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Research article

Finite-time contraction stability of a stochastic reaction-diffusion dengue model with impulse and Markov switching

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Abstract: From the perspective of prevention and treatment of dengue, it is important to minimize the number of infections within a limited time frame. That is, the study of finite time contraction stability (FTCS) of dengue system is a meaningful topic. This article proposes a dengue epidemic model with reaction-diffusion, impulse and Markov switching. By constructing an equivalent system, the well-posedness of the positive solution is proved. The main result is that sufficient conditions to guarantee the finite time contraction stability of the dengue model are acquired based on the average pulse interval method and the bounded pulse interval method. Furthermore, the numerical findings indicate the influences of impulse, control strategies and noise intensity on the FTCS.

Keywords: reaction-diffusion; finite-time contraction stability; bounded pulse interval method; Markov switching; average pulse interval method

1. Introduction

Dengue fever is one of the most dangerous diseases spread by mosquitoes around the world. According to the statistics from the WHO, almost 50% of people on earth reside in areas at risk of dengue. Each year, up to 400 million people get infected with dengue, of which 100 million exhibit clinical symptoms of varying severity, resulting in about 40,000 deaths [1]. In recent years, the incidence of dengue fever has increased dramatically, from 505,430 cases in 2000 to 4.2 million cases in 2019 [2]. As a result, dengue fever has seriously endangered people's safety and hindered economic growth. Understanding the pathogenesis of dengue is essential for early intervention.

Mathematical modeling has become an extremely effective method to understand the mechanisms of dengue transmission and predict the development trend of dengue. Recently, various dengue models considering different factors have been widely proposed [3–6]. For example, Zhu and Xu [5] developed a dengue model that takes into account both spatial heterogeneity and temporal periodicity to discuss the asymptotic stability of periodic solutions. In [4], Li et al. studied a study on the optimal control problem of a dengue system that incorporates reaction-diffusion. Unfortunately, the above models ignored the effect of spraying insecticides and sterile mosquitoes for dengue. These human behaviors can be characterized by impulsive differential equations. So far, some results have been obtained for modeling impulsive dengue models [7–10]. For instance, Pang et al. [7] investigated a stage-structured impulsive model to explore the feasibility of controlling wild mosquito populations by periodically releasing sterile mosquitoes. Yang and Nie [8] conducted a dengue model that involved culling mosquitoes to investigate the effect of the impulsive strategies. However, they did not discuss the influence of noise.

In fact, it is widely recognized that external environmental factors have a significant impact on the spread of dengue fever. The use of random differential equations driven by white noise to construct dengue models have attracted widespread attention [11–14]. For example, Liu and Din [11] introduced the random disturbance and information intervention factors into the dengue model to analyze the stability of the positive solution. Chang et al. [12] established a stochastic dengue model related to white noise to investigate the stationary distribution and optimal control. However, white noise can only be applied to continuous random disturbances and cannot describe sudden changes in the environment. Dengue may be impacted by unforeseen weather changes, like temperature and rainfall, which will lead to switching from one environmental system to another. Frequently, the switch occurs without memory and the waiting period until the next transition is determined by the exponential distrnibution [15]. Therefore, some scholars have introduced continuous-time Markov chains to describe the regime switch [16–19]. Nevertheless, Markov switching is hardly considered in the dengue models, only in [16] Liu et al. analyzed the stationary distribution of a stochastic dengue model considering immune responses and Markov switching.

It is worth noting that the existing dengue models only consider the long-term dynamical behavior. When addressing the concern of controlling dengue, it makes more sense to focus on limiting the number of infected individuals within a specific time frame to reduce within a specified threshold and eventually make it lower than the original number. This dynamic property is referred to as finite-time contraction stability (FTCS). As far as we know, there have been intriguing findings concerning FTCS [20–23]. However, no studies have analyzed the finite-time contraction stability of dengue.

In view of the above discussion, a stochastic dengue epidemic model with reaction-diffusion, impulse and Markov switching is constructed. Compared with the latest research results on the dengue model that only consider the long-term behaviors of the system [24, 25], our results not only consider FTCS of dengue model for the first time but also investigate the effects of control strategies, impulse and noise intensity on FTCS. The coexistence of diffusion, pulse and Markov switching makes the model complex, which brings challenges to analyze the stability of the system by Lyapunov stability theory. Moreover, it is difficult to get a suitable range of control variables for the system to achieve FTCS. To solve these problems, we use the average and bounded pulse interval method combined with the comparison method to effectively obtain sufficient conditions for FTCS. Especially, the highlights are listed as follows:

- Considering the use of pesticides and the release of sterile mosquitoes, as well as parameters uncertainties, a hybrid impulse dengue model with diffusion and Markov switching is proposed, which is an extension of the model in [12].

- Less conservative sufficient conditions for FTCS of the stochastic dengue system are presented based on two impulsive representations. In addition, the effects of control strategies, impulse and noise intensity on FTCS are also analyzed.

The remainder of this document is structured in the following manner: In Section 2, Some preliminary preparations are introduced. According to an equivalent system, Section 3 yields the well-posedness of positive solution of the system. In Section 4, the average and bounded pulse interval method are used to provide the criteria to ensure FTCS. Section 5 gives various numerical simulations to effectively illustrate the theoretical discoveries, as well as to display the impacts of impulse, control strategies and noise intensity on FTCS. The last section gives the conclusion of this paper and discusses further work.

2. Model development and preliminary preparation

For the purpose of subsequent analysis, a reaction-diffusion dengue model with impulse and Markov switching is established and give some notations.

2.1. Model development

The rapid development of transportation has created convenient conditions for the spread of mosquitoes and people, which increases the risk of contracting the virus. Considering the existence of spatial diffusion, Chang et al. [12] proposed the following system:

$$\begin{cases} \dfrac{\partial S_H(x,t)}{\partial t} = D_1 \triangle S_H(x,t) + \mu_h(x) N_H - \dfrac{\beta_H(x) b}{N_H + m} S_H(x,t) I_v(x,t) - \mu(x) S_H(x,t), \\ \dfrac{\partial I_H(x,t)}{\partial t} = D_2 \triangle I_H(x,t) + \dfrac{\beta_H(x) b}{N_H + m} S_H(x,t) I_v(x,t) - (\mu(x) + \gamma_H(x)) I_H(x,t), \\ \dfrac{\partial R_H(x,t)}{\partial t} = D_3 \triangle R_H(x,t) + \gamma_H(x) I_H(x,t) - \mu(x) R_H(x,t), \\ \dfrac{\partial S_v(x,t)}{\partial t} = D_4 \triangle S_v(x,t) + A - \dfrac{\beta_v(x) b}{N_H + m} S_v(x,t) I_H(x,t) - v(x) S_v(x,t), \\ \dfrac{\partial I_v(x,t)}{\partial t} = D_5 \triangle I_v(x,t) + \dfrac{\beta_v(x) b}{N_H + m} S_v(x,t) I_H(x,t) - v(x) I_v(x,t), \end{cases} \tag{2.1}$$

with initial value

$$(S_H(x,0), I_H(x,0), R_H(x,0), S_v(x,0), I_v(x,0)) = (S_{H,0}(x), I_{H,0}(x), R_{H,0}(x), S_{v,0}(x), I_{v,0}(x)). \tag{2.2}$$

and boundary condition

$$\frac{\partial S_H(x,t)}{\partial \nu} = \frac{\partial I_H(x,t)}{\partial \nu} = \frac{\partial R_H(x,t)}{\partial \nu} = \frac{\partial S_v(x,t)}{\partial \nu} = \frac{\partial I_v(x,t)}{\partial \nu} = 0, \quad x \in \partial\Gamma, t > 0. \tag{2.3}$$

Below is an explanation of the variables used in the equation.

- S_H, I_H, R_H: the density of susceptible, infectious and immune individuals, respectively. Meanwhile, $N_H = S_H + I_H + R_H$.

- S_v, I_v: the density of susceptible and infectious mosquitoes, respectively.
- D_1, D_2, D_3: the diffusion coefficient of individuals, respectively. (per square kilometer per day)
- D_4, D_5: the diffusion coefficient of mosquitoes, respectively. (per square kilometer per day)
- γ_H, μ_h, μ: the recovery, birth, and death rate of individuals, respectively. (per day)
- m, A: the density of alternative hosts that can serve as blood sources and the recruitment of mosquitoes, respectively
- b, v: the biting rate and natural mortality rate of mosquitoes, respectively. (per day)
- β_H: the probability of infected mosquitoes transmitting to susceptible individuals. (per bite)
- β_v: the probability of infected individuals transmitting to susceptible mosquitoes.(per bite)

At the same time, there is evidence that the vector control techniques (e.g., the development of Wolbachia-infected mosquitoes, the use of insecticides and so on) are effective ways to avoid the spread of dengue fever [26, 27]. Therefore, we consider the following control measures on the basis of system (2.1).

(i) We define $\varnothing_{1k}$ and $\varnothing_{2k}$ as the impulse intensities that affect susceptible mosquitoes and infected mosquitoes, respectively. Among them, $\varnothing_{1k} > 0$ indicates the release of sterile mosquitoes, and $\varnothing_{1k} < 0, \varnothing_{2k} > 0$ indicates the killing of mosquitoes. Because of the need to maintain ecological stability, mosquitoes cannot be completely killed. Thus, we have reason to assume that $-1 < \varnothing_{1k} < \varnothing_{1m}$, $0 < \varnothing_{2k} < \varnothing_{2m}$, where $\varnothing_{1m}$ and $\varnothing_{2m}$ are the maximum allowable impulse on susceptible mosquitoes and infected mosquitoes, respectively.

(ii) Treating infected individuals and spraying insecticides on mosquitoes can reduce the number of infected. δ_1 is the recovery rate of dengue infection individuals due to treatment, δ_2 is the culling rate of mosquitoes, $\delta_1(x)I_H(x,t)$ indicates the proportion of infected individuals recovered and $\delta_2(x)S_v(x,t), \delta_2(x)I_v(x,t)$ represents the proportion of mosquitoes eliminated.

Based on the above discussions, we tend to unintentionally introduce pulse perturbations into system (2.1), and obtain the following equations with control variables:

$$\begin{cases} \left.\begin{aligned} dS_H(x,t) &= [D_1 \triangle S_H(x,t) + \mu_h(x)N_H - \frac{\beta_H(x)b}{N_H+m} S_H(x,t)I_v(x,t) - \mu(x)S_H(x,t)]dt \\ dI_H(x,t) &= [D_2 \triangle I_H(x,t) + \frac{\beta_H(x)b}{N_H+m} S_H(x,t)I_v(x,t) - (\mu(x)+\gamma_H(x))I_H(x,t) - \delta_1(x)I_H(x,t)]dt \\ dR_H(x,t) &= [D_3 \triangle R_H(x,t) + \gamma_H(x)I_H(x,t) + \delta_1(x)I_H(x,t) - \mu(x)R_H]dt, \\ dS_v(x,t) &= [D_4 \triangle S_v(x,t) + A - \frac{\beta_v(x)b}{N_H+m} S_v(x,t)I_H(x,t) - v(x)S_v(x,t) - \delta_2(x)S_v(x,t)]dt \\ dI_v(x,t) &= [D_5 \triangle I_v(x,t) + \frac{\beta_v(x)b}{N_H+m} S_v(x,t)I_H(x,t) - v(x)I_v(x,t) - \delta_2(x)I_v(x,t)]dt, \end{aligned}\right\} & \begin{array}{l} t \neq t_k, \\ t > 0, \\ x \in \Gamma \end{array} \\ \left.\begin{aligned} S_H(x,t_k^+) &= S_H(x,t_k), \\ I_H(x,t_k^+) &= I_H(x,t_k), \\ R_H(x,t_k^+) &= R_H(x,t_k), \\ S_v(x,t_k^+) &= (1+\varnothing_{1k})S_v(x,t_k), \\ I_v(x,t_k^+) &= (1-\varnothing_{2k})I_v(x,t_k) \end{aligned}\right\} t = t_k. \end{cases} \tag{2.4}$$

which with initial values (2.2) and boundary condition (2.3). We set the admissible control set $\mathcal{U}_\delta = \left\{\delta(x) = (\delta_1(x), \delta_2(x)) \in L^\infty\left([0,T]; R^2\right) \mid 0 \leq \delta_i(x) \leq \delta_{i\max} < 1, \forall t \in [0,T]\right\}$. Impulsive sequence $\{t_k\}$, $k \in \mathbb{N}^+$, represents the spraying of pesticides and the release of sterile mosquitoes at time point $t = t_k$, satisfies $0 = t_0 < t_1 < t_2 < \ldots < t_k < \ldots$, as well as $\lim_{k\to+\infty} t_k = +\infty$ and $z(x, t_k^+) = \lim_{t\to t_k^+} z(x,t)$.

