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Research article

Dynamics study of nonlinear discrete predator-prey system with Michaelis-Menten type harvesting

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Abstract: In this paper, we study a discrete predator-prey system with Michaelis-Menten type harvesting. First, the equilibrium points number, local stability and boundedness of the system are discussed. Second, using the bifurcation theory and the center manifold theorem, the bifurcation conditions for the system to go through flip bifurcation and Neimark-Sacker bifurcation at the interior equilibrium point are obtained. A feedback control strategy is used to control chaos in the system, and an optimal harvesting strategy is introduced to obtain the optimal value of the harvesting coefficient. Finally, the numerical simulation not only shows the complex dynamic behavior, but also verifies the correctness of our theoretical analysis. In addition, the results show that the system causes nonlinear behaviors such as periodic orbits, invariant rings, chaotic attractors, and periodic windows by bifurcation.

Keywords: discrete system; stability; flip bifurcation; Neimark-Sacker bifurcation; optimal harvesting

1. Introduction

Population is a collection of individuals of the same species living together within a certain spatial range. The study of population dynamics mainly describes the dynamic relationship between population communities in predation systems and food web systems. Food web refers to the complex relationship network between predation, competition, cooperation and reciprocity between biological populations in biological systems. The study of food web population dynamics can give humans an important understanding of the basic nature of ecosystems, promoting the evolution of life, protecting the natural environment, and maintaining ecological balance at the macro level, and give certain guiding opinions on the protection of endangered species at the micro level. Therefore, studying the interactions between different biological populations through mathematical modeling is an important area of research for ecosystems.

In 1976, May showed in [1] that the first-order discrete equation model, although simple, can produce a series of extremely complex dynamic behaviors, such as from stable points to unstable bifurcation levels, and eventually produce chaos. However, in continuous-time models, achieving such complex dynamic behavior requires differential equations of three dimensions and more than three dimensions to cause chaos [2]. It can be seen that discrete systems described by the difference equation are richer in dynamical phenomena. Consequently, the discrete dynamical system model has attracted the attention and research of more scholars [3–11].

In [12], Gan et al. studied the stability in a simple food chain system with Michaelis-Menten functional responses and nonlocal delays, using the Lyapunov functional to derive sufficient conditions for global stability of positive steady state and semi-trivial steady state. In [13], Clark et al. described mathematical models of exploited fish stocks, assuming that certain stocks can be obtained through dynamic aggregation processes. The effects of aggregation on yield-effort relationships, abundance indices, and fishery dynamics are discussed, as well as various management approaches for these models. On the other hand, with the increasing demand for food and other resources, the exploitation of biological resources is also increasing. More ecologists are very interested in studying these types of models, such as predator-prey models, and consider the impact of exploiting (harvesting) resources to protect the sustainable use of biological resources [14–17]. To do this, they applied optimal capture and control strategies to achieve the recyclability of biological resources.

In [20], we studied the behavioral analysis of a class of discrete dynamical system with linear harvest rates, and obtained many complex dynamic phenomena. Considering the finite nature of resources and space, the linear capture rate has no upper bound, so we will further study the Michaelis-Menten [13] type capture on the basis of the original, and this type of capture is a kind of harvest that gradually rises until the saturation state with the increase of the number of captured objects, which is bounded and more in line with the realistic ecological environment. So, in this paper we consider a discrete-time predator-prey model with Michaelis-Menten type harvesting in preys, which is given by

$$\begin{cases} u_{n+1} = u_n \exp[r_1(1-\frac{u_n}{K}) - \frac{r_2 v_n}{c+u_n} - \frac{qE}{m_1E+m_2u_n}], \\ v_{n+1} = v_n \exp[a + \frac{r_2 d u_n}{c+u_n} - b v_n - h_2], \end{cases} \tag{1.1}$$

where the meanings of all parameters of model (1.1) are shown in Table1.

The paper is organized as follows. In Section 2, we analyze the dynamics of system (1.1), including the existence and local stability of the equilibrium points, and the boundedness of system (1.1). In Section 3, using the bifurcation theory and the center manifold theorem, the flip bifurcation and Neimark-Sacker bifurcation are analyzed, and the conditions for determining the bifurcation direction and the stability of the bifurcation periodic solution are obtained. In Section 4, a feedback control strategy is used to control chaos in the system, and an optimal harvesting strategy is introduced to obtain the optimal value of the harvesting coefficient. In Section 5, we verify our analytical results through numerical simulations. In the last Section, the article is ended with a brief conclusion.

2. Model dynamics

Lemma 1 Solutions of system (1.1) with nonnegative initial conditions remain nonnegative. If $u_0 = 0$, then $u_n = 0$ for all $n \geq 0$. If $v_0 = 0$, then $v_n = 0$ for all $n \geq 0$. If $u_0 > 0$ and $v_0 \geq 0$, then $u_n > 0$

for all $n \geq 0$. If $u_0 \geq 0$ and $v_0 > 0$, then $v_n > 0$ for all $n \geq 0$.

Table 1. The interpretation of parameters.

parameters	interpretation
$u,\ v$	the densities of prey and predator population, respectively
r_1	the intrinsic growth rates of prey
a	the intrinsic growth rates of predator
K	the environmental carrying capacity of prey
r_2	the consumption rate of prey
b	the competition between individuals due to overcrowding of predator
c	the half saturation constant
d	the conversion rate of predator
h_2	the capture rate of predator
$m_i, i = 1, 2$	suitable constants
q	the catchability co-efficients of prey
E	the degree of harvest effort

Proof. It can be directly demonstrated by system (1.1) structure. □

Lemma 2

(I) System (1.1) always has a trivial equilibrium point $E_0(0, 0)$. At this moment, the two species will go extinct.

(II) If $q < r_1 m_1$, then system (1.1) always has a positive semi-trivial equilibrium point $E_1(\frac{-\alpha+\sqrt{\Delta}}{2}, 0)$, where $\alpha = \frac{m_1 E}{m_2} - K, \beta = \frac{qEK - r_1 EKm_1}{r_1 m_1}$, $\Delta = \alpha^2 - 4\beta > 0$. At this time, the prey population reaches $\frac{-\alpha+\sqrt{\Delta}}{2}$, and the predator tends to go extinct.

(III) If $a > h_2$, then system (1.1) always has a positive semi-trivial equilibrium point $E_2(0, \frac{a-h_2}{b})$. At this time, the prey population tend to go extinct, and the predator population converge to $\frac{a-h_2}{b}$.

(IV) System (1.1) has a positive nontrivial equilibrium point $E^*(u^*, \frac{1}{b}(a + \frac{r_2 du^*}{u^*+c} - h_2))$, where $a > h_2$, and u is the positive solution to the quartic equation of one variabie

$$A_1 u^4 + A_2 u^3 + A_3 u^2 + A_4 u + A_5 = 0,$$

where

$$\begin{aligned}
&A_1 = -bm_2,\ \ A_2 = bm_2(r_1 K - 2c) - bm_1 E,\\
&A_3 = r_1 bK(m_1 E + 2cm_2) - bc(2m_1 E + cm_2) + r_2 Km_2(h_2 - a) - dr_2^2 Km_2 - bqEK,\\
&A_4 = r_1 bcK(2m_1 E + cm_2) - bc^2 m_1 E + r_2 Km_1 E(h_2 - a) + r_2 cKm_2(h_2 - a)\\
&\quad - EK(dr_2^2 m_1 + 2bcq),\\
&A_5 = cEKm_1(r_1 bc + r_2 h_2 - r_2 a) + bc^2 qEK.
\end{aligned}$$

When system (1.1) has a positive interior equilibrium point E^*, two species coexist, i.e. two species do not go extinct.

Proof. Direct computation. □

Lemma 3 [18] The equilibrium point (u, v) is called

(I) Sink (locally asymptotically stable) if $|\lambda_1| < 1$ and $|\lambda_2| < 1$;
(II) Source (locally unstable) if $|\lambda_1| > 1$ and $|\lambda_2| > 1$;
(III) Saddle if $|\lambda_1| > 1$ and $|\lambda_2| < 1$ (or $|\lambda_1| < 1$ and $|\lambda_2| > 1$);
(IV) Non-hyperbolic if $|\lambda_1| = 1$ or $|\lambda_2| = 1$.
Jacobian matrix can be evaluated at $E_0(0,0)$ as

$$J(E_0) = \begin{pmatrix} e^{r_1 - \frac{q}{m_1}} & 0 \\ 0 & e^{a-h_2} \end{pmatrix}. \tag{2.1}$$

The eigenvalues of $J(E_0)$ are $\lambda_1 = e^{r_1 - \frac{q}{m_1}}$ and $\lambda_2 = e^{a-h_2}$. The results regarding dynamical behaviors are listed in Table 2.

Table 2. Properties of equilibrium point $E_0(0,0)$.

Conditions		Eigenvalues		Properties
		$\lambda_1 = e^{r_1 - \frac{q}{m_1}}$	$\lambda_2 = e^{a-h_2}$	
$r_1m_1 < q$	$a < h_2$	$\lvert\lambda_1\rvert < 1$	$\lvert\lambda_2\rvert < 1$	Sink
	$a > h_2$		$\lvert\lambda_2\rvert > 1$	Saddle
	$a = h_2$		$\lvert\lambda_2\rvert = 1$	Non-hyperbolic
$r_1m_1 > q$	$a < h_2$	$\lvert\lambda_1\rvert > 1$	$\lvert\lambda_2\rvert < 1$	Saddle
	$a > h_2$		$\lvert\lambda_2\rvert > 1$	Source
	$a = h_2$		$\lvert\lambda_2\rvert = 1$	Non-hyperbolic
$r_1m_1 = q$	$a < h_2$	$\lvert\lambda_1\rvert = 1$	$\lvert\lambda_2\rvert < 1$	Non-hyperbolic
	$a > h_2$		$\lvert\lambda_2\rvert > 1$	Non-hyperbolic
	$a = h_2$		$\lvert\lambda_2\rvert = 1$	Non-hyperbolic

From Table 2, we can get the following theorem.

Theorem 1. When $r_1m_1 < q$ and $a < h_2$ are satisfied, the trivial equilibrium point $E_0(0,0)$ is locally asymptotically stable.

