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Research article

Distributed convex optimization of bipartite containment control for high-order nonlinear uncertain multi-agent systems with state constraints

Yuhang Yao[1], **Jiaxin Yuan**[1,*], **Tao Chen**[2], **Xiaole Yang**[1] **and Hui Yang**[1]

[1] School of Air Transportation, Shanghai University of Engineering Science, Shanghai 201620, China.
[2] College of Engineering, China Agricultural University-East Campus, Beijin 100083, China

* **Correspondence:** Email: yuanke2964@sjtu.edu.cn.

Abstract: This article investigates a penalty-based distributed optimization algorithm of bipartite containment control for high-order nonlinear uncertain multi-agent systems with state constraints. The proposed method addresses the distributed optimization problem by designing a penalty function in the form of a quadratic function, which is the sum of the global objective function and the consensus constraint. Moreover, the observer is presented to address the unmeasurable state of each agent. Radial basis function neural networks (RBFNN) are employed to approximate the unknown nonlinear functions. Then, by integrating RBFNN and dynamic surface control (DSC) techniques, an adaptive backstepping controller based on the barrier Lyapunov function (BLF) is proposed. Finally, the effectiveness of the suggested control strategy is verified under the condition that the state constraints are not broken. Simulation results indicate that the output trajectories of all agents remain within the upper and lower boundaries, converging asymptotically to the global optimal signal.

Keywords: multi-agent systems; distributed optimization; bipartite containment control; state constraints; dynamic surface control

1. Introduction

Recently, due to its remarkable computational performance and scalability, distributed convex optimization has garnered significant attention from researchers [1, 2]. Distributed convex optimization algorithms effectively address optimization problems in complex and large-scale network environments by integrating traditional optimization theories with recently developed theoretical techniques for coordinated control of multi-agent systems (MASs) [3, 4]. Compared to centralized optimization, distributed convex optimization offers improved scalability, robustness, privacy and flexibility, making it widely applicable in various fields, such as economic dispatching of smart grid systems [5, 6], resource assignment for sensor networks [7], large-scale machine learning [8] and distributed source location and estimation [9].

In distributed convex optimization, agents are assigned local objective functions, and the global objective function is constructed by aggregating these local objective functions. Through continuous information exchange in the communication network, the optimal solution to the global objective function is eventually obtained. As an extension of the consensus problem in MASs, the consensus problem in distributed optimization has garnered increased attention from researchers. In engineering practices, designing distributed protocols to achieve both consensus and performance optimization in MASs is highly reasonable. In UAV formation control, aircraft should consume a small amount of fuel when the speed and attitude are consistent. Discrete-time distributed consensus optimization algorithms for MASs have been studied in [10, 11]. However, to meet the requirements of practical applications, research on a distributed consensus in continuous-time MASs [12, 13] is more prevalent.

Actually, most distributed optimization problems can be reformulated as optimization problems with consensus constraints. For example, in [14], a distributed consensus optimization algorithm based on a penalty function was proposed for a first-order system. For first-order [15, 16] and second-order [17–19] MASs, several distributed optimization algorithms have been presented. For instance, a non-uniform gradient gain was proposed in [20], and a continuous-time zero-gradient-sum algorithm was designed in [21]. However, in comparison to the articles [20–22], a more concise distributed optimization algorithm based on the penalty function was proposed in [23], which avoided the involvement of distributed estimations.

The objective of this paper is to devise a distributed controller that facilitates the collaborative convergence of all agents towards the optimal solution. For high-order MASs, there are various control methods for solving distributed optimization. For example, a distributed output feedback integral controller is proposed to address the output consensus problem for the distributed optimization in [24]. A projection-based second-order control algorithm was proposed in [19]. A Non-smooth embedded control framework was designed in [25]. In [26], the penalty function method and the additive power integrator technique were combined. However, when the states are unmeasurable, the aforementioned distributed optimization problem cannot be solved. Therefore, the unmeasurable system states highly affect the running of high-order nonlinear MASs. Compared to other control methods, adaptive control is widely used and exhibits a robustness to system uncertainties.

As suggested by the general approximation theory, radial basis function neural networks (RBFNN) and fuzzy logic systems are applicable in addressing the uncertainty of nonlinear systems. In [27], a fuzzy state observer based on radial basis functions was presented to estimate unavailability states. In [28], for a class of nonlinear MASs distributed control with strict feedback, adaptive neural networks were employed to approximate uncertain states. Moreover, RBFNN and backstepping techniques were combined to construct an adaptive controller in [29]. Dynamic surface control (DSC) was employed to avoid the complexity explosion in the backstepping method [29–31]. For high-order nonlinear uncertain MASs, the utilization of the adaptive neural network backstepping control method to solve distributed optimization problems is relatively rare. Therefore, designing an adaptive controller to address the distributed optimization problem for high-order nonlinear MASs with unknown states is a meaningful task.

Furthermore, these above protocols are designed with little regard to the control performance and state constraints, which is impractical in systems with either limited resources or actuators. As a matter of fact, the problem of being state-constrained is common in practical systems. The barrier Lyapunov function (BLF) method has been presented by domestic and foreign scholars [32], keeping the system state within constraints. In [33], the employment of BLF ensured the boundedness of the entire system state and constrained all closed-loop signals within a compact set. In [34], the log-asymmetric BLF was proposed, combined with the DSC technique and an adaptive backstepping controller is designed to solve the constraints. With the extension of the nonlinear system with state constraints, the BLF is applied more widely. However, distributed optimization for high-order nonlinear uncertain MASs with state constraints have not been studied, which provides the research motivation of this paper.

Based on previous observations and discussions, this paper focuses on investigating the distributed optimization problem of high-order MASs with state constraints by using the penalty function method. First, by integrating RBFNN and DSC techniques, an adaptive backstepping controller is proposed. From the second step, the introduction of the BLF guarantees the preservation of state constraints. The unknown state of the system is observed by a state observer. The key research contributions of the article are outlined below in comparison to earlier works.

1) Differing from the adaptive controller designed only for the consensus problem [35, 36] and the containment problem [37], we take the distributed optimization problem into account to enhance the system performance. Besides, compared to the bipartite consensus control studied in [38, 39], we introduce two virtual leaders to achieve a bipartite containment effect. The trajectory of the optimal solution in the distributed optimization remains within the convex hulls delineated by the upper and lower reference signals.
2) Unlike [40] and [26], in which the distributed optimization for high-order nonlinear MASs is studied, in this paper, for purpose of meeting the practical requests of the system, we consider state constraints in distributed optimization problems. The introduction of the BLF guarantees the preservation of state constraints. Distributed optimization for high-order nonlinear uncertain MASs with state constraints have not been studied before.
3) In contrast to [14, 15, 41], in which the distributed optimization algorithm is proposed for the fist-order nonlinear MASs, in this paper, we focus on the high-order MASs with unknown variables, which have an increased engineering application value. Combined with RBFNN technology, the observer is designed to estimate the states of each agent.

The structure of the remainder of this paper is presented below. The second part introduces some reserve knowledge, such as the basic theory of the multi-agent and distributed optimization principle. In Section 3, an adaptive observer is prepared and we create a BLF-based adaptive backstepping controller, and a DSC technique is utilized to update virtual control law constantly. In Section 4, simulations are carried out to verify the validity of the controller. The final section summarizes the whole text and draws some conclusions.

2. Preliminaries

2.1. Graph theory

Based on the information interaction between multiple agents, we use digraph $G = \left(w, \chi, \bar{A}\right)$ to represent the relationship between agents. $w = \{n_1, ..., n_N\}$ represents a set of node and $\chi = \left\{\left(n_i, n_j\right)\right\} \in w \times w$ is defined as a set of edges. Information can be transferred between the agent i and agent j. There exists $N_i = \left\{ j \left| \left(n_i, n_j\right) \in \chi \right.\right\}$, which expresses the set of neighboring agents for agents i. $\bar{A} = \left\{a_{ij}\right\} \in R^{N\times N}$ is the adjacency matrix and the element a_{ij} of it has two possible values. When $\left(n_i, n_j\right) \in \chi$, $a_{ij} = 1$; in other cases, $a_{ij} = 0$. It is presumed that $a_{ij} = 0$. Then, we design $D = diag\left(d_1, ..., d_N\right)$ as the diagonal matrix, $d_i = \sum_{j\in n=N_i} a_{ij}$. The Laplacian matrix can be defined as $L = D - \bar{A}$.

2.2. Nonlinear multi-agent system

Consider the following nonlinear high-order multi-agent system:

$$\begin{cases} \dot{x}_{i,p} = x_{i,p+1} + h_{i,p}(x_{i,1}, \cdots, x_{i,p}) \\ \dot{x}_{i,n} = u_i(t) + h_{i,n}(x_{i,1}, \cdots, x_{i,n}) \\ y_i = x_{i,1} \end{cases} \tag{2.1}$$

where y_i represents the system output and $h_{i,p}(\cdot), (p = 1, 2, ..., n-1)$ are unknown nonlinear functions. The control input is defined as u_i. $X_{i,p} = (x_{i,1}, \cdots, x_{i,p})^T \in R^p$ is defined as the system state vectors. Rewrite the system (2.1) into the form below:

$$\dot{X}_{i,n} = A_i X_{i,n} + T_i y_i + \sum_{p=1}^{n} B_{i,p}\left[h_{i,p}(X_{i,p})\right] + B_i u_i \tag{2.2}$$

where Hurwitz matrix $A_i = \begin{bmatrix} -\varepsilon_{i,1} & & & \\ \vdots & & I_{n-1} & \\ -\varepsilon_{i,n} & 0 & \cdots & 0 \end{bmatrix}$, $T_i = \begin{bmatrix} T_{i,1} \\ \vdots \\ T_{i,n} \end{bmatrix}$, $B_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix}$, $B_{i,p} = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$. $Q_i^T = Q_i$ and $P_i^T = P_i$ are predetermined positive matrix,

$$A_i^T P_i + P_i A_i = -2Q_i. \tag{2.3}$$

2.3. Barrier lyapunov function

In this paper, consider the BLF of the *i*th agent as follows:

$$V_{i,1} = \frac{1}{2}\log\frac{{k_{i,b1}}^2}{{k_{i,b1}}^2 - {s_{i,1}}^2} \tag{2.4}$$

where $k_{i,b1}$ is the constraint on $s_{i,1}$ and $\left|s_{i,1}\right| \le k_{i,b1}$.

Lemma 1. *([42, 43]) For any given positive constant $k_{i,b1}$ and $s_{i,1}$ satisfying $\left|s_{i,1}\right| \le k_{i,b1}$, we have*

$$\log\frac{{k_{i,b1}}^2}{{k_{i,b1}}^2 - {s_{i,1}}^2} < \frac{{s_{i,1}}^2}{{k_{i,b1}}^2 - {s_{i,1}}^2}. \tag{2.5}$$

2.4. Problem formulation

To facilitate subsequent calculations, we refer to the following lemmas.

