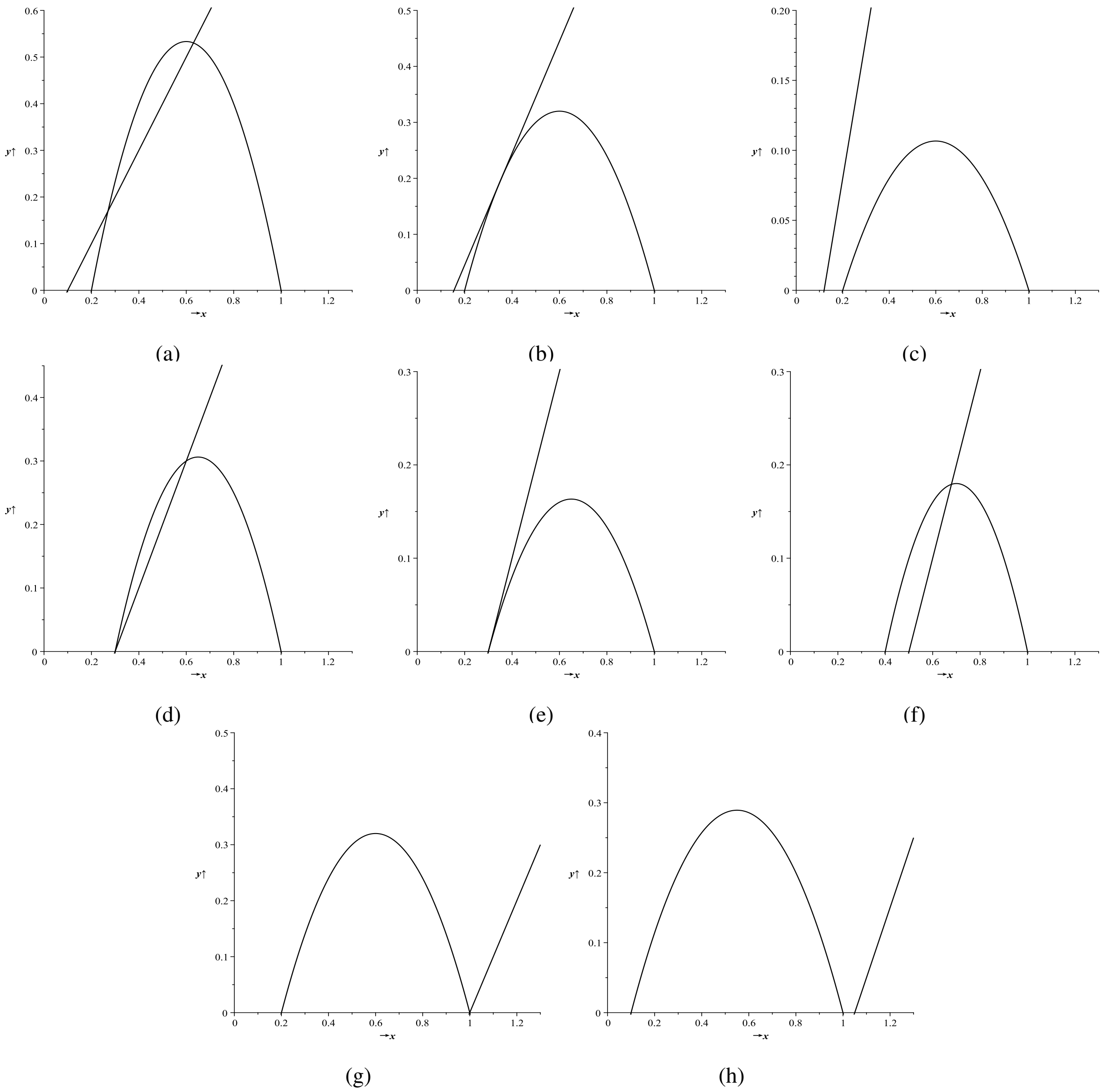


Figure 1. Graphical representation of predator and prey isoclines. (a) $p - c < a$, $\frac{1-a}{m} > 1$, $\Delta > 0$. (b) $p - c < a$, $\frac{1-a}{m} > 1$, $\Delta = 0$. (c) $p - c < a$, $0 < \frac{1-a}{m} \leq 1$. (d) $p - c = a$, $\frac{1-a}{m} > 1$. (e)$p - c = a$, $0 < \frac{1-a}{m} \leq 1$. (f) $a < p - c < 1$. (g) $p - c = 1$. (h) $p - c > 1$.

Theorem 2.1. *The existence of the positive equilibria of system* (1.5) *are classified as follows.*

(1) *Assume that* $0 < m < (1 - \sqrt{a})^2$ *and* $c > c_1^*$, *or* $(1 - \sqrt{a})^2 \leq m < 1 - a$; *then,*

(1.a) *if* $p < p^*$ *or* $p \geq 1 + c$, *system* (1.5) *has no positive equilibrium;*

(1.b) *if* $p = p^*$, *system* (1.5) *has a unique positive equilibrium* $E_*(x_*, y_*)$, *where* $y_* = x_* + c - p$;

(1.c) *if* $p^* < p < p^{**}$, *system* (1.5) *has two positive equilibria* $E_4(x_1^*, y_1^*)$ *and* $E_5(x_2^*, y_2^*)$, *where* $y_1^* = x_1^* + c - p$ *and* $y_2^* = x_2^* + c - p$;

(1.d) *if* $p^{**} \leq p < 1 + c$, *system* (1.5) *has a unique positive equilibria* $E_5(x_2^*, y_2^*)$.

(2) *Assume that* $0 < m < (1 - \sqrt{a})^2$ *and* $c \le c_1^*$*; then,*

(2.a) *if* $p \ge 1 + c$*, system* (1.5) *has no positive equilibrium;*
(2.b) *if* $p^{**} \le p < 1 + c$*, system* (1.5) *has a unique positive equilibrium* $E_5(x_2^*, y_2^*)$*;*
(2.c) *if* $p < p^{**}$*, system* (1.5) *has two positive equilibria* $E_4(x_1^*, y_1^*)$ *and* $E_5(x_2^*, y_2^*)$*.*

(3) *Assume that* $m \ge 1 - a$*; then,*

(3.a) *if* $0 < p \le p^{**}$ *or* $p \ge 1 + c$*, system* (1.5) *has no positive equilibrium;*
(3.b) *if* $p^{**} < p < 1 + c$*, system* (1.5) *has a unique positive equilibrium* $E_5(x_2^*, y_2^*)$*.*

3. Stability of equilibria

Now we discuss the stability of system (1.5). The Jacobian matrix of any equilibrium of system (1.5) is

$$J(E) = \begin{bmatrix} (1-x)(x-a) - my - x(2x - a - 1) & -mx \\ \frac{sy^2}{(x+c)^2} & sy(\frac{2}{y+p} - \frac{y}{(y+p)^2} - \frac{2}{x+c}) \end{bmatrix}. \tag{3.1}$$

The local stability of equilibria is determined by the eigenvalues of the Jacobian matrix (3.1) at each equilibrium.

Theorem 3.1. *The boundary equilibrium* $E_3(0, c - p)$ *is always a stable node if* $c > p$*.*

Proof. The Jacobian matrix of system (1.5) at $E_3(0, c - p)$ is

$$J(E_3) = \begin{bmatrix} -a - m(c-p) & 0 \\ \frac{s(c-p)^2}{c^2} & -\frac{s(c-p)^2}{c^2} \end{bmatrix}.$$

Obviously, $J(E_3)$ has two eigenvalues $\lambda_1 = -a - m(c - p) < 0$ and $\lambda_2 = -\frac{s(c-p)^2}{c^2} < 0$, which implies that E_3 is a stable node (see Figure 2(a)).

Theorem 3.2. *For equilibrium* E_0*, we have the following conditions:*

(1) *if* $p < c$*,* E_0 *is an attracting saddle node, including a hyperbolic sector in the upper half-plane;*
(2) *if* $p = c$*,* E_0 *is a stable degenerate node;*
(3) *if* $p > c$*,* E_0 *is an attracting saddle node, including a parabolic sector in the upper half-plane.*

Proof. The Jacobian matrix of system (1.5) at $E_0(0, 0)$ is

$$J(E_0) = \begin{bmatrix} -a & 0 \\ 0 & 0 \end{bmatrix}.$$

Obviously, $J(E_0)$ has one zero eigenvalue, which means that E_0 is a degenerate equilibrium. Performing a Taylor expansion at the origin and applying $d\tau = -adt$ (still denoting τ as t), then system (1.5) becomes

$$\begin{aligned} \dot{x} &= x - \frac{(a+1)x^2}{a} + \frac{myx}{a} + \frac{x^3}{a}, \\ \dot{y} &= \frac{s(p-c)y^2}{pca} - \frac{sy^2x}{c^2a} + \frac{sy^3}{p^2a} + o(|x, y|^3). \end{aligned} \tag{3.2}$$

According to Theorem 7.1 in Chapter 2 in [26], if $p < c$, E_0 is an attracting saddle node, with a hyperbolic sector in the upper half-plane (see Figure 2(a)). If $p > c$, E_0 is an attracting saddle node, with a parabolic sector in the upper half-plane (see Figure 2(c)). If $p = c$, system (3.2) becomes

$$\begin{aligned} \dot{x} &= x - \frac{(a+1)x^2}{a} + \frac{myx}{a} + \frac{x^3}{a}, \\ \dot{y} &= -\frac{s y^2 x}{c^2 a} + \frac{s y^3}{c^2 a} + o(|x, y|^3). \end{aligned} \tag{3.3}$$

From the center manifold theorem, we assume that $x = \alpha_1 y^2 + \beta_1 y^3 + o(|y|^3)$. Substituting this into the first equation of system (3.3), we get that $\alpha_1 = 0$ and $\beta_1 = 0$. Substituting $x = o(|y|^3)$ into the second equation of system (3.3), we can obtain the reduced system, restricted to the center manifold:

$$\dot{y} = \frac{s y^3}{c^2 a} + o(|y|^3).$$

Clearly, $\frac{s}{c^2 a} > 0$, which implies that E_0 is a stable degenerate node by Theorem 7.1 in [26] (see Figure 2(b)). The proof is completed.

Theorem 3.3. *For the equilibrium E_1, we have the following conditions:*

(1) *if $p < 1 + c$, E_1 is an attracting saddle node, with a hyperbolic sector in the upper half-plane;*
(2) *if $p = 1 + c$, E_1 is a stable degenerate node;*
(3) *if $p > 1 + c$, E_1 is an attracting saddle node, with a parabolic sector in the upper half-plane.*

Proof. The Jacobian matrix of system (1.5) at $E_1(1, 0)$ is

$$J(E_1) = \begin{bmatrix} a-1 & -m \\ 0 & 0 \end{bmatrix},$$

which means that $J(E_1)$ has one zero eigenvalue. So, E_1 is a degenerate equilibrium. Making the transformation $(x_1, y_1) = (x - 1, y)$ to move E_1 to the origin, system (1.5) becomes

$$\begin{aligned} \dot{x}_1 &= (a-1)x_1 - my_1 - mx_1 y_1 + (-2+a)x_1^2 - x_1^3, \\ \dot{y}_1 &= \frac{s(c+1-p)y_1^2}{p(c+1)} + \frac{s y_1^2 x_1}{(c+1)^2} - \frac{s y_1^3}{p^2} + o(|x, y|^3). \end{aligned} \tag{3.4}$$

Letting

$$x_1 = \frac{1}{a-1}x_2 + \frac{m}{a-1}y_2, \quad y_1 = y_2, \quad t = \frac{1}{a-1}\tau,$$

system (3.4) becomes

$$\begin{aligned} \dot{x}_2 &= x_2 + a_{20}x_2^2 + a_{11}x_2 y_2 + a_{02}y_2^2 + a_{30}x_2^3 + a_{12}x_2 y_2^2 + a_{21}x_2^2 y_2 + a_{03}y_2^3 + o(|x_2, y_2|^3), \\ \dot{y}_2 &= b_{02}y_2^2 + b_{12}x_2 y_2^2 + b_{03}y_2^3 + o(|x_2, y_2|^3), \end{aligned} \tag{3.5}$$

where

$$
\begin{aligned}
a_{20} &= \frac{(-2+a)\,m}{(a-1)^2}, \qquad a_{11} = \frac{(a-3)\,m}{(a-1)^2}, \qquad a_{02} = -\frac{(c+1-p)(a-1)\,s+(c+1)\,mp}{(a-1)^2\,(c+1)p},\\
a_{30} &= -\frac{1}{(a-1)^3}, \qquad a_{12} = -\frac{(3c^2m+as+6cm+3m-s)m}{(a-1)^3(1+c)^2}, \qquad a_{21} = -\frac{3m}{(a-1)^3},\\
a_{03} &= \frac{((a-1)^2s-m^2p^2)(c+1)^2m-(a-1)p^2m^2s}{(a-1)^3(1+c)^2p^2}, \qquad b_{02} = \frac{(1+c-p)s}{(a-1)(1+c)p},\\
b_{12} &= \frac{s}{(a-1)^2(1+c)^2}, \qquad b_{03} = -\frac{((c+1)^2(a-1)-m\,p^2)s}{(a-1)^2(1+c)^2p^2}.
\end{aligned}
$$

According to Theorem 7.1 in [26], if $b_{02} \neq 0$, i.e., $p \neq 1+c$, E_1 is an attracting saddle node (see Figure 2(a), (b) and (c)). If $p = 1+c$, system (3.5) is reduced to the following system:

$$
\begin{aligned}
\dot{\bar{x}}_2 &= \bar{x}_2 + a_{20}\bar{x}_2^2 + a_{11}\bar{x}_2\bar{y}_2 + \bar{a}_{02}\bar{y}_2^2 + a_{30}\bar{x}_2^3 + a_{12}\bar{x}_2\bar{y}_2^2 + a_{21}\bar{x}_2^2\bar{y}_2 + \bar{a}_{03}\bar{y}_2^3 + o(|\bar{x}_2,\bar{y}_2|^3),\\
\dot{\bar{y}}_2 &= b_{12}\bar{x}_2\bar{y}_2^2 + \bar{b}_{03}\bar{y}_2{}^3 + o(|\bar{x}_2,\bar{y}_2|^3),
\end{aligned} \tag{3.6}
$$

where

$$
\begin{aligned}
\bar{a}_{02} &= -\frac{m^2}{(a-1)^2}, \qquad \bar{a}_{03} = \frac{(a-1)(a-m-1)sm-m^3(c+1)^2}{(a-1)^3(c+1)^2},\\
\bar{b}_{03} &= -\frac{(a-m-1)s}{(c+1)^2(a-1)^2}.
\end{aligned}
$$

Based on the center manifold theorem, let $\bar{x}_2 = \eta_1\bar{y}_2^2 + o(|\bar{y}_2|^2)$, and substitute it into the first equation of system (3.6); then, we have that $\eta_1 = \frac{m^2}{(a-1)^2}$. Substituting $\bar{x}_2 = \frac{m^2}{(a-1)^2}\bar{y}_2^2 + o(\bar{y}_2^2)$ into the second equation of system (3.3), we obtain the reduced system restricted to the center manifold, as follows:

$$
\dot{\bar{y}}_2 = \frac{(1-a+m)s}{(c+1)^2(a-1)^2}\bar{y}_2^3 + o(|\bar{y}_2|^3).
$$

E_1 is a stable degenerate node (see Figure 2(d)) since $a < 1$ (see [26]). The proof is completed.