The Jacobian matrix computed at $E_1(\frac{-\alpha+\sqrt{\Delta}}{2}, 0)$ is

$$J(E_1) = \begin{pmatrix} 1 - \frac{r_1(\Delta^2-\alpha)}{2K} + \frac{2qEm_2(\Delta^2-\alpha)}{4m_1^2E^2+m_2^2(\Delta^2-\alpha)^2+4m_1m_2E(\Delta^2-\alpha)} & -\frac{r_2(\Delta^2-\alpha)}{(\Delta^2-\alpha)+2c} \\ 0 & \exp[a + \frac{r_2d(\Delta^2-\alpha)}{(\Delta^2-\alpha)+2c} - h_2] \end{pmatrix}.$$

The eigenvalues of the Jacobian are $\lambda_1 = 1 - \frac{r_1(\Delta^2-\alpha)}{2K} + \frac{2qEm_2(\Delta^2-\alpha)}{4m_1^2E^2+m_2^2(\Delta^2-\alpha)^2+4m_1m_2E(\Delta^2-\alpha)}$ and $\lambda_2 = \exp[a + \frac{r_2d(\Delta^2-\alpha)}{(\Delta^2-\alpha)+2c} - h_2]$. The properties of semi-trivial equilibrium point $E_1(\frac{-\alpha+\sqrt{\Delta}}{2}, 0)$ are summarized in Table 3.

From Table 3, we can have the following theorem.

Theorem 2. When $\frac{r_1(\Delta^2-\alpha)}{2K} - \frac{2qEm_2(\Delta^2-\alpha)}{4m_1^2E^2+m_2^2(\Delta^2-\alpha)^2+4m_1m_2E(\Delta^2-\alpha)} > 2$ and $a + \frac{r_2d(\Delta^2-\alpha)}{(\Delta^2-\alpha)+2c} > h_2$ are satisfied, the semi-trivial equilibrium point $E_1(\frac{-\alpha+\sqrt{\Delta}}{2}, 0)$ is locally unstable.

Jacobian matrix can be evaluated at $E_2(0, \frac{a-h_2}{b})$ as

$$J(E_2) = \begin{pmatrix} \exp[r_1 - \frac{r_2(a-h_2)}{bc} - \frac{q}{m_1}] & 0 \\ \frac{r_2d(a-h_2)}{bc} & 1-(a-h_2) \end{pmatrix}. \tag{2.2}$$

The eigenvalues of the Jacobian are $\lambda_1 = \exp[r_1 - \frac{r_2(a-h_2)}{bc} - \frac{q}{m_1}]$ and $\lambda_2 = 1-(a-h_2)$ at semi-trivial equilibrium point $E_2(0, \frac{a-h_2}{b})$. The results of dynamical behaviors are listed in Table 4.

Table 3. Properties of semi-trivial equilibrium point $E_1(\frac{-\alpha+\sqrt{\Delta}}{2}, 0)$.

Conditions		Eigenvalues λ_1	Eigenvalues λ_2	Properties
$0 < \frac{r_1(\Delta^2-\alpha)}{2K} - \frac{2qEm_2(\Delta^2-\alpha)}{4m_1^2E^2+m_2^2(\Delta^2-\alpha)^2+4m_1m_2E(\Delta^2-\alpha)} < 2$	$a + \frac{r_2d(\Delta^2-\alpha)}{(\Delta^2-\alpha)+2c} = h_2$	$\lvert\lambda_1\rvert < 1$	$\lvert\lambda_2\rvert = 1$	Non-hyperbolic
	$a + \frac{r_2d(\Delta^2-\alpha)}{(\Delta^2-\alpha)+2c} < h_2$		$\lvert\lambda_2\rvert < 1$	Sink
	$a + \frac{r_2d(\Delta^2-\alpha)}{(\Delta^2-\alpha)+2c} > h_2$		$\lvert\lambda_2\rvert > 1$	Saddle
$\frac{r_1(\Delta^2-\alpha)}{2K} - \frac{2qEm_2(\Delta^2-\alpha)}{4m_1^2E^2+m_2^2(\Delta^2-\alpha)^2+4m_1m_2E(\Delta^2-\alpha)} > 2$	$a + \frac{r_2d(\Delta^2-\alpha)}{(\Delta^2-\alpha)+2c} = h_2$	$\lvert\lambda_1\rvert > 1$	$\lvert\lambda_2\rvert = 1$	Non-hyperbolic
	$a + \frac{r_2d(\Delta^2-\alpha)}{(\Delta^2-\alpha)+2c} < h_2$		$\lvert\lambda_2\rvert < 1$	Saddle
	$a + \frac{r_2d(\Delta^2-\alpha)}{(\Delta^2-\alpha)+2c} > h_2$		$\lvert\lambda_2\rvert > 1$	Source
$\frac{r_1(\Delta^2-\alpha)}{2K} - \frac{2qEm_2(\Delta^2-\alpha)}{4m_1^2E^2+m_2^2(\Delta^2-\alpha)^2+4m_1m_2E(\Delta^2-\alpha)} = 2$	$a + \frac{r_2d(\Delta^2-\alpha)}{(\Delta^2-\alpha)+2c} = h_2$	$\lvert\lambda_1\rvert = 1$	$\lvert\lambda_2\rvert = 1$	Non-hyperbolic
	$a + \frac{r_2d(\Delta^2-\alpha)}{(\Delta^2-\alpha)+2c} < h_2$		$\lvert\lambda_2\rvert < 1$	Non-hyperbolic
	$a + \frac{r_2d(\Delta^2-\alpha)}{(\Delta^2-\alpha)+2c} > h_2$		$\lvert\lambda_2\rvert > 1$	Non-hyperbolic

Table 4. Properties of semi-trivial equilibrium point $E_2(0, \frac{a-h_2}{b})$.

Conditions		Eigenvalues $\lambda_1 = \exp[r_1 - \frac{r_2(a-h_2)}{bc} - \frac{q}{m_1}]$	$\lambda_2 = 1-(a-h_2)$	Properties
$r_1 > \frac{r_2(a-h_2)}{bc} + \frac{q}{m_1}$	$0 < a-h_2 < 2$	$\lvert\lambda_1\rvert > 1$	$\lvert\lambda_2\rvert < 1$	Saddle
	$a-h_2 > 2$		$\lvert\lambda_2\rvert > 1$	Source
	$a-h_2 = 2$		$\lvert\lambda_2\rvert = 1$	Non-hyperbolic
$r_1 < \frac{r_2(a-h_2)}{bc} + \frac{q}{m_1}$	$0 < a-h_2 < 2$	$\lvert\lambda_1\rvert < 1$	$\lvert\lambda_2\rvert < 1$	Sink
	$a-h_2 > 2$		$\lvert\lambda_2\rvert > 1$	Saddle
	$a-h_2 = 2$		$\lvert\lambda_2\rvert = 1$	Non-hyperbolic
$r_1 = \frac{r_2(a-h_2)}{bc} + \frac{q}{m_1}$	$0 < a-h_2 < 2$	$\lvert\lambda_1\rvert = 1$	$\lvert\lambda_2\rvert < 1$	Non-hyperbolic
	$a-h_2 > 2$		$\lvert\lambda_2\rvert > 1$	Non-hyperbolic
	$a-h_2 = 2$		$\lvert\lambda_2\rvert = 1$	Non-hyperbolic

From Table 4, we can get the following theorem.

Theorem 3 The semi-trivial equilibrium point $E_2(0, \frac{a-h_2}{b})$ is always locally asymptotically stable when $r_1 < \frac{r_2(a-h_2)}{bc} + \frac{q}{m_1}$ and $0 < a-h_2 < 2$ are satisfied.

$J|_{(u,v)}$ evaluated at the positive equilibrium point $E^*(u^*, v^*)$ is

$$J(E^*) = \begin{pmatrix} 1 - \frac{r_1u^*}{K} + \frac{r_2u^*v^*}{(u^*+c)^2} + \frac{qEm_2u^*}{(m_1E+m_2u^*)^2} & -\frac{r_2u^*}{c+u^*} \\ \frac{r_2dcv^*}{(c+u^*)^2} & 1-bv^* \end{pmatrix}. \tag{2.3}$$

Then characteristic equation of $J(E^*)$ is given by

$$\lambda^2 - \mathrm{Tr}A\lambda + \mathrm{Det}A = 0, \tag{2.4}$$

where

$$T = \mathrm{Tr}A = 2 - \frac{r_1 u^*}{K} + \frac{r_2 u^* v^*}{(u^*+c)^2} + \frac{qEm_2u^*}{(m_1E+m_2u^*)^2} - bv^*,$$

$$D = \mathrm{Det}A = \left[1 - \frac{r_1 u^*}{K} + \frac{r_2 u^* v^*}{(u^*+c)^2} + \frac{qEm_2u^*}{(m_1E+m_2u^*)^2}\right](1-bv^*) + \frac{r_2^2 dcu^*v^*}{(c+u^*)^3}.$$

Lemma 4 [18] Suppose that $F(\lambda) = \lambda^2 - T\lambda + D$, and $F(1) > 0$, λ_1 and λ_2 are roots of $F(\lambda) = 0$. Then the following results hold true:

(I) $|\lambda_1| < 1$ and $|\lambda_2| < 1$ if and only if $F(-1) > 0$ and $D < 1$;

(II) $|\lambda_1| < 1$ and $|\lambda_2| > 1$ (or $|\lambda_1| > 1$ and $|\lambda_2| < 1$) if and only if $F(-1) < 0$;

(III) $|\lambda_1| > 1$ and $|\lambda_2| > 1$ if and only if $F(-1) > 0$ and $D > 1$;

(IV) $\lambda_1 = -1$ and $|\lambda_2| \neq 1$ if and only if $F(-1) = 0$ and $D \neq 0, 2$;

(V) λ_1 and λ_2 are complex and $|\lambda_1| = |\lambda_2| = 1$ if and only if $T^2 - 4D < 0$ and $D = 1$.

Lemma 5 [11] Let $E^*(u^*, v^*)$ be the unique positive equilibrium point of system (1.1), then the following propositions hold:

(1.) It is a sink if and only if

$$|T| < D + 1 \quad \text{and} \quad D < 1.$$

(2.) It is a source if and only if

$$|T| < D + 1 \quad \text{and} \quad D > 1, \quad \text{or} \quad |T| > D + 1.$$

(3.) It is a saddle if and only if

$$T^2 > 4D \quad \text{and} \quad |T| > |D + 1|.$$

(4.) It is non-hyperbolic if and only if

$$|T| = |D + 1|, \quad \text{or} \quad D = 1 \quad \text{and} \quad |T| \leq 2.$$

To sum up, we have the following theorem.