In real life, it is widely understood that there are a variety of environmental factors (for instance rainfall, temperature change and so on) that can affect the spread of dengue fever. Assume that the transmission rate β_H and β_v are subject to stochastic fluctuations. That is

$$\beta_H \to \beta_H + \rho_1 dB_1(t), \ \ \beta_v \to \beta_v + \rho_2 dB_2(t).$$

where $B_i(t)(i = 1,2)$ is standard Brownian movements, and ρ_i is its intensity. Furthermore, these sudden environmental changes can also cause the parameters in the system to not be fixed constants. These parameters are assumed to be a stochastic process that satisfies Markov switching. Let $\{\zeta_t\}_{t\geq 0}$ be a right-continuous Markov chain taking values in a finite state $S = \{1,2,...,M\}$. Its generator $o = (q_{ij})_{M\times M}$ is given by the following formula.

$$\mathbb{P}(\zeta_{t+\Delta} = j|\zeta_t = i) = \begin{cases} q_{ij}\Delta + o(\Delta), & i \neq j, \\ 1 + q_{ii}\Delta + o(\Delta), & i = j, \end{cases}$$

where $o(\Delta)$ satisfies $\lim_{\Delta\to 0}\frac{o(\Delta)}{\Delta} = 0$, $\Delta > 0$. If $i \neq j$, $q_{ij} > 0$ is the transition rate from i to j while $q_{ij} = -\sum_{i\neq j} q_{ij}$. Then, system (2.4) can be written as:

$$\begin{cases} \left.\begin{aligned} dS_H(x,t) &= [D_1(\zeta(t))\triangle S_H(x,t) + \mu_h(\zeta(t))N_H(\zeta(t)) - \frac{\beta_H(\zeta(t))b(\zeta(t))}{N_H(\zeta(t)) + m(\zeta(t))}S_H(x,t)I_v(x,t) \\ &\quad - \mu(\zeta(t))S_H(x,t)]dt - \frac{b(\zeta(t))}{N_H(\zeta(t)) + m(\zeta(t))}\rho_1(\zeta(t))S_H(x,t)I_v(x,t)dB_1(t), \\ dI_H(x,t) &= [D_2(\zeta(t))\triangle I_H(x,t) + \frac{\beta_H(\zeta(t))b(\zeta(t))}{N_H(\zeta(t)) + m(\zeta(t))}S_H(x,t)I_v(x,t) - (\mu(\zeta(t)) + \gamma_H(\zeta(t)))I_H(x,t) \\ &\quad - \delta_1(\zeta(t))I_H(x,t)]dt + \frac{b(\zeta(t))}{N_H(\zeta(t)) + m(\zeta(t))}\rho_1(\zeta(t))S_H(x,t)I_v(x,t)dB_1(t), \\ dR_H(x,t) &= [D_3(\zeta(t))\triangle R_H(x,t) + \gamma_H(\zeta(t))I_H(x,t) + \delta_1(\zeta(t))I_H(x,t) - \mu(\zeta(t))R_H]dt, \\ dS_v(x,t) &= [D_4(\zeta(t))\triangle S_v(x,t) + A(\zeta(t)) - \frac{\beta_v(\zeta(t))b(\zeta(t))}{N_H(\zeta(t)) + m(\zeta(t))}S_v(x,t)I_H(x,t) - v(\zeta(t))S_v(x,t) \\ &\quad - \delta_2(\zeta(t))S_v(x,t)]dt - \frac{b(\zeta(t))}{N_H(\zeta(t)) + m(\zeta(t))}\rho_2(\zeta(t))S_v(x,t)I_H(x,t)dB_2(t), \\ dI_v(x,t) &= [D_5(\zeta(t))\triangle I_v(x,t) + \frac{\beta_v(\zeta(t))b(\zeta(t))}{N_H(\zeta(t)) + m(\zeta(t))}S_v(x,t)I_H(x,t) - v(\zeta(t))I_v(x,t) \\ &\quad - \delta_2(\zeta(t))I_v(x,t)]dt + \frac{b(\zeta(t))}{N_H(\zeta(t)) + m(\zeta(t))}\rho_2(\zeta(t))S_v(x,t)I_H(x,t)dB_2(t), \end{aligned}\right\} & \begin{aligned} &t \neq t_k, \\ &t > 0, \\ &x \in \Gamma \end{aligned} \\ \left.\begin{aligned} &S_H(x,t_k^+) = S_H(x,t_k), \\ &I_H(x,t_k^+) = I_H(x,t_k), \\ &R_H(x,t_k^+) = R_H(x,t_k), \\ &S_v(x,t_k^+) = (1+\varnothing_{1k})S_v(x,t_k), \\ &I_v(x,t_k^+) = (1-\varnothing_{2k})I_v(x,t_k) \end{aligned}\right\} t = t_k. \end{cases} \tag{2.5}$$

Next, in order to study subsequent conclusions, the following preparatory knowledge needs to be given.

2.2. Preliminaries

Assign

$$V = H^1(\Gamma) \equiv \Big\{\varphi|\varphi \in L^2(\Gamma), \frac{\partial\varphi}{\partial x_i} \in L^2(\Gamma), \ i = 1,2,3\Big\},$$

where the dual space of V is $V^{-1} = H^{-1}(\Gamma)$. Define C_+^b as a group of functions that are both bounded and continuous. $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\le t\le T}, P)$ be a complete probability space equipped with a filtration $\{\mathcal{F}_t\}_{0\le t\le T}$ satisfying the usual conditions (i.e., it is right continuous and $\mathcal{F}_0$ contains all P-null sets). $\mathbb{E}$ denotes the probability expectation corresponding to $\mathbb{P}$. Let $z(x,t) = (S_H(x,t), I_H(x,t), R_H(x,t), S_v(x,t), I_v(x,t))$. The set $\mathbb{M}_+ = L^2\left(\Gamma\times[0,\infty), \mathbb{R}_+^5\right)$ is made up of square integrable functions defined on $\Gamma\times[0,\infty)$. These functions are equipped with a norm denoted by $\|\cdot\|$. For any function $z(x,t)$ in $\mathbb{M}_+$, the norm is defined as $\|z(x,t)\| = \left(\int_\Gamma z(x,t)z^T(x,t)dx\right)^{\frac{1}{2}}$.

Before continuing, let's first determine the following symbols:

(A1) For any $i\in\mathbb{S}$, hypothesis

$$\check{f} = \max_{i\in\mathbb{S}}\{f(i)\}, \quad \hat{f} = \min_{i\in\mathbb{S}}\{f(i)\}$$

where $f(i)$ illustrates the parameter of system (2.5) in the i-th state.

(A2) $\underline{h}_2$ is the lower bound of $h_2(t)$.

3. Well-posedness

To start with, using a method similar to [28], to prove the well-posedness of global positive solution of system (2.5), it is imperative that we first scrutinize the ensuing system:

$$\begin{cases} dy_1(x,t) = [D_1(\zeta(t))\triangle y_1(x,t) + \mu_h(\zeta(t))N_H(\zeta(t)) - \dfrac{\beta_H(\zeta(t))b(\zeta(t))}{N_H(\zeta(t))+m(\zeta(t))}y_1(x,t)h_2(t)y_5(x,t) \\ \qquad - \mu(\zeta(t))y_1(x,t)]dt - \dfrac{b(\zeta(t))}{N_H(\zeta(t))+m(\zeta(t))}\rho_1(\zeta(t))y_1(x,t)h_2(t)y_5(x,t)dB_1(t), \\ dy_2(x,t) = [D_2(\zeta(t))\triangle y_2(x,t) + \dfrac{\beta_H(\zeta(t))b(\zeta(t))}{N_H(\zeta(t))+m(\zeta(t))}y_1(x,t)h_2(t)y_5(x,t) - (\mu(\zeta(t)) + \gamma_H(\zeta(t)))y_2(x,t) \\ \qquad - \delta_1(\zeta(t))y_2(x,t)]dt + \dfrac{b(\zeta(t))}{N_H(\zeta(t))+m(\zeta(t))}\rho_1(\zeta(t))y_1(x,t)h_2(t)y_5(x,t)dB_1(t), \\ dy_3(x,t) = [D_3(\zeta(t))\triangle y_3(x,t) + \gamma_H(\zeta(t))y_2(x,t) + \delta_1(\zeta(t))y_2(x,t) - \mu(\zeta(t))y_3(x,t)]dt, \\ dy_4(x,t) = [D_4(\zeta(t))\triangle y_4(x,t) + A(\zeta(t))h_1^{-1}(t) - \dfrac{\beta_v(\zeta(t))b(\zeta(t))}{N_H(\zeta(t))+m(\zeta(t))}y_4(x,t)y_2(x,t) - \delta_2(\zeta(t))y_4(x,t) \\ \qquad - (v(\zeta(t)) - \ln(1+\varnothing_{1k}))y_4(x,t)]dt - \dfrac{b(\zeta(t))\rho_2(\zeta(t))}{N_H(\zeta(t))+m(\zeta(t))}y_4(x,t)y_2(x,t)dB_2(t), \\ dy_5(x,t) = [D_5(\zeta(t))\triangle y_5(x,t) + \dfrac{\beta_v(\zeta(t))b(\zeta(t))}{N_H(\zeta(t))+m(\zeta(t))}h_2^{-1}(t)h_1(t)y_4(x,t)y_2(x,t) - \delta_2(\zeta(t))y_5(x,t) \\ \qquad - (v(\zeta(t)) - \ln(1-\varnothing_{2k}))y_5(x,t)]dt + \dfrac{b(\zeta(t))\rho_2(\zeta(t))}{N_H(\zeta(t))+m(\zeta(t))}h_2^{-1}(t)h_1(t)y_4(x,t)y_2(x,t)dB_2(t), \end{cases} \tag{3.1}$$

with initial value

$$(y_1(x,0), y_2(x,0), y_3(x,0), y_4(x,0), y_5(x,0)) = (S_{H,0}(x), I_{H,0}(x), R_{H,0}(x), S_{v,0}(x), I_{v,0}(x)), \tag{3.2}$$

and boundary condition

$$\frac{\partial y_1(x,t)}{\partial \nu} = \frac{\partial y_2(x,t)}{\partial \nu} = \frac{\partial y_3(x,t)}{\partial \nu} = \frac{\partial y_4(x,t)}{\partial \nu} = \frac{\partial y_5(x,t)}{\partial \nu} = 0, \quad x\in\partial\Gamma, t>0. \tag{3.3}$$

And, $h_i(t)$, $i = 1, 2$, are left continuous can be expressed as follows:

$$h_1(t) = \begin{cases} (1 + \varnothing_{1k})^{[t]-t} & t \neq t_k,\ k \in \mathbb{N},\ t > 0 \\ (1 + \varnothing_{1k})^{-1} & t = t_k. \end{cases}$$

$$h_2(t) = \begin{cases} (1 - \varnothing_{2k})^{[t]-t} & t \neq t_k,\ k \in \mathbb{N},\ t > 0 \\ (1 - \varnothing_{2k})^{-1} & t = t_k. \end{cases}$$

In fact, the relationship between system (2.5) and system (3.1) has the following lemma.

Lemma 3.1. *The system* (3.1) *described by initial value* (3.2) *and boundary condition* (3.3) *can be transformed into an equivalent system* (2.5) *represented by boundary condition* (2.3).

Proof. Denote

$$(S_H(x,t), I_H(x,t), R_H(x,t), S_v(x,t), I_v(x,t)) = (y_1(x,t), y_2(x,t), y_3(x,t), h_1(t)y_4(x,t), h_2(t)y_5(x,t)).$$

It is easy to get that $y_i(x,t)$, $i = 1, 2, 3, 4$ is continuous on $(t_k, t_{k+1}) \subset [0, +\infty)$. For every $t \neq t_k$,

$$\begin{aligned} dS_v(x,t) &= h_1'(t)y_4(x,t)dt + h_1(t)dy_4(x,t) \\ &= -h_1(t)\ln(1+\varnothing_{1k})y_4(x,t)dt + h_1(t)\{[D_4(\zeta(t))\triangle y_4(x,t) + A(\zeta(t))h_1^{-1}(t) \\ &\quad - \frac{\beta_v(\zeta(t))b(\zeta(t))}{N_H(\zeta(t)) + m(\zeta(t))} y_4(x,t)y_2(x,t) - (v(\zeta(t)) - \ln(1+\varnothing_{1k})y_4(x,t) \\ &\quad - \delta_2(\zeta(t))y_4(x,t)]dt - \frac{b(\zeta(t))}{N_H(\zeta(t)) + m(\zeta(t))}\rho_2(\zeta(t))y_4(x,t)y_2(x,t)dB_2(t)\} \\ &= [D_4(\zeta(t))\triangle S_v(x,t) + A(\zeta(t)) - \frac{\beta_v(\zeta(t))b(\zeta(t))}{N_H(\zeta(t)) + m(\zeta(t))} S_v(x,t)I_H(x,t) - v(\zeta(t))S_v(x,t) \\ &\quad - \delta_2(\zeta(t))S_v(x,t)]dt - \frac{b(\zeta(t))}{N_H(\zeta(t)) + m(\zeta(t))}\rho_2(\zeta(t))S_v(x,t)I_H(x,t)dB_2(t). \end{aligned}$$

Moreover, for $t_k \geq 0$,

$$S_v(x,t^-) = \lim_{t \to t_k^-} h_1(t)y_4(x,t) = (1+\varnothing_{1k})^{(t_k-1)-t_k} y_4(x,t_k) = (1+\varnothing_{1k})^{-1} y_4(x,t_k) = S_v(x,t_k),$$

and

$$S_v(x,t_k^+) = \lim_{t \to t_k^+} h_1(t)y_4(x,t) = (1+\varnothing_{1k})^{t_k-t_k} y_4(x,t_k) = y_4(x,t_k).$$

It follows that

$$S_v(x,t_k^+) = (1+\varnothing_{1k})S_v(x,t_k), \quad \text{for } t = t_k.$$

For $I_v(x,t)$, we can follow the same process as we did for $S_v(x,t)$.