Theorem 3.4. *For the equilibrium E_2, we have the following conclusions.*

(1) *When $p > a+c$, E_2 is a repelling saddle node, with a hyperbolic sector in the upper half-plane.*
(2) *When $p = a+c$,*
   (a) *E_2 is an unstable degenerate node if $m > 1-a$;*
   (b) *E_2 is a repelling saddle node, with a parabolic sector in the upper half-plane if $m = 1-a$;*
   (c) *E_2 is a degenerate saddle if $m < 1-a$.*
(3) *If $p < a+c$, E_2 is a repelling saddle node, with a parabolic sector in the upper half-plane.*

Proof. The Jacobian matrix of system (1.5) at $E_2(a,0)$ is

$$
J(E_2) = \begin{bmatrix} (1-a)a & -am \\ 0 & 0 \end{bmatrix}.
$$

The two eigenvalues of $J(E_2)$ are $\lambda_1 = (1-a)a$ and $\lambda_2 = 0$; that is, E_2 is a degenerate equilibrium. By the transformation $(x_1, y_1) = (x-a, y)$, we have

$$\begin{aligned} \dot{x}_1 &= (1-a)ax_1 - may_1 - mx_1y_1 + (-2a+1)x_1^2 - x_1^3, \\ \dot{y}_1 &= \tfrac{s(c+a-p)y_1^2}{p(c+a)} + \tfrac{sy_1^2x_1}{(c+a)^2} - \tfrac{sy_1^3}{p^2} + \tfrac{sy_1^2x_1^2}{(a+c)^3} + \tfrac{sy_1^4}{p^3} + o(|x_1, y_1|^4). \end{aligned} \tag{3.7}$$

Next, making the transformation

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} am & am \\ 0 & (1-a)a \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix},$$

and applying $d\tau = (1-a)adt$ (still rewriting τ as t), system (3.7) becomes the following system:

$$\begin{aligned} \dot{x}_2 &= x_2 + c_{20}x_2^2 + c_{11}x_2y_2 + c_{02}y_2^2 + c_{30}x_2^3 + c_{12}x_2y_2^2 + c_{21}x_2^2y_2 + c_{03}y_2^3 \\ &\quad + c_{22}x_2^2y_2^2 + c_{13}x_2y_2^3 + c_{04}y_2^4 + o(|x_2, y_2|^4), \\ \dot{y}_2 &= d_{02}y_2^2 + d_{12}x_2y_2^2 + d_{03}y_2^3 + d_{22}x_2^2y_2^2 + d_{13}x_2y_2^3 + d_{04}y_2^4 + o(|x_2, y_2|^4), \end{aligned} \tag{3.8}$$

where

$$\begin{aligned} c_{20} &= \frac{(2a-1)\,m}{a-1}, \qquad c_{02} = \frac{amp(a+c) - s(a-1)(a+c-p)}{(a+c)(a-1)\,p}, \qquad c_{11} = \frac{(3a-1)\,m}{a-1}, \\ c_{12} &= \frac{(3a^2m + 6acm + 3c^2m - sa + s)\,am}{(a+c)^2(a-1)}, \qquad c_{21} = \frac{3a\,m^2}{a-1}, \qquad c_{30} = \frac{a\,m^2}{a-1}, \\ c_{03} &= -\frac{((a-1)^2(a+c)^2s + ((a-1)s - (a+c)^2m)\,p^2m)\,a}{(a-1)(a+c)^2p^2}, \qquad c_{22} = \frac{s\,a^2m^2}{(a+c)^3}, \\ c_{13} &= \frac{2s\,a^2m^2}{(a+c)^3}, \qquad c_{04} = -\frac{((a-1)^2(a+c)^3 - m^2p^3)a^2s}{(a+c)^3p^3}, \qquad d_{02} = \frac{(c+a-p)\,s}{(c+a)\,p}, \\ d_{12} &= \frac{asm}{(a+c)^2}, \qquad d_{03} = \frac{((a-1)(a+c)^2 + m\,p^2)\,as}{(a+c)^2p^2}, \qquad d_{22} = -\frac{s\,a^2m^2}{(a+c)^3}, \\ d_{13} &= -\frac{2s\,a^2m^2}{(a+c)^3}, \qquad d_{04} = \frac{((a-1)^2(a+c)^3 - m^2p^3)a^2s}{(a+c)^3p^3}. \end{aligned}$$

From Theorem 7.1 in [26], if $d_{02} \neq 0$, i.e., $p \neq a+c$, E_2 is a repelling saddle node (see Figure 2(a), (b) and (d)). Assume that $p = a+c$; system (3.8) becomes

$$\begin{aligned} \dot{\bar{x}}_2 &= \bar{x}_2 + c_{20}\bar{x}_2^2 + c_{11}\bar{x}_2\bar{y}_2 + \bar{c}_{02}\bar{y}_2^2 + c_{30}\bar{x}_2^3 + c_{12}\bar{x}_2\bar{y}_2^2 + c_{21}\bar{x}_2^2\bar{y}_2 + \bar{c}_{03}\bar{y}_2^3 \\ &\quad + c_{22}\bar{x}_2^2\bar{y}_2^2 + c_{13}\bar{x}_2\bar{y}_2^3 + \bar{c}_{04}\bar{y}_2^4 + o(|\bar{x}_2, \bar{y}_2|^4), \\ \dot{\bar{y}}_2 &= d_{12}\bar{x}_2\bar{y}_2^2 + \bar{d}_{03}\bar{y}_2^3 + d_{22}\bar{x}_2^2\bar{y}_2^2 + d_{13}\bar{x}_2\bar{y}_2^3 + \bar{d}_{04}\bar{y}_2^4 + o(|\bar{x}_2, \bar{y}_2|^4), \end{aligned} \tag{3.9}$$

where

$$\begin{aligned} \bar{c}_{02} &= \frac{am}{a-1}, \qquad \bar{c}_{03} = \frac{(m^2(a+c)^2 - (a-1)(a-1+m)s)\,a}{(a-1)(a+c)^2}, \qquad \bar{d}_{03} = \frac{(a+m-1)\,as}{(a+c)^2}, \\ \bar{c}_{04} &= -\frac{(a^2 - m^2 - 2a + 1)a^2s}{(a+c)^3}, \qquad \bar{d}_{04} = \frac{(a^2 - m^2 - 2a + 1)a^2s}{(a+c)^3}. \end{aligned}$$

By using the center manifold theorem and the first equation of system (3.9), we have

$$\bar{x}_2 = \frac{am}{a-1}\bar{y}_2^2 + \frac{a((a-1)^2(a-1+m)s + 2a\,m^2(a+c)^2)}{(a-1)^2(a+c)^2}\bar{y}_2^3 + o(\bar{y}_2^3). \tag{3.10}$$

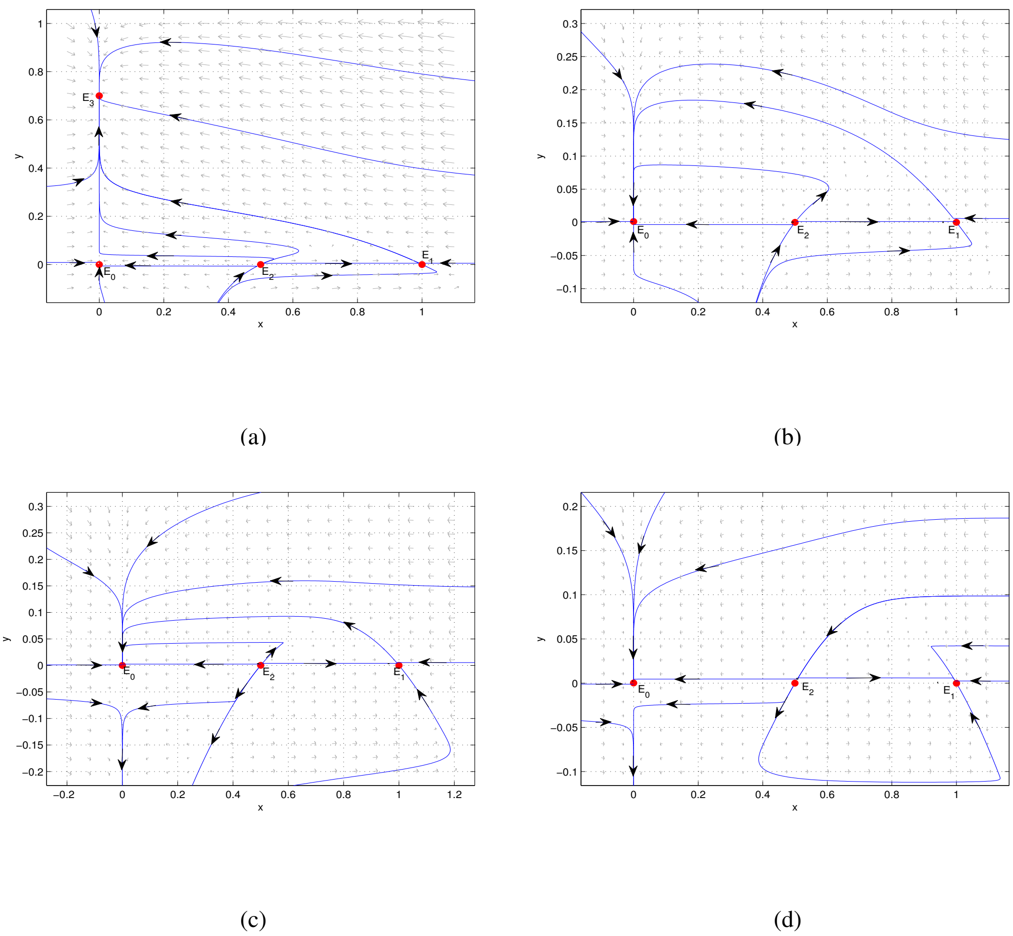
Substituting (3.10) into the second equation of system (3.9), we obtain the reduced system, restricted to the center manifold, as follows:

$$\dot{\bar{y}}_2 = \frac{(a+m-1)as}{(a+c)^2}\bar{y}_2^3 + \frac{a^2s((1-2a-c)m^2+(a-1)^3)}{(a+c)^3(a-1)}\bar{y}_2^4 + o(\bar{y}_2^4). \tag{3.11}$$

Since $a < 1$, E_2 is an unstable degenerate node (or degenerate saddle) if $m > 1-a$ (or $m < 1-a$). When $m = 1-a$, (3.11) becomes

$$\dot{\bar{y}}_2 = \frac{a^2s(1-a)}{(a+c)^2}\bar{y}_2^4 + o(\bar{y}_2^4).$$

Hence, E_2 is a repelling saddle node since $\frac{a^2s(1-a)}{(a+c)^2} > 0$ (see [26]). The proof is completed.



(a) (b) (c) (d)

Figure 2. (a) E_0, E_1 are both attracting saddle nodes, E_2 is a repelling saddle node and E_3 is a stable node with $a = 0.5$, $m = 0.8$, $s = 0.6$, $c = 1.2$, $p = 0.5$. (b) E_0 is a stable node, E_1 is an attracting saddle node and E_2 is a repelling saddle node with $a = 0.5$, $m = 0.8$, $s = 0.6$, $c = 0.5$, $p = 0.5$. (c) E_0, E_1 is an attracting saddle node and E_2 is an unstable node with $a = 0.5$, $m = 0.8$, $s = 0.6$, $c = 0.5$, $p = 1$. (d) E_0 is an attracting saddle node, E_1 is a stable node and E_2 is a repelling saddle node with $a = 0.5$, $m = 0.8$, $s = 0.6$, $c = 0.5$, $p = 1.5$.

Next, we study the stability of the positive equilibria of system (1.5). The Jacobian matrix of system (1.5) at positive equilibria is

$$J(E) = \begin{bmatrix} -x(2x-a-1) & -mx \\ \frac{(x+c-p)^2 s}{(x+c)^2} & -\frac{(x+c-p)^2 s}{(x+c)^2} \end{bmatrix}. \tag{3.12}$$

The determinant and trace of (3.12) are, respectively,

$$Det(J(E)) = -\frac{xs(x+c-p)^2(-2x+a+1-m)}{(x+c)^2}$$

and

$$Tr(J(E)) = -x(2x-a-1) - \frac{(x+c-p)^2 s}{(x+c)^2}.$$

By simple calculation, we get

$$Det(J(E_4)) = -\frac{x_1^* s(x_1^*+c-p)^2\sqrt{\Delta}}{(x_1^*+c)^2} < 0,$$

which means that E_4 is a saddle. Hence, we have the following theorem about the stability of positive equilibria E_4 and E_5.

Theorem 3.5. *Assume that E_4 exists; then, E_4 is a saddle.*

Define

$$\bar{s} = \frac{x_2^*(a-2x_2^*+1)(x_2^*+c)^2}{(x_2^*+c-p)^2}.$$

Theorem 3.6. *Assume that E_5 exists; then, E_5 is*

(1) *stable if $\bar{s} \le 0$ or $s > \bar{s} > 0$,*
(2) *unstable if $0 < s < \bar{s}$,*
(3) *a focus or center if $s = \bar{s} > 0$.*

Proof. By simple calculation, we obtain

$$Det(J(E_5)) = \frac{x_2^* s(x_2^*+c-p)^2\sqrt{\Delta}}{(x_2^*+c)^2} > 0,$$

and

$$Tr(J(E_5)) = \frac{(x_2^*+c-p)^2}{(x_2^*+c)^2}(\bar{s}-s).$$

So, if $\bar{s} \le 0$, we get that $Tr(J(E_5)) < 0$; that is, E_5 is stable. If $0 < s < \bar{s}$, we get that $Tr(J(E_5)) > 0$, which means that E_5 is unstable. If $s > \bar{s} > 0$, we get that $Tr(J(E_5)) < 0$, which implies that E_5 is stable. If $s = \bar{s} > 0$, we get that $Tr(J(E_5)) = 0$, which implies that the eigenvalues of $J(E_5)$ are a pair of conjugate complex roots and E_5 is a focus or center. The proof is completed.

Define

$$s_* = \frac{2m^3(-m+1+a)(-m+1+a+2c)^2}{(a^2-m^2-2a+1)^2}.$$

If the conditions of Theorem 2.1(1.b) hold, then $s_* > 0$. When $p = p^*$, by simple computation, we obtain

$$Det(J(E_*)) = 0$$

and

$$Tr(J(E_*)) = \frac{(a^2-m^2-2a+1)^2}{4m^2(-m+1+a+2c)^2}(s_* - s),$$

which implies that $J(E_*)$ has at least one zero eigenvalue. From the conditions of Theorem 2.1(1.b), we have that $a^2 - m^2 - 2a + 1 = (a - 1 - m)(a - 1 + m) > 0$ and $-m + 1 + a + 2c > 0$. Then, the sign of $Tr(J(E_*))$ is determined by $s_* - s$.

Now, the following theorem shows that E_* is a saddle node if $s \neq s_*$.

Theorem 3.7. *Assume that the conditions of Theorem* 2.1(1.b) *hold. Moreover,*

(1) *if $s > s_*$, E_* is an attracting saddle node;*
(2) *if $s < s_*$, E_* is a repelling saddle node.*

Proof. We move E_* to the origin by applying the transformations $x_1 = x - x_*$ and $y_1 = y - y_*$; then, system (1.5) becomes

$$\begin{aligned} \dot{x}_1 &= e_{10}x_1 + e_{01}y_1 + e_{20}x_1^2 + e_{11}x_1y_1 + o(|x_1, y_1|^2), \\ \dot{y}_1 &= f_{10}x_1 + f_{01}y_1 + f_{20}x_1^2 + f_{11}x_1y_1 + f_{02}y_1^2 + o(|x_1, y_1|^2), \end{aligned} \tag{3.13}$$

where

$$\begin{aligned} e_{10} &= \frac{m(-m+1+a)}{2}, \quad e_{01} = -\frac{m(-m+1+a)}{2}, \quad e_{20} = \frac{3m-a-1}{2}, \quad e_{11} = -m, \\ f_{10} &= \frac{s(a^2-m^2-2a+1)^2}{4m^2(-m+1+a+2c)^2}, \quad f_{01} = -\frac{s(a^2-m^2-2a+1)^2}{4m^2(-m+1+a+2c)^2}, \\ f_{20} &= -\frac{s(a^2-m^2-2a+1)^2}{2m^2(-m+1+a+2c)^3}, \quad f_{11} = \frac{2s(a^2-m^2-2a+1)}{m(-m+1+a+2c)^2}, \\ f_{02} &= \frac{s(a^2-m^2-2a+1)((a-1)^2 - m(4a+8c-3m+4))}{2m^2(-m+1+a+2c)^3}. \end{aligned}$$

Assume that $s \neq s_*$, that is $Tr(J(E_*)) \neq 0$. By applying the transformation

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 1 & \frac{m(-m+1+a)}{2} \\ 1 & \frac{s(a^2-m^2-2a+1)^2}{4m^2(-m+1+a+2c)^2} \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$$