Theorem 4 System (1.1) at the the positive equilibrium point $E^*(u^*, v^*)$ is local asymptotically stable when the conditions

$$4 - \frac{r_1 u^*(2+bv^*)}{K} + \frac{r_2(2-bv^*)u^*v^*}{(c+u^*)^2} + \frac{qEm_2u^*(2-bv^*)}{(m_1E+m_2u^*)^2} - 2bv^* + \frac{r_2^2 dcu^*v^*}{(u^*+c)^3} > 0$$

and

$$-\frac{r_1 u^*(1+bv^*)}{K} + \frac{r_2(1-bv^*)u^*v^*}{(c+u^*)^2} + \frac{qEm_2u^*(1-bv^*)}{(m_1E+m_2u^*)^2} - bv^* + \frac{r_2^2 dcu^*v^*}{(u^*+c)^3} < 0$$

hold.

Proof. According to Lemma 3 and Lemma 4, $E^*(u^*, v^*)$ is local asymptotically stable if and only if $F(1) > 0$, $F(-1) > 0$ and $D < 0$, the conclusion of Theorem 4 obtained by calculation holds.

Lemma 6 [19] Assume that u_t satisfies $u_0 > 0$, and $u_{t+1} \leq u_t \exp[B(1 - Cu_t)]$ for $t \in [t_1, \infty)$, where C is a positive constant. Then $\limsup\limits_{t\to\infty} u_t \leq \frac{1}{BC}\exp(B-1)$.

Theorem 5 Every positive solution $\{(u_n, v_n)\}$ of system (1.1) is uniformly bounded.

Proof. Suppose that $\{(u_n, v_n)\}$ be an arbitrary positive solution corresponding to system (1.1). Then, from first part of system (1.1), one has

$$\begin{aligned} u_{n+1} &= u_n \exp[r_1(1-\frac{u_n}{K}) - \frac{r_2 v_n}{c+u_n} - \frac{qE}{m_1E+m_2u_n}] \\ &\leq u_n \exp[r_1(1-\frac{u_n}{K}) - \frac{r_2 v_n}{c+u_n}] \\ &\leq u_n \exp[r_1(1-\frac{u_n}{K})], \end{aligned}$$

for all $n = 0, 1, 2, \cdots$. Suppose that $u_0 > 0$, then according to Lemma 6, we gain

$$\limsup_{n\to\infty} \ u_n \leq \frac{K}{r_1}\exp(r_1 - 1) := M_1.$$

From the second part of system (1.1), we have

$$\begin{aligned} v_{n+1} &= v_n \exp[a + \frac{r_2 d u_n}{c+u_n} - b v_n - h_2] \\ &\leq v_n \exp[a + \frac{r_2 d u_n}{c+u_n} - b v_n] \\ &\leq v_n \exp[a + \frac{r_2 d M_1}{c+M_1} - b v_n]. \end{aligned}$$

Assume that $v_0 > 0$, then using Lemma 6, we obtain

$$\limsup_{n\to\infty} \ v_n \leq b\exp(\frac{a(M_1+c) + r_2 d M_1}{M_1 + c} - 1) := M_2.$$

That is to say that $\limsup_{n\to\infty} (u_n, v_n) \leq M$, where $M = \max\{M_1, M_2\}$. This completes the proof.

3. Bifurcation analysis

3.1. Flip bifurcation

The characteristic equation related to system (1.1) at the unique positive interior equilibrium point $E^*(u^*, v^*)$ is

$$F(\lambda) = \lambda^2 - T(u^*, v^*)\lambda + D(u^*, v^*) = 0,$$

where

$$\begin{aligned} &T(u^*, v^*) = 1 - \tfrac{r_1 u^*}{K} + \Phi + \Psi, \\ &D(u^*, v^*) = \Psi\left[1 - \tfrac{r_1 u^*}{K} + \Phi\right] + \Theta, \\ &\Psi := 1 - bv^*, \quad \Theta := \tfrac{r_2 d c u^* v^*}{(u^*+c)^3}, \\ &\Phi := \tfrac{r_2 u^* v^*}{(u^*+c)^2} + \tfrac{qEm_2u^*}{(m_2u^*+m_1E)^2}. \end{aligned}$$

Assume that $T^2(u^*, v^*) > 4D(u^*, v^*)$, that is,

$$\left(1 - \tfrac{r_1 u^*}{K} + \Phi + \Psi\right)^2 > 4\Psi\left[1 - \tfrac{r_1 u^*}{K} + \Theta\right] + 4\Theta \tag{3.1}$$

and $T(u^*, v^*) + D(u^*, v^*) = -1$, that is to say

$$r_1 = \frac{K}{u^*(1+\Psi)}\left(2 + 2\Psi + \Phi(1+\Psi) + \Theta\right). \tag{3.2}$$

Then eigenvalue of $F(\lambda) = 0$ are $\lambda_1 = -1$ and $\lambda_2 = 2 + \Phi + \Psi - \frac{r_1 u^*}{K}$. The condition $|\lambda_2| \neq 1$ indicates that

$$\Psi\left[1 - \frac{r_1 u^*}{K} + \Phi\right] + \Theta \neq \pm 1. \tag{3.3}$$

Consider the following set

$$A_F = \left\{(a, b, c, d, K, r_1, r_2, m_1, m_2, h_2, q, E) \in \mathbb{R}_+^{12} : (3.1),\ (3.2) \text{ and } (3.3) \text{ are satisfied}\right\}.$$

When the perturbation parameter changes within a small field of A_F, the system (1.1) will have flip bifurcation at E^*. Let parameters $(a, b, c, d, K, r_1, r_2, m_1, m_2, h_2, q, E) \in A_F$ and consider the following systems:

$$\begin{cases} u_{n+1} = u_n \exp[(r_1 + r^*)(1 - \frac{u_n}{K}) - \frac{r_2 v_n}{c+u_n} - \frac{qE}{m_1 E + m_2 u_n}], \\ v_{n+1} = v_n \exp[a + \frac{r_2 d u_n}{c+u_n} - b v_n - h_2], \end{cases}$$

where r^* is a small perturbation parameter and $|r^*| \ll 1$.

Let $x = u - u^*$ and $y = v - v^*$. Then we gain

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} f(x, y, r^*) \\ g(x, y, r^*) \end{pmatrix}, \tag{3.4}$$

where

$$\begin{aligned} f(x, y, r^*) &= W_{13}x^2 + W_{14}xy + W_{15}y^2 + W_{16}x^3 + W_{17}x^2y + W_{18}xy^2 + W_{19}y^3 + Z_1 x r^* + Z_2 y r^* \\ &\quad + Z_3 r^{*2} + Z_4 xy r^* + Z_5 x^2 r^* + Z_6 y^2 r^* + Z_7 x r^{*2} + Z_8 y r^{*2} + Z_9 r^{*3} + O((|x|, |y|, |r^*|)^4), \\ g(x, y, r^*) &= W_{23}x^2 + W_{24}xy + W_{25}y^2 + W_{26}x^3 + W_{27}x^2y + W_{28}xy^2 + W_{29}y^3 + O((|x|, |y|, |r^*|)^4), \\ W_{11} &= 1 - \frac{r_1 u^*}{K} + \frac{r_2 u^* v^*}{(u^* + c)^2} + \frac{qEm_2}{(m_1 E + m_2 u^*)^2} = 1 + \Omega, \quad W_{12} = -\frac{r_2 u^*}{(c + u^*)}, \end{aligned}$$

$$W_{13} = \frac{\Omega}{2} - \frac{r_1}{2K} - \frac{r_1 u^* \Omega}{2K} + \frac{r_2 v^*(c - u^*)}{2(u^* + c)^3} + \frac{r_2 u^* v^* \Omega}{2(u^* + c)^2} + \frac{qEm_2(m_1 E - m_2 u^*)}{2(m_1 E + m_2 u^*)^3} + \frac{qEm_2 u^* \Omega}{2(m_1 E + m_2 u^*)^2},$$

$$W_{14} = -\frac{r_2}{c + u^*} + \frac{r_1 r_2 u^*}{(u^* + c)K} - \frac{r_2 qEm_2}{(c + u^*)(m_1 E + m_2 u^*)^2}, \quad W_{15} = -\frac{r_1^2 u^*}{2(c + u^*)^2},$$

$$W_{16} = \frac{\Omega_u + \Omega^2}{6} - \frac{r_1 \Omega}{3K} - \frac{r_1 u^* \Omega \Omega_u}{6K} - \frac{r_2 c v^*}{2(u^* + c)} + \frac{r_2 c \Omega v^*}{6(c + u^*)^3} - \frac{c - 2u^*}{6(c + u^*)^4} - \frac{qE^2 m_1 m_2}{2(m_1 E + m_2 u^*)^4} - \frac{u^* \Omega}{6(c + u^*)^3} + \frac{r_2 v(c - u^*)\Omega}{6(u^* + c)^3} + \frac{r_2 u^* v^* \Omega \Omega_u}{6(c + u^*)^2} + \frac{qEm_2(m_1 E - m_2 u^*)\Omega}{6(m_1 E + m_2 u^*)^3} + \frac{qEm_2 u^* \Omega \Omega_u}{6(m_1 E + m_2 u^*)^2} + \frac{qE^2 m_1 m_2 \Omega}{6(m_1 E + m_2 u^*)^3} - \frac{qEm_2^2 u^* \Omega}{6(m_1 E + m_2 u^*)^3} - \frac{qEm_2^2(m_1 E - m_2 u^*)}{6(m_1 E + m_2 u^*)^4},$$

$$W_{17} = \frac{r_1 r_2(1 + u^*)}{2K(c + u^*)} + \frac{r_2 u^*}{2(c + u^*)^2} + \frac{r_2(c - u^*) - r_2^2 u^* v^*}{2(c + u^*)^3} - \frac{r_1^2 r_2 u^2}{2K^2(c + u^*)} - \frac{r_1 r_2 u^{*2}}{K(c + u^*)^2} + \frac{r_1 r_2^2 u^{*2} v^*}{(u^* + c)^3} + \frac{2r_2^2 u^{*2} v^* - r_2^2 v^*(c - u^*)}{2(c + u^*)^4} - \frac{r_2^3 u^{*2} v^{*2}}{2(c + u^*)^5} - \frac{r_2 qEm_2}{2(u^* + c)(m_1 E + m_2 u^*)^2} + \frac{r_2 qEm_1 u^*(1 + u^*)}{2(c + u^*)^2(m_2 u^* + m_1 E)^2} - \frac{r_2^2 qEm_2 u^* v^*(1 + u^*)}{2(c + u^*)^3(m_2 u^* + m_1 E)^2} - \frac{r_2 qEm_2(m_1 E - m_2 u^*)}{2(c + u^*)(m_2 u^* + m_1 E)^3} + \frac{r_1 r_2 qEm_2(1 + u^*)u^*}{2K(c + u^*)(m_1 Em_2 u^*)^2} - \frac{r_2 q^2 m_2^2 E^2 u^*}{2(c + u^*)(m_2 u^* + m_1 E)^4},$$