Lemma 2. *([44]) For an undirected connected communication topology, distributed optimization problems can be*

$$\min_{x\in\mathbb{R}^{Nm}} \sum_{i=1}^{N} f_i\left(x_i\right), s.t. \left(L \otimes I_m\right) x = 0_{Nm} \tag{2.6}$$

where $x = [x_1{}^T, \cdots, x_N{}^T]^T$. *Utilizing the principles of the penalty function theory, the approximate optimization problem is formulated as follows:*

$$\min_{x\in\mathbb{R}^{Nm}} \sum_{i=1}^{N} f_i(x_i) + \frac{1}{2}\vartheta x^T (L\otimes I_m)\, x \tag{2.7}$$

where $\vartheta > 0$ *is a constant penalty parameter and* $\frac{1}{2}\vartheta x^T (L\otimes I_m)\, x$ *is the penalty term for violating the consensus constraint* $L\otimes I_m x = 0_{Nm}$.

Lemma 3. *([45]) The inequality relationship shown below is valid*

$$x^T y \leq \frac{n^a}{a}\|x\|^a + \frac{1}{bn^b}\|y\|^b \tag{2.8}$$

where $x, y \in R^n, a, b > 1, n > 0$, *and* $(a-1)(b-1) = 1$.

The distributed optimization problem is the subject of this paper, so for N agents, the global objective function f is the sum of strictly convex local objective functions f_i:

$$f(x_{i,1}) = \sum_{i=1}^{N} f_i(x_{i,1}). \tag{2.9}$$

Define $x_1 = \begin{bmatrix} x_{1,1} & x_{2,1} & \cdots & x_{N,1}\end{bmatrix}^T$. According to [46], 1_N is eigenvector for eigenvalue 0 of Laplacian matrix; for some $\alpha \in R$, if $x_1 = \alpha \cdot 1_N$, we obtain

$$Lx_1 = 0 \tag{2.10}$$

$$x_1^T L x_1 = 0. \tag{2.11}$$

It should be noted that in this paper $x \in \mathbb{R}^N$, therefore, based on Lemma 2, we construct the following penalty function:

$$P(x_1) = \sum_{i=1}^{N} f_i(x_{i,1}) + x_1^T L x_1. \tag{2.12}$$

Our target is as follows

$$\left(x_{1,1}^*, \ldots, x_{N,1}^*\right) = \underset{\left(x_{1,1},\ldots,x_{N,1}\right)}{\arg\min}\; P(x_1). \tag{2.13}$$

The local cost function of the ith agent $f_i(x_{i,1})$ can be designed as:

$$\begin{aligned} f_i(x_{i,1}) &= a_{i,1}(x_{i,1} - x_{d1})^2 + a_{i,2}(x_{i,1} - x_{d2})^2 + c \\ &= a_i x_{i,1}^2 + b_i x_{i,1} + c_i \end{aligned} \tag{2.14}$$

where x_{d1} and x_{d2} are the upper and lower bound of the trajectory of motion, $a_i = a_{i,1} + a_{i,2}$, $a_i > 0$, $b_i = -2a_{i,1}x_{d1} - 2a_{i,2}x_{d2}$, $c_i = a_{i,1}x_{d1}{}^2 + a_{i,2}x_{d2}{}^2 + c$ and a_i, c are scalars.

Remark 1. *Based on the strong convexity of the quadratic function, the global objective function designed in this paper is composed of quadratic functions. To solve the optimization problem of the bipartite containment control, it involves multiple virtual leaders. By constructing a penalty function, all followers are converged to the optimal solution of the distributed optimization problem, which lies within the convex hull of the trajectories of each virtual leader and their opposite trajectories.*

Control objectives: The purpose of this paper is to design a distributed optimization controller, so that the optimal solution for the trajectory moving between the upper and lower bounds of the virtual leaders under state constraints can be found.

3. Main results

3.1. Observer design

Assumption 1. *The unknown functions $h_{i,p}\left(X_{i,p}\right)$, $(i=1,\ \cdots,\ n)$ can be shown as follows:*

$$h_{i,p}\left(X_{i,p}|\theta_{i,p}\right)=\theta_{i,p}^{T}\Psi_{i,p}\left(X_{i,p}\right),1\leq i\leq n \tag{3.1}$$

in which $\Psi_{i,p}\left(X_{i,p}\right)$ delegates the Gaussian basis function vector, and $\theta_{i,p}$ represents the ideal constant vector.

In this paper, an observer is placed to estimate the agent's unmeasurable states, since the assumption that the state variables given in (2.2) are unavailable. The observer is presented as follows:

$$\begin{aligned}\dot{\widehat{X}}_{i,n}&=A_i\widehat{X}_{i,n}+T_iy_i+\sum_{p=1}^{n}B_{i,p}\left[\hat{h}_{i,p}\left(\widehat{X}_{i,p}|\theta_{i,p}\right)\right]+B_iu_i\\ \hat{y}_i&=C_i\widehat{X}_{i,n}\end{aligned} \tag{3.2}$$

where $C_i=[1\ldots0\ldots0]$, $\widehat{X}_{i,p}=\left(\hat{x}_{i,1},\hat{x}_{i,2},...,\hat{x}_{i,p}\right)^{T}$ are the estimated values of $X_{i,p}$.

Let $e_i=X_{i,n}-\widehat{X}_{i,n}$ be state observation errors of system (2.1). Combining Eqs (2.2) and (3.2), we get

$$\dot{e}_i=A_ie_i+\sum_{p=1}^{n}B_{i,p}\left[h_{i,p}\left(\widehat{X}_{i,p}\right)-\hat{h}_{i,p}\left(\widehat{X}_{i,p}\left|\theta_{i,p}\right.\right)+\Delta h_{i,p}\right] \tag{3.3}$$

where $\Delta h_{i,p}=h_{i,p}\left(X_{i,p}\right)-h_{i,p}\left(\widehat{X}_{i,p}\right)$.

By Assumption 1, we can obtain

$$\hat{h}_{i,p}\left(\widehat{X}_{i,p}\left|\theta_{i,p}\right.\right)=\theta_{i,p}^{T}\Psi_{i,p}\left(\widehat{X}_{i,p}\right). \tag{3.4}$$

The optimal parameters are set as

$$\theta_{i,p}^{*}=\arg\min_{\theta_{i,p}\in\Omega_{i,p}}\left[\sup_{\widehat{X}_{i,p}\in U_{i,p}}\left|\hat{h}_{i,l}\left(\widehat{X}_{i,p}\left|\theta_{i,p}\right.\right)-h_{i,p}\left(\widehat{X}_{i,p}\right)\right|\right] \tag{3.5}$$

where $1\leq p\leq n$, $\Omega_{i,p}$ and $U_{i,p}$ are tight regions for $\theta_{i,p}$, $X_{i,p}$ and $\widehat{X}_{i,p}$.

Define errors of the optimal approximation $\xi_{i,p}$ and parameter estimation $\widetilde{\theta}_{i,p}$ as

$$\begin{aligned}\xi_{i,p}&=h_{i,p}\left(\widehat{X}_{i,p}\right)-\hat{h}_{i,p}\left(\widehat{X}_{i,p}\left|\theta_{i,p}^{*}\right.\right)\\ \widetilde{\theta}_{i,p}&=\theta_{i,p}^{*}-\theta_{i,p}.\end{aligned} \tag{3.6}$$

Assumption 2. *([47, 48]) The boundedness of the optimal approximation errors is ensured by the existence of positive constants ξ_{i0}, such that $\left|\xi_{i,p}\right|\leq\xi_{i0}$.*

Assumption 3. *The set of constants γ_i satisfies the following relationship*

$$\left|h_{i,p}\left(X_{i,p}\right)-h_{i,p}\left(\widehat{X}_{i,p}\right)\right|\le\gamma_{i,p}\left\|X_{i,p}-\widehat{X}_{i,p}\right\|. \tag{3.7}$$

From Eqs (3.2) and (3.3), we obtain

$$\begin{aligned}\dot{e}_i&=A_ie_i+\sum_{p=1}^{n}B_{i,p}\left[\xi_{i,p}+\Delta h_{i,p}+\widetilde{\theta}_{i,p}^T\Psi_{i,p}\left(\widehat{X}_{i,p}\right)\right]\\&=A_ie_i+\Delta h_i+\xi_i+\sum_{p=1}^{n}B_{i,p}\left[\widetilde{\theta}_{i,p}^T\Psi_{i,p}\left(\widehat{X}_{i,p}\right)\right]\end{aligned} \tag{3.8}$$

where $\xi_i=[\xi_{i,1},...,\xi_{i,n}]^T$, $\Delta h_i=[\Delta h_{i,1},...,\Delta h_{i,n}]^T$.

Constructing the Lyapunov function:

$$V_0=\sum_{i=1}^{N}V_{i,0}=\sum_{i=1}^{N}\frac{1}{2}e_i^TP_ie_i.$$

Then, by derivation, we have

$$\begin{aligned}\dot{V}_0&\le\sum_{i=1}^{N}\left\{\frac{1}{2}e_i^T\left(P_iA_i^T+A_iP_i\right)e_i+e_i^TP_i\left(\xi_i+\Delta h_i\right)+\sum_{p=1}^{n}e_i^TP_iB_{i,p}\left[\widetilde{\theta}_{i,p}^T\Psi_{i,p}\left(\widehat{X}_{i,p}\right)\right]\right\}\\&\le\sum_{i=1}^{N}\left\{-e_i^TQ_ie_i+e_i^TP_i\left(\xi_i+\Delta h_i\right)+e_i^TP_i\sum_{p=1}^{n}B_{i,p}\widetilde{\theta}_{i,p}^T\Psi_{i,p}\left(\widehat{X}_{i,p}\right)\right\}.\end{aligned} \tag{3.9}$$

According to Lemma 3 and Assumption 3, we obtain

$$\begin{aligned}e_i^TP_i\left(\xi_i+\Delta h_i\right)&\le\left|e_i^TP_i\xi_i\right|+\left|e_i^TP_i\Delta h_i\right|\\&\le\|e_i\|^2+\frac{1}{2}\|P_i\xi_i\|^2+\frac{1}{2}\|P_i\|^2\sum_{p=1}^{n}\left|\Delta h_{i,p}\right|^2\\&\le\|e_i\|^2+\frac{1}{2}\|e_i\|^2\|P_i\|^2\sum_{p=1}^{n}\gamma_{i,p}^2+\frac{1}{2}\|P_i\xi_i\|^2\\&\le\|e_i\|^2\left(1+\frac{1}{2}\|P_i\|^2\sum_{p=1}^{n}{\gamma_{i,p}}^2\right)+\frac{1}{2}\|P_i\xi_i\|^2\end{aligned} \tag{3.10}$$

and

$$\begin{aligned}e_i^TP_i\sum_{p=1}^{n}B_{i,p}\widetilde{\theta}_{i,p}^T\Psi_{i,p}\left(\widehat{X}_{i,p}\right)&\le\frac{1}{2}e_i^TP_i^TP_ie_i+\frac{1}{2}\sum_{p=1}^{n}\widetilde{\theta}_{i,p}^T\Psi_{i,p}\left(\widehat{X}_{i,p}\right)\Psi_{i,p}^T\left(\widehat{X}_{i,p}\right)\widetilde{\theta}_{i,p}\\&\le\frac{1}{2}\lambda_{i,\max}^2\left(P_i\right)\|e_i\|^2+\frac{1}{2}\sum_{p=1}^{n}\widetilde{\theta}_{i,p}^T\widetilde{\theta}_{i,p}.\end{aligned} \tag{3.11}$$

Due to the positive definite matrix P_i, the maximum eigenvalue is proposed as $\lambda_{i,max}(P_i)$. Based on Eqs (3.9), (3.10) and (3.11), we have

$$\dot{V}_0 \leq \sum_{i=1}^{N}\left(-q_{i,0}\|e_i\|^2 + \frac{1}{2}\|P_i\xi_i\|^2 + \frac{1}{2}\sum_{p=1}^{n}\widetilde{\theta}_{i,p}^T\widetilde{\theta}_{i,p}\right) \tag{3.12}$$

where $0 < \Psi_{i,p}(\cdot)\Psi_{i,p}^T(\cdot) \leq 1$ and $q_{i,0} = \lambda_{i,\min}(Q_i) - \left(1 + \frac{1}{2}\|P_i\|^2 \sum\limits_{p=1}^{n}\gamma_{i,p}^2 + \frac{1}{2}\lambda_{i,\max}^2(P_i)\right)$.