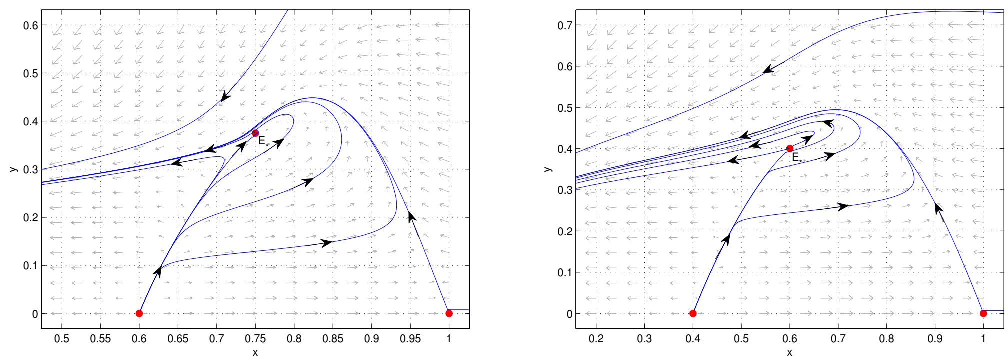
and $d\tau = \frac{s_*-s}{4m^2(-m+1+a+2c)^2}dt$ (still denoting τ as t), system (3.13) becomes

$$\begin{aligned} \dot{x}_2 &= \bar{e}_{20}x_2^2 + \bar{e}_{11}x_2y_2 + \bar{e}_{02}y_2^2 + o(|x_2, y_2|^2), \\ \dot{y}_2 &= y_2 + \bar{f}_{20}x_2^2 + \bar{f}_{11}x_2y_2 + \bar{f}_{02}y_2^2 + o(|x_2, y_2|^2), \end{aligned}$$

where

$$\begin{aligned}
\bar{e}_{20} &= \frac{(m-1+a)^2(-m+1+a)(-m-1+a)^2 s}{8m^2(-m+1+a+2c)^2(s-s_*)^2},\\
\bar{e}_{11} &= \frac{s(a^2-m^2-2a+1)^2A_1}{16m^3(-m+1+a+2c)^5(s-s_*)^2}, \quad \bar{f}_{11} = \frac{A_3}{4m^4(-m+1+a+2c)^5(s-s_*)^2},\\
\bar{e}_{02} &= -\frac{(-m+1+a)s(a^2-m^2-2a+1)^2A_2}{64m^5(-m+1+a+2c)^7(s-s_*)^2}, \quad \bar{f}_{20} = \frac{m-1-a}{2},\\
\bar{f}_{02} &= \frac{A_4}{32m^6(-m+1+a+2c)^7(s-s_*)^2}
\end{aligned}$$

and the coefficients $A_i\,(i = 1, 2, 3, 4)$ are given in Appendix A. Using the conditions of Theorem 2.1(1.b), we get that $\bar{e}_{20} > 0$. From Theorem 7.1 in [26], E_* is a saddle node. Considering the time variable, E_* is an attracting saddle node if $s > s_*$ (see Figure 3(a)), and a repelling saddle node if $0 < s < s_*$ (see Figure 3(b)). The proof is completed.



(a) (b)

Figure 3. (a) E_* is an attracting saddle node with $a = \frac{3}{5}$, $m = \frac{1}{10}$, $s = 2$, $c = 1$, $p = \frac{11}{8}$. (b) E_* is a repelling saddle node with $a = \frac{2}{5}$, $m = \frac{1}{5}$, $s = \frac{4}{5}$, $c = \frac{6}{5}$, $p = \frac{7}{5}$.

When the conditions of Theorem 2.1(1.b) hold and $s = s_*$, that is, $Tr(J(E_*)) = 0$, $J(E_*)$ has two zero eigenvalues. Using the following two lemmas, E_* is a cusp of codimension 2 or 3.

Lemma 3.1 ([27]). *The system given by*

$$\begin{aligned}
\dot{x} &= y + Ax^2 + Bxy + Cy^2 + o(|x,y|^2),\\
\dot{y} &= Dx^2 + Exy + Fy^2 + o(|x,y|^2)
\end{aligned}$$

is equivalent to

$$\begin{aligned}
\dot{x} &= y,\\
\dot{y} &= Dx^2 + (E+2A)xy + o(|x,y|^2)
\end{aligned}$$

by some nonsingular transformations in the neighborhood of $(0, 0)$.

Lemma 3.2 ([27]). *The system given by*

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= x^2 + a_{30}x^4 + y(a_{21}x^2 + a_{31}x^3) + y^2(a_{12}x + a_{22}x^2) + o(|x, y|^4) \end{aligned}$$

is equivalent to

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= x^2 + Gx^3y + o(|x, y|^4) \end{aligned}$$

by some nonsingular transformations in the neighborhood of $(0, 0)$*, where* $G = a_{31} - a_{30}a_{21}$.

Define

$$\begin{aligned} c_2^* &= -\frac{(1 + a - m)((3a + 3)m^2 + (a - 1)^2(a + 1 - 4m))}{2(a - 1)^2(a + 1 - 2m) + 2m^2(3a - 2m + 3)}, \\ \Delta_1 &= 4(a - 1)^2(a^2 - 14a + 1), \\ M_1 &= \frac{4\,(a - 1)^2 + \sqrt{\Delta_1}}{6a + 6}. \end{aligned}$$

Theorem 3.8. *Assume that* $p = p^*$ *and* $s = s_*$*.* E_* *is a cusp of codimension 2 if one of the following conditions holds:*

(1) $0 < a < 7 - 4\sqrt{3}$*, and either*
  (1.a) $0 < m \leq \frac{1+a}{2}$ *and* $c > c_1^*$*;*
  (1.b) $\frac{1+a}{2} < m < (1 - \sqrt{a})^2$*,* $c > c_1^*$ *and* $c \neq c_2^*$*;*
  (1.c) $(1 - \sqrt{a})^2 \leq m < M_1$ *and* $c \neq c_2^*$*;*
  (1.d) $M_1 \leq m < 1 - a$*;*

(2) $7 - 4\sqrt{3} \leq a < 1$.

Proof. If $s = s_*$, then $J(E_*)$ has two zero eigenvalues and system (3.13) can be written as follows:

$$\begin{aligned} \dot{x}_1 &= g_{10}x_1 + g_{01}y_1 + g_{20}x_1^2 + g_{11}x_1y_1 + o(|x_1, y_1|^2), \\ \dot{y}_1 &= h_{10}x_1 + h_{01}y_1 + h_{20}x_1^2 + h_{11}x_1y_1 + h_{02}y_1^2 + o(|x_1, y_1|^2), \end{aligned} \tag{3.14}$$

where

$$\begin{aligned} g_{10} &= \frac{m(-m + 1 + a)}{2}, \quad g_{01} = -\frac{m(-m + 1 + a)}{2}, \quad g_{20} = \frac{3m - a - 1}{2}, \\ g_{11} &= -m, \quad h_{10} = \frac{m(-m + 1 + a)}{2}, \quad h_{01} = -\frac{m(-m + 1 + a)}{2}, \\ h_{20} &= -\frac{m(-m + 1 + a)}{-m + 1 + a + 2c}, \quad h_{11} = \frac{4m^2(-m + 1 + a)}{a^2 - m^2 - 2a + 1}, \\ h_{02} &= \frac{(-m + 1 + a)m((a - 1)^2 - m(4a + 8c - 3m + 4))}{(a^2 - m^2 - 2a + 1)(-m + 1 + a + 2c)}. \end{aligned}$$

Taking the transformation

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} -\frac{m(-m+1+a)}{2} & 0 \\ -\frac{m(-m+1+a)}{2} & 1 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix},$$

we get

$$\begin{array}{rcl}\dot{X} & = & Y + \bar{g}_{20}X^2 + \bar{g}_{11}XY + o(|X,Y|^2),\\ \dot{Y} & = & \bar{h}_{20}X^2 + \bar{h}_{11}XY + \bar{h}_{02}Y^2 + o(|X,Y|^2),\end{array} \tag{3.15}$$

where

$$\bar{g}_{20} = \frac{m(-m+1+a)^2}{4}, \quad \bar{g}_{11} = -m, \quad \bar{h}_{20} = \frac{m^2(-m+1+a)^3}{8},$$
$$\bar{h}_{11} = \frac{m^2(m-1-a)[2c((a-1)^2 - m(4a-3m+4)) + (-m+1+a)(3(a-1)^2 - m(4a-m+4))]}{2(m-1+a)(a-m-1)(-m+1+a+2c)},$$
$$\bar{h}_{02} = \frac{m(-m+1+a)(a^2 - 4ma - 8mc + 3m^2 - 2a - 4m + 1)}{(m-1+a)(a-m-1)(-m+1+a+2c)}.$$

According to Lemma 3.1, system (3.15) is equivalent to

$$\begin{array}{rcl}\dot{X} & = & Y,\\ \dot{Y} & = & \bar{h}_{20}X^2 + EXY + o(|X,Y|^2),\end{array}$$

where

$$E = \bar{h}_{11} + 2\bar{g}_{20} = \frac{m(-m+1+a)(c-c_2^*)A_5}{2(m-1+a)(a-m-1)(-m+1+a+2c)},$$

with

$$A_5 = 2(a-1)^2(a+1-2m) + 2m^2(3a+3-2m),$$
$$g(m) = (3a+3)\,m^2 - 4(a-1)^2\,m + (a+1)(a-1)^2,$$
$$c_2^* = -\frac{(-m+1+a)\,g(m)}{A_5}.$$

If the conditions of Theorem 2.1(1.b) hold, we have that $\frac{m(-m+1+a)}{2(m-1+a)(a-m-1)(-m+1+a+2c)} > 0$ and $\bar{h}_{20} > 0$.

Define

$$\tilde{A}_5 = (10a+2)m^2 - 4(a-1)^2m + 2(a-1)^2(a+1).$$

By simple calculation, the discriminant of $\tilde{A}_5$ is $\Delta_{\tilde{A}_5} = -64(a-1)^2(a+2)a < 0$, that is, $\tilde{A}_5 > 0$. When $0 < m < 1-a$, we have that $A_5 - \tilde{A}_5 = 4m^2(1-a-m) > 0$, i.e., $A_5 > 0$. Hence, the sign of c_2^* depends on $g(m)$. If $g(m) < 0\,(=0, >0)$, then $c_2^* > 0\,(=0, <0)$. The discriminant of $g(m)$ is

$$\Delta_1 = 4\,(a-1)^2\left(a^2 - 14a + 1\right).$$

If $\Delta_1 \le 0$, that is, $7 - 4\sqrt{3} \le a < 1$, we have that $g(m) \ge 0$. Then, $E > 0$, that is, E_* is a cusp of codimension 2 for $7 - 4\sqrt{3} \le a < 1$ [28].

If $\Delta_1 > 0$, that is, $0 < a < 7 - 4\sqrt{3}$, $g(m)$ has two positive roots $M_0 \triangleq \frac{4\,(a-1)^2 - \sqrt{\Delta_1}}{6a+6}$ and $M_1 \triangleq \frac{4\,(a-1)^2 + \sqrt{\Delta_1}}{6a+6}$. When $0 < a < 7 - 4\sqrt{3}$, we obtain the following: $g(\frac{a+1}{2}) = -\frac{1}{4}(a+1)(a^2 - 14a + 1) < 0$, $g((1-\sqrt{a})^2) = -4\sqrt{a}(a - 4\sqrt{a} + 1)(\sqrt{a} - 1)^2 < 0$, $g(1-a) = 8a(a-1)^2 > 0$, $(1-\sqrt{a})^2 - \frac{a+1}{2} = \frac{a - 4\sqrt{a} + 1}{2} > 0$. Therefore, $M_0 < \frac{a+1}{2} < (1-\sqrt{a})^2 < M_1 < 1-a$ for $0 < a < 7 - 4\sqrt{3}$.

By calculation, we have

$$c_1^* - c_2^* = \frac{(a+1-2m)(a-1+m)^2(a-m-1)^2}{2mA_5}.$$

Assume that $0 < a < 7 - 4\sqrt{3}$ holds. According to the conditions of Theorem 2.1(1.b), $E \neq 0$ if one of the following conditions holds:

(1) $0 < m \leq \frac{1+a}{2}$, $c > c_1^*$;
(2) $\frac{1+a}{2} < m < (1-\sqrt{a})^2$, $c > c_1^*$, $c \neq c_2^*$;
(3) $(1-\sqrt{a})^2 \leq m < M_1$, $c \neq c_2^*$;
(4) $M_1 \leq m < 1-a$,

which implies that E_* is a cusp of codimension 2 [28] (see Figure 4(a)). The proof is completed.

From the proof of Theorem 3.8, if $0 < a < 7 - 4\sqrt{3}$, $\frac{1+a}{2} < m < M_1$ and $c = c_2^*$, we have that $E = 0$. Define

$$\begin{aligned} B_0 \triangleq & -12m^5 + 32(a+1)m^4 - (41a^2 + 2a + 41)m^3 + 2(a+1)(17a^2 - 30a + 17)m^2 \\ & - 15(a-1)^2(a+1)^2 m + 2(a-1)^2(a+1)^3. \end{aligned}$$

Hence, the following theorem shows that E_* is a cusp of codimension of at least 4.

Theorem 3.9. *Assume that* $0 < a < 7 - 4\sqrt{3}$, $\frac{1+a}{2} < m < M_1$, $p = p^*$, $s = s_*$ *and* $c = c_2^*$ *hold. Moreover,*
(1) if $B_0 \neq 0$, E_* *is a cusp of codimension 3;*
(2) if $B_0 = 0$, E_* *is a cusp of codimension of at least 4.*

Proof. If $c = c_2^*$, system (3.14) becomes

$$\begin{aligned} \dot{x}_1 &= \tfrac{m(a+1-m)}{2}x_1 + \tfrac{m(a+1-m)}{2}y_1 - mx_1y_1 - \tfrac{(a-3m+1)}{2}x_1^2 - x_1^3, \\ \dot{y}_1 &= b_{10}x_1 + b_{01}y_1 + b_{20}x_1^2 + b_{11}x_1y_1 + b_{02}y_1^2 + b_{30}x_1^3 + b_{12}x_1y_1^2 + b_{21}x_1^2y_1 \\ &\quad + b_{03}y_1^3 + b_{40}x_1^4 + b_{22}x_1^2y_1^2 + b_{31}x_1^3y_1 + b_{04}y_1^4 + o(|x_1, y_1|^4), \end{aligned} \tag{3.16}$$

where

$$\begin{aligned} b_{10} &= \frac{m(a+1-m)}{2}, \quad b_{01} = -\frac{m(a+1-m)}{2}, \quad b_{20} = -\frac{A_5}{4(a^2-m^2-2a+1)}, \\ b_{11} &= \frac{4m^2(a+1-m)}{a^2-m^2-2a+1}, \quad b_{02} = \frac{(a-1)^2(a+1-2m) + m^2(6m-5a-5)}{2(a^2-m^2-2a+1)}, \\ b_{30} &= \frac{A_5^2}{8m(a+1-m)(a^2-m^2-2a+1)^2}, \quad b_{21} = -\frac{2mA_5}{(a^2-m^2-2a+1)^2}, \\ b_{12} &= \frac{8m^3(a+1-m)}{(a^2-m^2-2a+1)^2}, \quad b_{40} = -\frac{A_5^3}{16m^2(a+1-m)^2(a^2-m^2-2a+1)^3}, \\ b_{03} &= -\frac{(a+1-2m)^2}{2m(a+1-m)}, \quad b_{31} = \frac{A_5^2}{(a+1-m)(a^2-m^2-2a+1)^3}, \\ b_{22} &= -\frac{4m^2A_5}{(a^2-m^2-2a+1)^3}, \quad b_{40} = \frac{(1+a-2m)^2A_5}{4m^2(a+1-m)^2(a^2-m^2-2a+1)}. \end{aligned}$$