It can be shown that the well-posedness of positive solution for the system (3.1) by proving system (2.5) has a unique positive solution. The required lemma is provided below.

Lemma 3.2. *For any initial value* (3.2)*, the solution of system* (2.5)*, satisfies that*

$$\limsup_{t\to\infty} \int_\Gamma (S_H(x,t) + I_H(x,t) + R_H(x,t) + S_V(x,t) + I_V(x,t))\,dx < B,$$

where $B = \frac{(\mu_h N_H + A)|\Gamma|}{\mu}$ *and* $|\Gamma|$ *represents the volume of* Γ.

The proof procedure is omitted, the interested reader can refer to [29]. Based on this Lemma 3.2, the following theorem holds.

Theorem 3.1. *For any initial value* (3.2) *and* $t \geq 0$*, system* (2.5) *has a unique positive solution* $z(x,t)$*.*

The proof is given in the Appendix. This is the basis of the whole paper, which makes the subsequent analysis meaningful.

4. Finite-time contraction stability

Before giving sufficient conditions to ensure FTCS, the following necessary lemma is introduced.

Lemma 4.1. *[30] Assume that there exist positive constants* g_r*,* $r = 1, 2, \dots, s$*, such that* $x \in \Gamma$ *and* $|x_r| < g_r$*. Given a function* $z(x) \in \mathbb{R}^n$ *which belongs to* $C^2(\Gamma)$ *and vanishes on* $\partial\Gamma$*, has*

$$\int_\Gamma z^\top(x)\frac{\partial^2 z(x)}{\partial x^2}dx \leq -\sum_{r=1}^{s}\frac{1}{g_r^2}\int_\Gamma z^\top(x)z(x)dx.$$

Furthermore, to facilitate subsequent studies, it is necessary to give the following definitions.

Definition 4.1. Given positive numbers T, B_1 and B_2 with $B_1 > B_2$, system (2.5) is guaranteed to be finite-time stable concerning (T, B_1, B_2), if for any $t \in [0, T]$,

$$\sup\Big(\int_\Gamma S_H^2(x,0) + I_H^2(x,0) + R_H^2(x,0) + S_v^2(x,0) + I_v^2(x,0)dx\Big) \leq B_1,$$

we have

$$E\Big(\int_\Gamma S_H^2(x,t) + I_H^2(x,t) + R_H^2(x,t) + S_v^2(x,t) + I_v^2(x,t)dx\Big) \leq B_2.$$

Definition 4.2. Given positive constants T, B_1, B_2, B_3, ω with $B_2 > B_1 > B_3$, $\omega \in (0, T)$, system (2.5) is said to be finite-time contraction stability with respect to $(B_1, B_2, B_3, \omega, T)$, if

$$\sup\Big(\int_\Gamma S_H^2(x,0) + I_H^2(x,0) + R_H^2(x,0) + S_v^2(x,0) + I_v^2(x,0)dx\Big) \leq B_1,$$

we have

$$E\Big(\int_\Gamma S_H^2(x,t) + I_H^2(x,t) + R_H^2(x,t) + S_v^2(x,t) + I_v^2(x,t)dx\Big) \leq B_2 \quad \forall t \in [0, T]. \tag{4.1}$$

Moreover,

$$E\Big(\int_\Gamma S_H^2(x,t) + I_H^2(x,t) + R_H^2(x,t) + S_v^2(x,t) + I_v^2(x,t)dx\Big) \leq B_3. \quad \forall t \in [T-\omega, T]. \tag{4.2}$$

Definition 4.3. [22] For any $T \geq t \geq 0$, the average impulsive interval of the sequence t_k ($k \in \mathbb{N}^+$) on $(t, T]$ is l, when $N_0 \in \mathbb{N}^+$ and $l \in \mathbb{N}$, there is

$$\frac{T-t}{l} - N_0 \leq n(T,t) \leq \frac{T-t}{l} + N_0,$$

where $n(T, t)$ denotes the number of pulse moments.

Remark 4.4. Note that for a given pulse sequence, N_0 is not unique. As N_0 increases, the number of impulsive instant sequences it contains also increases. In particular, for the case $N_0 = 1$, it is required that there be no less than one pulse control input on each interval of l.

Definition 4.5. [22] For any $t \geq s,\ t,\ s \in [0, T]$, it can be concluded that the pulse number function $n(t, s)$ of the bounded pulse interval satisfies

$$\frac{t-s}{l_M} - 1 \leq n(t, s) \leq \frac{t-s}{l_m},$$

where $l_M = \max_{k\in K}(t_k - t_{k-1})$ and $l_m = \min_{k\in K}(t_k - t_{k-1})$, $k \in K = \{1, 2, \ldots, n(T, 0)\}$.

Lemma 4.2. *[31] Consider a stochastic differential equation in the form of Markov switching*

$$dX(t) = f(X(t), t, \zeta(t))dt + g(X(t), t, \zeta(t))dB(t),\ X(0) = x_0,$$

where $B(\cdot)$ and $\zeta(\cdot)$ are the d-dimensional Brownian motion and the right continuous Markov chain, respectively, and $f(\cdot,\cdot) : \mathbb{R}^n \times \mathbb{S} \to \mathbb{R}^n$, $g(\cdot,\cdot) : \mathbb{R}^n \times \mathbb{S} \to \mathbb{R}^{n\times d}$. If $V \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{S}; \mathbb{R}_+)$, define an operator $\mathcal{L}V$ from $\mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{S}$ to $\mathbb{R}$ by

$$\mathcal{L}V(x, t, i) = V_t(x, t, i) + V_x(x, t, i)f(x, t, i) + \frac{1}{2}\,\mathrm{trace}\left[g^{\mathrm{T}}(x, t, i)V_{xx}(x, t, i)g(x, t, i)\right] + \sum_{j=1}^{N}\gamma_{ij}V(x, t, j),$$

$$dV(X(t), t, \zeta(t)) = \mathcal{L}V(X(t), t, \zeta(t))dt + V_x(X(t), t, \zeta(t))g(X(t), t, \zeta(t))dB(t).$$

Lemma 4.3. *[32] Comparison theorem: Let $V \in$ class V_0 and suppose that*

$$D^+V(t, x) \leq g(t, V(t, x)),\quad t \neq t_k$$
$$V(t, x + I_k(x)) \leq \Psi_k(V(t, x)),\quad t = t_k,$$

where $g : R_+ \times R_+ \to R$ is continuous, $\lim_{(t,y)\to(t_k^+,x)} g(t, y) = g\left(t_k^+, x\right)$ exists and $\Psi_k : R_+ \to R_+$ is nondecreasing. Let $u(t)$ be the maximal solution of system (4.3)

$$\begin{cases} \dot{u} = g(t, u),\quad t \neq t_k, \\ u(t_k^+) = \Psi_k(u(t_k)),\quad t = t_k, \\ u(t_0^+) = u_0 \geq 0. \end{cases} \tag{4.3}$$

For any $t \geq t_0$, $V(t_0, x_0) \leq u_0$ is established, then $V(t, x(t)) \leq u(t)$.

4.1. Finite-time contractive stability with average impulsive interval

The sufficient conditions for FTCS are established by means of the average pulse interval method. Assign

$$r_1 = \check{Q}(\check{\mu}_h^2\check{N}_H^2 + \check{A}^2)|\Gamma|,\ r_{21} = 1 - 2\check{\mu} - 2\tfrac{\check{\beta}_H\check{b}}{\check{N}_H+\check{m}}B - 2\hat{D}_1\sum_{i=1}^{r}\tfrac{1}{l_i^2},\ \check{Q} = \max_{i\in\mathbb{S}}\{Q(i)\},\ \hat{Q} = \min_{i\in\mathbb{S}}\{Q(i)\}$$
$$r_{22} = \tfrac{\check{b}}{\check{N}_H+\check{m}}(\check{\beta}_H + \check{\beta}_v)B + \tfrac{2\check{b}^2\check{\rho}_2^2}{(\check{N}_H+\check{m})^2}B^2 - 2\check{\mu} - \check{\gamma}_H - 2\hat{D}_2\sum_{i=1}^{r}\tfrac{1}{l_i^2},\ r_{23} = \check{\gamma}_H - 2\check{\mu} - 2\hat{D}_3\sum_{i=1}^{r}\tfrac{1}{l_i^2}$$
$$r_{24} = 1 - 2\check{v} - 2\tfrac{\check{\beta}_v\check{b}}{\check{N}_H+\check{m}}B - 2\hat{D}_4\sum_{i=1}^{r}\tfrac{1}{l_i^2},\ r_{25} = \tfrac{\check{b}}{\check{N}_H+\check{m}}(\check{\beta}_H + \check{\beta}_v)B + \tfrac{2\check{b}^2\check{\rho}_1^2}{(\check{N}_H+\check{m})^2}B^2 - 2\check{v} - 2\hat{D}_5\sum_{i=1}^{r}\tfrac{1}{l_i^2},$$
$$r_2 = \max\left\{r_{21}, r_{22}, r_{23}, r_{24}, r_{25}\right\} + q_{ii} + \textstyle\sum_{j\neq i}^{m} q_{ij}\check{Q}\hat{Q}^{-1},\ \varphi = \max\{1, (1 + \varnothing_i)^2, i = 1, 2\}.$$

Theorem 4.6. *The system* (2.5) *is FTCS about* $(B_1, B_2, B_3, \omega, T)$*, when* $\varphi \geq 1$*, if one of the following conditions is satisfied*

$\mathbb{C}1.$ $\frac{\ln\varphi}{l} + r_2 \leq \hat{\delta} \leq min\{r_2 - \frac{r_1}{B_1 - \hat{Q}B_2\varphi^{-N_0}e^{-\frac{\ln\varphi}{l}T}}, r_2 - \frac{r_1}{B_1 - \hat{Q}B_3\varphi^{-N_0}e^{-\frac{\ln\varphi}{l}T}}\}.$

$\mathbb{C}2.$ $0 < \hat{\delta} \leq min\{r_2 - \frac{r_1}{\varphi^{-N_0}\hat{Q}B_2e^{-(\frac{\ln\varphi}{l}+r_2)T} - B_1}, r_2 - \frac{r_1}{\varphi^{-N_0}\hat{Q}B_3e^{-(\frac{\ln\varphi}{l}+r_2)T} - B_1}, r_2 + \frac{\ln\varphi}{l}\}.$

Proof. Choose

$$W(x,t,i) = Q(i)(\int_\Gamma S_H^2(x,t) + I_H^2(x,t) + R_H^2(x,t) + S_v^2(x,t) + I_v^2(x,t)dx).$$

Application Itô formula can get the result

$$\begin{aligned}
&dW(x,t,i)\\
&= 2Q(i)\int_\Gamma S_H(x,t)\Big[[D_1(i)\triangle S_H(x,t) + \mu_h(i)N_H(i) - \frac{\beta_H(i)b(i)}{N_H(i)+m(i)}S_H(x,t)I_v(x,t) - \mu(i)S_H(x,t)]dt\\
&- \frac{b(i)}{N_H(i)+m(i)}\rho_1(i)S_H(x,t)I_v(x,t)dB_1(t)\Big]dx + 2Q(i)\int_\Gamma I_H(x,t)\Big[[D_2(i)\triangle I_H(x,t) + \frac{\beta_H(i)b(i)}{N_H(i)+m(i)}\\
&\times S_H(x,t)I_v(x,t) - (\mu(i)+\gamma_H(i)+\delta_1(i))I_H(x,t)]dt + \frac{b(i)}{N_H(i)+m(i)}\rho_1(i)S_H(x,t)I_v(x,t)dB_1(t)\Big]dx,\\
&+ 2Q(i)\int_\Gamma R_H(x,t)\Big[D_3(i)\triangle R_H(x,t) + \gamma_H(i)I_H(x,t) + \delta_1(i)I_H(x,t) - \mu(i)R_H(x,t)dt\Big]dx + 2Q(i)\\
&\times \int_\Gamma S_v(x,t)\Big[[D_4(i)\triangle S_v(x,t) + A(i) - \frac{\beta_v(i)b(i)}{N_H(i)+m(i)}S_v(x,t)I_H(x,t) - \delta_2(i)S_v(x,t) - v(i)S_v(x,t)]dt\\
&- \frac{b(i)}{N_H(i)+m(i)}\rho_2(i)S_v(x,t)I_H(x,t)dB_2(t)\Big]dx + 2Q(i)I_v(x,t)\Big[(\int_\Gamma D_5(i)\triangle I_v(x,t) + \frac{\beta_v(i)b(i)}{N_H(i)+m(i)}\\
&\times S_v(x,t)I_H(x,t) - (\delta_2(i)+v(i))I_v(x,t)]dt + \frac{b(i)}{N_H(i)+m(i)}\rho_2(i)S_v(x,t)I_H(x,t)dB_2(t)\Big]dx + Q(i)\\
&\times (\int_\Gamma \frac{2(b(i)\rho_1(i))^2}{(N_H(i)+m(i))^2}S_H^2(x,t)I_v^2(x,t) + \frac{2(b(i)\rho_2(i))^2}{(N_H(i)+m(i))^2}S_v^2(x,t)I_H^2(x,t))dxdt + \sum_{j=1}^m q_{ij}W(x,t,i)dt.
\end{aligned}$$