Then, (3.12) turns to

$$\dot{V}_0 \leq -q_0\|e\|^2 + \frac{1}{2}\|P\xi\|^2 + \sum_{i=1}^{N}\sum_{p=1}^{n}\frac{1}{2}\widetilde{\theta}_{i,p}^T\widetilde{\theta}_{i,p} \tag{3.13}$$

where $q_0 = \sum_{i=1}^{N} q_{i,0}$.

3.2. Controller design

Theorem 1. *For systems* (2.1) *where Assumptions 1–3 hold, combining observer* (3.2)*, virtual control laws* (3.36)*,* (3.48) *and* (3.60)*, adaptive laws* (3.37)*,* (3.49)*,* (3.61)*,* (3.70) *and control input* (3.69) *together, signals $x_{i,1}$ which converge to the distributed optimization problem's optimal solution x_1^*, remain semi-global uniformly ultimately bounded (SGUUB) in the closed-loop system.*

Define the virtual control laws

$$\begin{cases} x_{i,2}^* = -c_{i,1}[2a_{i,1}(x_{i,1} - x_{d1}) + 2a_{i,2}(x_{i,1} - x_{d2}) + \sum_{j\in N_i} a_{ij}(x_{i,1} - x_{j,1})] - \theta_{i,1}^T\Psi_{i,1}(\hat{X}_{i,1}) \\ x_{i,3}^* = -c_{i,2}s_{i,2} - \left(\frac{1}{2\delta_1} + \frac{5}{2}\right)s_{i,2} - \theta_{i,2}^T\Psi_{i,2}(\hat{X}_{i,2}) + \frac{x_{i,2}^* - v_{i,2}}{\lambda_{i,2}} \\ x_{i,m+1}^* = -c_{i,m}s_{i,m} - (\frac{1}{2\delta_m} + \frac{5}{2})s_{i,m} - \theta_{i,m}^T\Psi_{i,m}(\hat{X}_{i,m}) + \frac{x_{i,m}^* - v_{i,m}}{\lambda_{i,m}} \end{cases} \tag{3.14}$$

adaptive laws

$$\begin{cases} \dot{\theta}_{i,1} = \sigma_{i,1}\Psi_{i,1}(\hat{X}_{i,1})[2a_{i,1}(x_{i,1} - x_{d1}) + 2a_{i,2}(x_{i,1} - x_{d2}) + \sum_{j\in N_i} a_{ij}(x_{i,1} - x_{j,1})] - \rho_{i,1}\theta_{i,1} \\ \dot{\theta}_{i,2} = \sigma_{i,2}\delta_1\Psi_{i,2}(\hat{X}_{i,2})s_{i,2} - \rho_{i,2}\theta_{i,2} \\ \dot{\theta}_{i,m} = \sigma_{i,m}\delta_m\Psi_{i,m}(\hat{X}_{i,m})s_{i,m} - \rho_{i,m}\theta_{i,m} \\ \dot{\theta}_{i,n} = \sigma_{i,n}\delta_n\Psi_{i,n}(\hat{X}_{i,n})s_{i,n} - \rho_{i,n}\theta_{i,n} \end{cases} \tag{3.15}$$

control input

$$u_i = -c_{i,n}s_{i,n} - \left(\frac{1}{2\delta_n} + \frac{3}{2}\right)s_{i,n} - \theta_{i,n}^T\Psi_{i,n}(\hat{X}_{i,n}) + \frac{x_{i,n}^* - v_{i,n}}{\lambda_{i,n}} \tag{3.16}$$

where $c_{i,1} = 3 + \frac{\gamma_{i,1}^2}{2}$, $c > 0$, $\rho > 0$, $\sigma > 0$, $\delta = \frac{1}{k_{i,b}^2 - s_i^2}$, $|s_i| \leq k_{i,b}$.

Proof. Specify the error variables in the following manner:

$$\begin{aligned} s_{i,1} &= x_{i,1} - x_{i,1}^* \\ s_{i,p} &= \hat{x}_{i,p} - v_{i,p} \\ w_{i,p} &= v_{i,p} - x_{i,p}^* \quad p = 2, \cdots, n \end{aligned} \tag{3.17}$$

where $s_{i,p}$ represents the tracking error, $v_{i,p}$ is a state variable that can be obtained using a filter with the virtual controller $x^*_{i,p}$, $w_{i,p}$ denotes the error between $v_{i,p}$ and $x^*_{i,p}$, and $\hat{x}_{i,p}$ is the estimation of $x_{i,p}$.

Through the DSC technique, this paper constructs the following filter

$$\lambda_{i,p}\dot{v}_{i,p} + v_{i,p} = x^*_{i,p}, \quad v_{i,p}(0) = x^*_{i,p}(0) \tag{3.18}$$

where p is the order of the multi-agent model and $2 \le p \le n$. Combining (3.17) with (3.18), we have

$$\begin{aligned} \dot{w}_{i,p} &= \dot{v}_{i,p} - \dot{x}^*_{i,p} \\ &= -\frac{v_{i,p} - x^*_{i,p}}{\lambda_{i,p}} - \dot{x}^*_{i,p} \\ &= -\frac{w_{i,p}}{\lambda_{i,p}} + B_{i,p} \end{aligned} \tag{3.19}$$

where $\lambda_{i,p}$ is the parameter we set. $B_{i,p} = -\dot{x}^*_{i,p}$. According to [49] and [50], there exist constants $M_{i,p} > 0$, $|B_{i,p}| \le M_{i,p}$.

Step 1. First, the gradient of the penalty function (2.12) is calculated as follows

$$\frac{\partial P(x_1)}{\partial x_1} = vec\left(\frac{\partial f_i\left(x_{i,1}(t)\right)}{\partial x_{i,1}}\right) + Lx_1 \tag{3.20}$$

where $vec\left(\frac{\partial f_i\left(x_{i,1}(t)\right)}{\partial x_{i,1}}\right)$ is a column vector. To find the extremum of the penalty function, we need to locate the point where its derivative is zero

$$\frac{\partial P(x^*_1)}{\partial x^*_1} = 0.$$

Combining (2.12) and (3.20), we obtain:

$$\frac{\partial f_i(x^*_{i,1}(t))}{\partial x^*_{i,1}} + \sum_{j\in N_i} a_{ij}(x^*_{i,1} - x^*_{j,1}) = 0. \tag{3.21}$$

According to (2.14) and (3.21),we have

$$2a_{i,1}\left(x^*_i - x_{d1}\right) + 2a_{i,2}\left(x^*_i - x_{d2}\right) + \sum_{j\in N_i} a_{ij}(x^*_{i,1} - x^*_{j,1}) = 0. \tag{3.22}$$

Then according to (3.17) and (3.22), we have

$$\begin{aligned} \frac{\partial P(x_1)}{\partial x_{i,1}} =& \frac{\partial f_i\left(x_{i,1}(t)\right)}{\partial x_{i,1}} + \sum_{j\in N_i} a_{ij}(x_{i,1} - x_{j,1}) \\ =& 2a_{i,1}\left(x_{i,1} - x_{d1}\right) + 2a_{i,2}\left(x_{i,1} - x_{d2}\right) + \sum_{j\in N_i} a_{ij}(x_{i,1} - x_{j,1}) \\ &- 2a_{i,1}\left(x^*_i - x_{d1}\right) - 2a_{i,2}\left(x^*_i - x_{d2}\right) + \sum_{j\in N_i} a_{ij}(x^*_{i,1} - x^*_{j,1}) \\ =& 2a_i s_{i,1} + \sum_{j\in N_i} a_{ij}(s_{i,1} - s_{j,1}). \end{aligned} \tag{3.23}$$

Let $s_1 = [s_{1,1} \ \cdots \ s_{N,1}]^T$, $\mathcal{A} = diag\{2a_i\}$, $H = \mathcal{A} + L$. According to (3.23), we have

$$\frac{\partial P(x_1)}{\partial x_1} = Hs_1.$$

Construct the following Lyapunov function:

$$\begin{aligned} V_1 =& V_0 + \frac{1}{2}\left(\frac{\partial P(x_1)}{\partial x_1}\right)^T H^{-1}\left(\frac{\partial P(x_1)}{\partial x_1}\right) + \sum_{i=1}^{N}\frac{1}{\sigma_{i,1}}\widetilde{\theta}_{i,1}^T\widetilde{\theta}_{i,1} \\ =& V_0 + \frac{1}{2}s_1^T H s_1 + \sum_{i=1}^{N}\frac{1}{\sigma_{i,1}}\widetilde{\theta}_{i,1}^T\widetilde{\theta}_{i,1} \end{aligned} \tag{3.24}$$

where $s_1 = [s_{1,1} \ \cdots \ s_{N,1}]^T$ and $\sigma_{i,1}$ are the parameter we set. According to (2.1), (3.2) and (3.17), we have

$$\dot{s}_{i,1} = \hat{x}_{i,2} + \theta_{i,1}^T\Psi_{i,1} + \widetilde{\theta}_{i,1}^T\Psi_{i,1} + \Delta h_{i,1} + \xi_{i,1} + e_{i,2}. \tag{3.25}$$