Letting

$$\begin{aligned} x_2 &= x_1, \\ y_2 &= \tfrac{m(a+1-m)}{2}x_1 + \tfrac{m(a+1-m)}{2}y_1 - mx_1y_1 - \tfrac{(a-3m+1)}{2}x_1^2 - x_1^3, \end{aligned}$$

system (3.16) can be rewritten as

$$\begin{aligned}\dot{x}_2 &= y_2,\\ \dot{y}_2 &= c_{20}x_2^2 + c_{02}y_2^2 + c_{30}x_2^3 + c_{12}x_2y_2^2 + c_{21}x_2^2y_2 + c_{03}y_2^3 + c_{40}x_2^4 + c_{13}x_2y_2^3\\ &\quad + c_{22}x_2^2y_2^2 + c_{31}x_2^3y_2 + c_{04}y_2^4 + o(|x_2, y_2|^4),\end{aligned} \tag{3.17}$$

where the coefficients of system (3.17) are given in Appendix B. Next, applying the transformations $x_3 = x_2$ and $y_3 = y_2(1 - c_{02}x_2)$ and $d\tau = \frac{1}{1-c_{02}x_2}dt$ (rewritten τ as t), system (3.17) becomes

$$\begin{aligned}\dot{x}_3 &= y_3,\\ \dot{y}_3 &= d_{20}x_3^2 + d_{30}x_3^3 + d_{12}x_3y_3^2 + d_{21}x_3^2y_3 + d_{03}y_3^3 + d_{40}x_3^4 + d_{13}x_3y_3^3 + d_{22}x_3^2y_3^2\\ &\quad + d_{31}x_3^3y_3 + d_{04}y_3^4 + o(|x_3, y_3|^4),\end{aligned} \tag{3.18}$$

where

$$\begin{aligned}d_{20} &= c_{20},\ d_{30} = c_{30} - 2c_{20}c_{02},\ d_{12} = c_{12} - c_{02}^2,\ d_{21} = c_{21},\ d_{03} = c_{03},\ d_{04} = c_{04},\\ d_{22} &= c_{22} - c_{02}^3,\ d_{40} = c_{02}^2c_{20} - 2c_{02}c_{30} + c_{40},\ d_{13} = c_{02}c_{03} + c_{13},\ d_{31} = c_{31} - c_{21}c_{02}.\end{aligned}$$

Through the following two transformations:

$$\begin{aligned}&x_3 = x_4 + \frac{d_{03}}{2}x_4^2y_4 + \frac{d_{13}}{6}x_4^3y_4 + \frac{d_{04}}{2}x_4^2y_4^2, \quad y_3 = y_4 + d_{03}x_4y_4^2 + \frac{d_{13}}{2}x_4^2y_4^2 + d_{04}x_4y_4^3,\\ &x_4 = x_5, \quad y_4 = y_5 + \frac{d_{20}d_{03}}{2}x_5^4,\end{aligned}$$

system (3.18) is transformed into

$$\begin{aligned}\dot{x}_5 &= y_5,\\ \dot{y}_5 &= e_{20}x_5^2 + e_{30}x_5^3 + e_{12}x_5y_5^2 + e_{21}x_5^2y_5 + e_{40}x_5^4 + e_{22}x_5^2y_5^2 + e_{31}x_5^3y_5 + o(|x_5, y_5|^4),\end{aligned} \tag{3.19}$$

where

$$e_{20} = d_{20},\ e_{30} = d_{30},\ e_{12} = d_{12},\ e_{21} = d_{21},\ e_{40} = d_{40},\ e_{22} = d_{22},\ e_{31} = d_{31} - 3d_{20}d_{03}.$$

Note that $e_{20} = -\frac{m(-m+1+a)^2}{4} < 0$. Making the following transformation

$$x_6 = -x_5,\ y_6 = -\frac{y_5}{\sqrt{-e_{20}}},\ \tau = \sqrt{-e_{20}}t,$$

system (3.19) becomes

$$\begin{aligned}\dot{x}_6 &= y_6,\\ \dot{y}_6 &= x_6^2 + \alpha_{30}x_6^3 + \alpha_{12}x_6y_6^2 + \alpha_{21}x_6^2y_6 + \alpha_{40}x_6^4 + \alpha_{22}x_6^2y_6^2 + \alpha_{31}x_6^3y_6 + o(|x_6, y_6|^4),\end{aligned} \tag{3.20}$$

where

$$\alpha_{30} = -\frac{e_{30}}{e_{20}},\ \alpha_{12} = e_{12},\ \alpha_{21} = \frac{e_{21}}{\sqrt{-e_{20}}},\ \alpha_{40} = \frac{e_{40}}{e_{20}},\ \alpha_{22} = -e_{22},\ \alpha_{31} = -\frac{e_{31}}{\sqrt{-e_{20}}}.$$

From Lemma 3.2, system (3.20) is equivalent to

$$\begin{aligned} \dot{x}_7 &= y_7, \\ \dot{y}_7 &= x_7^2 + G x_7^3 y_7 + o(|x_7, y_7|^4), \end{aligned}$$

where

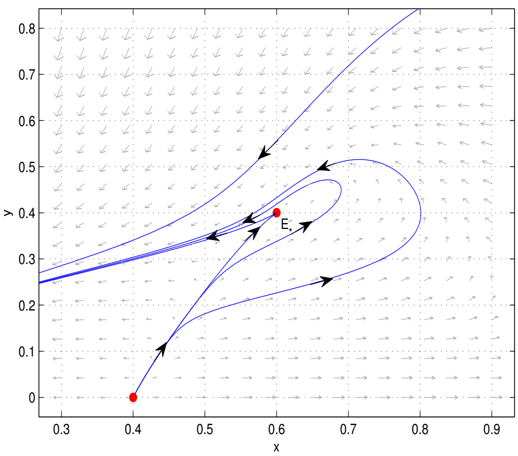
$$G = -\frac{B_0}{\sqrt{m}\,(-1-m+a)(m-1+a)(-m+1+a)^3 m^2}.$$

Clearly, $-\frac{1}{\sqrt{m}\,(-1-m+a)(m-1+a)(-m+1+a)^3 m^2} < 0$. Then, the sign of G is determined by B_0. In fact, from [28], E_* is a cusp of codimension 3 if $B_0 \neq 0$ (see Figure 4(b)). E_* is a cusp of codimension of at least 4 if $B_0 = 0$.

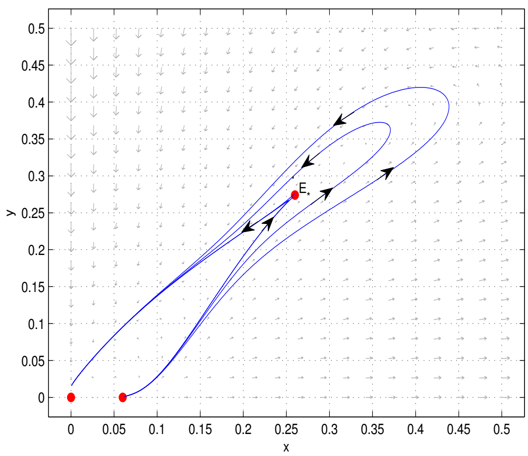
Remark 3.1. *Because B_0 is a complicated polynomial with respect to a and m, it is difficult to discuss whether B_0 is zero. Next, by numerical simulation, we find that the value of B_0 can be zero or nonzero. Let $a = 0.01$; then, $\frac{1+a}{2} = 0.505$, $M_1 = 0.949947725$ and*

$$B_0 = -12m^5 + 32.32m^4 - 41.0241m^3 + 33.737434m^2 - 14.99700015m + 2.0195960202.$$

When $a = 0.01$, we have that $0.505 < m < 0.949947725$. By calculation, we find that $B_0 < 0$ if $0.505 < m < 0.849804962$; $B_0 = 0$ if $m = 0.849804962$ and $B_0 > 0$ if $0.849804962 < m < 0.949947725$. Therefore, $B_0 = 0$ may occur and E_* is a cusp of codimension of at least 4 under some suitable conditions.



(a)



(b)

Figure 4. (a) E_* is a cusp of codimension 2 with $a = \frac{2}{5}$, $m = \frac{1}{5}$, $s = \frac{243}{400}$, $c = \frac{3}{10}$, $p = \frac{1}{2}$. (b) E_* is a cusp of codimension 3 with $a = \frac{3}{50}$, $m = \frac{27}{50}$, $s = \frac{31524548679}{215853160000}$, $c = \frac{4537}{232300}$, $p = \frac{1369}{250884}$.

4. Bifurcation

In this section, we will discuss some bifurcation phenomena that occur in system (1.5), such as saddle-node bifurcation, Hopf bifurcation and Bogdanov-Takens bifurcation.

4.1. Saddle-node bifurcation

From Theorem 2.1, when $p < p^*$, $p = p^*$ and $p^* < p < p^{**}$, system (1.5) has 0, 1 and 2 positive equilibria, respectively. Therefore, selecting the Allee threshold $p = p_{SN} = p_*$ as the bifurcation parameter, and by using Sotomayor's theorem in [28], we verify that system (1.5) undergoes a saddle-node bifurcation around the positive equilibrium E_*.

Theorem 4.1. *Assume that* $0 < m < (1 - \sqrt{a})^2$ *and* $c > c_1^*$, *or that* $(1 - \sqrt{a})^2 \leq m < 1 - a$. *System* (1.5) *undergoes a saddle-node bifurcation around* E_* *if* $p = p_{SN}$.

Proof. The Jacobian matrix of the positive equilibrium E_* can be expressed as

$$J(E_*; p_{SN}) = \begin{bmatrix} \frac{m(-m+1+a)}{2} & -\frac{m(-m+1+a)}{2} \\ \frac{s(a^2-m^2-2a+1)^2}{4m^2(-m+1+a+2c)^2} & -\frac{s(a^2-m^2-2a+1)^2}{4m^2(-m+1+a+2c)^2} \end{bmatrix}.$$

Then, $Det(J(E_*; p_{SN})) = 0$ and $Tr(J(E_*; p_{SN})) \neq 0$. So, $J(E_*; p_{SN})$ has a zero eigenvalue. Let V and W be the eigenvectors corresponding to the zero eigenvalues of the matrix $J(E_*; p_{SN})$ and $J(E_*; p_{SN})^T$, respectively. By computation, we obtain

$$V = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad W = \begin{pmatrix} W_1 \\ W_2 \end{pmatrix} = \begin{pmatrix} \frac{s(a^2-m^2-2a+1)^2}{4m^2(-m+1+a+2c)^2} \\ -\frac{m(-m+1+a)}{2} \end{pmatrix}.$$

Let

$$F(x, y) = \begin{pmatrix} F_1(x, y) \\ F_2(x, y) \end{pmatrix} = \begin{pmatrix} x(1-x)(x-a) - mxy \\ sy\left(\frac{y}{y+p} - \frac{y}{x+c}\right) \end{pmatrix}.$$

Hence,

$$F_p(E_*; p_{SN}) = \begin{pmatrix} 0 \\ -\frac{s(a^2-m^2-2a+1)^2}{4m^2(-m+1+a+2c)^2} \end{pmatrix},$$

$$D^2F(E_*; p_{SN})(V, V) = \begin{pmatrix} \frac{\partial^2 F_1}{\partial x^2} V_1^2 + 2\frac{\partial^2 F_1}{\partial x \partial y} V_1 V_2 + \frac{\partial^2 F_1}{\partial y^2} V_2^2 \\ \frac{\partial^2 F_2}{\partial x^2} V_1^2 + 2\frac{\partial^2 F_2}{\partial x \partial y} V_1 V_2 + \frac{\partial^2 F_2}{\partial y^2} V_2^2 \end{pmatrix}_{(E_*; p_{SN})} = \begin{pmatrix} m - 1 - a \\ 0 \end{pmatrix}.$$

Thus, we have

$$W^T \cdot F_P(E_*; p_{SN}) = \frac{s(a^2 - m^2 - 2a + 1)^2(-m + 1 + a)}{8m(-m + 1 + a + 2c)^2} \neq 0,$$
$$W^T \cdot D^2F(E_*; p_{SN})(V, V) = \frac{s(a^2 - m^2 - 2a + 1)^2(m - 1 - a)}{4m^2(-m + 1 + a + 2c)^2} \neq 0.$$

According to Sotomayor's theorem [26], system (1.5) undergoes a saddle-node bifurcation at E_*. The proof is completed.

4.2. Hopf bifurcation

It follows from Theorem 3.6 that the stability of the positive equilibrium E_5 is closely related to the value of $Tr(J(E_5))$. When $Tr(J(E_5)) = 0$, i.e., $s = \bar{s}$, $J(E_5)$ has a pair of pure imaginary roots, which implies that system (1.5) may undergo Hopf bifurcation at E_5. For simplicity, we apply the following (see [29, 30]):

$$\bar{x} = \frac{x}{x_2^*}, \quad \bar{y} = \frac{y}{y_2^*}, \quad \bar{t} = x_2^{*2}t, \quad \bar{a} = \frac{a}{x_2^*}, \quad \bar{k} = \frac{1}{x_2^*},$$
$$\bar{m} = \frac{my_2^*}{x_2^*}, \quad \bar{s} = \frac{s}{x_2^{*2}}, \quad \bar{p} = \frac{p}{y_2^*}, \quad \bar{n} = \frac{x_2^*}{y_2^*}, \quad \bar{c} = \frac{c}{y_2^*}.$$

Dropping the bar, system (1.5) becomes

$$\begin{aligned} \dot{x} &= x(k-x)(x-a) - mxy, \\ \dot{y} &= sy\left(\frac{y}{y+p} - \frac{y}{nx+c}\right), \end{aligned} \tag{4.1}$$

where $0 < a < k$ and all parameters are positive. Apparently, $\bar{E}_5(1,1)$ is an equilibrium of system (4.1), which yields that $m = (k-1)(1-a) > 0$ (i.e., $0 < a < 1 < k$) and $n = p + 1 - c > 0$. In addition, there exists another positive equilibrium $\bar{E}_4(\bar{x}_1, \bar{y}_1)$, with $\bar{x}_1$ satisfying the following equation:

$$x^2 + (mn - a - k)x + ka + mc - pm = 0. \tag{4.2}$$

Substituting $m = (k-1)(1-a)$ and $n = p + 1 - c$ into (4.2), we obtain

$$x^2 + ((k-1)(1-a)(p+1-c) - a - k)x + ka + (k-1)(1-a)(c-p) = 0.$$

Notice that $\bar{x}_1 < 1$. From Vieta's theorem, we can get

$$\bar{x}_1 \cdot 1 = ka + (k-1)(1-a)(c-p) < 1.$$

Introducing a time variation $dt = (y+p)(nx+c)d\tau$ and rewriting τ as t, system (4.1) becomes

$$\begin{aligned} \dot{x} &= (x(k-x)(x-a) - (k-1)(1-a)xy)(y+p)((p+1-c)x+c), \\ \dot{y} &= sy^2((p+1-c)x + c - p - y). \end{aligned} \tag{4.3}$$

The Jacobian matrix of system (4.2) at $\bar{E}_5(1,1)$ is

$$J(\bar{E}_5) = \begin{bmatrix} (-2+a+k)(p+1)^2 & -(k-1)(1-a)(p+1)^2 \\ s(p+1-c) & -s \end{bmatrix}.$$

Letting

$$s^* = (-2+a+k)(p+1)^2,$$

the determinant and trace of $J(\bar{E}_5)$ are, respectively,

$$Det(J(\bar{E}_5)) = (p+1)^2 s(1 - (ka + (k-1)(1-a)(c-p)))$$

and

$$Tr(J(\bar{E}_5)) = s^* - s.$$

Clearly, when $Det(J(\bar{E}_5)) > 0$, the stability of the equilibrium $\bar{E}_5$ is determined by the trace $Tr(J(\bar{E}_5))$. Hence, we have the following theorem.

Theorem 4.2. *Suppose that*

$$0 < a < 1 < k, \quad c < p + 1 \ \text{ and } \ 0 < ka + (k-1)(1-a)(c-p) < 1;$$

then, for system (4.3), $\bar{E}_5$ *is*

(1) *a stable hyperbolic focus or node if* $s^* \le 0$ *or* $s > s^* > 0$*;*
(2) *an unstable hyperbolic focus or node if* $0 < s < s^*$*;*
(3) *a center or fine focus if* $s = s^* > 0$.