Follows Lemma 3.2 and Lemma 4.1, we can obtain

$$\begin{aligned}
&dW(x,t,i)\\
&\leq Q(i)\{-2\sum_{i=1}^r \frac{1}{l_i^2}\int_\Gamma [D_1(i)S_H^2(x,t) + D_2(i)I_H^2(x,t) + D_3(i)R_H^2(x,t) + D_4(i)S_v^2(x,t) + D_5(i)I_v^2(x,t)]dxdt\\
&+ \int_\Gamma \check{\mu}_h^2\check{N}_H^2 + S_H^2(x,t) - 2\frac{\beta_H(i)b(i)}{N_H(i)+m(i)}BS_H^2(x,t) - 2\mu(i)S_H^2(x,t)dxdt + \int_\Gamma \frac{\beta_H(i)b(i)}{N_H(i)+m(i)}BI_H^2(x,t)\\
&+ \frac{\beta_H(i)b(i)}{N_H(i)+m(i)}BI_v^2(x,t)) - 2(\mu(i)+\gamma_H(i)+\delta_1(i))I_H^2(x,t)dxdt + \int_\Gamma (\gamma_H(i)+\delta_1(i))(I_H^2(x,t)+R_H^2(x,t))\\
&- 2\mu(i)R_H^2(x,t)dxdt + \int_\Gamma \check{A}^2 + S_v^2(x,t) - 2\frac{\beta_v(i)b(i)}{N_H(i)+m(i)}BS_v^2(x,t) - 2(\delta_2(i)+v(i))S_v^2(x,t)dxdt
\end{aligned}$$

$$
\begin{aligned}
&+ \int_\Gamma \frac{\beta_v(i)b(i)}{N_H(i)+m(i)} B(I_H^2(x,t)+I_v^2(x,t)) - 2(v(i)+\delta_2(i))I_v^2(x,t)dxdt + \int_\Gamma \frac{2(b(i)\rho_1(i))^2}{(N_H(i)+m(i))^2} B^2 I_v^2(x,t) \\
&+ \frac{2(b(i)\rho_2(i))^2}{(N_H(i)+m(i))^2} B^2 I_H^2(x,t)dxdt - 2\int_\Gamma \frac{b(i)\rho_1(i)}{N_H(i)+m(i)} S_H^2(x,t)I_v(x,t)dxdB_1(t) + 2\int_\Gamma \frac{b(i)\rho_1(i)}{N_H(i)+m(i)} \\
&\times S_H(x,t)I_H(x,t)I_v(x,t)dxdB_1(t) - 2\int_\Gamma \frac{b(i)\rho_2(i)}{N_H(i)+m(i)} S_v^2(x,t)I_H(x,t)dxdB_2(t) + 2\int_\Gamma \frac{b(i)\rho_2(i)}{N_H(i)+m(i)} \\
&\times S_v(x,t)I_v(x,t)I_H(x,t)dxdB_2(t)\} + (q_{ii} + \sum_{j\neq i}^{m} q_{ij}\check{Q}\hat{Q}^{-1})W(x,t,i)dt \\
&\leq \check{Q}\Big\{(\check{\mu}_h^2\check{N}_H^2 + \check{A}^2)|\Gamma| + \int_\Gamma (1 - 2\check{\mu} - 2\frac{\check{\beta}_H\check{b}}{\check{N}_H+\check{m}}B - 2\hat{D}_1\sum_{i=1}^{r}\frac{1}{l_i^2})S_H^2(x,t)dxdt + \int_\Gamma (\frac{\check{b}}{\check{N}_H+\check{m}}\check{\beta}_H B \\
&+ \frac{\check{b}}{\check{N}_H+\check{m}}\check{\beta}_v B + \frac{2\check{b}^2\check{\rho}_2^2}{(\check{N}_H+\check{m})^2}B^2 - 2\check{\mu} - \check{\gamma}_H - \check{\delta}_1 - 2\hat{D}_2\sum_{i=1}^{r}\frac{1}{l_i^2})I_H^2(x,t)dxdt + \int_\Gamma (\check{\gamma}_H - 2\check{\mu} \\
&- 2\hat{D}_3\sum_{i=1}^{r}\frac{1}{l_i^2})R_H^2(x,t)dxdt + \int_\Gamma (1 - 2\check{v} - 2\frac{\check{\beta}_v\check{b}}{\check{N}_H+\check{m}}B - 2\check{\delta}_2 - 2\hat{D}_4\sum_{i=1}^{r}\frac{1}{l_i^2})S_v^2(x,t)dxdt \\
&+ \int_\Gamma (\frac{\check{b}}{\check{N}_H+\check{m}}(\check{\beta}_H + \check{\beta}_v)B - 2\check{v} + \frac{2\check{b}^2\check{\rho}_1^2}{(\check{N}_H+\check{m})^2}B^2 - 2\check{\delta}_2 - 2\hat{D}_5\sum_{i=1}^{r}\frac{1}{l_i^2})I_v^2(x,t)dxdt\Big\} \\
&+ (q_{ii} + \sum_{j\neq i}^{m} q_{ij}\check{Q}\hat{Q}^{-1})W(x,t,i)dt + B(t) \\
&\leq [r_1 + (r_2 - \hat{\delta})W(x,t,i)]dt + B(t)
\end{aligned}
\tag{4.4}
$$

where

$$
\begin{aligned}
B(t) = &-2\int_\Gamma \frac{b(i)\rho_1(i)}{N_H(i)+m(i)} S_H^2(x,t)I_v(x,t)dxdB_1(t) + 2\int_\Gamma \frac{b(i)\rho_1(i)}{N_H(i)+m(i)} S_H(x,t)I_H(x,t)I_v(x,t)dxdB_1(t) \\
&- 2\int_\Gamma \frac{b(i)\rho_2(i)}{N_H(i)+m(i)} S_v^2(x,t)I_H(x,t)dxdB_2(t) + 2\int_\Gamma \frac{b(i)\rho_2(i)}{N_H(i)+m(i)} S_v(x,t)I_v(x,t)I_H(x,t)dxdB_2(t)
\end{aligned}
$$

For $t = t_k$, it is easy to derive that

$$
\begin{aligned}
&W(S_H(x,t_k^+), I_H(x,t_k^+), R_H(x,t_k^+), S_v(x,t_k^+), I_v(x,t_k^+)) \\
&= \int_\Gamma Q(i)S_H^2(x,t_k)dx + \int_\Gamma Q(i)I_H^2(x,t_k)dx + \int_\Gamma Q(i)R_H^2(x,t_k)dx \\
&\quad + \int_\Gamma Q(i)(1+\varnothing_{1k})^2 S_v^2(x,t_k)dx + + \int_\Gamma Q(i)(1+\varnothing_{2k})^2 I_v^2(x,t_k)dx \\
&\leq \varphi W(t_k),
\end{aligned}
\tag{4.5}
$$

where $\varphi = \max\{1, (1+\varnothing_i)^2, i = 1, 2\}$. Taking expectation on Eqs (4.4) and (4.5), we get

$$
\begin{aligned}
&dEW(t) \leq r_1 + (r_2 - \hat{\delta})EW(t) \;\; t \neq t_k, \\
&EW(t_k^+) \leq \varphi EW(t_k) \;\; t = t_k.
\end{aligned}
$$

Set $u(t)$ be the solution of system (4.6)

$$
\begin{cases}
du(t) = [r_1 + (r_2 - \hat{\delta})u(t)]dt \;\; t \neq t_k, \\
u(t_k^+) = \varphi u(t_k) \;\; t = t_k.
\end{cases}
\tag{4.6}
$$

Using the comparison thereom yields

$$EW(t) \leq u(t).$$

It can be seen in terms of the method of variation of constant on (4.6)

$$u(t) = -\frac{r_1}{r_2 - \hat{\delta}}\varphi^{n(t,0)} + (u(0) + \frac{r_1}{r_2 - \hat{\delta}})\beta(t, 0), \tag{4.7}$$

where

$$\beta(t, 0) = \varphi^{n(t,0)} e^{(r_2 - \hat{\delta})t} = e^{n(t,0) \ln \varphi + (r_2 - \hat{\delta})t}, \quad t \geq 0.$$

For any $t \in [0, T]$, we have

$$\frac{t}{l} - N_0 \leq n(t, 0) \leq \frac{t}{l} + N_0.$$

When $\varphi \geq 1$ i.e., $\frac{\ln \varphi}{l} > 0$, it can be calculated that

$$\beta(t, 0) = e^{n(t,0) \ln \varphi + (r_2 - \hat{\delta})t} \leq e^{(\frac{t}{l} + N_0) \ln \varphi + (r_2 - \hat{\delta})t} = e^{\left(\frac{\ln \varphi}{l} + r_2 - \hat{\delta}\right)t + N_0 \ln \varphi},$$
$$\varphi^{n(t,0)} = e^{n(t,0) \ln \varphi} \leq e^{(\frac{t}{l} + N_0) \ln \varphi} = e^{\frac{\ln \varphi}{l} t + N_0 \ln \varphi}.$$

Thus, we have

$$u(t) \leq -\varphi^{N_0} \frac{r_1}{r_2 - \hat{\delta}} e^{\frac{\ln \varphi}{l} t} + \varphi^{N_0} \left(u(0) + \frac{r_1}{r_2 - \hat{\delta}}\right) e^{\left(\frac{\ln \varphi}{l} + r_2 - \hat{\delta}\right)t}. \tag{4.8}$$

If $\frac{\ln \varphi}{l} + r_2 - \hat{\delta} \leq 0$, from (4.8), one obtains

$$Q(i)E(\int_{\Gamma} S_H^2(x, t) + I_H^2(x, t) + R_H^2(x, t) + S_v^2(x, t) + I_v^2(x, t)dx) \leq u(t)$$
$$\leq \varphi^{N_0}(u(0) - \frac{r_1}{r_2 - \hat{\delta}})e^{\frac{\ln \varphi}{l} t} \leq \varphi^{N_0}(B_1 - \frac{r_1}{r_2 - \hat{\delta}})e^{\frac{\ln \varphi}{l} T}.$$

For $\mathbb{C}1$, we obtain (4.1) and (4.2).

If $\frac{\ln \varphi}{l} + r_2 - \hat{\delta} > 0$, from (4.8), one obtains

$$EW(t) \leq u(t)$$
$$\leq \varphi^{N_0}(u(0) + \frac{r_1}{r_2 - \hat{\delta}})e^{\left(\frac{\ln \varphi}{l} + r_2 - \hat{\delta}\right)t} \leq \varphi^{N_0}(u(0) + \frac{r_1}{r_2 - \hat{\delta}})e^{\left(\frac{\ln \varphi}{l} + r_2 - \hat{\delta}\right)T}$$
$$\leq \varphi^{N_0}(B_1 + \frac{r_1}{r_2 - \hat{\delta}})e^{\left(\frac{\ln \varphi}{l} + r_2\right)T}.$$

Based on $\mathbb{C}2$, the inequality (4.1) and (4.2) are verified and this completes the proof.

From (4.1), the corollary concerning finite-time stability (FTS) is obtained.

Corollary 4.1. *The system* (2.5) *is FTS about* (B_1, B_2, T)*, when* $\varphi \geq 1$*, if one of the following conditions is satisfied*

$\mathbb{C} - 1.$ $\frac{\ln \varphi}{l} + r_2 \leq \hat{\delta} \leq r_2 - \frac{r_1}{B_1 - \hat{Q} B_2 \varphi^{-N_0} e^{-\frac{\ln \varphi}{l} T}}$.

$\mathbb{C} - 2.$ $0 < \hat{\delta} \leq min\{r_2 - \frac{r_1}{\varphi^{-N_0} \hat{Q} B_2 e^{-(\frac{\ln \varphi}{l} + r_2)T} - B_1}, r_2 + \frac{\ln \varphi}{l}\}$.

Remark 4.7. When dengue fever occurs, some parameters are known, such as the use of pesticides and the intensity and frequency of release of sterile mosquitoes. According to the result of this paper (Theorem 4.6), it can be judged whether the spread of dengue fever can be controlled within a limited time, which has a certain guiding role in the prevention of the disease. Moreover, it can be seen from the Theorem 4.6 and Corollary 4.1 that if the system is FTCS then the system must be FTS.

4.2. Finite-time contractive stability with bounded pulse interval

In this part, based on the bounded pulse interval method, a criterion for FTCS about $(B_1, B_2, B_3, \omega, T)$ is given.

Theorem 4.8. *The system* (2.5) *is FTCS about* $(B_1, B_2, B_3, \omega, T)$*, when* $\varphi \geq 1$*, if one of the following conditions is satisfied*
$\mathbb{C}'1$. $\frac{\ln\varphi}{l_m} + r_2 \leq \hat{\delta} \leq min\{r_2 - \frac{r_1}{B_1 - \hat{Q}B_2 e^{-\frac{\ln\varphi}{l_m}T}}, \hat{\delta} \leq r_2 - \frac{r_1}{B_1 - \hat{Q}B_3 e^{-\frac{\ln\varphi}{l_m}T}}\}$.
$\mathbb{C}'2$. $0 < \hat{\delta} \leq min\{r_2 - \frac{r_1}{\hat{Q}B_2 e^{-(\frac{\ln\varphi}{l_m}+r_2)T} - B_1}, r_2 - \frac{r_1}{\hat{Q}B_3 e^{-(\frac{\ln\varphi}{l_m}+r_2)T} - B_1}, r_2 + \frac{\ln\varphi}{l_m}\}$
where φ, r_1 and r_2 have the same definitions as in Theorem 4.6.