$$W_{18} = \frac{r_2^2}{2(c + u^*)^2} - \frac{r_2^2 u^*}{(c + u^*)^3} + \frac{r_2^3 u^* v^*}{2(u^* + c)^4} - \frac{r_1 r_2^2 u^*}{2K(c + u^*)^2} + \frac{r_2^2 qEm_2}{2(u^* + c)^2(m_1 E + m_2 u^*)^2},$$

$$W_{19} = -\frac{r_2^3 u^*}{6(c + u^*)^3}, \quad Z_1 = 1 - \frac{2u^*}{K} - \frac{r_1(K - u^*)u^*}{K^2} + \frac{r_2 u^* v^*(K - u^*)}{K(u^* + c)^2} + \frac{qEm_2(K - u^*)}{K(m_1 E + m_2 u^*)^2},$$

$$Z_2 = -\frac{r_2(K - u^*)u^*}{K(c + u^*)}, \quad Z_3 = \frac{u^*(K - u^*)^2}{2K^2}, \quad Z_6 = -\frac{r_2^2 u^*(K - u^*)}{2K(u^* + c)^2},$$

$$Z_4 = \frac{r_2(2u^* - K)}{K(c + u^*)} + \frac{r_1 r_2 u^*(K - u^*)}{K^2(u^* + c)} - \frac{r_2 qEm_2(K - u^*)}{K(c + u^*)(m_1 E + m_2 u^*)^2},$$

$$Z_5 = -\frac{u^* + 1}{2K} + \frac{r_1^2(K - u^*)u^{*2}}{2K^3} + \frac{r_1 u^*(3u^* + 1) - r_1 K(1 + u^*)}{2K^2} + \frac{r_2 v^*(c - u^*)(K - u^*)}{2K(c + u^*)^3} + \frac{qEm_2(u^* - K)(m_1 E - m_2 u^*)}{2K(m_1 E + m_2 u^*)^3} + \frac{r_2(K - 3u^*)u^* v^*}{2K(c + u^*)^2} + \frac{r_2 qEm_2 u^* v^*(K - u^*)(1 + u^*)}{2K(c + u^*)^2(m_1 E + m_2 u^*)^2} - \frac{r_1 r_2(K - u^*)u^{*2} v^*}{K^2(c + u^*)^2} + \frac{r_2^2(K - u^*)u^{*2} v^{*2}}{2K(c + u^*)^4} - \frac{r_1 qEm_2 u^*(K - u^*)(1 + u^*)}{2K^2(m_1 E + m_2 u^*)^2} + \frac{q^2 E^2 m_2^2(K - u^*)}{2K(m_1 E + m_2 u^*)^4} + \frac{qEm_2(K - 2u^* - u^{*2})}{2K(m_1 E + m_2 u^*)^2},$$

$$Z_7 = \frac{1}{2} - \frac{u^*}{2K} - \frac{3(K - u^*)u^*}{2K^2} - \frac{r_1 u^*(K - u^*)^2}{2K^3} + \frac{r_2 u^* v^*(K - u^*)^2}{2K^2(u^* + c)^2} + \frac{qEm_2(K - u^*)^2}{2K^2(m_1 E + m_2 u^*)^2},$$

$$Z_8 = -\frac{r_2 u^*(K - u^*)^2}{2K^2(c + u^*)}, \quad Z_9 = \frac{u^*(c + u^*)^3}{6K^3}, \quad W_{21} = \frac{r_2 dcv^*}{(c + u^*)^2}, \quad W_{22} = 1 - bv^*,$$

$$W_{23}=-\frac{r_2dcv^*}{(c+u^*)^3}+\frac{r_2^2d^2c^2}{2(u^*+c)^4},\quad W_{24}=\frac{r_2dc(1-bv^*)}{(u^*+c)^2},\quad W_{25}=\frac{b(bv^*-2)}{2},$$
$$W_{26}=\frac{r_2dcv^*}{(c+u^*)^4}-\frac{r_2^2d^2c^2v^{*2}}{(c+u^*)^5}+\frac{r_2^3d^3c^3v^{*3}}{6(c+u^*)^6},\quad W_{27}=\frac{r_2dc(bv^*-1)}{(c+u^*)^3}+\frac{r_2^2d^2c^2v^*(2-v^*)}{2(c+u^*)^4},$$
$$W_{28}=-\frac{r_2bdc}{(c+u^*)^2}+\frac{r_2b^2dcv^*}{2(c+u^*)^2},\quad W_{29}=\frac{b^2(3-bv^*)}{6}.$$

Construct a nonsingular matrix D_1 and consider the following translation:

$$\begin{pmatrix} x \\ y \end{pmatrix}=D_1\begin{pmatrix} u \\ v \end{pmatrix}, \tag{3.5}$$

where

$$D_1=\begin{pmatrix} W_{12} & W_{12} \\ -1-W_{11} & \lambda_2-W_{11} \end{pmatrix}.$$

Taking D_1^{-1} on both sides of Eq (3.5), we obtain

$$\begin{pmatrix} u \\ v \end{pmatrix}=\begin{pmatrix} -1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}+\begin{pmatrix} f_1(x,y,r^*) \\ g_1(x,y,r^*) \end{pmatrix}, \tag{3.6}$$

where

$$\begin{aligned}
f_1(x,y,r^*) = {} & \frac{[W_{13}(\lambda_2-W_{11})-W_{12}W_{23}]x^2}{W_{12}(\lambda_2+1)}+\frac{[W_{14}(\lambda_2-W_{11})-W_{12}W_{24}]xy}{W_{12}(\lambda_2+1)}+\frac{Z_1(\lambda_2-W_{11})xr^*}{W_{12}(\lambda_2+1)}\\
&+\frac{[W_{15}(\lambda_2-W_{11})-W_{12}W_{25}]y^2}{W_{12}(\lambda_2+1)}+\frac{[W_{16}(\lambda_2-W_{11})-W_{12}W_{26}]x^3}{W_{12}(\lambda_2+1)}+\frac{Z_2(\lambda_2-W_{11})yr^*}{W_{12}(\lambda_2+1)}\\
&+\frac{[W_{17}(\lambda_2-W_{11})-W_{12}W_{27}]x^2y}{W_{12}(\lambda_2+1)}+\frac{[W_{18}(\lambda_2-W_{11})-W_{12}W_{28}]xy^2}{W_{12}(\lambda_2+1)}+\frac{Z_3(\lambda_2-W_{11})r^{*2}}{W_{12}(\lambda_2+1)}\\
&+\frac{[W_{19}(\lambda_2-W_{11})-W_{12}W_{29}]y^3}{W_{12}(\lambda_2+1)}+\frac{Z_4(\lambda_2-W_{11})xyr^*}{W_{12}(\lambda_2+1)}+\frac{Z_5(\lambda_2-W_{11})x^2r^*}{W_{12}(\lambda_2+1)}\\
&+\frac{Z_6(\lambda_2-W_{11})y^2r^*}{W_{12}(\lambda_2+1)}+\frac{Z_7(\lambda_2-W_{11})xr^{*2}}{W_{12}(\lambda_2+1)}+\frac{Z_8(\lambda_2-W_{11})yr^{*2}}{W_{12}(\lambda_2+1)}+\frac{Z_9(\lambda_2-W_{11})r^{*3}}{W_{12}(\lambda_2+1)}\\
&+O((|x|,|y|,|r^*|)^4),\\
g_1(x,y,r^*) = {} & \frac{[W_{13}(\lambda_2+W_{11})+W_{12}W_{23}]x^2}{W_{12}(\lambda_2+1)}+\frac{[W_{14}(\lambda_2+W_{11})+W_{12}W_{24}]xy}{W_{12}(\lambda_2+1)}+\frac{Z_1(\lambda_2+W_{11})xr^*}{W_{12}(\lambda_2+1)}\\
&+\frac{[W_{15}(\lambda_2+W_{11})+W_{12}W_{25}]y^2}{W_{12}(\lambda_2+1)}+\frac{[W_{16}(\lambda_2+W_{11})+W_{12}W_{26}]x^3}{W_{12}(\lambda_2+1)}+\frac{Z_2(\lambda_2+W_{11})yr^*}{W_{12}(\lambda_2+1)}\\
&+\frac{[W_{17}(\lambda_2+W_{11})+W_{12}W_{27}]x^2y}{W_{12}(\lambda_2+1)}+\frac{[W_{18}(\lambda_2+W_{11})+W_{12}W_{28}]xy^2}{W_{12}(\lambda_2+1)}+\frac{Z_3(\lambda_2+W_{11})r^{*2}}{W_{12}(\lambda_2+1)}\\
&+\frac{[W_{19}(\lambda_2+W_{11})+W_{12}W_{29}]y^3}{W_{12}(\lambda_2+1)}+\frac{Z_4(\lambda_2+W_{11})xyr^*}{W_{12}(\lambda_2+1)}+\frac{Z_5(\lambda_2+W_{11})x^2r^*}{W_{12}(\lambda_2+1)}\\
&+\frac{Z_6(\lambda_2+W_{11})y^2r^*}{W_{12}(\lambda_2+1)}+\frac{Z_7(\lambda_2+W_{11})xr^{*2}}{W_{12}(\lambda_2+1)}+\frac{Z_8(\lambda_2+W_{11})yr^{*2}}{W_{12}(\lambda_2+1)}+\frac{Z_9(\lambda_2+W_{11})r^{*3}}{W_{12}(\lambda_2+1)}\\
&+O((|x|,|y|,|r^*|)^4),\\
x = {} & W_{12}(u+v),\quad y=(\lambda_2+W_{11})v-(1+W_{11})u.
\end{aligned}$$