Now, if Theorem 4.2(3) holds, system (4.3) may undergo Hopf bifurcation at $\bar{E}_5$. First, we check the transversality condition for the occurrence of Hopf bifurcation. By calculation, we get

$$\frac{d}{ds} Tr(J(\bar{E}_5))\bigg|_{s=s^*} = -1 \neq 0.$$

Next, we calculate the first Lyapunov coefficient that can determine the stability of limit cycles around $\bar{E}_5$. In biology, if two species coexist in the form of periodic oscillations, the system will have a limit cycle. Making the transformations $X = x - 1$ and $Y = y - 1$, we have

$$\begin{aligned} \dot{X} &= a_{10}X + a_{01}Y + a_{20}X^2 + a_{11}XY + a_{02}Y^2 + o(|X, Y|^2), \\ \dot{Y} &= b_{10}X + b_{01}Y + b_{20}X^2 + b_{11}XY + b_{02}Y^2 + o(|X, Y|^2), \end{aligned} \tag{4.4}$$

where

$$\begin{aligned} a_{10} &= (a+k-2)(p+1)^2, \qquad a_{01} = -(k-1)(1-a)(p+1)^2, \\ a_{20} &= -((a+k-2)c - (p+1)(2a+2k-5))(p+1), \qquad a_{02} = (k-1)(a-1)(p+1), \\ a_{11} &= -((k-1)(a-1)(c-2p) + (-2a+1)k + a)(p+1), \qquad b_{20} = 0, \\ b_{10} &= (a+k-2)(p+1-c)(p+1)^2, \qquad b_{01} = -(a+k-2)(p+1)^2, \\ b_{11} &= 2(a+k-2)(p+1-c)(p+1)^2, \qquad b_{02} = -2(a+k-2)(p+1)^2. \end{aligned}$$

Letting $D = a_{10}b_{01} - a_{01}b_{10}$, obviously,

$$D = (p+1)^4(a+k-2)(1 - (ka + (k-1)(1-a)(c-p))) > 0.$$

Taking the following transformation

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} -\frac{a_{01}\sqrt{D}}{a_{10}^2+D} & -\frac{a_{01}a_{10}}{a_{10}^2+D} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X_1 \\ Y_1 \end{pmatrix},$$

system (4.4) can be expressed as follows

$$\begin{aligned} \dot{X}_1 &= -\sqrt{D}Y_1 + H_1(X_1, Y_1), \\ \dot{Y}_1 &= \sqrt{D}X_1 + H_2(X_1, Y_1), \end{aligned}$$

where

$$H_1(X_1, Y_1) = c_{20}X_1^2 + c_{11}X_1Y_1 + c_{02}Y_1^2 + c_{30}X_1^3 + c_{12}X_1Y_1^2 + c_{21}X_1^2Y_1 + c_{03}Y_1^3 + o(|X_1, Y_1|^3),$$
$$H_2(X_1, Y_1) = d_{20}X_1^2 + d_{11}X_1Y_1 + d_{02}Y_1^2 + d_{30}X_1^3 + d_{12}X_1Y_1^2 + d_{21}X_1^2Y_1 + d_{03}Y_1^3 + o(|X_1, Y_1|^3),$$

and the coefficients of $H_1(X_1, Y_1)$ and $H_2(X_1, Y_1)$ are given in Appendix C.

The first-order Lyapunov number in [26] at $\bar{E}_5$ is given by the following formula

$$\begin{aligned} l_1 &= \frac{1}{16}\left[6c_{30} + 2c_{12} + 2d_{21} + 6d_{03} + \frac{1}{\sqrt{D}}(2c_{11}(c_{20} + c_{02}) - 2d_{11}(d_{20} + d_{02}) - 4c_{20}d_{20} + 4c_{02}d_{02})\right] \\ &= \frac{(p+1)^4 C_6}{8(-p-1+c)D}, \end{aligned}$$

where the coefficient C_6 is given in Appendix D. Clearly, the sign of l_1 is determined by C_6. Hence, we have the following theorem.

Theorem 4.3. *Assume that the condition of Theorem 4.2(3) holds.*

(1) *System* (4.3) *undergoes a subcritical Hopf bifurcation and an unstable limit cycle around* $\bar{E}_5$ *when* $C_6 < 0$ *(see Figure 5(a) and (b)).*
(2) *System* (4.3) *undergoes a supercritical Hopf bifurcation and a stable limit cycle around* $\bar{E}_5$ *when* $C_6 > 0$ *(see Figure 5(c) and (d)).*
(3) *System* (4.3) *undergoes a degenerate Hopf bifurcation and at least two limit cycles around* $\bar{E}_5$ *when* $C_6 = 0$ *(see Figure 5(e) and (f)).*

Because the first-order Lyapunov number l_1 is too complicated, we give an example to show that system (1.5) undergoes a degenerate Hopf bifurcation of codimension 3. Letting $a = \frac{1}{2}$ and $c = 1$, system (4.3) becomes

$$\begin{aligned} \dot{x} &= (y+p)(px+1)(x(k-x)(x-\tfrac{1}{2}) - (\tfrac{k}{2} - \tfrac{1}{2})xy), \\ \dot{y} &= s^*y(y(px+1) - y(y+p)), \end{aligned} \tag{4.5}$$

where $k > \frac{3}{2}$ and $p > \frac{2k-3}{k-1}$. Through a series of transformations and methods described in [26], the first and two Lyapunov numbers are obtained as follows:

$$L_1 = -\frac{C_7\sqrt{(p+1)^4(2k-3)(kp-2k-p+3)}}{4\,p\,(kp-2k-p+3)^2(2k-3)^2(p+1)^4},$$

$$L_2 = \frac{C_8\sqrt{(p+1)^4(2k-3)(kp-2k-p+3)}}{24\,p^3\,(kp-2k-p+3)^3(2k-3)^4(p+1)^6},$$

and the coefficients C_7 and C_8 are given in Appendix E.

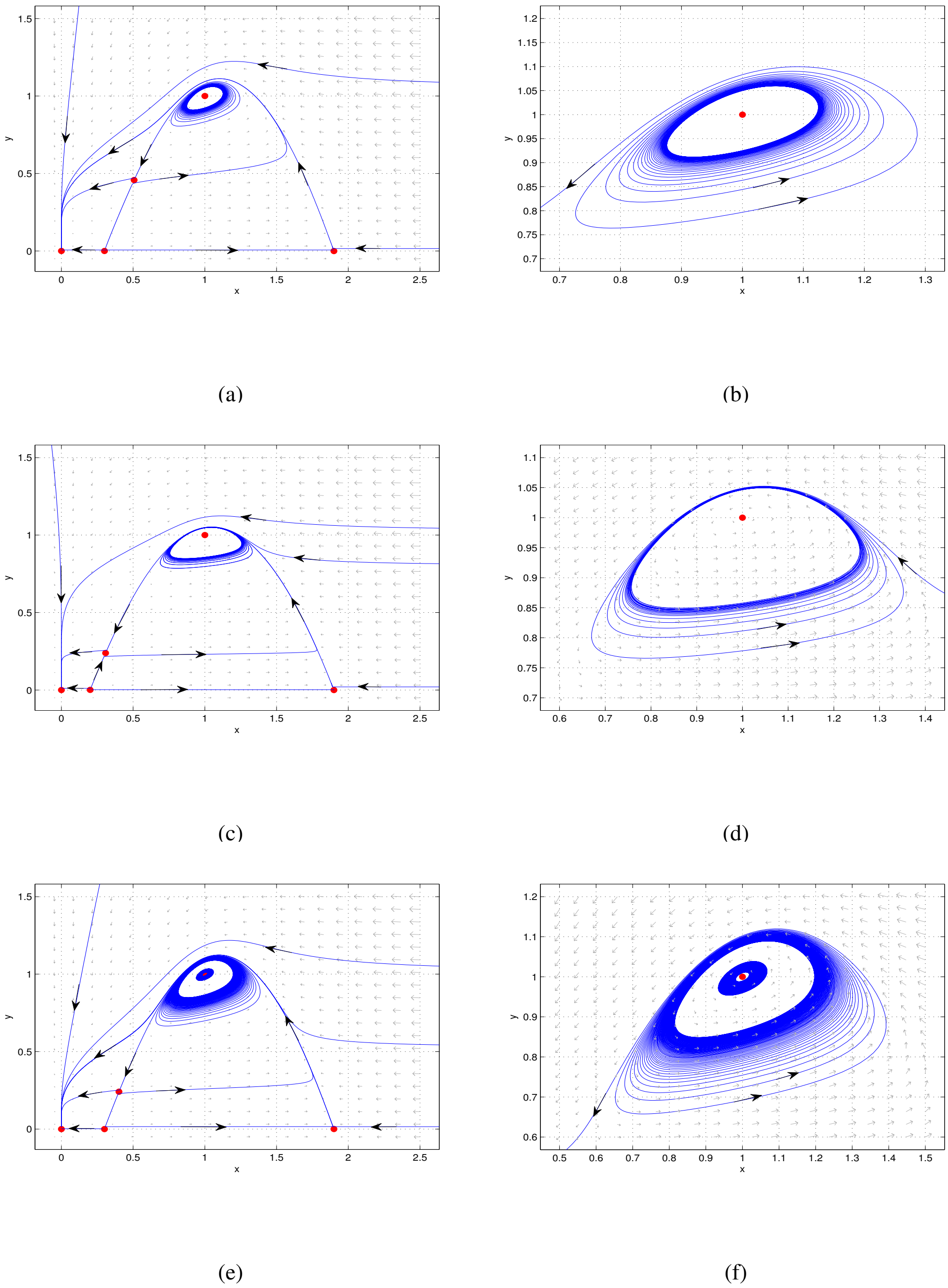


Figure 5. (a) Selecting $a = \frac{3}{10}$, $k = \frac{19}{10}$, $c = \frac{3}{5}$, $p = \frac{7}{10}$, $s = \frac{289}{500} + \frac{1}{100}$, system (4.3) undergoes a subcritical Hopf bifurcation and an unstable limit cycle around $\bar{E}_5$. (b) Amplified phase portrait of (a). (c) Selecting $a = \frac{1}{5}$, $k = \frac{19}{10}$, $c = \frac{3}{5}$, $p = \frac{7}{10}$, $s = \frac{289}{1000} - \frac{1}{100}$, system (4.3) undergoes a supercritical Hopf bifurcation and a stable limit cycle around $\bar{E}_5$. (d) Amplified phase portrait of (c). (e) Selecting $a = \frac{3}{10}$, $k = \frac{19}{10}$, $c = \frac{1031438}{438165} - \sqrt{\frac{113429142649}{175266}}$, $p = \frac{6}{10} + \frac{1}{10}$, $s = \frac{289}{500} + \frac{1}{1000}$, system (4.3) underdoes a degenerate Hopf bifurcation and multiple two-limit cycles (the inner one is stable and the outer one is unstable) around $\bar{E}_5$. (f) Amplified phase portrait of (e).

In order to investigate whether system (4.5) will undergo a degenerate Hopf bifurcation of codimension 3, we need to discuss whether L_1 and L_2 will be 0 at the same time. In fact, we need to analyze whether C_7 and C_8 have a common zero root under certain parameter conditions. Using the command "resultant" in Maple software, we obtain

$$C_{78} = \text{res}(C_7, C_8, p) = -128(10k^3 - 37k^2 + 51k - 26)(2k + 1)^2(3k - 4)^2(k - 1)^8(-3 + 2k)^8 C_9,$$

where the coefficient C_9 is given in Appendix E. Let $C_{78} = 0$, that is $k = 1.7017166155$. Applying $C_7 = 0$ and $C_8 = 0$, we obtain that $p = 2.6992221856$. Hence, $L_1 = L_2 = 0$ for $k = 1.7017166155$ and $p = 2.6992221856$. Selecting the parameters as

$$(a_1,\ c_1,\ k_1,\ p_1,\ s_1) = (0.5,\ 1,\ 1.6,\ 2.5495696920,\ 1.2599444999),$$

$$(a_2,\ c_2,\ k_2,\ p_2,\ s_2) = (0.5,\ 1,\ 1.7017166155,\ 2.6992221856,\ 2.7603395424),$$

we can get

$$L_1|_{(a,c,k,p,s)=(a_1,c_1,k_1,p_1,s_1)} = 0,\ \ L_2|_{(a,c,k,p,s)=(a_1,c_1,k_1,p_1,s_1)} = 0.5939540338,$$

$$L_1|_{(a,c,k,p,s)=(a_2,c_2,k_2,p_2,s_2)} = L_2|_{(a,c,k,p,s)=(a_2,c_2,k_2,p_2,s_2)} = 0,$$

and

$$\left.\frac{\partial\left(Tr(J(\bar{E}_5),\ L_1)\right)}{\partial(s,\ p)}\right|_{(a,c,k,p,s)=(a_1,c_1,k_1,p_1,s_1)} = 1.6413885919 \neq 0,$$

$$\left.\frac{\partial\left(Tr(J(\bar{E}_5),\ L_1,\ L_2)\right)}{\partial(s,\ p,\ k)}\right|_{(a,c,k,p,s)=(a_2,c_2,k_2,p_2,s_2)} = -0.4620942636 \neq 0.$$

Therefore, system (4.5) can undergo Hopf bifurcation of codimension 2 and 3. According to the above analysis, we can summarize the following remark.

Remark 4.1. *Assume that the condition of Theorem 4.2(3) holds.*

(1) $\bar{E}_5$ *is a weak focus of order 1 if* $L_1 \neq 0$.
(2) $\bar{E}_5$ *is a weak focus of order 2 if* $L_1 = 0$ *and* $L_2 \neq 0$.
(3) $\bar{E}_5$ *is a weak focus of order of at least 3 if* $L_1 = L_2 = 0$.

4.3. Bogdanov-Takens bifurcation

From Thoerem 3.8, the unique positive equilibrium E_* is a cusp of codimension 2 under some suitable conditions. In this section, we choose some suitable parameters as bifurcation parameters to show that system (1.5) undergoes a Bogdognov-Taken bifurcation of codimension 2.

Theorem 4.4. *Assume that the conditions of Theorem 3.8 hold. Choosing p and c as two bifurcation parameters, system* (1.5) *undergoes a Bogdanov-Takens bifurcation of codimension 2 around* E_*.

Proof. Letting $p = p^*$ and $s = s_*$, and substituting $p = p^* + \lambda_1$ and $c = c + \lambda_2$ into system (1.5), we get the following system:

$$\begin{aligned} \dot{x} &= x(1-x)(x-a) - mxy, \\ \dot{y} &= s_* y\left(\frac{y}{y + p^* + \lambda_1} - \frac{y}{x + c + \lambda_2}\right), \end{aligned} \tag{4.6}$$

where (λ_1, λ_2) is in a small neighborhood of $(0, 0)$.

Making the transformations $X = x - x_*$ and $Y = y - (x_* + c - p)$, system (4.6) becomes

$$\begin{aligned} \dot{X} &= \tilde{a}_{10}X + \tilde{a}_{01}Y + \tilde{a}_{20}X^2 + \tilde{a}_{11}XY + o(|X, Y|^2), \\ \dot{Y} &= \tilde{b}_{00} + \tilde{b}_{10}X + \tilde{b}_{01}Y + \tilde{b}_{20}X^2 + \tilde{b}_{11}XY + \tilde{b}_{02}Y^2 + o(|X, Y|^2), \end{aligned} \tag{4.7}$$

where

$$\begin{aligned} \tilde{a}_{10} &= \frac{m(a+1-m)}{2}, \quad \tilde{a}_{01} = -\frac{m(a+1-m)}{2}, \quad \tilde{a}_{20} = -\frac{a+1-3m}{2}, \\ \tilde{a}_{11} &= -m, \quad \tilde{a}_{30} = -1, \quad \tilde{b}_{00} = -\frac{m(a+1-m+2c)^2(a+1-m)(-\lambda_2+\lambda_1)}{2(-m+a+2c+1+2\lambda_1)(a+1-m+2c+2\lambda_2)}, \\ \tilde{b}_{01} &= \frac{(a+1-m+2c)^2(a+1-m)m\tilde{A}_1}{2(a-1+m)(a-m-1)(-m+a+2c+1+2\lambda_1)^2(a+1-m+2c+2\lambda_2)}, \\ \tilde{b}_{10} &= \frac{m(a+1-m)(a+1-m+2c)^2}{2(a+1-m+2c+2\lambda_2)^2}, \quad \tilde{b}_{20} = \frac{(a+1-m+2c)^2(a+1-m)m}{(a+1-m+2c+2\lambda_2)^3}, \\ \tilde{b}_{11} &= \frac{4m^2(a+1-m)(a+1-m+2c)^2}{(a-1+m)(a-m-1)(a+1-m+2c+2\lambda_2)^2}, \\ \tilde{b}_{02} &= \frac{m(a+1-m)(a+1-m+2c)^2\tilde{A}_2}{(a^2-m^2-2a+1)^2(-m+a+2c+1+2\lambda_1)^3(a+1-m+2c+2\lambda_2)}, \end{aligned}$$

and

$$\begin{aligned} \tilde{A}_1 = {} & 16\lambda_1^2 m + 8m\lambda_1(a+2c-2\lambda_2-m+1) + 2\lambda_2(3m^2-4m(a+2c+1)+(a-1)^2) \\ & + (a-1+m)(a-1-m)(-m+1+a+2c), \\ \tilde{A}_2 = {} & (-m+1+a+2c)[(a^2-m^2-2a+1)(3m^2-4m(a+2c+1)+(a-1)^2) \\ & + 8m\lambda_1(2m^2-(a+2c+1)m-(a-1)^2)] + 2\lambda_2(m^2-2m(a+2c+1)+(a-1)^2)^2 \\ & - 16m\lambda_2\lambda_1(m^2-2m(a+2c+1)+(a-1)^2) - 32\lambda_1^2 m^2(\lambda_1+a+2c-\lambda_2-m+1). \end{aligned}$$