Proof. On the basis of Definition 4.5, obviously

$$\frac{t}{l_M} - 1 \leq n(t, 0) \leq \frac{t}{l_m}.$$

When $\varphi \geq 1$ i.e., $\frac{\ln\varphi}{l} > 0$, it is seen that

$$\begin{aligned} \beta(t, 0) &= e^{n(t,0)\ln\varphi + (r_2 - \hat{\delta})t} \leq e^{\frac{t}{l_m}\ln\varphi + (r_2 - \hat{\delta})t} = e^{\left(\frac{\ln\varphi}{l_m} + r_2 - \hat{\delta}\right)t}, \\ \varphi^{n(t,0)} &= e^{n(t,0)\ln\varphi} \leq e^{\frac{t}{l_m}\ln\varphi} = e^{\frac{\ln\varphi}{l_m}t}. \end{aligned} \tag{4.9}$$

Similarly, combining Eqs (4.7) and (4.9) shows that

$$u(t) \leq -\frac{r_1}{r_2 - \hat{\delta}} e^{\frac{\ln\varphi}{l_m}t} + \left(u(0) + \frac{r_1}{r_2 - \hat{\delta}}\right) e^{\left(\frac{\ln\varphi}{l_m} + r_2 - \hat{\delta}\right)t}. \tag{4.10}$$

If $\frac{\ln\varphi}{l_m} + r_2 - \hat{\delta} \leq 0$, from (4.10), one obtains

$$\begin{aligned} &Q(i)E(\int_\Gamma S_H^2(x, t) + I_H^2(x, t) + R_H^2(x, t) + S_v^2(x, t) + I_v^2(x, t)dx) \leq z(t) \\ &\leq (u(0) - \frac{r_1}{r_2 - \hat{\delta}})e^{\frac{\ln\varphi}{l_m}t} \leq (B_1 - \frac{r_1}{r_2 - \hat{\delta}})e^{\frac{\ln\varphi}{l_m}T}. \end{aligned}$$

The condition $\mathbb{C}'1$ means that (4.1) and (4.2) hold.
If $\frac{\ln\varphi}{l_m} + r_2 - \hat{\delta} > 0$, from (4.10), one obtains

$$\begin{aligned} &EW(t) \leq z(t) \\ &\leq (u(0) + \frac{r_1}{r_2 - \hat{\delta}})e^{\left(\frac{\ln\varphi}{l_m} + r_2 - \hat{\delta}\right)t} \leq (u(0) + \frac{r_1}{r_2 - \hat{\delta}})e^{\left(\frac{\ln\varphi}{l_m} + r_2 - \hat{\delta}\right)T} \\ &\leq (B_1 + \frac{r_1}{r_2 - \hat{\delta}})e^{\left(\frac{\ln\varphi}{l_m} + r_2\right)T}. \end{aligned}$$

Then, under condition $\mathbb{C}'2$, (4.1) and (4.2) are true and this completes the proof.

Similarly, the FTS of the bounded pulse interval method can be deduced from (4.1).

Corollary 4.2. *The system* (2.5) *is FTS about* (B_1, B_2, T)*, when* $\varphi \geq 1$*, if one of the following conditions is satisfied*
$\mathbb{C} - 3$. $\frac{\ln\varphi}{l_m} + r_2 \leq \hat{\delta} \leq r_2 - \frac{r_1}{B_1 - \hat{Q}B_2 e^{-\frac{\ln\varphi}{l_m}T}}$.
$\mathbb{C} - 4$. $0 < \hat{\delta} \leq min\{r_2 - \frac{r_1}{\hat{Q}B_2 e^{-(\frac{\ln\varphi}{l_m}+r_2)T} - B_1}, r_2 + \frac{\ln\varphi}{l_m}\}$.

Remark 4.9. From Definition 4.5, it can be seen that the impulsive sequence is related to the minimum impulsive interval l_m. Then, combined with Theorem 4.8, it is known that for an unstable impulse effect, i.e., $\varphi \geq 1$, when other parameters remain the same, the larger minimal impulsive interval l_m (fewer pulses) means that finite-time contraction stability is easier to achieve.

5. Numerical simulations

Table 1. Description of parameters in system (2.5).

Parameters	Values	Parameters	Values	Parameters	Values	Parameters	Values
$D_1(1)$	0.12	$\mu(1)$	0.1	$D_1(2)$	0.11	$\mu(2)$	0.15
$D_2(1)$	0.32	$A(1)$	1.2	$D_2(2)$	0.3	$A(2)$	1.3
$D_3(1)$	0.15	$v(1)$	0.08	$D_3(2)$	0.13	$v(2)$	0.1
$D_4(1)$	0.15	$\gamma_H(1)$	0.048	$D_4(2)$	0.13	$\gamma_H(2)$	0.048
$D_5(1)$	0.12	$\beta_H(1)$	0.5	$D_5(2)$	0.11	$\beta_H(2)$	0.6
$b(1)$	0.3	$\beta_v(1)$	0.7	$b(2)$	0.5	$\beta_v(2)$	0.6
$\mu_h N_H(1)$	1	$\rho_1(1)$	1.6	$\mu_h N_H(2)$	1	$\rho_1(2)$	1.4
$(N_H + m)(1)$	6	$\rho_2(1)$	1	$(N_H + m)(2)$	6.1	$\rho_2(2)$	0.8
$\delta_1(1)$	0.3	$\delta_2(1)$	0.4	$\delta_1(2)$	0.2	$\delta_2(2)$	0.3

This section aims to demonstrate the validity of the theoretical findings we obtained in the previous sections. We assume that $r = 1$ and $x \in \Gamma = [-0.3, 0.3]$. The Markov process $\zeta(t)$ takes values in S $= \{1, 2\}$ with generator $\Gamma = \begin{pmatrix} -0.4 & 0.4 \\ 0.6 & -0.6 \end{pmatrix}$ (see Figure 1). The parameters of system (2.5) for the two states are given in Table 1 as in Refs. [29, 33, 34].

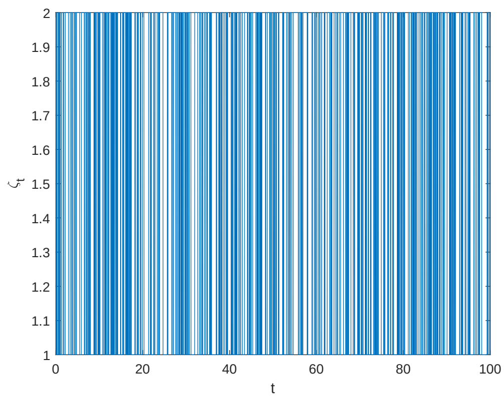


Figure 1. The sample paths of $\zeta(t)$.

5.1. Analysis of FTCS

In this part, the FTCS for system (2.5) are explained through two examples.

Example 5.1. The average pulse interval is used to study finite-time contraction stability. Take the impulsive sequence $\{t_k\} = \{0.6, 1.2, 1.8, 2.4, 3, 3.6\}$, i.e., $l = 0.6$. By setting $T = 3$ *years*, $B_1 = 6.5$,

$B_2 = 7$, $B_3 = 5.5$, $\phi_{1k} = \phi_{2k} = 0.1$ and selecting $\check{Q} = \hat{Q} = 1.5$, $N_0 = 1$, simple calculation shows that $r_1 = 1.2105$, $r_2 = 0.21$, $r_2 - \frac{r_1}{\varphi^{-N_0}\hat{Q}B_2 e^{-(\frac{\ln\varphi}{lm}+r_2)T} - B_1} = 0.467$, $r_2 - \frac{r_1}{\varphi^{-N_0}\hat{Q}B_3 e^{-(\frac{\ln\varphi}{lm}+r_2)T} - B_1} = 0.452$ and $r_2 + \frac{\ln\varphi}{l} = 0.526$, this implies that $\mathbb{C}2$ in Theorem 4.6 holds. Hence, system (2.5) is FTCS w.r.t (6.5,7,5.5,1,3). Figure 2 displays the initial concentrations of populations are less than B_1 and not more than B_2 in $[0, T]$ and reach B_3 in $[T - \omega, T]$. Namely, before reaching the terminal time, the system can reach a certain threshold which is smaller than the initial threshold, which means that the spread of dengue can be controlled effectively within a limited period of time.

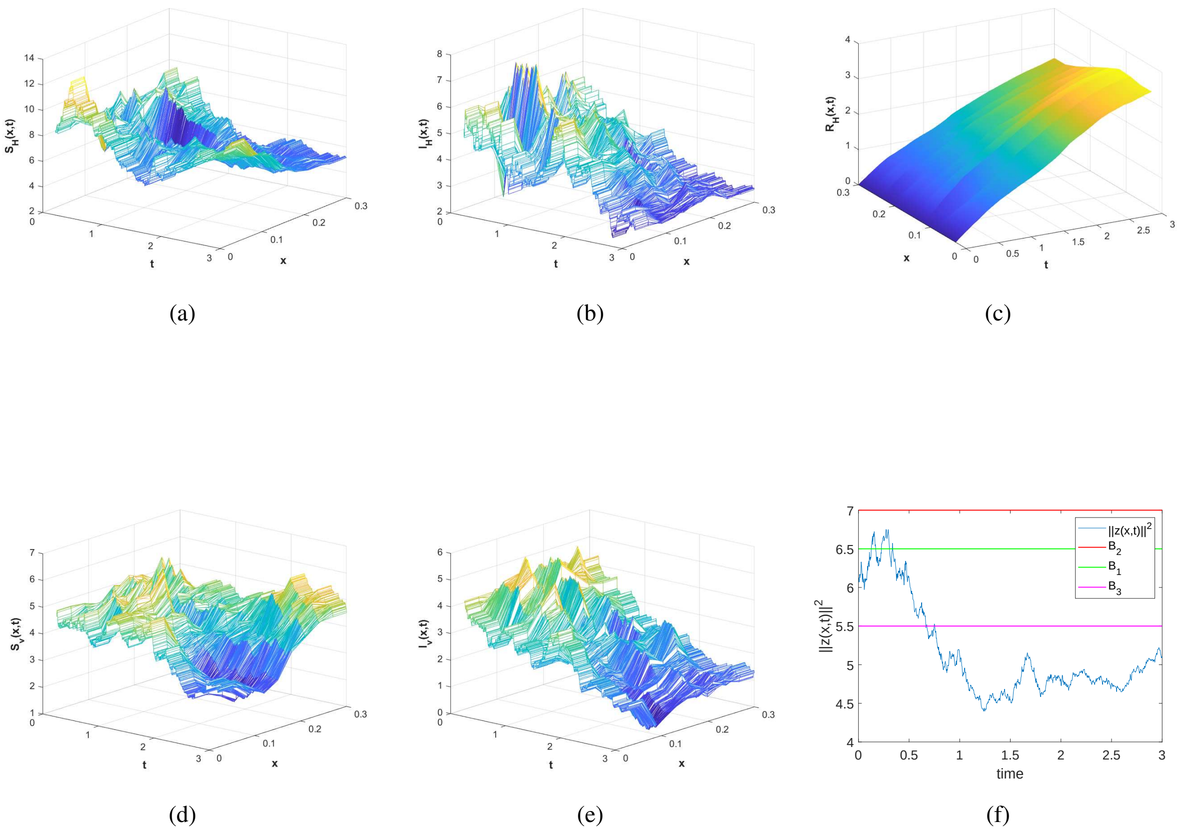
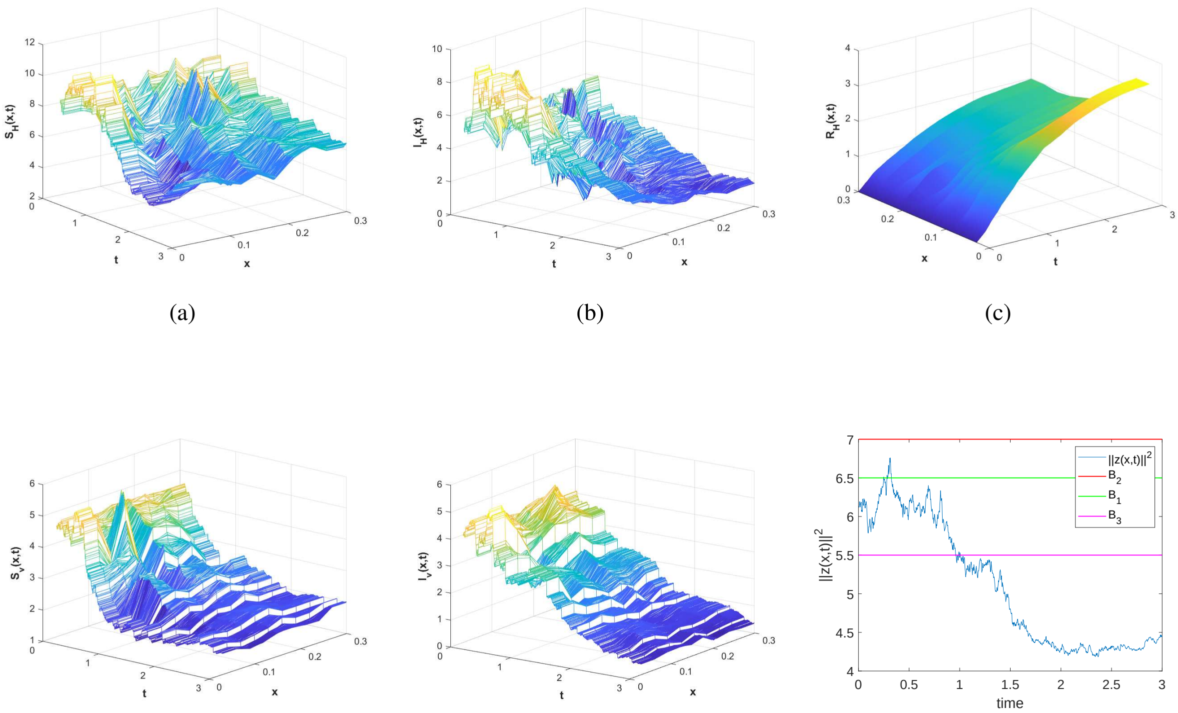


Figure 2. The state trajectories of system (2.5) with control and the average impulsive interval with initial value $(8, 5, 0, 4.5, 4)$.