Then, according to (3.24) and (3.25), we can obtain

$$\begin{aligned} \dot{V}_1 =& \dot{V}_0 + s_1^T H\dot{s}_1 + \sum_{i=1}^{N}\frac{1}{\sigma_{i,1}}\widetilde{\theta}_{i,1}^T\dot{\widetilde{\theta}}_{i,1} \\ =& \dot{V}_0 + s_1^T H\Big(\hat{x}_2 + vec\left(\theta_{i,1}^T\Psi_{i,1}\right) + vec\left(\widetilde{\theta}_{i,1}^T\Psi_{i,1}\right) + \Delta h_1 + \xi_1 + e_2\Big) + \sum_{i=1}^{N}\frac{1}{\sigma_{i,1}}\widetilde{\theta}_{i,1}^T\dot{\widetilde{\theta}}_{i,1} \\ =& \dot{V}_0 + s_1^T H\Big(s_2 + w_2 + x_2^* + vec\left(\theta_{i,1}^T\Psi_{i,1}\right) + vec\left(\widetilde{\theta}_{i,1}^T\Psi_{i,1}\right) + \Delta h_1 + \xi_1 + e_2\Big) + \sum_{i=1}^{N}\frac{1}{\sigma_{i,1}}\widetilde{\theta}_{i,1}^T\dot{\widetilde{\theta}}_{i,1} \\ =& \dot{V}_0 + s_1^T Hs_2 + s_1^T Hw_2 + s_1^T H\left(x_2^* + vec\Big(\theta_{i,1}^T\Psi_{i,1}\Big) + vec\left(\widetilde{\theta}_{i,1}^T\Psi_{i,1}\right)\right) + s_1^T H\Delta h_1 + s_1^T H\xi_1 \\ & + s_1^T He_2 - \sum_{i=1}^{N}\frac{1}{\sigma_{i,1}}\widetilde{\theta}_{i,1}^T\dot{\theta}_{i,1} \end{aligned} \tag{3.26}$$

where $s_2 = [s_{1,2} \ \cdots \ s_{N,2}]^T$, $w_2 = [w_{1,2} \ \cdots \ w_{N,2}]^T$, $x_2^* = [x_{1,2}^* \ \cdots \ x_{N,2}^*]^T$, $\Delta h_1 = [\Delta h_{1,1} \ \Delta h_{2,1} \ \cdots \ \Delta h_{N,1}]^T$, $\xi_1 = [\xi_{1,1} \ \xi_{2,1} \ \cdots \ \xi_{N,1}]^T$, $e_2 = [e_{1,2} \ e_{2,2} \ \cdots \ e_{N,2}]^T$, $vec\left(\theta_{i,1}^T\Psi_{i,1}\right)$ and $vec\left(\widetilde{\theta}_{i,1}^T\Psi_{i,1}\right)$ are column vectors. According to Lemma 3, we have

$$s_1^T Hs_2 \le \frac{1}{2}s_1^T HH^T s_1 + \frac{1}{2}s_2^T s_2 \tag{3.27}$$

$$s_1^T He_2 \le \frac{1}{2}s_1^T HH^T s_1 + \frac{1}{2}e_2^T e_2 \tag{3.28}$$

$$s_1^T Hw_2 \le \frac{1}{2}s_1^T HH^T s_1 + \frac{1}{2}w_2^T w_2 \tag{3.29}$$

$$s_1^T H\Delta h_1 \le \frac{1}{2}s_1^T H\gamma_1\gamma_1^T H^T s_1 + \frac{1}{2}e_1^T e_1 \tag{3.30}$$

$$s_1^T H\xi_1 \le \frac{1}{2}s_1^T HH^T s_1 + \frac{1}{2}\xi_1^T \xi_1 \tag{3.31}$$

where $\gamma_1 = diag[\gamma_{i,1}], e_1 = [e_{1,1}\ e_{2,1}\ \cdots\ e_{N,1}]^T$. Substituting (3.27)–(3.28) into (3.26), $\dot{V}_1$ turns to

$$\begin{aligned}\dot{V}_1 \leq& \dot{V}_0 + s_1^T H\left(x_2^* + vec\left(\theta_{i,1}^T \Psi_{i,1}\right) + vec\left(\widetilde{\theta}_{i,1}^T \Psi_{i,1}\right)\right) \\ &+ \frac{1}{2}s_1^T H H^T s_1 + \frac{1}{2}w_2^T w_2 + \frac{1}{2}s_1^T H H^T s_1 + \frac{1}{2}s_2^T s_2 \\ &+ \frac{1}{2}s_1^T H\gamma_1\gamma_1^T H^T s_1 + \frac{1}{2}e_1^T e_1 + \frac{1}{2}s_1^T H H^T s_1 + \frac{1}{2}\xi_1^T \xi_1 \\ &+ \frac{1}{2}s_1^T H H^T s_1 + \frac{1}{2}e_2^T e_2 - \sum_{i=1}^{N}\frac{1}{\sigma_{i,1}}\widetilde{\theta}_{i,1}^T \dot{\theta}_{i,1}.\end{aligned} \tag{3.32}$$

Due to $H = \mathcal{A} + L$, we can obtain

$$s_1^T H = \left[2a_1 s_{1,1} + \sum_{j\in N_i} a_{1j}(s_{1,1} - s_{j,1}),\ \cdots, 2a_N s_{N,1} + \sum_{j\in N_i} a_{Nj}(s_{N,1} - s_{j,1})\right]. \tag{3.33}$$

Then, we have

$$\begin{aligned}s_1^T H H^T s_1 =& \left(\frac{\partial P(x_1)}{\partial x_1}\right)^T \left(\frac{\partial P(x_1)}{\partial x_1}\right) \\ =& \sum_{i=1}^{N}\left[2a_{i,1}\left(x_{i,1} - x_{d1}\right) + 2a_{i,2}(x_{i,1} - x_{d2}) + \sum_{j\in N_i} a_{ij}(s_{i,1} - s_{j,1})\right]^2\end{aligned} \tag{3.34}$$

and

$$s_1^T H\gamma_1\gamma_1^T H^T s_1 = \sum_{i=1}^{N}\gamma_{i,1}^2\left[2a_{i,1}\left(x_{i,1} - x_{d1}\right) + 2a_{i,2}(x_{i,1} - x_{d2}) + \sum_{j\in N_i} a_{ij}(s_{i,1} - s_{j,1})\right]^2. \tag{3.35}$$

According to Theorem1,

$$x_{i,2}^* = -c_{i,1}[2a_{i,1}(x_{i,1} - x_{d1}) + 2a_{i,2}(x_{i,1} - x_{d2}) + \sum_{j\in N_i} a_{ij}(x_{i,1} - x_{j,1})] - \theta_{i,1}^T \Psi_{i,1}(\hat{X}_{i,1}) \tag{3.36}$$

$$\dot{\theta}_{i,1} = \sigma_{i,1}\Psi_{i,1}(\hat{X}_{i,1})[2a_{i,1}(x_{i,1} - x_{d1}) + 2a_{i,2}(x_{i,1} - x_{d2}) + \sum_{j\in N_i} a_{ij}(x_{i,1} - x_{j,1})] - \rho_{i,1}\theta_{i,1}. \tag{3.37}$$

Substituting (3.34)–(3.37) into (3.32), after (3.13), we have

$$\begin{aligned}\dot{V}_1 \leq& -q_0\|e\|^2 + \frac{1}{2}\|P\xi\|^2 + \sum_{i=1}^{N}\sum_{p=1}^{n}\frac{1}{2}\widetilde{\theta}_{i,p}^T\widetilde{\theta}_{i,p} + \frac{1}{2}e_2^T e_2 + \frac{1}{2}e_1^T e_1 + \frac{1}{2}\xi_1^T\xi_1 \\ &+ \sum_{i=1}^{N}\frac{\rho_{i,1}}{\sigma_{i,1}}\widetilde{\theta}_{i,1}^T\theta_{i,1} + \frac{1}{2}s_2^T s_2 + \frac{1}{2}w_2^T w_2 - \left(\frac{\partial P(x_1)}{\partial x_1}\right)^T\left(\frac{\partial P(x_1)}{\partial x_1}\right) \\ \leq& -q_1\|e\|^2 + \eta_1 + \sum_{i=1}^{N}\sum_{p=1}^{n}\frac{1}{2}\widetilde{\theta}_{i,p}^T\widetilde{\theta}_{i,p} + \sum_{i=1}^{N}\frac{\rho_{i,1}}{\sigma_{i,1}}\widetilde{\theta}_{i,1}^T\theta_{i,1} + \sum_{i=1}^{N}\frac{1}{2}s_{i,2}^2 + \sum_{i=1}^{N}\frac{1}{2}w_{i,2}^2 \\ &- \frac{2}{\lambda_{\max}\left(H^{-1}\right)}\left(\frac{\partial P\left(x_1\right)}{\partial x_1}\right)^T H^{-1}\left(\frac{\partial P\left(x_1\right)}{\partial x_1}\right)\end{aligned} \tag{3.38}$$

where $q_1 = q_0 - N$, $\eta_1 = \frac{1}{2}||P\xi||^2 + \frac{1}{2}\xi_1^T\xi_1$, $\lambda_{max}(H^{-1})$ is the maximum eigenvalue of the positive matrix H^{-1}.

Step 2. In accordance with (3.17), design $s_{i,2} = \hat{x}_{i,2} - v_{i,2}$. By (3.2) and (3.4), we obtain

$$\begin{aligned}\dot{s}_{i,2} =&\dot{\hat{x}}_{i,2} - \dot{v}_{i,2}\\ =&s_{i,3} + w_{i,3} + x^*_{i,3} + \varepsilon_{i,2}e_{i,1} + \widetilde{\theta}^T_{i,2}\Psi_{i,2} + \theta^T_{i,2}\Psi_{i,2} + \xi_{i,2} + \Delta h_{i,2} - \dot{v}_{i,2}.\end{aligned} \tag{3.39}$$

Starting from this step, according to Lemma 1, we add the state constraints condition and define $\delta_1 = \frac{1}{k^2_{i,b1} - s^2_{i,2}}$. Construct the BLF

$$\begin{aligned}V_2 &= V_1 + \sum_{i=1}^{N} V_{i,2}\\ &= V_1 + \frac{1}{2}\sum_{i=1}^{N}\left\{\delta_1 s^2_{i,2} + \frac{1}{\sigma_{i,2}}\widetilde{\theta}^T_{i,2}\widetilde{\theta}_{i,2} + w^2_{i,2}\right\}\end{aligned} \tag{3.40}$$

where $k_{i,b1}$ and $\sigma_{i,2}$ are parameters we set.

Then, we have

$$\dot{V}_2 = \dot{V}_1 + \sum_{i=1}^{N}\left\{\delta_1 s_{i,2}\dot{s}_{i,2} + \frac{1}{\sigma_{i,2}}\widetilde{\theta}^T_{i,2}\dot{\widetilde{\theta}}_{i,2} + w_{i,2}\dot{w}_{i,2}\right\}. \tag{3.41}$$

Substituting (3.39) into (3.41), $\dot{V}_2$ turns to

$$\begin{aligned}\dot{V}_2 =&\dot{V}_1 + \sum_{i=1}^{N}\Big[\delta_1 s_{i,2}\Big(w_{i,3} + s_{i,3} + x^*_{i,3} + \varepsilon_{i,2}e_{i,1} + \theta^T_{i,2}\Psi_{i,2} + \widetilde{\theta}^T_{i,2}\Psi_{i,2} + \xi_{i,2} + \Delta h_{i,2} - \dot{v}_{i,2}\Big)\\ &+ \frac{1}{\sigma_{i,2}}\widetilde{\theta}^T_{i,2}\dot{\widetilde{\theta}}_{i,2} + w_{i,2}\dot{w}_{i,2}\Big].\end{aligned} \tag{3.42}$$

According to Lemma 3, we obtain

$$s_{i,2}\varepsilon_{i,2}e_{i,1} \leq \frac{1}{2}s^2_{i,2} + \frac{1}{2}\varepsilon^2_{i,2}||e_{i,1}||^2 \tag{3.43}$$

$$s_{i,2}s_{i,3} + s_{i,2}w_{i,3} \leq s^2_{i,2} + \frac{1}{2}\left(s^2_{i,3} + w^2_{i,3}\right) \tag{3.44}$$

$$s_{i,2}\xi_{i,2} \leq \frac{1}{2}s^2_{i,2} + \frac{1}{2}||\xi_{i,2}||^2 \tag{3.45}$$

$$s_{i,2}\Delta h_{i,2} \leq \frac{1}{2}s^2_{i,2} + \frac{1}{2}\gamma^2_{i,2}||e_{i,2}||^2. \tag{3.46}$$