The center manifold theorem W^c is applied in a small field of $r^* = 0$ at the equilibrium point E_0. Then there exists $W^c(0)$ as follows:

$$W^c(0) = \left\{(x, y, r^*) \in \mathbb{R}^3 : y(x, r^*) = e_0 r^* + e_1 x^2 + e_2 x r^* + e_3 r^{*2} + O((|x| + |r^*|)^3)\right\}$$

and satisfies

$$H(y(x, r^*)) = y\left(-u + f_1(x, y(x, r^*), r^*)\right) - \lambda_2 y(x, r^*) - g_1(x, y(x, r^*), r^*) = 0,$$

and we have

$$\begin{aligned}
e_0 &= 0,\\
e_1 &= \frac{[W_{13}(1 + W_{11}) + W_{12}W_{23}]W_{12} - [W_{14}(1 + W_{11}) + W_{12}W_{24}](1 + W_{11})}{1 - \lambda_2^2}\\
&\quad + \frac{[W_{15}(1 + W_{11}) + W_{12}W_{25}](1 + W_{11})^2}{(1 - \lambda_2^2)W_{12}},\\
e_2 &= \frac{[W_{12}Z_1 - Z_2(1 + W_{11})](1 + W_{11})}{(1 - \lambda_2)^2},\\
e_3 &= \frac{Z_3(1 + W_{11})}{W_{12}(1 - \lambda_2)^2}.
\end{aligned}$$

Therefore, consider the following map on the center manifold $W^c(0)$:

$$G : x \to -x + s_1 x^2 + s_2 x r^* + s_3 x^2 r^* + n_4 x r^{*2} + s_5 x^3 + O((|x| + |r^*|)^4),$$

where

$$\begin{aligned}
s_1 =& \frac{[W_{13}(\lambda_2 - W_{11}) - W_{12}W_{23}]W_{12}}{1 + \lambda_2} - \frac{[W_{14}(\lambda_2 - W_{11}) - W_{12}W_{24}](1 + W_{11})}{1 + \lambda_2}\\
&+ \frac{[W_{15}(\lambda_2 - W_{11}) - W_{12}W_{25}](1 + W_{11})^2}{W_{12}(1 + \lambda_2)},\\
s_2 =& \frac{Z_1(\lambda_2 - W_{11})}{1 + \lambda_2} - \frac{Z_2(\lambda_2 - W_{11})(1 + W_{11})}{W_{12}(1 + \lambda_2)},\\
s_3 =& \frac{[W_{13}(\lambda_2 - W_{11}) - W_{12}W_{23}]2e_2W_{12}}{1 + \lambda_2} + \frac{[W_{14}(\lambda_2 - W_{11}) - W_{12}W_{24}](\lambda_2 - W_{11})e_2}{1 + \lambda_2}\\
&- \frac{2[W_{15}(\lambda_2 - W_{11}) - W_{12}W_{25}](1 + W_{11})(\lambda_2 - W_{11})e_2}{W_{12}(1 + \lambda_2)} + \frac{Z_1(\lambda_2 - W_{11})e_1}{1 + \lambda_2}\\
&+ \frac{(\lambda_2 - W_{11})^2 e_1}{W_{12}(1 + \lambda_2)} + \frac{W_{12}(\lambda_2 - W_{11})Z_5}{1 + \lambda_2} + \frac{Z_6(\lambda_2 - W_{11})(1 + W_{11})^2}{(1 + \lambda_2)W_{12}}\\
&- \frac{[W_{14}(\lambda_2 - W_{11}) - W_{12}W_{24}]e_2(1 + W_{11}) + Z_4(\lambda_2 - W_{11})(1 + W_{11})}{1 + \lambda_2},\\
s_4 =& \frac{[W_{13}(\lambda_2 - W_{11}) - W_{12}W_{23}]2e_2W_{12}}{1 + \lambda_2} + \frac{[W_{14}(\lambda_2 - W_{11}) - W_{12}W_{24}](\lambda_2 - W_{11})e_3}{\lambda_2 + 1}\\
&- \frac{[W_{14}(\lambda_2 - W_{11}) - W_{12}W_{24}](1 + W_{11})e_3}{\lambda_2 + 1} + \frac{(\lambda_2 - W_{11})(Z_1e_2 + Z_7)}{\lambda_2 + 1}\\
&- \frac{[W_{15}(\lambda_2 - W_{11}) - W_{12}W_{25}](1 + W_{11})(\lambda_2 - W_{11})e_3}{\lambda_2 + 1} + \frac{(\lambda_2 - W_{11})^2 Z_2 e_2}{W_{12}(\lambda_2 + 1)}\\
&- \frac{Z_8(\lambda_2 - W_{11})(W_{11} + 1)}{W_{12}(\lambda_2 + 1)},
\end{aligned}$$

$$s_5 = \frac{[W_{16}(\lambda_2 - W_{11}) - W_{12}W_{26}]W_{12}^2}{1+\lambda_2} - \frac{[W_{17}(\lambda_2 - W_{11}) - W_{12}W_{27}](1+W_{11})W_{12}}{\lambda_2+1}$$
$$- \frac{2[W_{15}(\lambda_2 - W_{11}) - W_{12}W_{25}](1+W_{11})(\lambda_2 - W_{11})e_1}{(1+\lambda_2)W_{12}} + \frac{[W_{13}(\lambda_2 - W_{11}) - W_{12}W_{23}]2e_2W_{12}}{1+\lambda_2}$$
$$- \frac{[W_{14}(\lambda_2 - W_{11}) - W_{12}W_{24}](1+W_{11})e_1}{\lambda_2+1} + \frac{[W_{14}(\lambda_2 - W_{11}) - W_{12}W_{24}](\lambda_2 - W_{11})e_1}{\lambda_2+1}$$
$$+ \frac{[W_{18}(\lambda_2 - W_{11}) - W_{12}W_{28}](1+W_{11})^2}{\lambda_2+1} - \frac{[W_{19}(\lambda_2 - W_{11}) - W_{12}W_{29}](1+W_{11})^3}{W_{12}(\lambda_2+1)}.$$

By flip bifurcation, we define two non-zero real numbers δ_1 and δ_2, where

$$\delta_1 = \left(\frac{\partial^2 G}{\partial x \partial r^*} + \frac{1}{2}\frac{\partial G}{\partial r^*}\frac{\partial^2 G}{\partial x^2}\right)\Bigg|_{(0,0)} = s_2, \quad \delta_2 = \left(\frac{1}{6}\frac{\partial^3 G}{\partial x^3} + \left(\frac{1}{2}\frac{\partial^2 G}{\partial x^2}\right)^2\right)\Bigg|_{(0,0)} = s_1^2 + s_5.$$

Based on the above analysis, the following theorem can be obtained.

Theorem 6 System (1.1) undergoes a flip bifurcation at the positive internal equilibrium point $E^*(u^*, v^*)$ if $\delta_1 \neq 0$, $\delta_2 \neq 0$ are satisfied and when parameter r^* changes within a small field of r_1. Moreover, if $\delta_2 > 0$ (resp., $\delta_2 < 0$), then the period-two orbits that bifurcate from equilibrium point $E^*(u^*, v^*)$ are stable (resp., unstable).

3.2. Neimark-Sacker bifurcation

Consider the characteristic equation at E^*, then $\mathbb{F}(\lambda) = 0$ has two complex conjugate roots with modulus one if the following conditions are satisfied:

$$(1 - \frac{r_1 u^*}{K} + \Phi)\Psi + \Theta = 1 \tag{3.7}$$

and

$$|T| = |1 - \frac{r_1 u^*}{K} + \Phi + \Psi| = |1 + D| < 2. \tag{3.8}$$

Let

$$A_{NS} = \{(a, b, c, d, K, r_1, r_2, m_1, m_2, h_2, q, E) \in \mathbb{R}_+^{12} : (3.7) \text{ and } (3.8) \text{ are satisfied}\}.$$

When the parameter changes in a small field of A_{NS}, system (1.1) will have Neimark-Sacker bifurcation at the unique positive equilibrium point $E^*(u^*, v^*)$. Select parameter $(a, b, c, d, K, r_1, r_2, m_1, m_2, h_2, q, E) \in A_{NS}$ and analyze the following system:

$$\begin{cases} u_{n+1} = u_n \exp[(r_1 + \bar{r})(1 - \frac{u_n}{K}) - \frac{r_2 v_n}{c+u_n} - \frac{qE}{m_1 E + m_2 u_n}], \\ v_{n+1} = v_n \exp[a + \frac{r_2 d u_n}{c+u_n} - b v_n - h_2], \end{cases}$$

where $\bar{r}$ is a small perturbation parameter and $|\bar{r}| \ll 1$.

Let $x = u - u^*$ and $y = v - v^*$. Then we gain

$$\begin{cases} x_{n+1} = W_{11}x + W_{12}y + W_{13}x^2 + W_{14}xy + W_{15}y^2 + W_{16}x^3 + W_{17}x^2y + W_{18}xy^2 + W_{19}y^3 \\ \quad +O((|x| + |y|)^4), \\ y_{n+1} = W_{21}x + W_{22}y + W_{23}x^2 + W_{24}xy + W_{25}y^2 + W_{26}x^3 + W_{27}x^2y + W_{28}xy^2 + W_{29}y^3 \\ \quad +O((|x| + |y|)^4), \end{cases} \tag{3.9}$$

where $W_{ij}(i=1,2,\ 1\le j\le 9)$ are given in (3.6) by substituting r_1 for $r_1+\bar{r}$.