Example 5.2. In this example, the pulse sequence is derived from the bounded pulse interval method. $\{t_k\} = \{0.6, 1.3, 1.9, 2.6, 3.3\}$, i.e., $l_m = 0.6$, $l_M = 0.7$. We also take $T = 3$ *years*, $B_1 = 6.5$, $B_2 = 7$, $B_3 = 5.5$ and other parameter values use the same parameter values as Example 5.1. The calculation yields $r_1 = 1.2105$, $r_2 = 0.21$, $r_2 - \frac{r_1}{\hat{Q}B_2 e^{-(\frac{\ln\varphi}{lm}+r_2)T} - B_1} = 0.489$, $r_2 - \frac{r_1}{\hat{Q}B_3 e^{-(\frac{\ln\varphi}{lm}+r_2)T} - B_1} = 0.462$ and $r_2 + \frac{\ln\varphi}{l} = 0.526$, this implies that $\mathbb{C}'2$ in Theorem 4.8 holds. Thus, system is FTCS w.r.t (6.5,7,5.5,1,3) (see Figure 3).



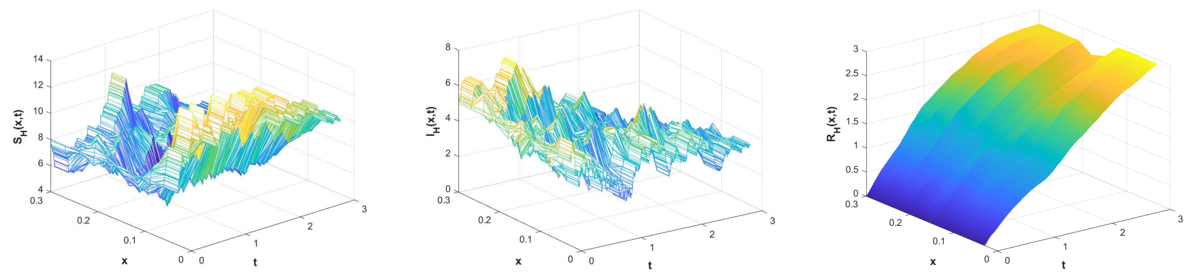
(a) (b) (c)

(d) (e) (f)

Figure 3. The trajectories of system (2.5) with initial value $(8, 5, 0, 4.5, 4)$ with control and the bounded impulsive interval.

5.2. *The influence of pulse*

To investigate the impact of pulse on (2.5), the same parameter values shown in Figure 2 are used and let $I_{ik} = 0$. By simple calculation, it can be found that all conditions of Theorem 4.8 are not satisfied, i.e., system (2.5) about (6.5,7,5.5,1,3) is not FTCS, as shown in Figure 4. For this case, compared to the conditions of Figure 2, we just alter the impulsive condition, but discover that the system cannot be FTCS without impulsive effects. It shows that under the same circumstances, impulsive perturbation is important to the system's stability for a finite period.



(a) (b) (c)

Continued on next page

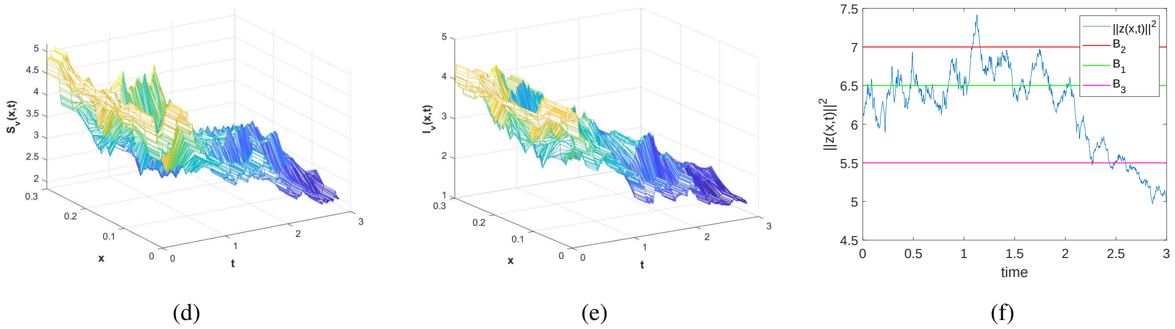


Figure 4. The state trajectories of system (2.5) without impulse with initial value $(8, 5, 0, 4.5, 4)$.

5.3. *The effect of the control variables*

According to Theorem 4.6 and Theorem 4.8, it can be noticed that the values of control variables are key to the FTS and FTCS of system (2.5). The following discussion is divided into two cases.

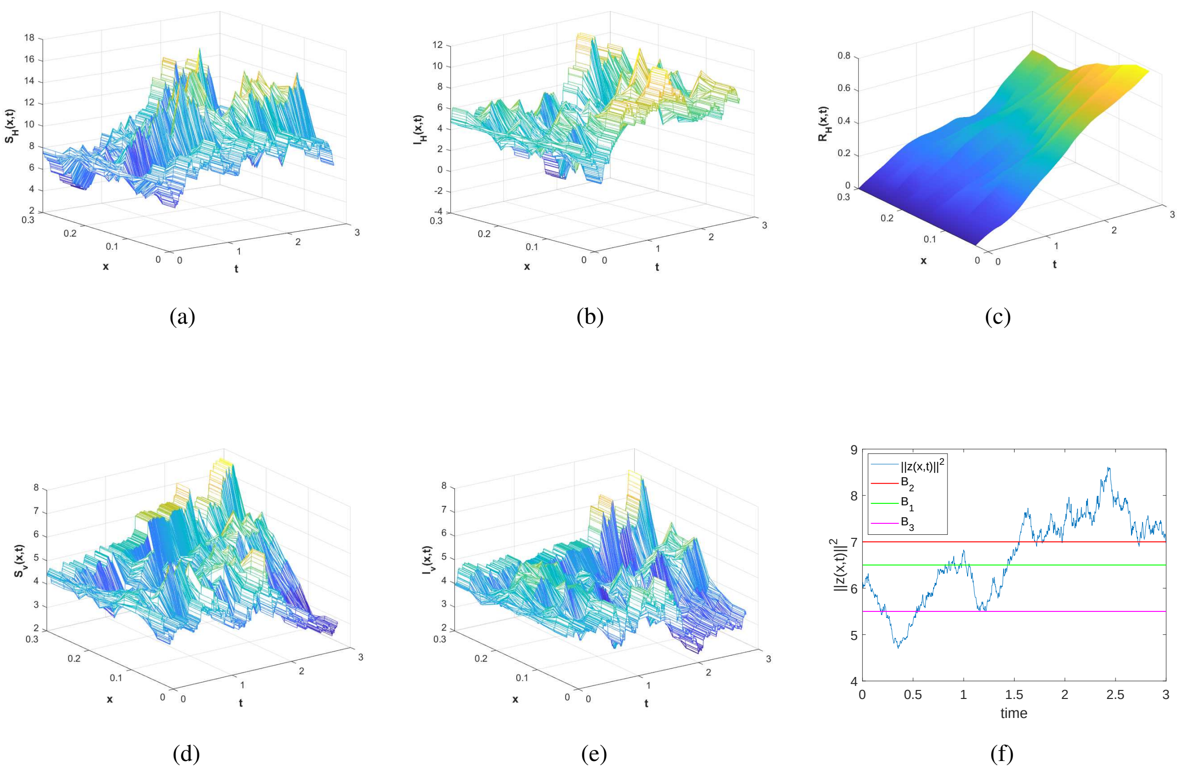
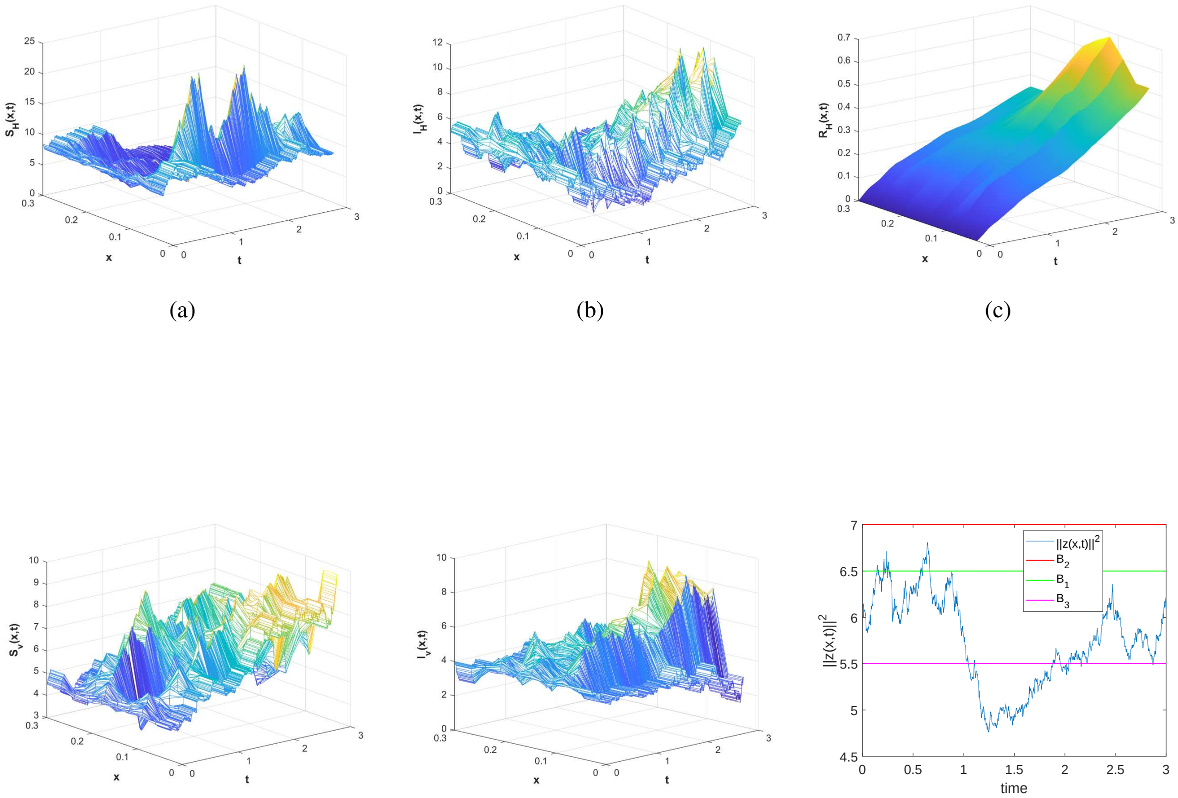


Figure 5. The state trajectories of system (2.5) with initial value $(8, 5, 0, 4.5, 4)$ without control.

Case 1: In this part, we can see by Figure 5 that if without controls (i.e., $\delta_1 = \delta_2 = 0$ and other parameters of the system are as same as in Figure 2), then it is neither FTS w.r.t (6.5, 7, 3) nor FTCS w.r.t (6.5, 7, 5.5, 1, 3).

Case 2: We choose the same parameters as in Figure 2 except $\delta_1(1) = 0.2, \delta_2(1) = 0.3, \delta_1(2) = 0.1, \delta_2(2) = 0.2$. Then system is FTS about (6.5, 7, 3) and not FTCS about (6.5, 7, 5.5, 1, 3) (see Figure 6).

Case 1 implied that system (2.5) itself is impossible to be stable in a finite period of time without control. Case 2 illustrates that if the control intensity is low, dengue fever cannot be finally controlled at a lower level (below the initial value). It can be observed that the stochastic dengue epidemic model can be FTCS only by choosing appropriate values of control variables.



(a) (b) (c) (d) (e) (f)

Figure 6. The trajectories of system (2.5) with initial value $(8, 5, 0, 4.5, 4)$ with $\delta_1(1) = 0.2, \delta_2(1) = 0.3, \delta_1(2) = 0.1, \delta_2(2) = 0.2$.

5.4. The influence of noise

Now consider the impact of environmental noise intensity on FTCS of (2.5). Based on the parameters from Table 1, choose $\rho_1(1) = 2, \rho_2(1) = 1.6, \rho_1(2) = 1.8, \rho_2(2) = 1.3$. Then Figure 7 illustrates system is not FTCS but FTS, which means that the finite-time behaviors of dengue is heavily influenced by external environmental disturbances.