Substituting (3.43)–(3.46) into (3.42), we have

$$\begin{aligned}\dot{V}_2 \leq&\dot{V}_1 + \sum_{i=1}^{N}\Big[\delta_1 s_{i,2}\Big(x^*_{i,3} + \theta^T_{i,2}\Psi_{i,2} + \widetilde{\theta}^T_{i,32}\Psi_{i,2} - \dot{v}_{i,2}\Big) + \frac{5\delta_1}{2}s^2_{i,2} + \frac{1}{2}\left(s^2_{i,3} + w^2_{i,3}\right)\\ &+ \frac{1}{2}\varepsilon^2_{i,2}||e_{i,1}||^2 + \frac{1}{2}||\xi_{i,2}||^2 + \frac{1}{2}\gamma^2_{i,2}||e_{i,2}||^2 - \frac{1}{\sigma_{i,2}}\widetilde{\theta}^T_{i,2}\dot{\theta}_{i,2} + w_{i,2}\dot{w}_{i,2}\Big].\end{aligned} \tag{3.47}$$

According to Theorem1,

$$x_{i,3}^{*} = -c_{i,2}s_{i,2} - \left(\frac{1}{2\delta_1} + \frac{5}{2}\right)s_{i,2} - \theta_{i,2}^{T}\Psi_{i,2}(\hat{X}_{i,2}) + \frac{x_{i,2}^{*} - v_{i,2}}{\lambda_{i,2}} \tag{3.48}$$

$$\dot{\theta}_{i,2} = \sigma_{i,2}\delta_1\Psi_{i,2}(\hat{X}_{i,2})s_{i,2} - \rho_{i,2}\theta_{i,2}. \tag{3.49}$$

Substitute (3.19), (3.48) and (3.49) into (3.47). According to Lemma 3, we have $w_{i,2}B_{i,2} \leq \frac{1}{2}w_{i,2}^2 + \frac{1}{2}M_{i,2}^2$; after (3.38), we can obtain

$$\begin{aligned} \dot{V}_2 \leq & -q_2\|e\|^2 + \eta_2 + \sum_{i=1}^{N}\sum_{p=1}^{n}\frac{1}{2}\widetilde{\theta}_{i,p}^{T}\widetilde{\theta}_{i,p} + \sum_{i=1}^{N}\frac{\rho_{i,1}}{\sigma_{i,1}}\widetilde{\theta}_{i,1}^{T}\theta_{i,1} + \sum_{i=1}^{N}\frac{\rho_{i,2}}{\sigma_{i,2}}\widetilde{\theta}_{i,2}^{T}\theta_{i,2} - \sum_{i=1}^{N}c_{i,2}s_{i,2}^2 \\ & - \sum_{i=1}^{N}\left(\frac{1}{\lambda_{i,2}} - 1\right)w_{i,2}^2 - \frac{2}{\lambda_{\max}(H^{-1})}\left(\frac{\partial P(x_1)}{\partial x_1}\right)^{T}H^{-1}\left(\frac{\partial P(x_1)}{\partial x_1}\right) \\ & + \sum_{i=1}^{N}\left[\frac{1}{2}M_{i,2}^2 + \frac{1}{2}\left(s_{i,3}^2 + w_{i,3}^2\right)\right] \end{aligned} \tag{3.50}$$

where $q_2 = q_1 - \frac{1}{2}\sum_{i=1}^{N}\left(\varepsilon_{i,2}^2 + \gamma_{i,2}^2\right), \eta_2 = \eta_1 + \frac{1}{2}\sum_{i=1}^{N}\|\xi_{i,2}\|^2$.

Step m. Design $s_{i,m} = \hat{x}_{i,m} - v_{i,m}$, by (3.2) and (3.4), we obtain

$$\dot{s}_{i,m} = \hat{x}_{i,m+1} + \varepsilon_{i,m}e_{i,1} + \theta_{i,m}^{T}\Psi_{i,m} + \widetilde{\theta}_{i,m}^{T}\Psi_{i,m} + \xi_{i,m} + \Delta h_{i,m} - \dot{v}_{i,m}. \tag{3.51}$$

According to Lemma 1, define $\delta_m = \frac{1}{k_{i,bm}^2 - s_{i,m}^2}$. Put forward the BLF

$$V_m = V_{m-1} + \frac{1}{2}\sum_{i=1}^{N}\left\{\delta_m s_{i,m}^2 + \frac{1}{\sigma_{i,m}}\widetilde{\theta}_{i,m}^{T}\widetilde{\theta}_{i,m} + w_{i,m}^2\right\} \tag{3.52}$$

where $k_{i,bm}$ and $\sigma_{i,m}$ are designed parameters.

After derivation,

$$\dot{V}_m = \sum_{i=1}^{N}\left\{\delta_m s_{i,m}\dot{s}_{i,m} + \frac{1}{\sigma_{i,m}}\widetilde{\theta}_{i,m}^{T}\dot{\widetilde{\theta}}_{i,m} + w_{i,m}\dot{w}_{i,m}\right\} + \dot{V}_{m-1}. \tag{3.53}$$

Substituting (3.51) into (3.53), refer to (3.17), we have

$$\begin{aligned} \dot{V}_m = & \dot{V}_{m-1} + \sum_{i=1}^{N}\Big[\delta_m s_{i,m}\Big(s_{i,m+1} + w_{i,m+1} + x_{i,m+1}^{*} + \varepsilon_{i,m}e_{i,1} + \theta_{i,m}^{T}\Psi_{i,m} + \widetilde{\theta}_{i,m}^{T}\Psi_{i,m} + \xi_{i,m} \\ & + \Delta h_{i,m} - \dot{v}_{i,m}\Big) + \frac{1}{\sigma_{i,m}}\widetilde{\theta}_{i,m}^{T}\dot{\widetilde{\theta}}_{i,m} + w_{i,m}\dot{w}_{i,m}\Big]. \end{aligned} \tag{3.54}$$

According to Lemma 3, we obtain

$$s_{i,m}\varepsilon_{i,m}e_{i,1} \leq \frac{1}{2}s_{i,m}^2 + \frac{1}{2}\varepsilon_{i,m}^2\|e_{i,1}\|^2 \tag{3.55}$$

$$s_{i,m}s_{i,m+1} + s_{i,m}w_{i,m+1} \le s_{i,m}^2 + \frac{1}{2}\left(s_{i,m+1}^2 + w_{i,m+1}^2\right) \tag{3.56}$$

$$s_{i,m}\xi_{i,m} \le \frac{1}{2}s_{i,m}^2 + \frac{1}{2}\|\xi_{i,m}\|^2 \tag{3.57}$$

$$s_{i,m}\Delta h_{i,m} \le \frac{1}{2}s_{i,m}^2 + \frac{1}{2}\gamma_{i,m}^2\|e_{i,m}\|^2. \tag{3.58}$$

Then, $\dot{V}_m$ turns to

$$\begin{aligned}\dot{V}_m \le & \dot{V}_{m-1} + \sum_{i=1}^{N}\Big[\delta_m s_{i,m}\Big(x_{i,m+1}^* + \theta_{i,m}^T\Psi_{i,m} + \widetilde{\theta}_{i,m}^T\Psi_{i,m} - \dot{v}_{i,m}\Big) + \frac{5\delta_m}{2}s_{i,m}^2 \\ & + \frac{1}{2}(s_{i,m+1}^2 + w_{i,m+1}^2) + \frac{1}{2}\varepsilon_{i,m}^2\|e_{i,1}\|^2 + \frac{1}{2}\|\xi_{i,m}\|^2 \\ & + \frac{1}{2}\gamma_{i,m}^2\|e_{i,m}\|^2 - \frac{1}{\sigma_{i,m}}\widetilde{\theta}_{i,m}^T\dot{\theta}_{i,m} + w_{i,m}\dot{w}_{i,m}\Big].\end{aligned} \tag{3.59}$$

According to Theorem1,

$$x_{i,m+1}^* = -c_{i,m}s_{i,m} - (\frac{1}{2\delta_m} + \frac{5}{2})s_{i,m} - \theta_{i,m}^T\Psi_{i,m}(\hat{X}_{i,m}) + \frac{x_{i,m}^* - v_{i,m}}{\lambda_{i,m}} \tag{3.60}$$

$$\dot{\theta}_{i,m} = \sigma_{i,m}\delta_m\Psi_{i,m}(\hat{X}_{i,m})s_{i,m} - \rho_{i,m}\theta_{i,m}. \tag{3.61}$$

According to Eqs (3.60), (3.61) and (3.19), by Lemma 3, we have $w_{i,m}B_{i,m} \le \frac{1}{2}w_{i,m}^2 + \frac{1}{2}M_{i,m}^2$. (3.59) can be rewritten as

$$\begin{aligned}\dot{V}_m \le & \dot{V}_{m-1} + \sum_{i=1}^{N}\Big[\delta_m s_{i,m}\Big(-c_{i,m}s_{i,m} - \left(\frac{1}{2\delta_m} + \frac{5}{2}\right)s_{i,m} - \theta_{i,m}^T\Psi_{i,m}(\hat{X}_{i,m}) \\ & + \frac{x_{i,m}^* - v_{i,m}}{\lambda_{i,m}} + \theta_{i,m}^T\Psi_{i,m} + \widetilde{\theta}_{i,m}^T\Psi_{i,m} - \dot{v}_{i,m}\Big) + \frac{5\delta_m}{2}s_{i,m}^2 + \frac{1}{2}(s_{i,m+1}^2 + w_{i,m+1}^2) \\ & + \frac{1}{2}\varepsilon_{i,m}^2\|e_{i,1}\|^2 + \frac{1}{2}\|\xi_{i,m}\|^2 + \frac{1}{2}\gamma_{i,m}^2\|e_{i,m}\|^2 - \frac{1}{\sigma_{i,m}}\widetilde{\theta}_{i,m}^T(\sigma_{i,m}\Psi_{i,m}s_{i,m} - \rho_{i,m}\theta_{i,m}) \\ & - \frac{w_{i,m}^2}{\lambda_{i,m}} + \frac{1}{2}w_{i,m}^2 + \frac{1}{2}M_{i,m}^2\Big].\end{aligned} \tag{3.62}$$

Combining (3.13), (3.38) and (3.50), we have

$$\begin{aligned}\dot{V}_m \le & -q_m\|e\|^2 + \eta_m + \sum_{i=1}^{N}\sum_{p=1}^{n}\frac{1}{2}\widetilde{\theta}_{i,p}^T\widetilde{\theta}_{i,p} - \frac{2}{\lambda_{\max}(H^{-1})}\left(\frac{\partial P(x_1)}{\partial x_1}\right)^T H^{-1}\left(\frac{\partial P(x_1)}{\partial x_1}\right) \\ & + \sum_{i=1}^{N}\Big[\sum_{p=1}^{m}\frac{\rho_{i,p}}{\sigma_{i,p}}\widetilde{\theta}_{i,p}^T\theta_{i,p} - \sum_{p=2}^{m}c_{i,p}s_{i,p}^2 - \sum_{p=2}^{m}\left(\frac{1}{\lambda_{i,p}} - 1\right)w_{i,p}^2 + \frac{1}{2}\sum_{p=2}^{m}M_{i,m}^2 + \frac{1}{2}\left(s_{i,m+1}^2 + w_{i,m+1}^2\right)\Big]\end{aligned} \tag{3.63}$$

where $q_m = q_{m-1} - \frac{1}{2}\sum_{i=1}^{N}\left(\varepsilon_{i,m}^2 + \gamma_{i,m}^2\right), \eta_m = \eta_{m-1} + \frac{1}{2}\sum_{i=1}^{N}\|\xi_{i,m}\|^2$.