Taking the following transformations

$$\begin{aligned} X_1 &= X, \\ Y_1 &= \tilde{a}_{10}X + \tilde{a}_{01}Y + \tilde{a}_{20}X^2 + \tilde{a}_{11}XY + o(|X, Y|^2), \end{aligned}$$

system (4.7) becomes

$$\begin{aligned} \dot{X}_1 &= Y_1, \\ \dot{Y}_1 &= \tilde{c}_{00} + \tilde{c}_{10}X_1 + \tilde{c}_{01}Y_1 + \tilde{c}_{20}X_1^2 + \tilde{c}_{11}X_1Y_1 + \tilde{c}_{02}Y_1^2 + o(|X_1, Y_1|^2), \end{aligned} \tag{4.8}$$

where

$$\tilde{c}_{00} = \tilde{a}_{01}\tilde{b}_{00}, \quad \tilde{c}_{10} = \tilde{a}_{01}\tilde{b}_{10} - \tilde{b}_{01}\tilde{a}_{10} + \tilde{a}_{11}\tilde{b}_{00}, \quad \tilde{c}_{01} = \tilde{b}_{01} + \tilde{a}_{10},$$

$$\begin{aligned}\tilde{c}_{11} &= \frac{(\tilde{b}_{11}+2\tilde{a}_{20})\tilde{a}_{01}-\tilde{a}_{10}(\tilde{a}_{11}+2\tilde{b}_{02})}{\tilde{a}_{01}}, \quad \tilde{c}_{02}=\frac{\tilde{a}_{11}+\tilde{b}_{02}}{\tilde{a}_{01}},\\ \tilde{c}_{20} &= \frac{\tilde{b}_{20}\,\tilde{a}_{01}^2+(-\tilde{b}_{11}\tilde{a}_{10}+\tilde{a}_{11}\tilde{b}_{10}-\tilde{b}_{01}\tilde{a}_{20})\tilde{a}_{01}+\tilde{b}_{02}\,\tilde{a}_{10}^2}{\tilde{a}_{01}}.\end{aligned}$$

Using the transformations $X_2 = X_1$, $Y_2 = Y_1(1-\tilde{c}_{02}X_1)$ and $dt = (1-\tilde{c}_{02}X_1)d\tau$, system (4.8) can be written as

$$\begin{aligned}\dot{X}_2 &= Y_2,\\ \dot{Y}_2 &= \tilde{d}_{00}+\tilde{d}_{10}X_2+\tilde{d}_{01}Y_2+\tilde{d}_{20}X_2^2+\tilde{d}_{11}X_2Y_2+o(|X_2,Y_2|^2),\end{aligned} \tag{4.9}$$

where

$$\begin{aligned}\tilde{d}_{00} &= \tilde{c}_{00}, \quad \tilde{d}_{10}=-2\tilde{c}_{00}\tilde{c}_{02}+\tilde{c}_{10}, \quad \tilde{d}_{01}=\tilde{c}_{01},\\ \tilde{d}_{11} &= -\tilde{c}_{01}\tilde{c}_{02}+\tilde{c}_{11}, \quad \tilde{d}_{20}=\tilde{c}_{00}\tilde{c}_{02}^2-2\tilde{c}_{02}\tilde{c}_{10}+\tilde{c}_{20}.\end{aligned}$$

When λ_1 and λ_2 are sufficiently small, we have

$$\tilde{d}_{20} = -\frac{(a+1-m)^2m}{4}+O(\lambda)<0.$$

Let

$$X_3 = X_2, \quad Y_3=\frac{Y_2}{\sqrt{-\tilde{d}_{20}}}, \quad \tau=\sqrt{-\tilde{d}_{20}}t;$$

then, system (4.9) becomes (still denoting τ as t)

$$\begin{aligned}\dot{X}_3 &= Y_3,\\ \dot{Y}_3 &= \tilde{e}_{00}+\tilde{e}_{10}X_3+\tilde{e}_{01}Y_3-X_3^2+\tilde{e}_{11}X_3Y_3+o(|X_3,Y_3|^2),\end{aligned} \tag{4.10}$$

where

$$\tilde{e}_{00}=-\frac{\tilde{d}_{00}}{\tilde{d}_{20}}, \quad \tilde{e}_{10}=-\frac{\tilde{d}_{10}}{\tilde{d}_{20}}, \quad \tilde{e}_{01}=\frac{\tilde{d}_{01}}{\sqrt{-\tilde{d}_{20}}}, \quad \tilde{e}_{11}=\frac{\tilde{d}_{11}}{\sqrt{-\tilde{d}_{20}}}.$$

Besides, taking the transformations $X_4 = X_3 - \frac{\tilde{e}_{10}}{2}$ and $Y_4 = Y_3$, system (4.10) becomes

$$\begin{aligned}\dot{X}_4 &= Y_4,\\ \dot{Y}_4 &= \tilde{f}_{00}+\tilde{f}_{01}Y_4-X_4^2+\tilde{f}_{11}X_4Y_4+o(|X_4,Y_4|^2),\end{aligned} \tag{4.11}$$

where

$$\tilde{f}_{00}=\tilde{e}+\frac{\tilde{e}_{10}^2}{4}, \quad \tilde{f}_{01}=\frac{\tilde{e}_{10}\tilde{e}_{11}+2\tilde{e}_{01}}{2}, \quad \tilde{f}_{11}=\tilde{e}_{11}.$$

Note that $c \neq c_2^*$. When λ_1 and λ_2 are sufficiently small,

$$\tilde{d}_{11} = -\frac{(c-c_2^*)A_5}{(a+1-m+2c)(a-m-1)(a-1+m)}+O(\lambda)\neq 0.$$

Finally, letting

$$X_5 = -\tilde{f}_{11}^2 X_4, \quad Y_5 = \tilde{f}_{11}^3 Y_4, \quad \tau = -\frac{t}{\tilde{f}_{11}},$$

and rewriting τ as t, system (4.11) becomes

$$\begin{aligned} \dot{X}_5 &= Y_5, \\ \dot{Y}_5 &= \mu_1 + \mu_2 Y_5 + X_5^2 + X_5 Y_5 + o(|X_5, Y_5|^2), \end{aligned} \tag{4.12}$$

where

$$\mu_1 = -\tilde{f}_{11}^4 \tilde{f}_{00}, \quad \mu_2 = -\tilde{f}_{11} \tilde{f}_{01}.$$

We express μ_1 and μ_2 in terms of λ_1 and λ_2, as follows:

$$\begin{aligned} \mu_1 &= \alpha_1 \lambda_1 + \alpha_2 \lambda_2 + o(|\lambda_1, \lambda_2|), \\ \mu_2 &= \alpha_3 \lambda_1 + \alpha_4 \lambda_2 + o(|\lambda_1, \lambda_2|), \end{aligned}$$

where

$$\begin{aligned} \alpha_1 &= -\frac{16(c - c_2^*)^4 A_5^4}{(a+1-m+2c)^4 (a+1-m)^4 (a-1+m)^4 (a-m-1)^4 m}, \\ \alpha_2 &= \frac{16(c - c_2^*)^4 A_5^4}{(a+1-m+2c)^4 (a+1-m)^4 (a-1+m)^4 (a-m-1)^4 m}, \\ \alpha_3 &= \frac{4(c - c_2^*) A_5 \Lambda_1}{(a+1-m)^3 (a-m-1)^3 (a-1+m)^3 (a+1-m+2c)^2}, \\ \alpha_4 &= -\frac{4(c - c_2^*) A_5 (\Lambda_1 - 2(a-m-1)^2 (a-1+m)^2 (a+1-m)^2)}{(a+1-m)^3 (a-m-1)^3 (a-1+m)^3 (a+1-m+2c)^2}, \end{aligned}$$

and

$$\begin{aligned} \Lambda_1 = {} & cm^3(28m^2 + 48a) + m^3(a^2+1)(40c - 57m) - 4m^2(a-1)^2(2a^2 + (c+10)a + c + 2) \\ & + m^3(a+1)(23m^2 + 14a - 62mc) + (a-1)^4[(3a+2c+3)(a+1) - (9a+4c+9)m] \\ & + m^3[50(a^3+1) - 2m^3 - 34am]. \end{aligned}$$

Note that the transversality condition for the existence of Bogdanov-Takens bifurcation, i.e.,

$$\left|\frac{\partial(\mu_1, \mu_2)}{\partial(\lambda_1, \lambda_2)}\right|_{\lambda_1=\lambda_2=0} = -\frac{128(c - c_2^*)^5 A_5^5}{m(a-m-1)^5 (a-1+m)^5 (a+1-m)^5 (a+1-m+2c)^6} \neq 0$$

holds. According to the result in [28], system (1.5) undergoes a Bogdanov-Takens bifurcation of codimension 2 when (λ_1, λ_2) is in a small neighborhood of $(0, 0)$. The proof is completed.

Theorem 4.5. *Assume that the conditions of Theorem 3.8 hold. System* (1.5) *undergoes a Bogdanov-Takens bifurcation of codimension 2 around E_* when (λ_1, λ_2) is in a small neighborhood of* $(0, 0)$. *Moreover, there are three bifurcation curves as depicted below.*

(I) *When* $0 < c < c_2^*$,

$$SN^+ = \{(\lambda_1,\lambda_2)\big|\,\lambda_1 = \lambda_2 + o(|\lambda_2|), \lambda_2 < 0\}, \quad SN^- = \{(\lambda_1,\lambda_2)\big|\,\lambda_1 = \lambda_2 + o(|\lambda_2|), \lambda_2 > 0\};$$

$$H = \{(\lambda_1,\lambda_2)|\,\lambda_1 = \lambda_2 + \frac{4(a+1-m)^2(a-m-1)^2(a-1+m)^2 m}{(c-c_2^*)^2 A_5^2}\lambda_2^2 + o(|\lambda_2|^2), \lambda_2 < 0\};$$

$$HL = \{(\lambda_1,\lambda_2)|\,\lambda_1 = \lambda_2 + \frac{196(a+1-m)^2(a-m-1)^2(a-1+m)^2 m}{25(c-c_2^*)^2 A_5^2}\lambda_2^2 + o(|\lambda_2|^2), \lambda_2 < 0\}.$$

(II) *When* $c > c_2^*$,

$$SN^+ = \{(\lambda_1,\lambda_2)\big|\,\lambda_1 = \lambda_2 + o(|\lambda_2|), \lambda_2 > 0\}, \quad SN^- = \{(\lambda_1,\lambda_2)\big|\,\lambda_1 = \lambda_2 + o(|\lambda_2|), \lambda_2 < 0\};$$

$$H = \{(\lambda_1,\lambda_2)|\,\lambda_1 = \lambda_2 + \frac{4(a+1-m)^2(a-m-1)^2(a-1+m)^2 m}{(c-c_2^*)^2 A_5^2}\lambda_2^2 + o(|\lambda_2|^2), \lambda_2 > 0\};$$

$$HL = \{(\lambda_1,\lambda_2)|\,\lambda_1 = \lambda_2 + \frac{196(a+1-m)^2(a-m-1)^2(a-1+m)^2 m}{25(c-c_2^*)^2 A_5^2}\lambda_2^2 + o(|\lambda_2|^2), \lambda_2 > 0\}.$$

SN, H and HL respectively denote the saddle-node bifurcation curve, Hopf bifurcation curve and homoclinic bifurcation curve of system (1.5) *around* E_*.

Proof. According to [28], the local bifurcation curve can be expressed as follows:

(i) The saddle-node bifurcation curve:

$$SN^+ = \{(\lambda_1,\lambda_2) : \mu_1(\lambda_1,\lambda_2) = 0, \mu_2(\lambda_1,\lambda_2) > 0\}, \quad SN^- = \{(\lambda_1,\lambda_2) : \mu_1(\lambda_1,\lambda_2) = 0, \mu_2(\lambda_1,\lambda_2) < 0\}.$$

(ii) The Hopf bifurcation curve:

$$H = \left\{(\lambda_1,\lambda_2) : \mu_1(\lambda_1,\lambda_2) < 0, \mu_2(\lambda_1,\lambda_2) = \sqrt{-\mu_1(\lambda_1,\lambda_2)}\right\}.$$

(iii) The homoclinic bifurcation curve:

$$HL = \left\{(\lambda_1,\lambda_2) : \mu_1(\lambda_1,\lambda_2) < 0, \mu_2(\lambda_1,\lambda_2) = \frac{5}{7}\sqrt{-\mu_1(\lambda_1,\lambda_2)}\right\}.$$

By the implicit function theorem, we can solve λ_1 and λ_2 from $\mu_1 = \mu_1(\lambda_1,\lambda_2,a,m,c)$ and $\mu_2 = \mu_2(\lambda_1,\lambda_2,a,m,c)$ in (4.12) as follows:

$$\begin{aligned} \lambda_1 &= \beta_1\mu_1 + \beta_2\mu_2 + o(|\mu_1,\mu_2|), \\ \lambda_2 &= \beta_3\mu_1 + \beta_4\mu_2 + o(|\mu_1,\mu_2|), \end{aligned} \tag{4.13}$$

where

$$\begin{aligned} \beta_1 &= \beta_3 - \frac{(a^2-m^2-2a+1)^4(a+1-m+2c)^4(a+1-m)^4 m}{16(c-c_2^*)^4 A_5^4}, \\ \beta_2 &= \frac{(a-m-1)(a-1+m)(a+1-m)(a+1-m+2c)^2}{8(c-c_2^*)A_5}, \end{aligned}$$

$$\beta_3 = \frac{m(a-m-1)^2(a-1+m)^2(a+1-m)^2(a+1-m+2c)^4\Lambda_1}{32(c-c_2^*)^4A_5^4},$$
$$\beta_4 = \frac{(a-m-1)(a-1+m)(a+1-m)(a+1-m+2c)^2}{8(c-c_2^*)A_5}$$

with Λ_1 being defined in Theorem 4.4.

First, we prove the case (I). When $0 < c < c_2^*$, we get that $\beta_4 < 0$. The saddle-node bifurcation curve is given by $\Gamma_1 \triangleq \mu_1(\lambda_1, \lambda_2) = 0$. From $\Gamma_1 = 0$, we can obtain a function $\lambda_1 = \lambda_2 + o(|\lambda_2|)$ which satisfies the conditions that $\lambda_1(0) = 0$ and $\Gamma_1(\lambda_1(\lambda_2), \lambda_2) = 0$, as follows:

$$\left.\frac{\partial\Gamma_1}{\partial\lambda_1}\right|_{\lambda=0} = -\frac{16(c-c_2^*)^4A_5^4}{(a+1-m+2c)^4(a+1-m)^4(a-1+m)^4(a-m-1)^4m} \neq 0.$$

On the curve $\Gamma_1 = 0$, it folllows from (4.13) that $\lambda_2 = \beta_4\mu_2 + o(|\mu_2|)$. Then, $\lambda_2 > 0$ (< 0) if $\mu_2 < 0$ (> 0). Hence, we have

$$SN^+ = \{(\lambda_1, \lambda_2)\big|\, \lambda_1 = \lambda_2 + o(|\lambda_2|), \lambda_2 < 0\}, \quad SN^- = \{(\lambda_1, \lambda_2)\big|\, \lambda_1 = \lambda_2 + o(|\lambda_2|), \lambda_2 > 0\}.$$

The Hopf bifurcation curve is given by $\Gamma_2 \triangleq \mu_1(\lambda_1, \lambda_2) + \mu_2^2(\lambda_1, \lambda_2) = 0$. Notice that

$$\left.\frac{\partial\Gamma_2}{\partial\lambda_1}\right|_{\lambda=0} = -\frac{16(c-c_2^*)^4A_5^4}{(a+1-m+2c)^4(a+1-m)^4(a-1+m)^4(a-m-1)^4m} \neq 0.$$

By the implicit function theorem, there exists a unique function

$$\lambda_1 = \lambda_2 + \frac{4(a+1-m)^2(a-m-1)^2(a-1+m)^2m}{(c-c_2^*)^2A_5^2}\lambda_2^2 + 0(|\lambda_2|^2),$$

which satisfies the conditions that $\lambda_1(0) = 0$ and $\Gamma_2(\lambda_1(\lambda_2), \lambda_2) = 0$. On the curve $\Gamma_2 = 0$, we get that $\lambda_2 = \beta_4\mu_2 + o(|\mu_2|) < 0$ if $\mu_2 > 0$. Therefore, the Hopf bifurcation curve can be expressed as

$$H = \{(\lambda_1, \lambda_2)|\, \lambda_1 = \lambda_2 + \frac{4(a+1-m)^2(a-m-1)^2(a-1+m)^2m}{(c-c_2^*)^2A_5^2}\lambda_2^2 + o(|\lambda_2|^2), \lambda_2 < 0\}.$$

The homoclinic bifurcation curve is given by $\Gamma_3 \triangleq \frac{25}{49}\mu_1(\lambda_1, \lambda_2) + \mu_2^2(\lambda_1, \lambda_2) = 0$. Note that

$$\left.\frac{\partial\Gamma_3}{\partial\lambda_1}\right|_{\lambda=0} = -\frac{400(c-c_2^*)^4A_5^4}{49(a+1-m+2c)^4(a+1-m)^4(a-1+m)^4(a-m-1)^4m} \neq 0.$$

From $\Gamma_3 = 0$ and the implicit function theorem, there exists a unique function

$$\lambda_1 = \lambda_2 + \frac{196(a+1-m)^2(a-m-1)^2(a-1+m)^2m}{25(c-c_2^*)^2A_5^2}\lambda_2^2 + o(|\lambda_2|^2)$$

satisfying that $\lambda_1(0) = 0$ and $\Gamma_3(\lambda_1(\lambda_2), \lambda_2) = 0$. On the curve $\Gamma_3 = 0$, we obtain that $\lambda_2 = \beta_4\mu_2 + o(|\mu_2|) < 0$ if $\mu_2 > 0$. Hence, the homoclinic bifurcation curve can be written as

$$HL = \{(\lambda_1, \lambda_2)|\, \lambda_1 = \lambda_2 + \frac{196(a+1-m)^2(a-m-1)^2(a-1+m)^2m}{25(c-c_2^*)^2A_5^2}\lambda_2^2 + o(|\lambda_2|^2), \lambda_2 < 0\}.$$

The proof of the case (II) is similar to that of the case (I), so we omit it here. The proof is completed.