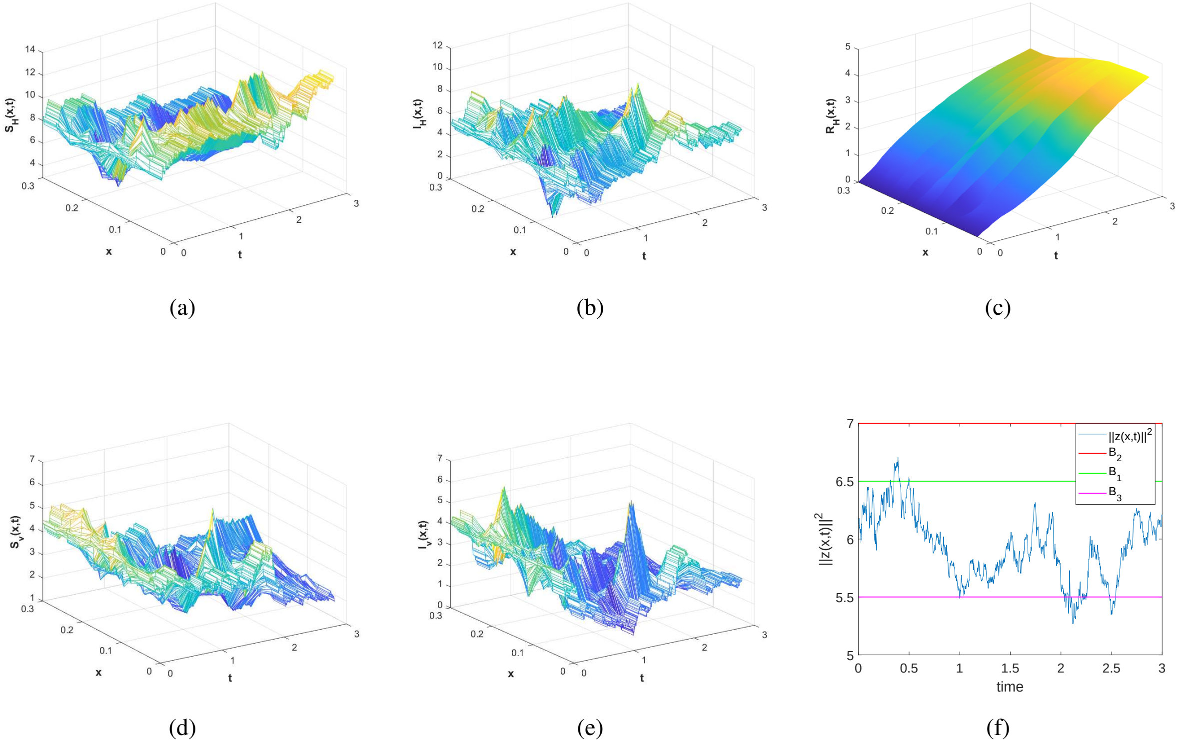


Figure 7. The state trajectories of system (2.5) with initial value $(8, 5, 0, 4.5, 4)$ with $\rho_1(1) = 2, \rho_2(1) = 1.6, \rho_1(2) = 1.8, \rho_2(2) = 1.3$.

6. Conclusions and discussion

This article investigated the FTCS of a reaction-diffusion dengue model with impulse and Markov switching. Sufficient conditions with respect to control variable for finite-time contraction stability are obtained via two representations of pulse sequences (i.e., the average pulse interval and bounded pulse interval) make use of the Lyapunov functional method and inequality techniques. All these conditions show the impacts of environmental noise intensity (ρ_1, ρ_2) and impulsive factor (φ, N_0, l, l_m) on the FTCS. FTCS can be realized only when appropriate control measures are selected, that is, the spread of dengue fever can be effectively controlled within a limited time. Moreover, delay [35–38] is also a widespread phenomenon that can lead to dramatic changes in dynamic behavior. It is challenging to consider the impact of time delays on FTCS, which will also be our future research direction.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

References

1. *Centers for Diease Control and Prevention*, Available from: http://www.cdc.gov/dengue/.
2. *World Health Organization (WHO)*, Available from: https://www.who.int/zh/news-room/fact-sheets/detail/dengue-and-severe-dengue.
3. Z. Xu, Y. Zhao, A reaction-diffusion model of dengue transmission, *Discrete Contin. Dyn. Syst. B*, **19** (2014), 2993–3018. https://doi.org/10.3934/dcdsb.2014.19.2993
4. Y. Li, Y. Wang, L. Liu, Optimal control of dengue vector based on a reaction-diffusion model, *Math. Comput. Simul.*, **203** (2023), 250–270. https://doi.org/10.1016/j.matcom.2022.06.026
5. M. Zhu, Y. Xu, A time-periodic dengue fever model in a heterogeneous environment, *Math. Comput. Simul.*, **155** (2018), 115–129. https://doi.org/10.1016/j.matcom.2017.12.008
6. M. Zhu, Z. Lin, L. Zhang, Spatial-temporal risk index and transmission of a nonlocal dengue model, *Nonlinear Anal. Real World Appl.*, **53** (2020), 103076–103097. https://doi.org/10.1016/j.nonrwa.2019.103076
7. Y. Pang, S. Wang, S. Liu, Dynamics analysis of stage-structured wild and sterile mosquito interaction impulsive model, *J. Biol. Dyn.*, **16** (2022), 464–479. https://doi.org/10.1080/17513758.2022.2079739
8. C. X. Yang, L. Nie, The effect of vector control strategy against dengue transmission between mosquitoes and humans, *Electron. J. Qual. Theory Differ. Equ.*, **17** (2017), 1–27. https://doi.org/10.14232/ejqtde.2017.1.17
9. R. Rao, Z. Lin, X. Ai, J. Wu, Synchronization of epidemic systems with Neumann boundary value under delayed impulse, *Mathematics*, **10** (2022), 2064. https://doi.org/10.3390/math10122064
10. G. D. Li, Y. Zhang, Y. J. Guan, W. J. Li, Stability analysis of multi-point boundary conditions for fractional differential equation with non-instantaneous integral impulse, *Math. Biosci. Eng.*, **20** (2023), 7020–7041. https://doi.org/10.3934/mbe.2023303
11. P. Liu, A. Din, Zenab, Impact of information intervention on stochastic dengue epidemic model, *Alexandria Eng. J.*, **60** (2021), 5725–5739. https://doi.org/10.1016/j.aej.2021.03.068
12. K. Chang, Q. Zhang, H. Yuan, Stationary distribution and control strategy of a stochastic dengue model with spatial diffusion, *J. Appl. Anal. Comput.*, **12** (2022), 153–178. https://doi.org/10.11948/20210094
13. Q. Liu, D. Jiang, T. Hayat, A. Alsaedi, Stationary distribution and extinction of a stochastic dengue epidemic model, *J. Franklin Inst.*, **355** (2018), 8891–8914. https://doi.org/10.1016/j.jfranklin.2018.10.003

14. M. Guo, L. Hu, L. F. Nie, Stochastic dynamics of the transmission of Dengue fever virus between mosquitoes and humans, *Int. J. Biomath.*, **14** (2021), 2150062. https://doi.org/10.1142/S1793524521500625

15. L. Zu, D. Jiang, D. O'Regan, Conditions for persistence and ergodicity of a stochastic Lotka-Volterra predator-prey model with regime switching, *Commun. Nonlinear Sci. Numer. Simul.*, **29** (2015), 1–11. https://doi.org/10.1016/j.cnsns.2015.04.008

16. Q. Liu, D. Jiang, T. Hayat, A. Alsaedi, Stationary distribution of a stochastic within-host dengue infection model with immune response and regime switching, *Physica A*, **526** (2019), 121057. https://doi.org/10.1016/j.physa.2019.121057

17. D. Kuang, Q. Yin, J. Li, The threshold of a stochastic SIRS epidemic model with general incidence rate under regime-switching, *J. Franklin Inst.*, **20** (2022), 48. https://doi.org/10.1016/j.jfranklin.2022.04.027

18. X. Mu, Q. Zhang, Near-optimal control for a stochastic multi-strain epidemic model with age structure and Markovian switching, *Int. J. Control*, **95** (2022), 1191–1205. https://doi.org/10.1080/00207179.2020.1843074

19. A. Gray, D. Greenhalgh, X. Mao, J. Pan, The SIS epidemic model with Markovian switching, *J. Math. Anal. Appl.*, **394** (2012), 496–516. https://doi.org/10.1016/j.jmaa.2012.05.029

20. S. Kazemi, M. Stommel, L. K. Cheng, W. Xu, Finite-time contraction control of a ring-shaped soft pneumatic actuator mimicking gastric pathologic motility conditions, *Soft Rob.*, **10** (2022). https://doi.org/10.1089/soro.2021.0167

21. R. Gan, C. Li, Finite-time stability of nonlinear time-varying systems with saturated impulse inputs, *Nonlinear Dyn.*, **111** (2023), 3497–3507. https://doi.org/10.1007/s11071-022-08024-y

22. K. N. Wu, M. Y. Na, L. Wang, X. Ding, B. Wu, Finite-time stability of impulsive reaction-diffusion systems with and without time delay, *Appl. Math. Comput.*, **363** (2019), 124591. https://doi.org/10.1016/j.amc.2019.124591

23. Y. Wu, J. Cao, A. Alofi, A. AL-Mazrooei, A. Elaiw, Finite-time boundedness and stabilization of uncertain switched neural networks with time-varying delay, *Neural Networks*, **69** (2015), 135–143. https://doi.org/10.1016/j.neunet.2015.05.006

24. H. R. Pandey, G. R. Phaijoo, Analysis of dengue infection transmission dynamics in Nepal using fractional order mathematical modeling, *Chaos, Solitons Fractals: X*, **11** (2023), 100098. https://doi.org/10.1016/j.csfx.2023.100098

25. J. Li, H. Wan, M. Sun, Modeling the impact of awareness programs on the transmission dynamics of dengue and optimal control, *Int. J. Biomath.*, **16** (2023), 2250072. https://doi.org/10.1142/S1793524522500723

26. S. T. Ogunlade, M. T. Meehan, A. I. Adekunle, E. S. McBryde, A systematic review of mathematical models of Dengue transmission and vector control: 2010–2020, *Viruses*, **15** (2023), 254. https://doi.org/10.3390/v15010254

27. C. Buhler, V. Winkler, S. Runge-Ranzinger, O. Horstick, Environmental methods for dengue vector control-A systematic review and meta-analysis, *PLoS Negl. Trop. Dis.*, **13** (2019), e0007420. https://doi.org/10.1371/journal.pntd.0007420

28. T. Pan, D. Jiang, T. Hayat, A. Alsaedi, Extinction and periodic solutions for an impulsive SIR model with incidence rate stochastically perturbed, *Physica A*, **505** (2018), 385–397. https://doi.org/10.1016/j.physa.2018.03.012

29. K. Chang, Q. Zhang, Sufficient and necessary conditions of near-optimal controls for diffusion dengue model with Levy noise, *J. Math. Anal. Appl.*, **514** (2022), 126044. https://doi.org/10.1016/j.jmaa.2022.126044

30. K. Wu, B. S. Chen, Synchronization of partial differential systems via diffusion coupling, *IEEE Trans. Circuits Syst. I Regul. Pap.*, **59** (2012), 2655–2668. https://doi.org/10.1109/tcsi.2012.2190670

31. X. Mao, Stochastic functional differential equations with Markovian switching, in *Stochastic Differential Equations & Applications*, (2011), 147–200. https://doi.org/10.1533/9780857099402.147

32. D. D. Bajnov, P. S. Simeonov, *Systems with Impulse Effect: Stability, Theory and Applications*, E. Horwood, Halsted Press, 1989.

33. R. Wu, X. Q. Zhao, A reaction-diffusion model of vector-borne disease with periodic delays, *J. Nonlinear Sci.*, **29** (2019), 29–64. https://doi.org/10.1007/s00332-018-9475-9

34. H. S. Rodrigues, M. T. T. Monteiro, D. F. M. Torres, Seasonality effects on dengue: basic reproduction number, sensitivity analysis and optimal control, *Math. Methods Appl. Sci.*, **39** (2016), 4671–4679. https://doi.org/10.1002/mma.3319

35. W. Hu, Q. Zhu, H. R. Karimi, Some improved Razumikhin stability criteria for impulsive stochastic delay differential systems, *IEEE Trans. Autom. Control*, **64** (2019), 5207–5213. https://doi.org/10.1109/TAC.2019.2911182

36. Y. Zhao, L. Wang, Practical exponential stability of impulsive stochastic food chain system with time-varying delays, *Mathematics*, **11** (2023), 147. https://doi.org/10.3390/math11010147

37. W. Hu, Q. Zhu, Stability criteria for impulsive stochastic functional differential systems with distributed-delay dependent impulsive effects, *IEEE Trans. Syst. Man Cybern. Syst.*, **51** (2019), 2027–2032. https://doi.org/10.1109/TSMC.2019.2905007

38. Y. Tang, L. Zhou, J. Tang, A. Alsaedi, Hybrid impulsive pinning control for mean square Synchronization of uncertain multi-link complex networks with stochastic characteristics and hybrid delays, *Mathematics*, **11** (2023), 1697. https://doi.org/10.1016/j.physa.2019.121057

Appendix A: The proof of Theorem 3.1

Proof. Due to the coefficients of (3.1) are locally Lipschitz continuous, for (3.2), system (3.1) admits a unique solution $y(x,t)$ on $[0,\tau_e)$, where τ_e is the explosion time.