Step n. Design $s_{i,n} = \hat{x}_{i,n} - v_{i,n}$, by (3.2) and (3.4), we obtain

$$\begin{aligned}\dot{s}_{i,n} =& \dot{\hat{x}}_{i,n} - \dot{v}_{i,n} \\ =& u_i + \varepsilon_{i,n} e_{i,1} + \theta_{i,n}^T \Psi_{i,n} + \widetilde{\theta}_{i,n}^T \Psi_{i,n} + \xi_{i,n} + \Delta h_{i,n} - \dot{v}_{i,n}.\end{aligned} \tag{3.64}$$

According to Lemma 1, define $\delta_n = \frac{1}{k_{i,bn}^2 - s_{i,n}^2}$. Put forward the BLF,

$$V_n = V_{n-1} + \frac{1}{2}\sum_{i=1}^{N}\left\{\delta_n s_{i,n}^2 + \frac{1}{\sigma_{i,n}}\widetilde{\theta}_{i,n}^T\widetilde{\theta}_{i,n} + w_{i,n}^2\right\} \tag{3.65}$$

where $k_{i,bn}$ and $\sigma_{i,n}$ are the parameter we set.

Then, we have

$$\dot{V}_n = \dot{V}_{n-1} + \sum_{i=1}^{N}\left\{\delta_n s_{i,n}\dot{s}_{i,n} + \frac{1}{\sigma_{i,n}}\widetilde{\theta}_{i,n}^T\dot{\widetilde{\theta}}_{i,n} + w_{i,n}\dot{w}_{i,n}\right\}. \tag{3.66}$$

Substituting (3.64) into (3.66), we obtain

$$\begin{aligned}\dot{V}_n =& \dot{V}_{n-1} + \sum_{i=1}^{N}\Big[\delta_n s_{i,n}\Big(u_i + \varepsilon_{i,m} e_{i,1} + \theta_{i,n}^T\Psi_{i,n} + \widetilde{\theta}_{i,n}^T\Psi_{i,n} + \xi_{i,n} + \Delta h_{i,n} - \dot{v}_{i,n}\Big) \\ &+ \frac{1}{\sigma_{i,n}}\widetilde{\theta}_{i,n}^T\dot{\widetilde{\theta}}_{i,n} + w_{i,n}\dot{w}_{i,n}\Big].\end{aligned} \tag{3.67}$$

According to Lemma 3 and the derivation principle of the previous steps, Eq (3.67) is formulated as

$$\begin{aligned}\dot{V}_n \le& \dot{V}_{n-1} + \sum_{i=1}^{N}\Big[\delta_n s_{i,n}\Big(u_i + \theta_{i,n}^T\Psi_{i,n} + \widetilde{\theta}_{i,n}^T\Psi_{i,n} - \dot{v}_{i,n}\Big) + \frac{3\delta_n}{2}s_{i,n}^2 \\ &+ \frac{1}{2}\varepsilon_{i,n}^2\|e_{i,1}\|^2 + \frac{1}{2}\|\xi_{i,n}\|^2 + \frac{1}{2}\gamma_{i,n}^2\|e_{i,n}\|^2 - \frac{1}{\sigma_{i,n}}\widetilde{\theta}_{i,n}^T\dot{\theta}_{i,n} + w_{i,n}\dot{w}_{i,n}\Big].\end{aligned} \tag{3.68}$$

According to Theorem1,

$$u_i = -c_{i,n}s_{i,n} - \left(\frac{1}{2\delta_n} + \frac{3}{2}\right)s_{i,n} - \theta_{i,n}^T\Psi_{i,n}(\hat{X}_{i,n}) + \frac{x_{i,n}^* - v_{i,n}}{\lambda_{i,n}} \tag{3.69}$$

$$\dot{\theta}_{i,n} = \sigma_{i,n}\delta_n\Psi_{i,n}(\hat{X}_{i,n})s_{i,n} - \rho_{i,n}\theta_{i,n}. \tag{3.70}$$

Substitute Eqs (3.69), (3.70) and (3.19) into (3.68). By Lemma 3, we have $w_{i,n}B_{i,n} \le \frac{1}{2}w_{i,n}^2 + \frac{1}{2}M_{i,n}^2$, then we can obtain

$$\begin{aligned}\dot{V}_n \le& -q_n\|e\|^2 + \eta_n + \sum_{i=1}^{N}\sum_{p=1}^{n}\frac{1}{2}\widetilde{\theta}_{i,p}^T\widetilde{\theta}_{i,p} - \frac{2}{\lambda_{\max}(H^{-1})}\left(\frac{\partial P(x_1)}{\partial x_1}\right)^T H^{-1}\left(\frac{\partial P(x_1)}{\partial x_1}\right) \\ &+ \sum_{i=1}^{N}\left[\sum_{p=1}^{n}\frac{\rho_{i,p}}{\sigma_{i,p}}\widetilde{\theta}_{i,p}^T\theta_{i,p} - \sum_{p=2}^{n}c_{i,p}s_{i,p}^2 - \sum_{p=2}^{n}\left(\frac{1}{\lambda_{i,p}} - 1\right)w_{i,p}^2 + \frac{1}{2}\sum_{p=2}^{n}M_{i,p}^2\right]\end{aligned} \tag{3.71}$$

where $q_n = q_{n-1} - \frac{1}{2}\sum_{i=1}^{N}\left(\varepsilon_{i,n}^2 + \gamma_{i,n}^2\right), \eta_n = \eta_{n-1} + \frac{1}{2}\sum_{i=1}^{N}\|\xi_{i,n}\|^2$.

Through Lemma 3, we have

$$\widetilde{\theta}_{*,p}^T \theta_{*,p} \leq -\frac{1}{2}\widetilde{\theta}_{*,p}^T \widetilde{\theta}_{*,p} + \frac{1}{2}\theta_{*,p}^{*T}\theta_{*,p}^{*}. \tag{3.72}$$

Define

$$\zeta = \eta_n + \sum_{i=1}^{N}\sum_{p=1}^{n}\frac{\rho_{i,p}}{2\sigma_{i,p}}\theta_{i,p}^{*T}\theta_{i,p}^{*} + \frac{1}{2}\sum_{p=2}^{n} M_{i,p}^2. \tag{3.73}$$

Thus, we rewrite (3.71) as follows

$$\begin{aligned}\dot{V}_n \leq & - q_n\|e\|^2 - \frac{2}{\lambda_{\max}\left(H^{-1}\right)}\left(\frac{\partial P\left(x_1\right)}{\partial x_1}\right)^T H^{-1}\left(\frac{\partial P\left(x_1\right)}{\partial x_1}\right) \\ & + \sum_{i=1}^{N}\left[-\sum_{p=2}^{n} c_{i,p}s_{i,p}^2 - \sum_{p=1}^{n}\left(\frac{\rho_{i,p}}{2\sigma_{i,p}} - \frac{1}{2}\right)\widetilde{\theta}_{i,p}^T\widetilde{\theta}_{i,p} - \sum_{p=2}^{n}\left(\frac{1}{\lambda_{i,p}} - 1\right)w_{i,p}^2\right] + \zeta\end{aligned} \tag{3.74}$$

where $c_{i,p} > 0, \left(\frac{\rho_{i,p}}{2\sigma_{i,p}} - \frac{1}{2}\right) > 0, \left(\frac{1}{\lambda_{i,p}} - 1\right) > 0, (p = 2, \cdots, n), \frac{2}{\lambda_{\max}\left(H^{-1}\right)} > 0$.

Define

$$C = min\left\{2\frac{q_n}{\lambda_{min}(P)}, 2c_{i,p}, 2\left(\frac{\rho_{i,p}}{2\sigma_{i,p}} - \frac{1}{2}\right), 2\left(\frac{1}{\lambda_{i,p}} - 1\right), -\frac{4}{\lambda_{\min}\left(H^{-1}\right)}\right\}. \tag{3.75}$$

Thus, we obtain

$$\dot{V}_n \leq - CV(x(t)) + \zeta. \tag{3.76}$$

According to the study conducted by [51], it is easily verified that in the closed-loop system, all of the signals from system (2.1) stay SGUUB.

Remark 2. *As is shown in Theorem1, compared to the papers that study the bipartite consensus control of MASs [38, 39], Eq* (3.22) *introduces two reference signals x_{d1} and x_{d2} in the design of a distributed optimization controller. Serving as virtual leaders, x_{d1} and x_{d2} achieve a bipartite containment effect, which aims to ensure that the trajectory of the optimal solution in the distributed optimization lies within the convex hulls defined by the upper and lower reference signals. These signals generate upper and lower convex hulls due to the negative signals emitted by agents.*

Remark 3. *For the distributed optimization problem of bipartite containment control in high-order nonlinear MASs with state constraints, by integrating RBFNN and DSC techniques, a BLF-based adaptive backstepping controller is designed. Moreover, a more concise optimization algorithm based on penalty functions is presented to minimize the global objective function and achieve optimal output.*

4. Simulations

In this session, we use two examples to show how successful the control mechanism is. The results prove that the distributed optimal control algorithm in this paper has practical applications.

4.1. Example 1

For a second-order system, the model is as follows

$$\begin{cases} \dot{x}_{i,1} = x_{i,2} + h_{i,1}(x_{i,1}) \\ \dot{x}_{i,2} = u_i + h_{i,2}(x_{i,1}, x_{i,2}) \\ y_i = x_{i,1}. \end{cases} \tag{4.1}$$

The unknown functions in system (4.1) are

$$\begin{aligned} h_{i,1} =& 0 \\ h_{1,2} =& x_{1,1} - 0.25x_{1,2} - x_{1,1}^3 + 0.3\cos(t) \\ h_{2,2} =& x_{2,1} - 0.25x_{2,2} - x_{2,1}^3 + 0.1\left(x_{2,1}^2 + x_{2,2}^2\right)^{1/2} + 0.3\cos(t) \\ h_{3,2} =& x_{3,1} - 0.25x_{3,2} - x_{3,1}^3 + 0.1\sin(t)\left(x_{3,1}^2 + 2x_{3,2}^2\right)^{1/2} + 0.3\cos(t) \\ h_{4,2} =& x_{4,1} - 0.25x_{4,2} - x_{4,1}^3 + 0.1\sin(t)\left(2x_{4,1}^2 + 2x_{4,2}^2\right)^{1/2} + 0.3\cos(t) \\ h_{5,2} =& x_{5,1} - 0.1x_{5,2} - x_{5,1}^3 + 0.2\sin(t)\left(x_{5,1}^2 + x_{5,2}^2\right)^{1/2} + 0.3\cos(t). \end{aligned}$$

The topology of Figure 1 shows the information exchange between five agents, with agent 4 receiving the opposite information. The initial states of five agents are set as $x_1(0) = [0.05, 0.05]$, $x_2(0) = [0.1, 0.1]$, $x_3(0) = [0.15, 0.15]$, $x_4(0) = [-0.2, -0.2]$, $x_5(0) = [0.25, 0.25]$. Define $x_{d1} = 0.2 * \sin(t) + 0.2$, $x_{d2} = 0.2 * \sin(t) + 0.4$ as the reference signals. Thus, the trajectory of the optimal signal is $x_1^* = 0.2 * \sin(t) + 0.3$. The local objective functions of agents are given as follows

$$\begin{aligned} f1 &= 8.5x_1^2 - (8x_{d1} + 9x_{d2})x_1 + 4{x_{d1}}^2 + 4.5{x_{d2}}^2 + 1 \\ f2 &= 16.5x_2^2 - (16x_{d1} + 17x_{d2})x_2 + 8{x_{d1}}^2 + 8.5{x_{d2}}^2 + 2 \\ f3 &= 13x_3^2 - (12x_{d1} + 14x_{d2})x_3 + 12{x_{d1}}^2 + 14{x_{d2}}^2 + 1 \\ f4 &= 15.2x_4^2 + (14.4x_{d3} + 16x_{d4})x_4 + 7.2{x_{d3}}^2 + 8{x_{d4}}^2 + 2 \\ f5 &= 9.5x_5^2 - (9x_{d1} + 10x_{d2})x_5 + 4.5{x_{d1}}^2 + 5{x_{d2}}^2 + 2. \end{aligned}$$

Regarding the observer, the parameters are chosen as $\varepsilon_{i,1} = 500$, $\varepsilon_{i,2} = 5,000$ and the initial states are selected as $\hat{x}_1 = [0.05, 0.05]$, $\hat{x}_2 = [0.1, 0.1]$, $\hat{x}_3 = [0.15, 0.15]$, $\hat{x}_4 = [-0.2, -0.2]$, $\hat{x}_5 = [0.25, 0.25]$.