The characteristic equation of system (3.9) at $(x,y)=(0,0)$ is as follows:

$$\lambda^2 - T(\bar{r})\lambda + D(\bar{r}) = 0,$$

where

$$T(\bar{r}) = 1 - \frac{(r_1+\bar{r})u^*}{K} + \Phi + \Psi,$$
$$D(\bar{r}) = (1 - \frac{(r_1+\bar{r})u^*}{K} + \Phi)\Psi + \Theta.$$

Since parameters $(a,b,c,d,K,r_1,r_2,m_1,m_2,h_2,q,E)\in A_{NS}$, the roots of the characteristic equation are

$$\lambda_{1,2} = \frac{T(\bar{r})}{2} \pm \frac{i}{2}\sqrt{4D(\bar{r}) - D^2(\bar{r})}$$

and we have

$$|\lambda_{1,2}| = \sqrt{D(\bar{r})}.$$

Suppose that

$$L = \frac{d|\lambda_{1,2}|}{d\bar{r}}\Big|_{\bar{r}=0} = -\frac{\Psi u^*}{2K\sqrt{D(0)}} \neq 0.$$

In addition, it is required that $\bar{r}=0,\ \lambda_{1,2}^l \neq 1\ (l=1,2,3,4)$ which is equal to $T(0)\neq$ -2, 0, -1, 2. Because $(a,b,c,d,K,r_1,r_2,m_1,m_2,h,q,E)\in A_{NS}$, thus $T(0)\neq$ -2, 2. We only require $T(0)\neq 0,-1$, so that

$$1 - \frac{r_1u^*}{K} + \Phi + \Psi \neq 0 \quad \text{and} \quad 2 - \frac{r_1u^*}{K} + \Phi + \Psi \neq 0. \tag{3.10}$$

Let $\eta = \frac{T(0)}{2},\ \omega = \frac{\sqrt{4D(0)-T^2(0)}}{2}$, we use the following transformation:

$$\begin{bmatrix} x \\ y \end{bmatrix} = D_2\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} W_{12} & 0 \\ \eta - W_{11} & -\omega \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix},$$

and system (3.10) becomes into

$$\begin{bmatrix} u_{t+1} \\ v_{t+1} \end{bmatrix} = \begin{bmatrix} \eta & -\omega \\ \omega & \eta \end{bmatrix}\begin{bmatrix} u_t \\ v_t \end{bmatrix} + \begin{bmatrix} \bar{f}(u,v) \\ \bar{g}(u,v) \end{bmatrix},$$

where

$$\begin{aligned}
\bar{f}(u,v) &= \frac{W_{13}}{W_{12}}x^2 + \frac{W_{14}}{W_{12}}xy + \frac{W_{15}}{W_{12}}y^2 + \frac{W_{16}}{W_{12}}x^3 + \frac{W_{17}}{W_{12}}x^2y + \frac{W_{18}}{W_{12}}xy^2 + \frac{W_{19}}{W_{12}}y^3 + O((|x|+|v|)^4),\\
\bar{g}(u,v) &= [\frac{W_{13}(\eta-W_{11})}{\omega W_{12}} - \frac{W_{23}}{\omega}]x^2 + [\frac{W_{14}(\eta-W_{11})}{\omega W_{12}} - \frac{W_{24}}{\omega}]xy + [\frac{W_{15}(\eta-W_{11})}{\omega W_{12}} - \frac{W_{25}}{\omega}]y^2\\
&\quad + [\frac{W_{16}(\eta-W_{11})}{\omega W_{12}} - \frac{W_{26}}{\omega}]x^3 + [\frac{W_{17}(\eta-W_{11})}{\omega W_{12}} - \frac{W_{27}}{\omega}]x^2y + [\frac{W_{18}(\eta-W_{11})}{\omega W_{12}} - \frac{W_{28}}{\omega}]xy^2\\
&\quad + [\frac{W_{19}(\eta-W_{11})}{\omega W_{12}} - \frac{W_{29}}{\omega}]y^3 + O((|x|+|y|)^4),\\
x &= W_{12}u, \quad y = (\eta - W_{11})u - \omega v.
\end{aligned}$$

System (3.9) undergoes the Neimark-Sacker bifurcation if the following quantity Λ is not zero

$$\Lambda = -\text{Re}\Big[\frac{(1-2\lambda_1)\lambda_2^2}{1-\lambda_1}P_{11}P_{12}\Big] - \frac{1}{2}|P_{11}|^2 - |P_{21}|^2 + \text{Re}(\lambda_2 P_{22}), \tag{3.11}$$

where

$$\begin{aligned}
P_{11} &= \frac{1}{4}\Big[(\overline{f}_{uu} + \overline{f}_{vv}) + i(\overline{g}_{uu} + \overline{g}_{vv})\Big],\\
P_{12} &= \frac{1}{8}\Big[(\overline{f}_{uu} - \overline{f}_{vv} + 2\overline{g}_{uv}) + i(\overline{g}_{uu} - \overline{g}_{vv} - 2\overline{f}_{uv})\Big],\\
P_{21} &= \frac{1}{8}\Big[(\overline{f}_{uu} - \overline{f}_{vv} - 2\overline{g}_{uv}) + i(\overline{g}_{uu} - \overline{g}_{vv} + 2\overline{f}_{uv})\Big],\\
P_{22} &= \frac{1}{16}\Big[(\overline{f}_{uuu} + \overline{f}_{uvv} + \overline{g}_{uuv} + \overline{g}_{vvv}) + i(\overline{g}_{uuu} + \overline{g}_{uvv} - \overline{f}_{uuv} - \overline{f}_{vvv})\Big].
\end{aligned}$$

If $\Lambda \neq 0$, Neimark-Sacker bifurcation will occur in system (1.1), and the following theorem holds:

Theorem 7 System (1.1) undergoes a Neimark-Sacker bifurcation at the positive equilibrium point $E^*(u^*, v^*)$ if conditions (3.10) are satisfied and $\Lambda \neq 0$. Moreover, if $\Lambda < 0$(resp., $\Lambda > 0$), an attracting (resp., repelling) invariant closed curve bifurcates from the steady state for $r_1 > \overline{r}$ (resp., $r_1 < \overline{r}$).

4. Chaos control and optimal harvesting policy

4.1. Chaos control

In this section, we will adopt the feedback control method [23–25] to stabilize the chaotic orbit at an unstable equilibrium point by adding a feedback control term to the system (1.1). Therefore, system (1.1) makes the following form:

$$\begin{cases} u_{n+1} = u_n \exp[r_1(1-\frac{u_n}{K}) - \frac{r_2 v_n}{c+u_n} - \frac{qE}{m_1E+m_2u_n}] - h(u_n, v_n) = \widetilde{f}(u_n, v_n),\\ v_{n+1} = v_n \exp[a + \frac{r_2 d u_n}{c+u_n} - b v_n - h_2] = \widetilde{g}(u_n, v_n), \end{cases} \tag{4.1}$$

where $h(u_n, v_n) = q_1(u_n - u^*) + q_2(v_n - v^*)$ is feedback controlling force, q_1 and q_2 are feedback gains. Furthermore, $\widetilde{f}(u^*, v^*) = u^*$, and $\widetilde{g}(u^*, v^*) = v^*$.

The Jacobian matrix corresponding to system (4.1) at interior equilibrium point (u^*, v^*) is as follows:

$$J(u^*, v^*) = \begin{bmatrix} W_{11} - q_1 & W_{12} - q_2 \\ W_{21} & W_{22} \end{bmatrix}.$$

Thus, the characteristic equation related to $A(u^*, v^*)$ is:

$$\lambda^2 - (W_{11} + W_{22} - q_1)\lambda + (W_{11} - q_1)W_{22} - (W_{12} - q_2)W_{21} = 0. \tag{4.2}$$

Let λ_1 and λ_2 be the eigenvalues of characteristic equation (4.2), then

$$\lambda_1 + \lambda_2 = W_{11} + W_{22} - q_1, \quad \lambda_1\lambda_2 = (W_{11} - q_1)W_{22} - (W_{12} - q_2)W_{21}. \tag{4.3}$$

Next, we must solve equations $\lambda_1 = \pm 1$ and $\lambda_1\lambda_2 = 1$ to gain the critical stability line. At the same time, it also ensures that the absolute value λ_1 and λ_2 are less than one.

Suppose that $\lambda_1\lambda_2 = 1$, then we gain

$$L^{'} : W_{11}W_{22} - W_{12}W_{21} - 1 = W_{22}q_1 - W_{21}q_2.$$

Assume that $\lambda_1 = 1$, then we have

$$L^{''} : W_{11} + W_{22} - W_{11}W_{22} + W_{12}W_{21} - 1 = (1 - W_{22})q_1 + W_{21}q_2.$$

Assume that $\lambda_1 = -1$, then we obtain

$$L^{'''} : W_{11} + W_{22} + W_{11}W_{22} - W_{12}W_{21} + 1 = (1 + W_{22})q_1 - W_{21}q_2.$$

Thus, the stable eigenvalues lie within the triangular region with the boundaries of the straight lines $L^{'}$, $L^{''}$, $L^{'''}$. In addition, when the control parameters q_1 and q_2 take values in the triangular region, system (4.1) will not create chaos phenomena.

4.2. Optimal harvesting policy

For the sustainable use of biological resources and the protection of the natural environment on which human beings depend. Therefore, the development of renewable resources must be reasonable and proportionate. Under the premise of achieving sustainable development of biological resources, pursue maximum yield or best economic benefits. The biological and economic equilibrium combines to form the bioeconomic equilibrium. Biological equilibrium [14–17] can be obtained by solving $u_{n+1} = u_n,\ v_{n+1} = v_n$ and when the economic rent is equal to zero (meaning that the total income equals the total cost), the economic equilibrium can be obtained. If $h_1,\ p$ are the cost of harvest per unit and unit price of the prey population, respectively, then the total cost is $TC = h_1E$ and total income as $TR = \frac{pqE}{m_1E+m_2u_n}$. Then the economic rent at the moment t can be expressed as $\Xi = TR - TC = (\frac{pq}{m_1E+m_2u_n} - h_1)E$. The bioeconomic equilibrium can be obtained by solving the following simultaneous equations:

$$\begin{cases} r_1(1 - \frac{u_n}{K}) - \frac{r_2v_n}{c+u_n} - \frac{qE}{m_1E+m_2u_n} = 0, \\ a + \frac{r_2du_n}{c+u_n} - bv_n - h_2 = 0, \\ \frac{pq}{m_1E+m_2u_n} - h_1 = 0. \end{cases} \tag{4.4}$$

At present, if $\frac{pq}{m_1E+m_2u_n} < h_1$, then we stop capturing because the cost of harvesting is greater than the revenue which also means losses. Similarly, if $\frac{pq}{m_1E+m_2u_n} > h_1$, then we will continue to capture because of the harvest cost less than revenue which means profit. In order to find the bioeconomic equilibrium point (u^*, v^*, E^*) from system (4.4), we can perform the following steps: first, we solve the value of u^* from the third equation, and then substitute the value of u^* into the second equation to get the value of v^*. Finally, substitute the values of u^* and v^* into the first equation to get the value of E^*.