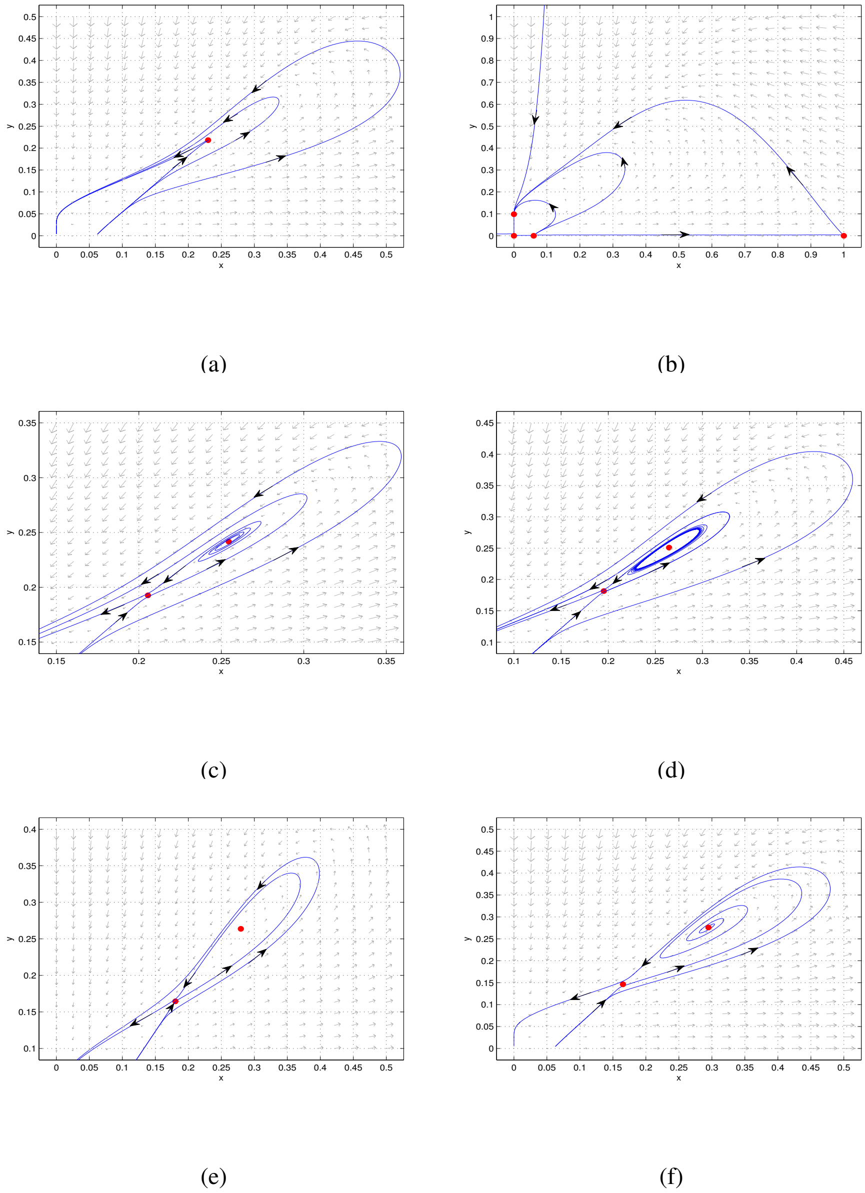


Figure 6. Phase portraits of system (4.6) with $a = \frac{3}{50}$, $m = \frac{3}{5}$, $c = \frac{1}{2}$, $p = \frac{3071}{6000}$, $s = \frac{13237236}{8567405}$. (a) A cusp of codimension 2 when $(\lambda_1, \lambda_2) = (0, 0)$. (b) Case of no positive equilibria when $(\lambda_1, \lambda_2) = (-0.05, 0.06)$. (c) Case of a saddle and an unstable focus when $(\lambda_1, \lambda_2) = (0.061, 0.06)$. (d) Case of an unstable limit cycle when $(\lambda_1, \lambda_2) = (0.062, 0.06)$. (e) Case of an unstable homoclinic loop when $(\lambda_1, \lambda_2) = (0.0641, 0.06)$. (f) Case of a saddle and a stable focus when $(\lambda_1, \lambda_2) = (0.067, 0.06)$.

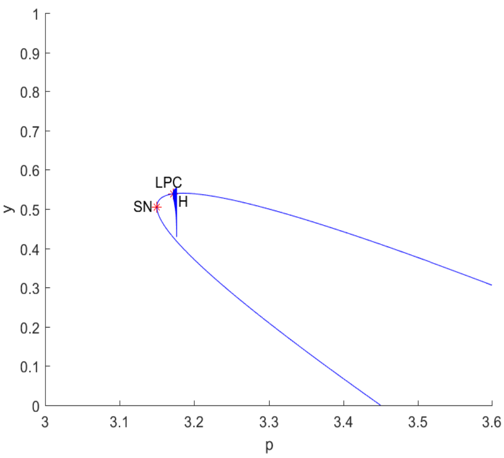
Assume that $a = \frac{3}{50}$, $m = \frac{3}{5}$ and $c = \frac{1}{2}$; we can get $p^* = \frac{3071}{6000}$ and $s_* = \frac{13237236}{8567405}$. When $(\lambda_1, \lambda_2) = (0, 0)$, E_* is a cusp of codimension 2; see Figure 6(a). When $(\lambda_1, \lambda_2) = (-0.05, 0.06)$, system (1.5) has no positive equilibrium and all trajectories converge to E_3; see Figure 6(b). When $(\lambda_1, \lambda_2) = (0.061, 0.06)$, system (1.5) has two positive equilibria, where one is a hyperbolic saddle and the other is a hyperbolic unstable focus; see Figure 6(c). When $(\lambda_1, \lambda_2) = (0.062, 0.06)$, system (1.5) undergoes a subcritical Hopf bifurcation and an unstable limit cycle appears around E_5; see Figure 6(d). When $(\lambda_1, \lambda_2) = (0.0641, 0.06)$, the unstable limit cycle expands to the unstable homoclinic loop; see Figure 6(e). When $(\lambda_1, \lambda_2) = (0.067, 0.06)$, system (1.5) has two positive equilibria, where one is a hyperbolic saddle and the other is a hyperbolic stable focus; see Figure 6(f).

5. Numerical simulations

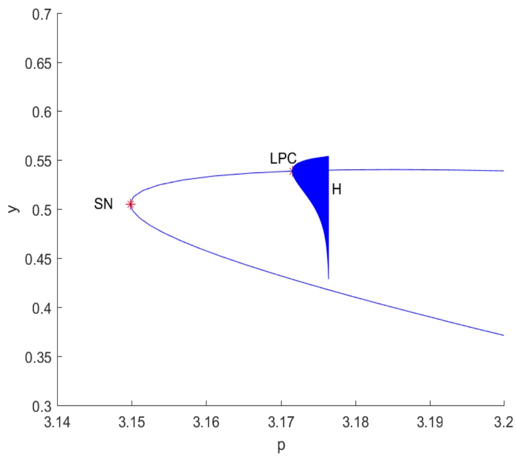
We discuss the influence of the Allee effect in the predator population on the dynamical behavior of system (1.5). Letting $a = 0.45$, $m = 0.14$, $c = 3$ and $s = 1$, Figure 7 shows the bifurcation diagram in the (p, y)-plane of system (1.5). We find that there exist four Allee thresholds: $p = p^* \approx 3.14982$, $p = p_H \approx 3.17156$, $p = p^{**} = 3.45$ and $p = 1 + c \triangleq p_* = 4$. According to Theorem 2.1 and the bifurcation diagram, if the Allee effect parameter satisfies that $p < p^*$ or $p \geq p_*$, system (1.5) has no positive equilibrium. If the Allee effect parameter satisfies that $p = p^*$, system (1.5) has a unique positive equilibrium E_*. If $p^* < p < p^{**}$, system (1.5) has two positive equilibria E_4 and E_5, where E_4 is always a saddle and E_5 is unstable if $p^* < p < p_H$, and stable if $p_H < p < p^{**}$. If the Allee effect parameter satisfies that $p^{**} \leq p < p_*$, system (1.5) has a unique positive equilibria E_5, which is stable. Also, we give the two-parameter bifurcation diagram of system (1.5) in the (a, p)-plane, as shown in Figure 8.

We selected p as the control parameter and plotted the phase portraits of system (1.5) at different values (see Figure 9). If $p = 0$, that is, the predator population exists without the Allee effect, system (1.5) has no positive equilibrium and the boundary equilibrium E_3 is globally asymptotically stable, which means that the predator can survive and the prey will tend to extinction (see Figure 9(a)). When $p = 3.1\,(p < p^*)$, system (1.5) has no positive equilibrium and the origin is globally asymptotically stable. That is, the predator will become extinct with the influence of the Allee effect on the predator population, which means that both predator and prey will tend to extinction (see Figure 9(b)). When $p = 3.165\,(p^* < p < p_H)$, system (1.5) has two positive equilibria, where E_4 is a saddle and E_5 is an unstable focus; see Figure 9(c). In this case, the predator and prey will still become extinct. When $p = 3.1725\,(p_H < p < p^{**})$, E_4 is still a saddle but E_5 becomes a stable focus; also, an unstable limit cycle appears around E_5; see Figure 6(d). Hence, the unstable limit cycle acts as a separatrix between the attraction of the origin and E_5. When $p = 3.17642\,(p_H < p < p^{**})$, there exists an unstable homoclinic loop in system (1.5); see Figure 9(e). When $p = 3.1767\,(p_H < p < p^{**})$, the homoclinic loop disappears and system (1.5) has a hyperbolic saddle E_4 and a hyperbolic stable focus E_5; see Figure 9(f). That is, the two stable manifolds of saddle E_4 act as a separatrix between the attraction of the origin and E_5. When $p = 3.45\,(p^{**} \leq p < p_*)$, system (1.5) has a unique positive stable equilibrium E_5 and a degenerate saddle E_2; see Figure 9(g). Obviously, the stable manifold of degenerate saddle E_2 is taken as a separatrix between the attraction of the origin and E_5. When $p = 4\,(p \geq p_*)$, system (1.5) has no positive equilibrium and a repelling saddle node E_2; see Figure 9(h). From Figure 9(h), the stable manifold of E_2 acts as a separatrix between the attraction of the origin and E_1.

As shown in Figure 9(a), if the predator has no Allee effect, the prey will tend to extinction, but the predator can survive because they have alternative food. When the Allee constant p increases, the alternative food source does not guarantee the survival of the predator. Then, both the predator and prey will become extinct; see Figure 9(b)–(c). However, when the Allee effect on the predator is strong (i.e., the Allee effect constant p is large), there is a bistable phenomenon (see Figure 9(d)–(g)). The unstable limit cycle or the stable manifold of E_2 acts as a separatrix between the origin and E_5. That is, the prey and predator may be able to coexist. Finally, under the condition that the Allee effect on the predator is strong enough, Figure 9(h) shows that the predator will become extinct, whereas the prey may become extinct or survive, depending on the initial value. On the whole, when the Allee effect in the predator population is strong enough, the predator will become extinct, whereas the prey will survive or become extinct, depending on the initial value. Hence, in contrast to the dynamic behavior of the predator without the Allee effect, a strong Allee effect can lead to the extinction of the predator and the increase of the survival rate of the prey.



(a)



(b)

Figure 7. (a) Bifurcation diagram in (p, y)-plane for system (1.5) with $a = 0.45$, $m = 0.14$, $c = 3$, $s = 1$. SN, H and LPC represent the saddle node, Hopf point and limit point of cycles, respectively. (b) Amplified phase portrait of (a).

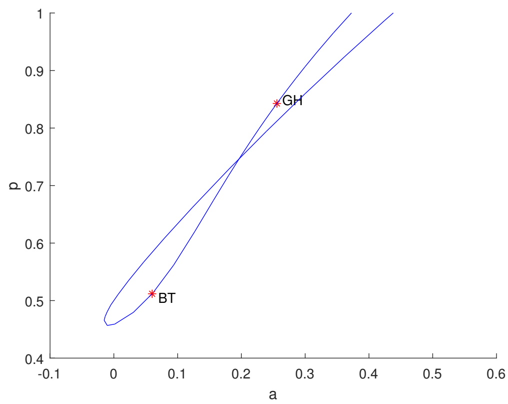
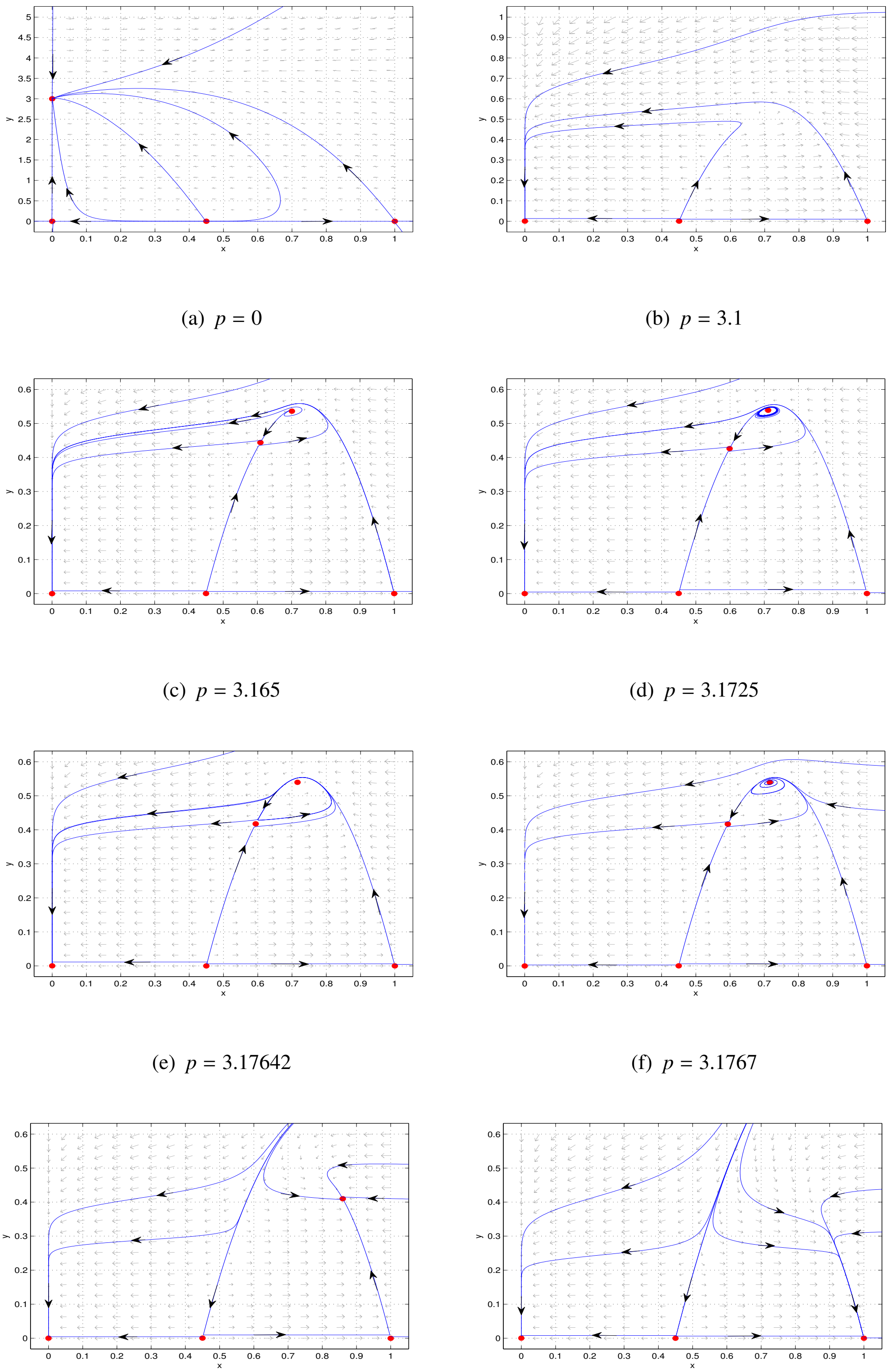


Figure 8. Bifurcation diagram in (a, p)-plane for system (1.5) with $m = \frac{3}{5}$, $c = \frac{14}{25}$, $s = \frac{13237236}{8567405}$. GH and BT represent degenerate Hopf bifurcation and Bogdanov-Takens bifurcation, respectively.