According to the virtual control law $x_{i,2}^*$ (3.36), adaptive law $\theta_{i,1}$ (3.37), $\theta_{i,n}$ (3.70) and control input u_i (3.69) in Theorem 1, we chose control parameters as $c_{i,1} = 3.5$, $c_{i,2} = 30$, $\sigma_{i,1} = \sigma_{i,2} = 1$, $\rho_{i,1} = \rho_{i,2} = 80$, $\lambda_{i,2} = 0.05$. To guarantee the state constraints are not violated, $kb_1 = 0.3$ is given.

In this simulation, Figures 2–7 display the simulation outcomes. Figure 2 describes the trajectories of x_1^* and $x_{i,1}$, verifying that each agent can track the optimal signal x_1^*. Besides, x_{d1} and x_{d2} act as virtual leaders to achieve a bipartite containment effect. In Figure 3, we use $x_{i,1}$ as an example via comparing the true value and the estimated value. It can be seen that the designed observer performs well and can approximately observe the unknown states. Figure 4 shows the trajectories of $s_{i,1}$, which clearly demonstrates how the tracking error can rapidly approach zero by the designed controller. Together, Figures 2 and 4 illustrate that there is a good distributed optimization consensus tracking effect and the tracking error is within 0.05. Figure 5 displays the trajectories of $x_{i,2}$, the state in this paper is constrained. Figure 6 gives the trajectories of control input u_i. Figure 7 shows that $s_{i,2}$ are all in the range of -0.3 to 0.3, satisfying $|s_{i,2}| \leq kb_1$. From Figures 5 and 7, we can draw the conclusion that based on the BLF, $s_{i,2}$ and $x_{i,2}$ can be limited successfully under the designed parameters. The state constraints are not violated, and the tracking error can converge to the compact sets.

Remark 4. *Compared to the simulation example in our previous work [37], under the unchanged initial conditions and parameters, Figure 8 is the trajectory tracking graph of all agents, which confirms that the agent outputs converge inside both the positive and negative convex hulls defined by the reference leader signals. Based on this, the distributed optimization problem is considered in our paper. We design a distributed optimization algorithm based on a penalty function. It can be clearly seen that the control protocol proposed in this paper enables all agents to track the optimal solution. The controller we designed can ensure that all agents have a good distributed optimization consensus tracking effect for high-order nonlinear uncertain MASs.*

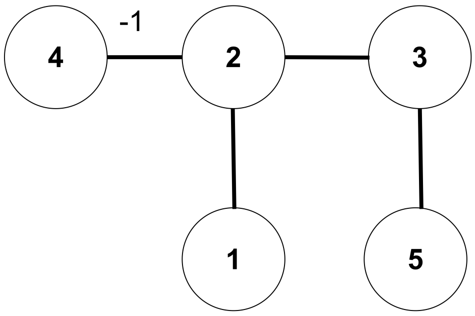

Figure 1. Communication graph in simulation.

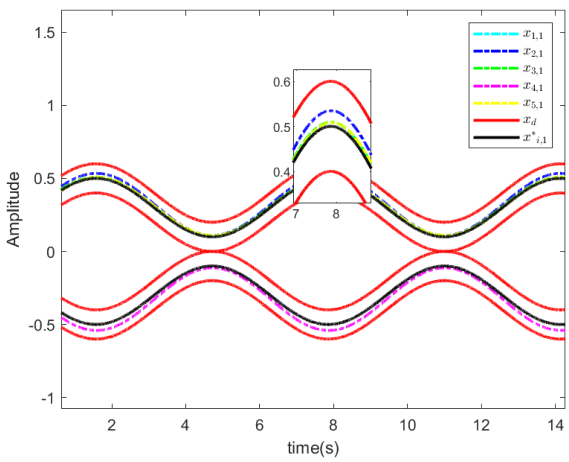

Figure 2. Tracking performance of $x_{i,1}$ ($i = 1, \cdots, 5$).

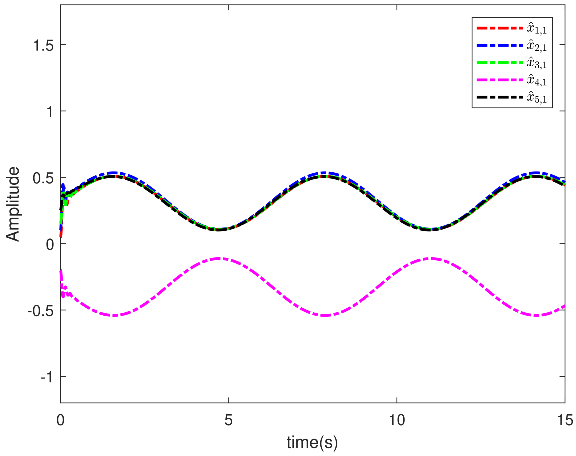

Figure 3. Trajectories of $x_{i,1}$ $(i = 1, \cdots, 5)$ estimation.

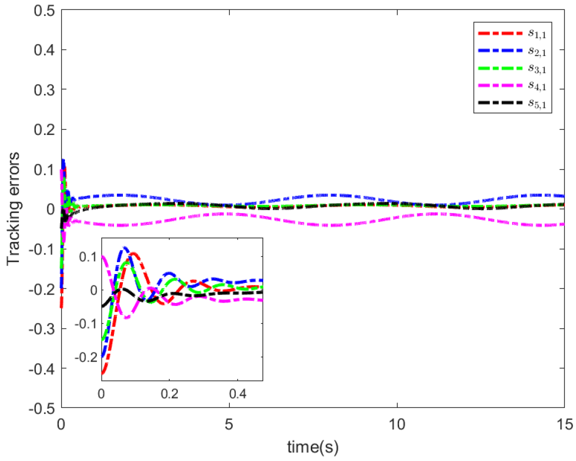

Figure 4. The curves of error $s_{i,1}$ $(i = 1, \cdots, 5)$.

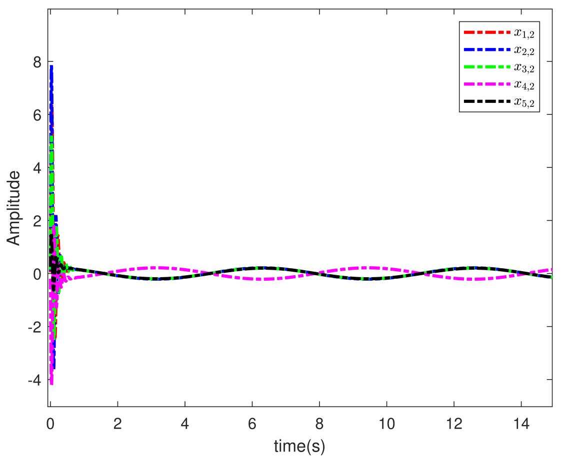

Figure 5. Constraint trajectories of $x_{i,2}$ $(i = 1, \cdots, 5)$.

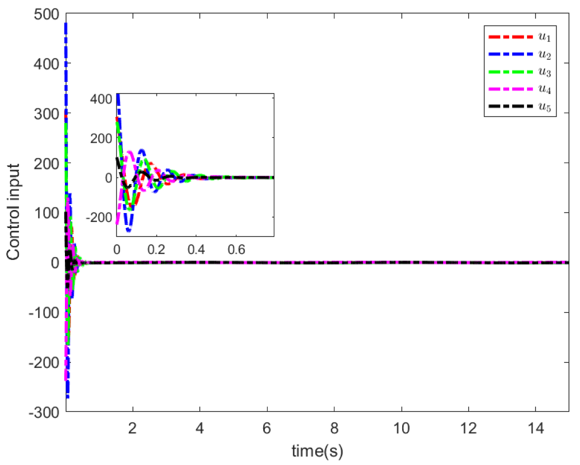

Figure 6. Control input u_i.

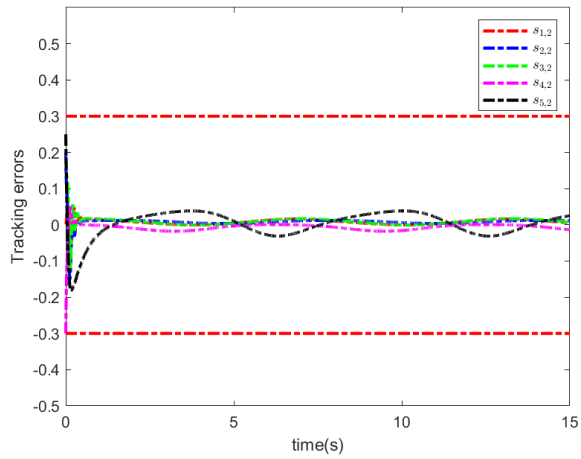

Figure 7. Constraint error $s_{i,2}$ $(i = 1, \cdots, 5)$.

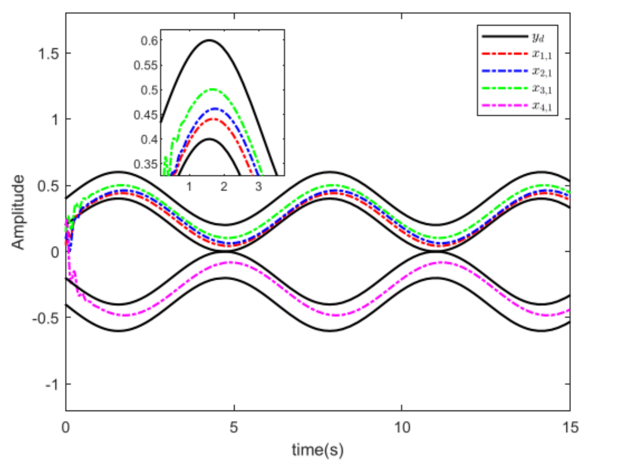

Figure 8. Trajectories of $x_{i,1}$ $(i = 1, \cdots, 4)$ estimation in the previous work.