Next, we aim to maximize net income while maintaining ecological balance. Define the net income function $J = \sum \exp(-\delta t)(\frac{pq}{m_1E+m_2u_n} - h_1)E$, where δ is the discount rate and $\exp(-\delta t)$ is the discount factor. At the same time, we use the discrete Pontryagin maximum principle [21] to acquire the optimal capture effort. So the optimal capture problem is

$$\max \sum_{t=1}^{n} \exp(-\delta t)\left(\frac{pq}{m_1E_t + m_2u_t} - h_1\right)E_t,$$

$$s.t.\begin{cases} u_{t+1} = u_t \exp[r_1(1-\frac{u_t}{K}) - \frac{r_2 v_t}{c+u_t} - \frac{qE_t}{m_1E_t+m_2u_t}], \\ v_{t+1} = v_t \exp[a + \frac{r_2 d u_t}{c+u_t} - bv_t - h_2], \\ u_1 = u_0,\ v_1 = v_0 \ and\ 0 \le E_t \le E_{\max},\ for\ t = 0, 1, \ldots, N-1, \end{cases}$$

where u_t, v_t are state variables and E_t is the control variable. The Hamiltonian function of the correlation at this moment is

$$H_t = \exp(-\delta t)\left(\frac{pq}{m_1E_t + m_2u_t} - h_1\right)E_t + \lambda_{1(t+1)}\left[u_t \exp\left(r_1(1-\frac{u_t}{K}) - \frac{r_2v_t}{c+u_t} - \frac{qE_t}{m_1E_t+m_2u_t}\right)\right] \\ + \lambda_{2(t+1)}\left[v_t \exp\left(a + \frac{r_2du_t}{c+u_t} - bv_t - h_2\right)\right].$$

Where $\lambda_{1(t+1)}$ and $\lambda_{2(t+1)}$ are adjoint variables. In addition, the necessary condition for the optimal problem is that $\frac{\partial H_t}{\partial u_t} = 0$, $\frac{\partial H_t}{\partial v_t} = 0$ and $\frac{\partial H_t}{\partial e_t} = 0$ are valid at the same time. Optimal harvest E_t^* is available at optimal population size level (u_t^*, v_t^*).

$$\frac{\partial H_t}{\partial u_t^*} = \exp(-\delta t)\frac{-pqm_2E_t^*}{(m_1E_t^* + m_2u_t^*)^2} + \lambda_{1(t+1)}\left[1 - \frac{r_1u_t^*}{K} + \frac{r_2u_t^*v_t^*}{(c+u_t^*)^2} + \frac{qm_2u_t^*E_t^*}{(m_1E_t^*+m_2u_t^*)^2}\right] \\ + \lambda_{2(t+1)}\frac{r_2dcv_t^*}{(c+u_t^*)^2} = 0,$$
$$\frac{\partial H_t}{\partial v_t^*} = -\lambda_{1(t+1)}\frac{r_2u_t^*}{c+u_t^*} + \lambda_{2(t+1)}(1 - bv_t^*) = 0,$$
$$\frac{\partial H_t}{\partial E_t^*} = \exp(-\delta t)\left(\frac{pqm_2E_t^*}{(m_1E_t^*+m_2u_t^*)^2} - h_1\right) - \lambda_{1(t+1)}\frac{qm_2u_t^{*2}}{(m_1E_t^*+m_2u_t^*)^2} = 0.$$

As a consequence, we first solve the value of $\lambda_{1(t+1)} = \frac{\exp(-\delta t)[pqm_2E_t^* - h_1(m_1E_t^*+m_2u_t^*)^2]}{qm_2u_t^{*2}}$ from $\frac{\partial H_t}{\partial E_t^*} = 0$, and then substitute the value of $\lambda_{1(t+1)}$ into the second equation $\frac{\partial H_t}{\partial v_t^*} = 0$ to get the value of $\lambda_{2(t+1)} = \frac{\exp(-\delta t)[pqm_2E_t^* - h_1(m_1E_t^*+m_2u_t^*)^2]r_2}{qm_2u_t^*(c+u_t^*)(1-bv_t^*)}$. Finally, substitute the values of $\lambda_{1(t+1)}$ and $\lambda_{2(t+1)}$ into the first equation $\frac{\partial H_t}{\partial u_t^*} = 0$ to obtain the value of E_t^*.

5. Numerical simulations

This section will show the bifurcation diagram, phase diagram and maximum Lyapunov exponent diagram of system (1.1) to verify the correctness of theoretical analysis.

Suppose that the parameters $(a, b, c, d, E, K, r_2, q, m_1, m_2, h_2) = (0.4, 0.2, 2, 0.1, 0.4, 5, 0.5, 0.1, 0.1, 0, 0.01) \in A_F$, r_1 as the bifurcation parameter, and the initial value is (3, 1). Meanwhile, when $r_1 < 1.5$, the interior equilibrium point does not exist; when $r_1 > 1.5$, there is a unique interior equilibrium point, and when $r_1 = 1.5$, the system (1.1) has a transcritical bifurcation at the boundary equilibrium point E_2. According to Theorem 6, when $r_1 = 3.32$, the system (1.1) will have a flip bifurcation at the interior equilibrium point (3.188, 2.104). The bifurcation diagram and the maximum Lyapunov exponent diagram are shown from Figure 1. Combined with Figures 1 and 2, when $r_1 < 3.32$, the interior equilibrium point is stable. However, when $r_1 > 3.32$, the interior equilibrium loses its stability, and orbits with periods of 2, 4, 8 appear. As the r_1 increases, the maximum Lyapunov exponent value is greater than zero, and it can be seen from Figure 1(c) and Figure 2(c),(f) that system (1.1) will generate chaos.

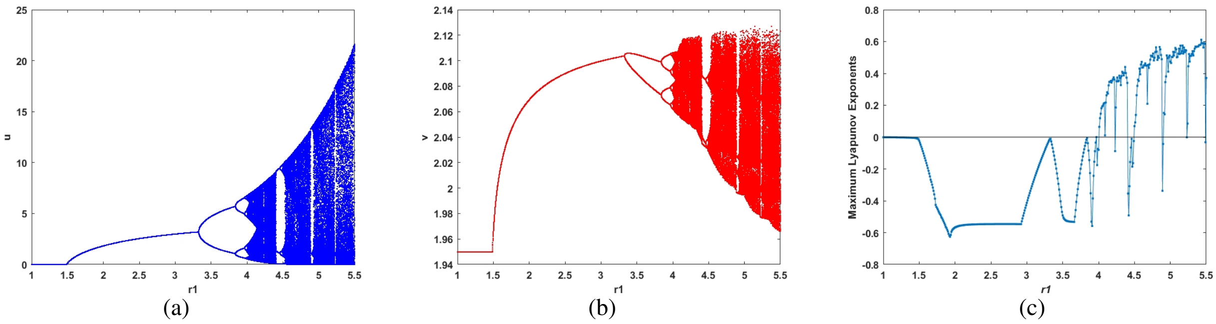


Figure 1. Flip bifurcation and MLE diagram.

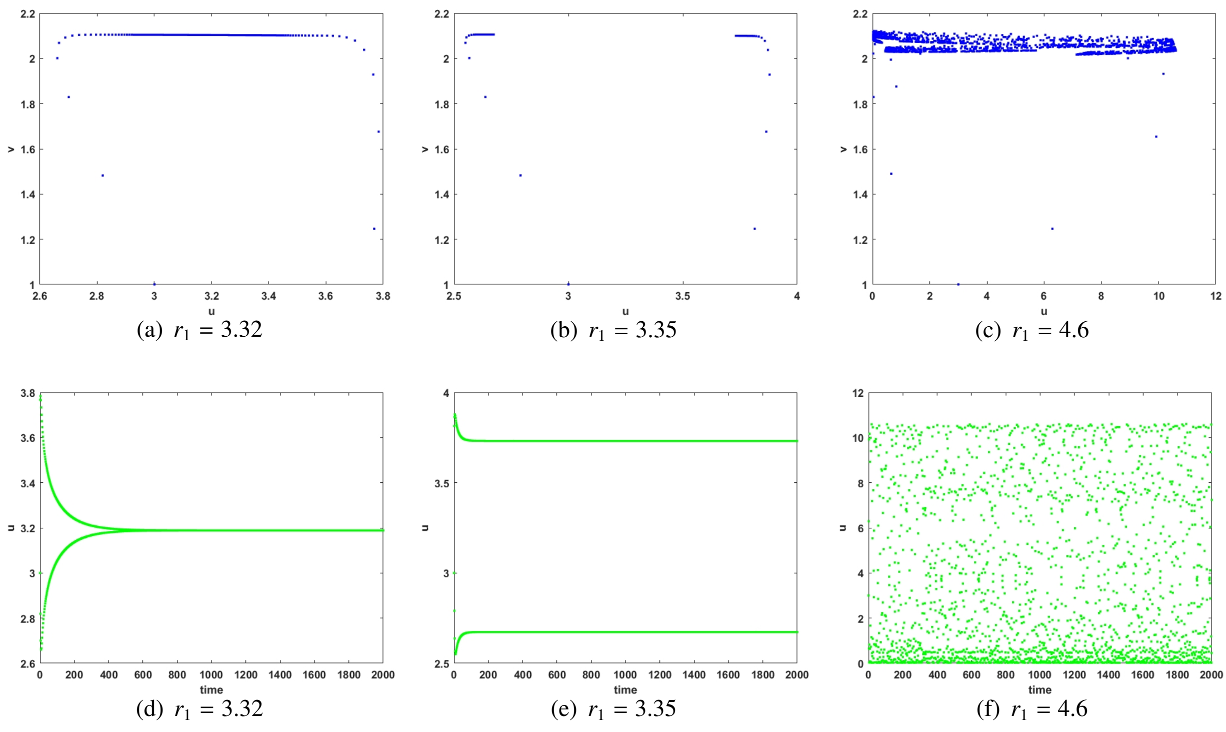
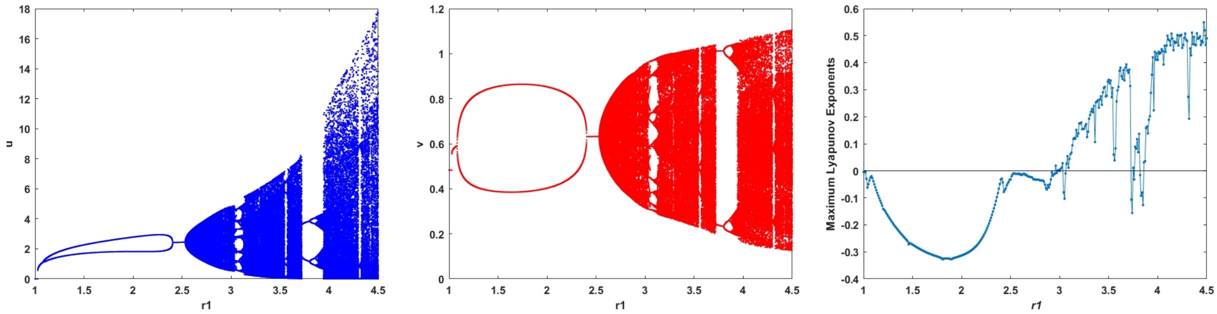