(a) $p = 0$ (b) $p = 3.1$

(c) $p = 3.165$ (d) $p = 3.1725$

(e) $p = 3.17642$ (f) $p = 3.1767$

(g) $p = 3.45$ (h) $p = 4$

Figure 9. Phase portraits of system (1.5) with $a = 0.45$, $m = 0.14$, $c = 3$ and $s = 1$.

6. Conclusions

In this manuscript, a modified Leslie-Gower predator-prey model with Allee effect on both prey and predator is proposed. We showed that the boundary equilibrium $E_3(0, c-p)$ is a stable node, while E_0, E_1 and E_2 are unstable if $c > p$. Hence, E_3 is globally asymptotically stable if system (1.5) has no positive equilibrium. That is, a weak Allee effect on the predator is conducive to the survival of the predator. However, if $p > 1 + c$, that is, the Allee effect in the predator population is strong, system (1.5) has no positive equilibrium by Theorem 2.1. Then, the predator and prey do not coexist, which means that a strong Allee effect on the predator is detrimental to the survival of both predator and prey. Moreover, the other three boundary equilibria E_0, E_1 and E_2 are non-hyperbolic (see Figure 2). We proved that the unique positive equilibrium E_* is a saddle node or a cusp of codimension 3 (see Figures 3 and 4). Further, because the expression of B_0 is complicated, we showed that E_* is a cusp of codimension of at least 4 by concrete example (see Remark 3.1).

We showed that system (1.5) undergoes saddle-node bifurcation, Hopf bifurcation and Bogdanov-Takens bifurcation. In more detail, system (1.5) can undergo a degenerate Hopf bifurcation for some suitable parameter values and result in two limit cycles (the inner one is stable and the outer one is unstable; see Figure 5). Biologically, this indicates the bistable phenomenon, where the predator and prey will oscillate periodically or become extinct, depending on the initial values. That is, the predator will coexist and oscillate periodically if the initial values lie within the unstable limit cycle. However, the predator and prey will tend to extinction if the initial values lie outside of the unstable limit cycle.

In addition, we give an example to illustrate that system (1.5) has a weak focus of order of at least 3 and can undergo a degenerate Hopf bifurcation of codimension 3. Moreover, we proved that, within system (1.5), there is a Bogdanov-Takens bifurcation of codimension 2; we also presented its bifurcation curves.

In the absence of an Allee effect on the predator, that is, $p = 0$, system (1.5) reduces to system (1.3). Arancibia-Ibarra [15] proved the existence of separatrices in the phase plane separating basins of attraction. They showed that system (1.3) has at most two positive equilibria, where the smaller positive equilibrium is always a saddle, whereas the larger positive equilibrium can be either an attractor or a repeller surrounded by a limit cycle. They showed that system (1.3) undergoes Hopf bifurcation and Bogdanov-Takens bifurcation without rigorous mathematical proof.

Incorporating the Allee effect on the predator into system (1.3), we investigated the stability and bifurcation of system (1.5) by using the Allee effect as a threshold condition. When the Allee effect on the predator is weak (i.e., $p < c$), E_3 is globally asymptotically stable if system (1.5) has no positive equilibrium (see Figure 9(a)). As the Allee effect constant on predator increases, E_3 disappears, which means that the origin is globally asymptotically stable (see Figure 9(b)–(c)). However, in [15], the predator always can always survive due to alternative food. Hence, we showed that, even if the predator has the alternative food source, as long as the Allee effect on the predator is strong enough, both prey and predator will become extinct, which is different from [15]. When the Allee effect on the predator is sufficiently strong, the predator will tend to extinction and the extinction and existence of the prey depend on the initial value. Therefore, a strong Allee effect on the predator is beneficial to the survival of the prey, but detrimental to the survival of the predator. Unlike [15], we give rigorous mathematical proof to prove that E_* is a cusp of codimension 3, and that system (1.5) undergoes degenerate Hopf bifurcation and Bogdanov-Takens bifurcation of codimension 2. We also showed that system (1.5) has a cusp of codimension of at least 4 and can undergo a degenerate Hopf bifurcation of codimension 3 by concrete examples. Therefore, compared with system (1.3), the Allee effect on the predator greatly affects the dynamical behavior of the system, resulting in more complex dynamical behavior. This enriches the dynamics of the modified Leslie-Gower predator-prey model with the double Allee effect.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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Appendix A. Coefficients in the proof of Theorem 3.7

$$
\begin{aligned}
A_1 &= (a^2-m^2-2a+1)(1-m+a+2c)(5m^2+2a-a^2-4am-4m-1)(s+s_*)+2[m^2(3m-a\\
&\quad -1)(-m+1+a+2c)^3+(a^2-m^2-2a+1)(s_*(a^2-m^2-2a+1)-s(a^2-4am-8cm\\
&\quad +3m^2-2a-4m+1))](1-m+a),\\
A_2 &= 2m^3(1-m+a+2c)^3\{m^2(1-m+a)(1-m+a+2c)[2m(1-m+a)+(3m-a-1)(1-m\\
&\quad +a+2c)]-s(a^2-m^2-2a+1)(a^2+4am-5m^2-2a+4m+1)\}-s^2(a^2-4am-8cm\\
&\quad +3m^2-2a-4m+1)(a^2-m^2-2a+1)^3,\\
A_3 &= -2m^5(1-m+a+2c)^5(1-m+a)(a-2m+1)+sm^3[(1-m+a+2c)^3(a^2-m^2-2a\\
&\quad +1)(m(5m-4-4a)-(a-1)^2)+2(1-m+a+2c)^2(1-m+a)(a^2-m^2-2a+1)^2]\\
&\quad -s^2(a^2-m^2-2a+1)^3(m^2+(-2a-4c-2)m+(-1+a)^2),\\
A_4 &= 4m^8(1-m+a)^2(3m-a-1)(1-m+a+2c)^7-s^3(a^2-m^2-2a+1)^5(a^2-4am-8cm\\
&\quad +3m^2-2a-4m+1)-8cm^6s(a^2-m^2-2a+1)^2(1-m+a)(1-m+a+2c)^4\\
&\quad -8m^4s^2(1-m+a+2c)^3(a^2-m^2-2a+1)^3(1-m+a).
\end{aligned}
$$

Appendix B. Coefficients in the proof of Theorem 3.9

$$
\begin{aligned}
c_{20} &= -\frac{m(-m+1+a)^2}{4}, \quad c_{02} = -\frac{(a-1)^2(a+1-4m)+m^2(8m-5a-5)}{m(-m+1+a)(a^2-m^2-2a+1)},\\
c_{30} &= \frac{(-m+1+a)(-3m+a+1)}{2}, \quad c_{12} = \frac{B_1}{m^2(-m+1+a)^2(a^2-m^2-2a+1)^2},
\end{aligned}
$$

$$c_{21} = \frac{m^2((6a+6-4m)m+7a^2-34a+7)-14(a+1)(a-1)^2m+5a^4-10a^2+5}{-2m(-m+1+a)(a^2-m^2-2a+1)},$$

$$c_{03} = -\frac{2(a-2m+1)^2}{m^3(-m+1+a)^3}, \quad c_{22} = \frac{-B_2}{m^3(-m+1+a)^3(a^2-m^2-2a+1)^3},$$

$$c_{40} = -\frac{(22m-19a-19)(a-1)^2m+(3a+3-10m)m^3+4a^4-8a^2+4}{4m(a^2-m^2-2a+1)},$$

$$c_{13} = \frac{8(a^2+3m^2-2a+1)(a-2m+1)^2}{m^4(-m+1+a)^3(a^2-m^2-2a+1)}, \quad c_{31} = \frac{B_3}{m^2(1+a-m)^2(a^2-m^2-2a+1)^2},$$

$$c_{04} = -\frac{4((a-1)^2(a+1-2m)+m^2(3a+3-2m))(a-2m+1)^2}{m^5(-m+1+a)^5(-1-m+a)(m-1+a)},$$

$$B_1 = (a+1)(a-1)^4(3a+3-10m)+2(a-1)^2m^2(6(a+1)m-a^2-10a-1)-m^4(20m^2 -10(3a+3)m+13(a+1)^2),$$

$$B_2 = (a+1)^2(a-1)^6(9a+9-45m)+(a-1)^4m^2((134a-37a^2-37)m+73a^3-85a^2 -85a+73)+(a-1)^2m^4((213a^2+122a+213)m-89a^3-59a^2-59a-89) +m^6(36m^3-(76a+76)m^2+(89a^2+162a+89)m-173a^3+41a^2+41a-173),$$

$$B_3 = (a+1)^2(a-1)^4(5a+5-27m)+(a-1)^2m^2((50a^2-92a+50)(a+1)-(34a^2 -44a+34)m)+m^4(4m^3-(16a+16)m^2+(25a^2+18a+25)m-7(a+1)^3).$$

Appendix C. Coefficients in the proof of Theorem 4.2

$$c_{20} = \frac{((a+k-2)(c-2p-2)+p+1)\sqrt{D}}{(p+1)(a+k-2)(c-p-1)}, \quad c_{02} = \frac{s^*(p+1)C_2}{(p+1-c)\sqrt{D}}, \quad c_{11} = \frac{(p+1)C_1}{p+1-c},$$

$$c_{30} = \frac{(c-(p+1-c)(a+k-4))D}{(-p-1+c)^2(a+k-2)^2(p+1)^3}, \quad c_{21} = \frac{\sqrt{D}C_4}{(p+1)^2(a+k-2)(-p-1+c)^2},$$

$$c_{12} = \frac{-C_3}{(-p-1+c)^2}, \quad c_{03} = -\frac{(a+k-2)(p+1)^2C_5}{(-p-1+c)^2\sqrt{D}}, \quad d_{20} = 0, \quad d_{11} = 2\sqrt{D},$$

$$d_{02} = 0, \quad d_{12} = \sqrt{D}, \quad d_{21} = 0, \quad d_{30} = 0, \quad d_{03} = 0,$$

$$C_1 = (p+1)(1-ak)(3c-2p-2)+(a+k-2)(5pc+2c+(1-4p)(p+1))-2p-2 +(k-1)(a-1)c^2,$$

$$C_2 = (p+1)(1-ak)(5c-3p-3)+(a+k-2)(5pc+3c-3p(p+1))-p-1 +2(k-1)(a-1)c^2,$$

$$C_3 = (p+1)(1-ak)(6c^2-9(p+1)c+4(p+1)^2)+(a+k-2)[(p^2+8p+5)c^2 -(2p^2+13p+2)(p+1)c+(p+1)^2(p^2+6p-2)]-(p+1)(5c-8p-8) +(k-1)(a-1)c^3,$$

$$C_4 = (p+1)(ak-1)(c-p-1)^2-(cp-p^2+3p+4)(a+k-2)(c-p-1) +(p+1)(4c-7p-7),$$

$$C_5 = (p+1)(1-ak)(5c^2-7(p+1)c+3(p+1)^2)+(a+k-2)[(5p+4)c^2 -(7p+3)(p+1)c+3p(p+1)^2]-(p+1)(2c-3p-3)+(k-1)(a-1)c^3.$$

Appendix D. Coefficient C_6 in the proof of Theorem 4.2

$$\begin{aligned} C_6 = {} & [((a+k-2)^2p^2+(p+1)(k-1)(a-1)(a+k-4))(k-1)(a-1)]c^2 \\ & +[-2(k-1)(a-1)(a+k-2)^2p^3-((ak-a-k+2)^2+a^2k^2)(a+k-2)p^2 \\ & -6(a+k-ak)akp^2+2(a+k)^2p^2-2p^2-(k-1)(a-1)(-(3p+1)(a+k-6)^2+4 \\ & +2(2(ak-5)p+ak-3)(a+k-5))]c+(ak-1)(a+k-2)(ak(p+1)+5p+12)p^2 \\ & +2(a+k-2)(ak-a-k+2)^2p^2-(a+k-2)^2(2p+1)p^2-2(ak-1)(2(ak-1)p \\ & +6ak-5)p^2+(k-1)(a-1)(ak(a+k-6)(3p+1)-p(a+k)(a+k-5)+2p+2). \end{aligned}$$

Appendix E. Coefficients C_7, C_8 and C_9 in the proof of Theorem 4.3

$$\begin{aligned} C_7 = {} & 4k^3p^4-10k^3p^3-16k^2p^4+2k^3p^2+43k^2p^3+21kp^4+4k^3p-17k^2p^2-58kp^3-9p^4 \\ & -28k^2p+31kp^2+26p^3-4k^2+37kp-16p^2+2k-13p+2, \\ C_8 = {} & 256k^6p^8+320k^6p^7-2208k^5p^8-3928k^6p^6-3264k^5p^7+7808k^4p^8+2376k^6p^5 \\ & +36780k^5p^6+13224k^4p^7-14488k^3p^8+5528k^6p^4-24076k^5p^5-141438k^4p^6 \\ & -27536k^3p^7+14872k^2p^8-872k^6p^3-55700k^5p^4+100314k^4p^5+286221k^3p^6 \\ & +31130k^2p^7-8004kp^8-2768k^6p^2+15172k^5p^3+226154k^4p^4-218561k^3p^5 \\ & -321839k^2p^6-18088kp^7+1764p^8-768k^6p+37872k^5p^2-86238k^4p^3-472315k^3p^4 \\ & +262673k^2p^5+191032kp^6+4218p^7+12560k^5p-188056k^4p^2+232883k^3p^3 \\ & +535400k^2p^4-165765kp^5-46792p^6+768k^5-65992k^4p+452172k^3p^2-327614k^2p^3 \\ & -314876kp^4+43410p^5-5184k^4+157020k^3p-562997k^2p^2+228497kp^3+76337p^4 \\ & +12000k^3-180054k^2p+343616kp^2-61624p^3-9360k^2+90240kp-79839p^2 \\ & -1608k-13006p+3384. \\ C_9 = {} & 46448640k^{20}-1616621568k^{19}+24638291968k^{18}-226672889856k^{17}+1437092961792k^{16} \\ & -6740635675392k^{15}+24417648365952k^{14}-70226640103872k^{13}+163306362727264k^{12} \\ & -310683308489888k^{11}+486912516445288k^{10}-630448297873396k^9+673792412980232k^8 \\ & -591555007568776k^7+422673274021358k^6-242089444966568k^5+108573246582396k^4 \\ & -36754214233392k^3+8836596249839k^2-1345620472722k+97623820593. \end{aligned}$$