4.2. Example 2

Given a single-link manipulator that includes motor dynamics [52], the dynamic equation of the system is as follows:

$$\begin{cases} D_i\ddot{q}_i + B_i\dot{q}_i + N_i \sin q_i = \tau_i \\ M_i\dot{\tau}_i + H_i\tau_i = u_i - K_i\dot{q}_i \end{cases} \tag{4.2}$$

where q_i , $\dot{q}$ and $\ddot{q}$ represent the link position, velocity and acceleration for the ith mechanical system. τ_i is the torque produced by the electrical subsystem, respectively. $D_i = 1kg \cdot m^2$ is the inertia, $B_i = 1N \cdot m \cdot s/rad$ represents the viscous friction coefficient at the joint and $N_i = 2$ is a constant that is related to the coefficient of gravity and the mass of the load. $M_i = 1H$ denotes the armature inductance. $H_i = 1\Omega$ represents armature resistance. $K_i = 2N \cdot m/A$ signifies the back electromotive force coefficient. u_i is the input signal. Letting $x_{i,1} = q_i$, $x_{i,2} = \dot{q}_i$, $x_{i,3} = \tau_i$, system (4.2) can be expressed as:

$$\begin{cases} \dot{x}_{i,1} = x_{i,2} \\ \dot{x}_{i,2} = x_{i,3} - N_i \sin x_{i,1} - B_i x_{i,2} \\ \dot{x}_{i,3} = u_i - K_i x_{i,2} - H_i x_{i,3}. \end{cases} \tag{4.3}$$

Consider the topology of Figure 1 including five agents, whose initial states are set as $x_1(0) = [0.3, 0.3, 0.3]$, $x_2(0) = [0.15, 0.15, 0.15]$, $x_3(0) = [0.15, 0.15, 0.15]$, $x_4(0) = [-0.2, -0.2, -0.2]$, $x_5(0) = [0.25, 0.25, 0.25]$. Define $x_{d1} = 0.2 * \sin(t) + 0.2$, $x_{d2} = 0.2 * \sin(t) + 0.4$ as the reference signals. The optimal signal is $x_1^* = 0.2 * \sin(t) + 0.3$.

The local objective functions of agents are given as follows

$$\begin{aligned} f1 &= 8.5x_1^2 - (8x_{d1} + 9x_{d2}) * x_1 + 4{x_{d1}}^2 + 4.5{x_{d2}}^2 + 1 \\ f2 &= 13.5x_2^2 - (12x_{d1} + 15x_{d2})x_2 + 6{x_{d1}}^2 + 7.5{x_{d2}}^2 + 2 \\ f3 &= 13x_3^2 - (12x_{d1} + 14x_{d2})x_3 + 6{x_{d1}}^2 + 7{x_{d2}}^2 + 1 \\ f4 &= 14.2x_4^2 + (14x_{d3} + 14.4x_{d4})x_4 + 7{x_{d3}}^2 + 7.2{x_{d4}}^2 + 2 \\ f5 &= 9.5x_5^2 - (9x_{d1} + 10x_{d2})x_5 + 4.5{x_{d1}}^2 + 5{x_{d2}}^2 + 2. \end{aligned}$$

According to the virtual control laws $x_{i,2}^*$ (3.36), $x_{i,3}^*$ (3.48), adaptive laws $\theta_{i,1}$ (3.37), $\theta_{i,2}$ (3.49), $\theta_{i,n}$ (3.70) and control input u_i (3.69) in Theorem 1, we chose control parameters as $c_{i,1} = 3.5$, $c_{i,2} = 2$, $c_{i,3} = 60$, $\sigma_{i,1} = \sigma_{i,2} = \sigma_{i,3} = 1$, $\rho_{i,1} = \rho_{i,2} = 80$, $\rho_{i,3} = 8$ $\lambda_{i,2} = \lambda_{i,3} = 0.01$. To guarantee the state constraints are not violated, $kb_1 = 4$ and $kb_2 = 4$ are given.

In this simulation, Figures 9–13 display the simulation outcomes. Figure 9 shows the trajectories of x_d and $x_{i,1}$, each agent can track the optimal signal $x_{i,1}^*$. Meanwhile, x_{d1} and x_{d2} act as virtual leaders to achieve a bipartite containment effect. Figure 10 shows the trajectories of $s_{i,1}$ which clearly demonstrates how the tracking error can rapidly approach to near zero, proving that there is a good distributed optimization consensus tracking effect and the tracking error is within 0.05. Figure 11 gives the trajectories of the control input u_i. The common barrier Lyapunov method mainly forms the error constraint, Figures 12 and 13 show that $s_{i,2}$ and $s_{i,3}$ are all in the range of –4 to 4, respectively, which indicates that the BLF can solve the state constraints problem of high-order nonlinear uncertain MASs well.

According to the simulation results, the distributed optimization controller proposed in this research can make agents converge to the optimal solution, which lies within the convex hull of the trajectories of each virtual leader and their opposite trajectories. Meanwhile, the tracking error gradually converges to a small range of the origin, which indicates the control performance is good. In addition, the barrier Lyapunov method is generally used to impose constraints on errors. From step 2, the BLF-based control scheme transforms the original state constraints into a new bound on the tracking error and achieves the state constraints by constraining the error surfaces.

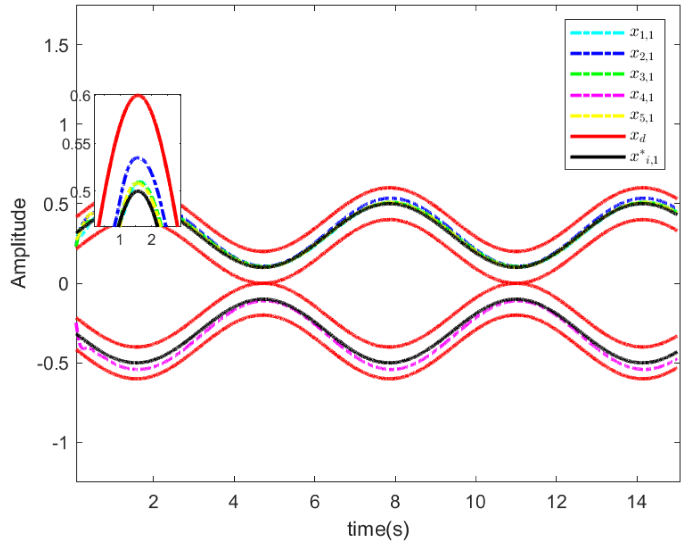

Figure 9. Tracking performance of $x_{i,1}$ $(i = 1, \cdots, 5)$.

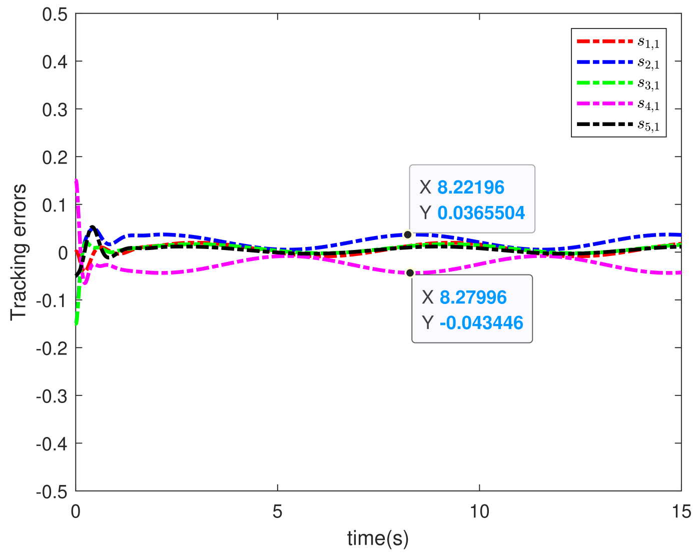

Figure 10. The curves of error $s_{i,1}$ $(i = 1, \cdots, 5)$.

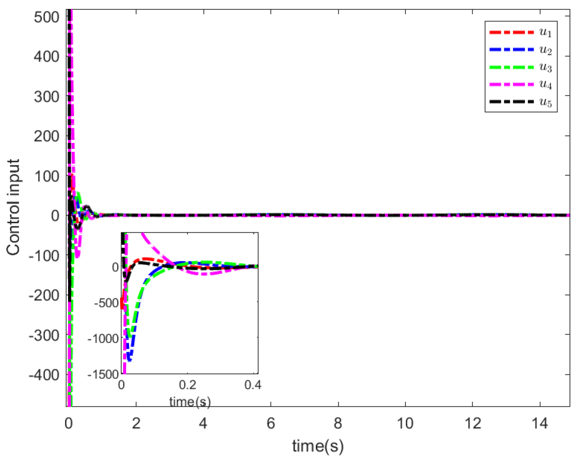

Figure 11. Control input u_i.

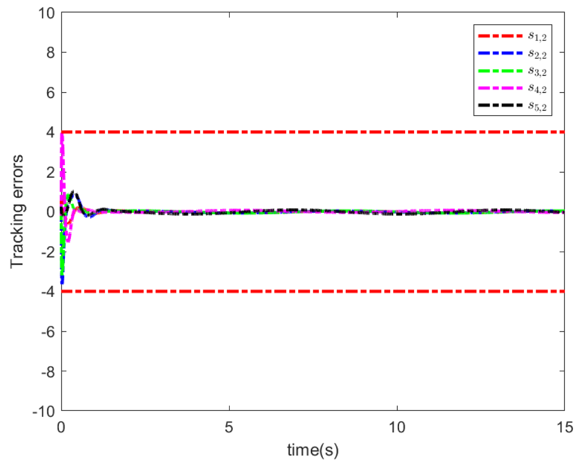

Figure 12. Constraint error $s_{i,2}$ $(i = 1, \cdots, 5)$.

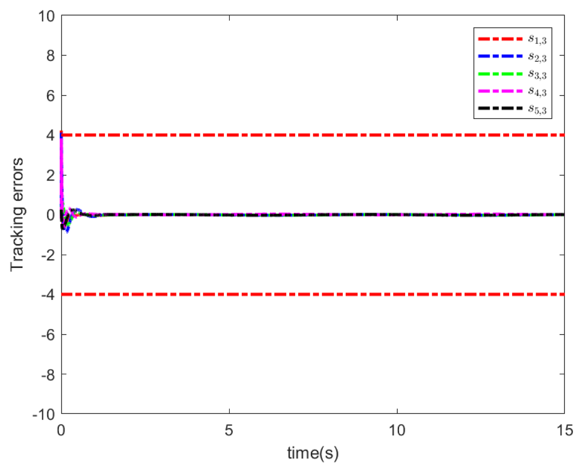

Figure 13. Constraint error $s_{i,3}$ $(i = 1, \cdots, 5)$.

5. Conclusions

This paper researches the bipartite containment distributed optimization problem of high-order MASs with uncertain nonlinear functions. Combining the consensus condition of MASs with the global objective function, we designed a penalty function, which is constructed by combining the bipartite containment definition. By integrating RBFNN and DSC techniques, an adaptive backstepping controller is proposed to avoid the complexity explosion and a distributed optimal consensus is accurately achieved. In addition, introducing state constraints in the distributed optimization holds significant practical importance in engineering applications. Besides, we will consider addressing the problem of prescribed-time distributed optimization of high-order nonlinear stochastic MASs with disturbances.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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