Let $q_0 > 0$ be sufficiently large such that $y(x,0)$ lies within $[\frac{1}{q_0}, q_0]$. For each $q \geq q_0$, define a stopping time

$$\tau_q = inf\{t \in [0,\tau_e] : min\{y_i(x,t)\} \leq \frac{1}{q} \\ or \;\; max\{y_i(x,t)\} \geq q\}, \; i = 1,2,3,4.$$

Set $\inf \emptyset = \infty$. It can be seen that τ_q is increasing as $q \to +\infty$. Let $\tau_\infty = \lim_{t\to+\infty}$, then $\tau_\infty \leq \tau_e$ $a.s.$ and $y(x,t) > 0$. For arbitrary $T > 0$, $t \in [0, t_q \wedge T)$, let

$$V(x,t,i) = G(i)(\int_\Gamma y_1^2(x,t) + y_2^2(x,t) + y_3^2(x,t) + y_4^2(x,t) + y_5^2(x,t)dx).$$

According to Itô formula we have

$$\begin{aligned}
&dV(x,t,i) \\
&= 2G(i)\int_\Gamma y_1(x,t)\Big[[D_1(i)\triangle y_1(x,t) + \mu_h(i)N_H(i) - \frac{\beta_H(i)b(i)}{N_H(i)+m(i)}y_1(x,t)h_2(t)y_5(x,t) - \mu(i)y_1(x,t)]dt \\
&\quad - \frac{b(i)}{N_H(i)+m(i)}\rho_1(i)y_1(x,t)h_2(t)y_5(x,t)dB_1(t)\Big]dx + 2G(i)\int_\Gamma y_2(x,t)\Big[[D_2(i)\triangle y_2(x,t) + \frac{\beta_H(i)b(i)}{N_H(i)+m(i)} \\
&\quad \times y_1(x,t)h_2(t)y_5(x,t) - (\mu(i)+\gamma_H(i))y_2(x,t) - \delta_1(i))y_2(x,t)]dt + \frac{b(i)}{N_H(i)+m(i)}\rho_1(i)y_1(x,t)h_2(t) \\
&\quad \times y_5(x,t)dB_1(t)\Big]dx + 2G(i)\int_\Gamma y_3(x,t)\Big[D_3(i)\triangle y_3(x,t) + (\gamma_H(i) + +\delta_1(i))y_2(x,t) - \mu(i)y_3(x,t)dt\Big]dx \\
&\quad + 2G(i)\int_\Gamma y_4(x,t)\Big[[D_4(i)\triangle y_4(x,t) + A(i)h_1^{-1}(t) - \frac{\beta_v(i)b(i)}{N_H(i)+m(i)}y_4(x,t)y_2(x,t) - \delta_2(i)y_4(x,t) \\
&\quad - (v(i) - \ln(1+\varnothing_{1k}))y_4(x,t)]dt - \frac{b(i)}{N_H(i)+m(i)}\rho_2(i)y_4(x,t)y_2(x,t)dB_2(t)\Big]dx + 2G(i)\int_\Gamma y_5(x,t) \\
&\quad \times \Big[[D_5(i)\triangle y_5(x,t) + \frac{\beta_v(i)b(i)}{N_H(i)+m(i)}h_2^{-1}(t)h_1(t)y_4(x,t)y_2(x,t) - v(i)y_5(x,t) + \ln(1-\varnothing_{2k})y_5(x,t) \\
&\quad - \delta_2(i)y_5(x,t)]dt + \frac{b(i)}{N_H(i)+m(i)}\rho_2(i)h_2^{-1}(t)h_1(t)y_4(x,t)y_2(x,t)dB_2(t)\Big]dx + G(i)(\int_\Gamma \frac{2(b(i)\rho_1(i))^2}{(N_H(i)+m(i))^2} \\
&\quad \times y_1^2(x,t)h_2^2(x,t)y_5^2(x,t) + \frac{(b(i)\rho_2(i))^2}{(N_H(i)+m(i))^2}y_4^2(x,t)y_2^2(x,t) + \frac{(b(i)\rho_2(i))^2}{(N_H(i)+m(i))^2}h_2^{-2}(t)h_1^2(t)y_4^2(x,t) \\
&\quad \times y_2^2(x,t))dxdt + \sum_{j=1}^m q_{ij}G(i)(\int_\Gamma y_1^2(x,t) + y_2^2(x,t) + y_3^2(x,t) + y_4^2(x,t) + y_5^2(x,t))dxdt.
\end{aligned} \tag{6.1}$$

Assign

$$
\begin{aligned}
&\mathcal{L}V(x,t,i)\\
&= 2G(i)\{\int_\Gamma y_1(x,t)\Big[[D_1(i)\triangle y_1(x,t) + \mu_h(i)N_H(i) - \frac{\beta_H(i)b(i)}{N_H(i)+m(i)}y_1(x,t)h_2(t)y_5(x,t) - \mu(i)y_1(x,t)]dt\\
&- \frac{b(i)}{N_H(i)+m(i)}\rho_1(i)y_1(x,t)h_2(t)y_5(x,t)dB_1(t)\Big]dx + \int_\Gamma y_2(x,t)\Big[[D_2(i)\triangle y_2(x,t) + \frac{\beta_H(i)b(i)}{N_H(i)+m(i)}\\
&\times y_1(x,t)h_2(t)y_5(x,t) - (\mu(i)+\gamma_H(i))y_2(x,t) - \delta_1(i))y_2(x,t)]dt + \frac{b(i)}{N_H(i)+m(i)}\rho_1(i)y_1(x,t)h_2(t)\\
&\times y_5(x,t)dB_1(t)\Big]dx + \int_\Gamma y_3(x,t)\Big[D_3(i)\triangle y_3(x,t) + (\gamma_H(i) + +\delta_1(i))y_2(x,t) - \mu(i)y_3(x,t)dt\Big]dx\\
&+ \int_\Gamma y_4(x,t)\Big[[D_4(i)\triangle y_4(x,t) + A(i)h_1^{-1}(t) - \frac{\beta_v(i)b(i)}{N_H(i)+m(i)}y_4(x,t)y_2(x,t) - \delta_2(i)y_4(x,t)\\
&- (v(i) - \ln(1+\varnothing_{1k}))y_4(x,t)]dt - \frac{b(i)}{N_H(i)+m(i)}\rho_2(i)y_4(x,t)y_2(x,t)dB_2(t)\Big]dx + \int_\Gamma y_5(x,t)\\
&\times\Big[[D_5(i)\triangle y_5(x,t) + \frac{\beta_v(i)b(i)}{N_H(i)+m(i)}h_2^{-1}(t)h_1(t)y_4(x,t)y_2(x,t) - v(i)y_5(x,t) + \ln(1-\varnothing_{2k})y_5(x,t)\\
&- \delta_2(i)y_5(x,t)]dt + \frac{b(i)}{N_H(i)+m(i)}\rho_2(i)h_2^{-1}(t)h_1(t)y_4(x,t)y_2(x,t)dB_2(t)\Big]dx\} + \sum_{j=1}^m q_{ij}G(i)\\
&\times(\int_\Gamma y_1^2(x,t) + y_2^2(x,t) + y_3^2(x,t) + y_4^2(x,t) + y_5^2(x,t))dxdt.
\end{aligned}
\tag{6.2}
$$

In view of the partial integral formula, some basic inequalities, we deduce that

$$
\begin{aligned}
&\mathcal{L}V(x,t,i)\\
&\leq G(i)\{\int_\Gamma -2D_1(i)(\nabla y_1(x,t))^2 - 2D_2(i)(\nabla y_2(x,t))^2 - 2D_3(i)(\nabla y_3(x,t))^2 - 2D_4(i)(\nabla y_4(x,t))^2\\
&- 2D_5(i)(\nabla y_5(x,t))^2]dxdt + \int_\Gamma \check{\mu}_h^2\check{N}_H^2 + y_1^2(x,t)dxdt + \int_\Gamma \frac{\beta_H(i)b(i)}{N_H(i)+m(i)}B(y_1^2(x,t)+y_2^2(x,t))\\
&+ \int_\Gamma(\gamma_H(i)+\delta_1(i))(y_2^2(x,t)+y_3^2(x,t))dxdt + \int_\Gamma \check{A}^2\underline{h}_1^{-2} + y_4^2(x,t) + 2\ln(1+\varnothing_{1k})y_4^2(x,t))dxdt\\
&+ \int_\Gamma \frac{\beta_v(i)b(i)}{N_H(i)+m(i)}B(y_2^2(x,t) + \underline{h}_2^{-2}y_5^2(x,t)) + 2\ln(1-\varnothing_{2k})y_5^2(x,t))dxdt + \int_\Gamma \frac{2(b(i)\rho_1(i))^2}{(N_H(i)+m(i))^2}\\
&\times B^2y_1^2(x,t) + \frac{(b(i)\rho_2(i))^2}{(N_H(i)+m(i))^2}B^2y_4^2(x,t) + \frac{(b(i)\rho_2(i))^2}{(N_H(i)+m(i))^2}B^2\underline{h}_2^{-2}y_2^2(x,t)dxdt\}\\
&+ (q_{ii} + \sum_{j\neq i}^m q_{ij}\check{G}\hat{G}^{-1})G(i)(\int_\Gamma y_1^2(x,t) + y_2^2(x,t) + y_3^2(x,t) + y_4^2(x,t) + y_5^2(x,t))dxdt\\
&\leq \check{G}\{(\check{\mu}_h^2\check{N}_H^2 + \check{A}^2\underline{h}_1^{-2})|\Gamma| + \int_\Gamma(1 + \frac{\check{\beta}_H\check{b}}{\check{N}_H+\check{m}}B + \frac{2\check{b}^2\check{\rho}_1^2}{(\check{N}_H+\check{m})^2}B^2)y_1^2(x,t)dxdt\\
&+ \int_\Gamma(\frac{\check{b}}{\check{N}_H+\check{m}}B(\check{\beta}_H+\check{\beta}_v) + \check{\gamma}_H + \check{\delta}_1 + \frac{\check{b}^2\check{\rho}_2^2}{(\check{N}_H+\check{m})^2}B^2\underline{h}_2^{-2})y_2^2(x,t)dxdt\\
&+ \int_\Gamma(\check{\gamma}_H+\check{\delta}_1)y_3^2(x,t) + \int_\Gamma(1 + 2\ln(1+\varnothing_{1k}) + \frac{\check{b}^2\check{\rho}_2^2}{(\check{N}_H+\check{m})^2}B^2)y_4^2(x,t)dxdt
\end{aligned}
$$

$$+\int_{\Gamma}(\frac{\check{\beta}_v\check{b}}{\check{N}_H+\check{m}}B\underline{h}_2^{-2}+2\ln(1-\varnothing_{2k}))y_5^2(x,t)dxdt\}+(q_{ii}+\sum_{j\neq i}^{m}q_{ij}\check{G}\hat{G}^{-1})V(x,t,j)dt$$

$$\leq c_0+c_1V(x,t,i)\,dt$$

where

$$c_0=\check{G}(\check{\mu}_h^2\check{N}_H^2+\check{A}^2\underline{h}_1^{-2})|\Gamma|,$$
$$c_1=\max\Big\{1+\tfrac{\check{\beta}_H\check{b}}{\check{N}_H+\check{m}}B+\tfrac{2\check{b}^2\check{\rho}_1^2}{(\check{N}_H+\check{m})^2}B^2,\ \tfrac{\check{b}}{\check{N}_H+\check{m}}B(\check{\beta}_H+\check{\beta}_v)+\check{\gamma}_H+\check{\delta}_1+\tfrac{\check{b}^2\check{\rho}_2^2}{(\check{N}_H+\check{m})^2}B^2\underline{h}_2^{-2},$$
$$1+2\ln(1+\varnothing_{1k})+\tfrac{\check{b}^2\check{\rho}_2^2}{(\check{N}_H+\check{m})^2}B^2,\ \tfrac{\check{\beta}_v\check{b}}{\check{N}_H+\check{m}}B\underline{h}_2^{-2}+2\ln(1-\varnothing_{2k})\Big\}.$$

Therefore, we can know that

$$\begin{aligned}dV(x,t,i)=\mathcal{L}V(x,t,i)-2G(i)[\int_{\Gamma}(y_1(x,t)-y_2(x,t))\frac{b(i)\rho_1(i)}{N_H(i)+m(i)}y_1(x,t)h_2(x,t)y_5(x,t)dB_1(t)dx\\+\int_{\Gamma}y_5(x,t)\frac{b(i)\rho_2(i)}{N_H(i)+m(i)}h_2^{-1}(t)h_1(t)y_4(x,t)y_2(x,t)dB_2(t)dx\\-\int_{\Gamma}\frac{b(i)\rho_2(i)}{N_H(i)+m(i)}y_4^2(x,t)y_2(x,t)dB_2(t)dx]\end{aligned}\tag{6.3}$$

Moreover,

$$\begin{aligned}&E[V(\tau_q\wedge T)]\\&\leq V(0)+c_0+c_1V(t)\,dt\\&\leq c_2+c_1\int_0^{\tau_q\wedge T}V(t)\,dt\\&\leq c_2+c_1\int_0^{T}E[V(\tau_q\wedge t)]dt.\end{aligned}$$

Using Gronwall inequality yields that

$$E[V(\tau_q\wedge T)]\leq c_2e^{c_1T}.\tag{6.4}$$

Denote

$$\mu_q=\inf\Big\{V(x,t,i),\|y(x,t)\|\geq q\Big\}.$$

It can be obtained that

$$\lim_{q\to+\infty}\mu_q=+\infty.\tag{6.5}$$

Then from (6.4), we have

$$\mu_q\mathbb{P}(\tau_q\leq T)\leq c_2e^{c_1T}.$$

Therefore, by (6.5) and choosing $q\to+\infty$ yields that

$$P(\tau_q\leq T)=0,$$

thus $P(\tau_\infty>T)=1$.