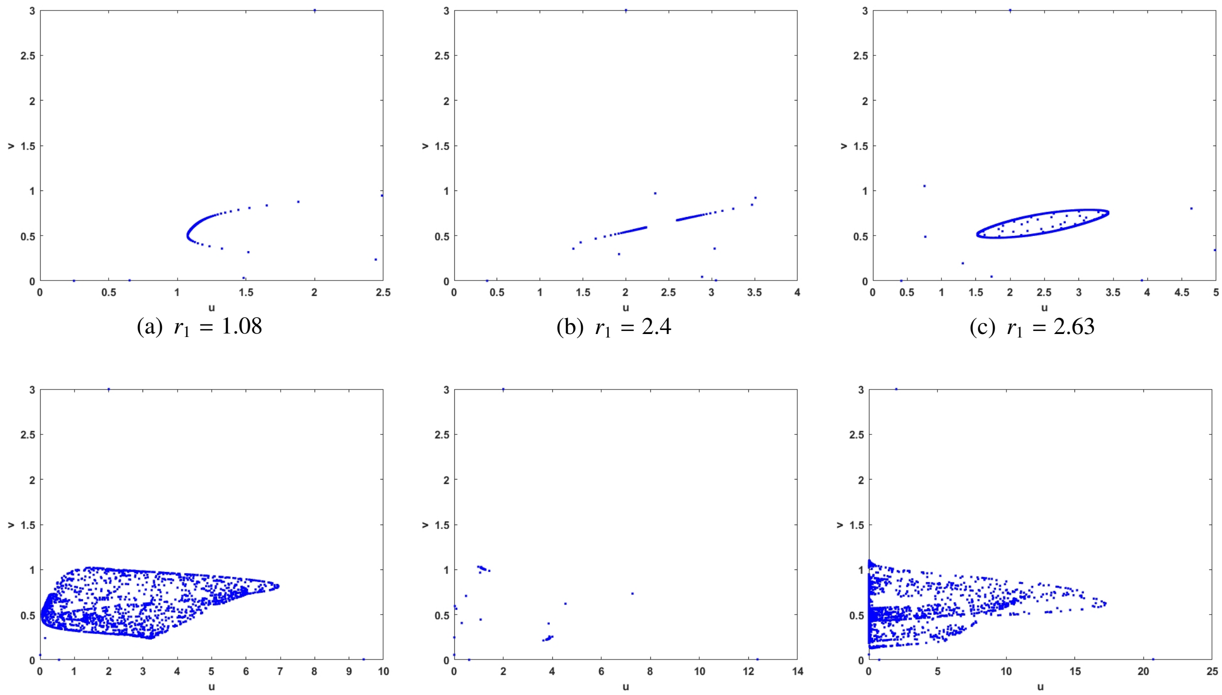
Figure 2. Phase and solution diagram related to Figure 1 when r_1 takes different values.

Assume that the parameters $(a, b, c, d, E, K, r_2, q, m_1, m_2, h_2) = (1.7, 3.5, 1.2, 0.3, 2, 3, 2.6, 0.1, 5, 2, 0.01)$, r_1 as the bifurcation parameter, and the initial value is (2, 3). The bifurcation diagram and the maximum Lyapunov exponent diagram are shown from Figure 3. Combined with Figures 4 and 5, it is clear from the figure that for smaller values of r_1 the system (1.1) is stable, with the increase of r_1 the system (1.1) stability disappears and a flip bifurcation with a period of 2 occurs, which subsequently disappears and tends to stabilize. Furthermore, when $2.43 < r_1 < 2.59$, the equilibrium point is stable. However, when $r_1 > 2.59$, the system loses its stability, and a stable invariant loop appears. At this moment, the system (1.1) produces Neimark-Sacker bifurcation and periodic solution. When r_1 increases, system (1.1) produces quasi-periodic solutions and chaotic phenomena. As the r_1 continues to increase, the maximum Lyapunov exponent value is greater than zero, the system (1.1) will generate chaos.



(a) (b) (c)

Figure 3. Mixed bifurcation and MLE diagram.

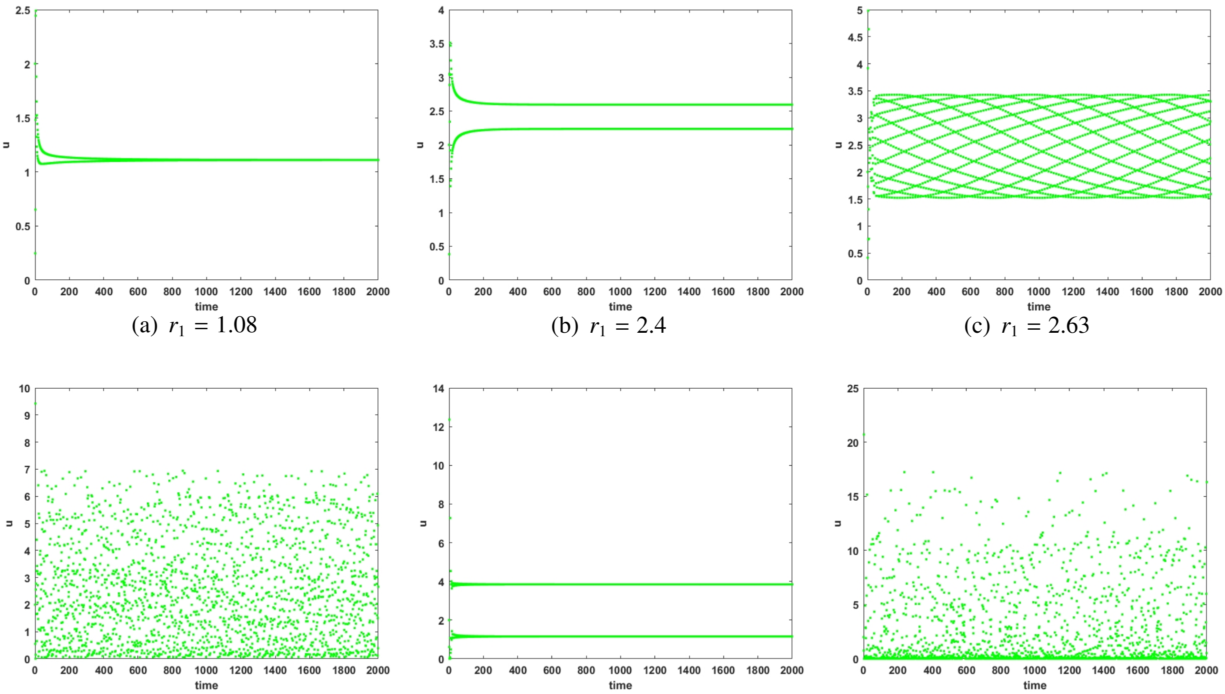


(a) $r_1 = 1.08$ (b) $r_1 = 2.4$ (c) $r_1 = 2.63$

(d) $r_1 = 3.5$ (e) $r_1 = 3.8$ (f) $r_1 = 4.45$

Figure 4. Phase diagram related to Figure 3 when r_1 takes various values.

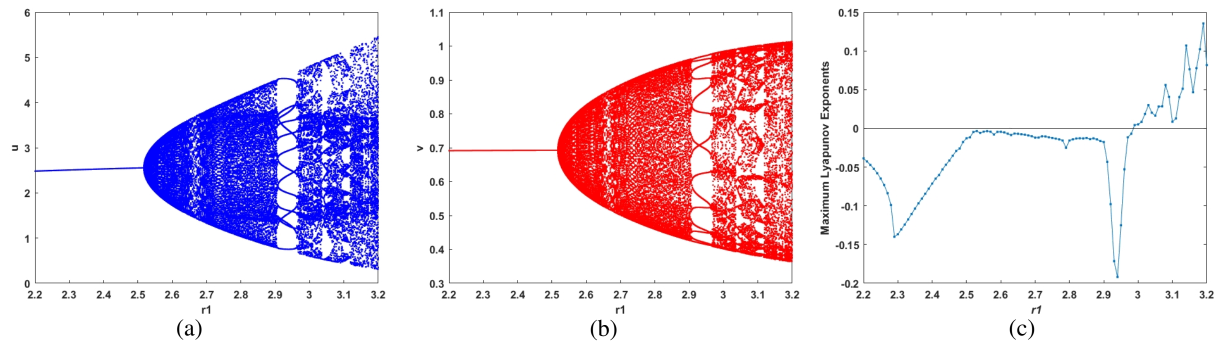
Considering the parameter values $(a, b, c, d, E, K, r_2, q, m_1, m_2, h_2) = (1.7, 3, 1.2, 0.3, 1.5, 3, 1.9, 0.1, 1, 2, 0.01) \in A_{NS}$ with the initial value is (2, 3), and r_1 as the bifurcation parameter. According to Theorem 7, when $r_1 = 2.518$, the system (1.1) has Neimark-Sacker bifurcation at the interior equilibrium point (2.52, 0.69). Figure 6 is the bifurcation and MLE graph corresponding to u and v with $r_1 \in [2.2, 3.2]$, and Figure 7 is the phase graph related to Figure 6(a). It can be seen from Figures 6 and 7 that when $r_1 < 2.518$, the equilibrium point is stable; when $r_1 > 2.518$, the equilibrium point loses its stability, and a stable invariant loop appears. At this moment, system (1.1) arises a periodic solution. When r_1 increases, system (1.1) generates quasi-periodic solutions and chaotic phenomena. Furthermore, as the r_1 continues to increase, the maximum Lyapunov exponent value is greater than zero, the system (1.1) will generate chaos.



(a) $r_1 = 1.08$ (b) $r_1 = 2.4$ (c) $r_1 = 2.63$

(d) $r_1 = 3.5$ (e) $r_1 = 3.8$ (f) $r_1 = 4.45$

Figure 5. Solution diagram corresponding to Figure 4 when r_1 takes various values.

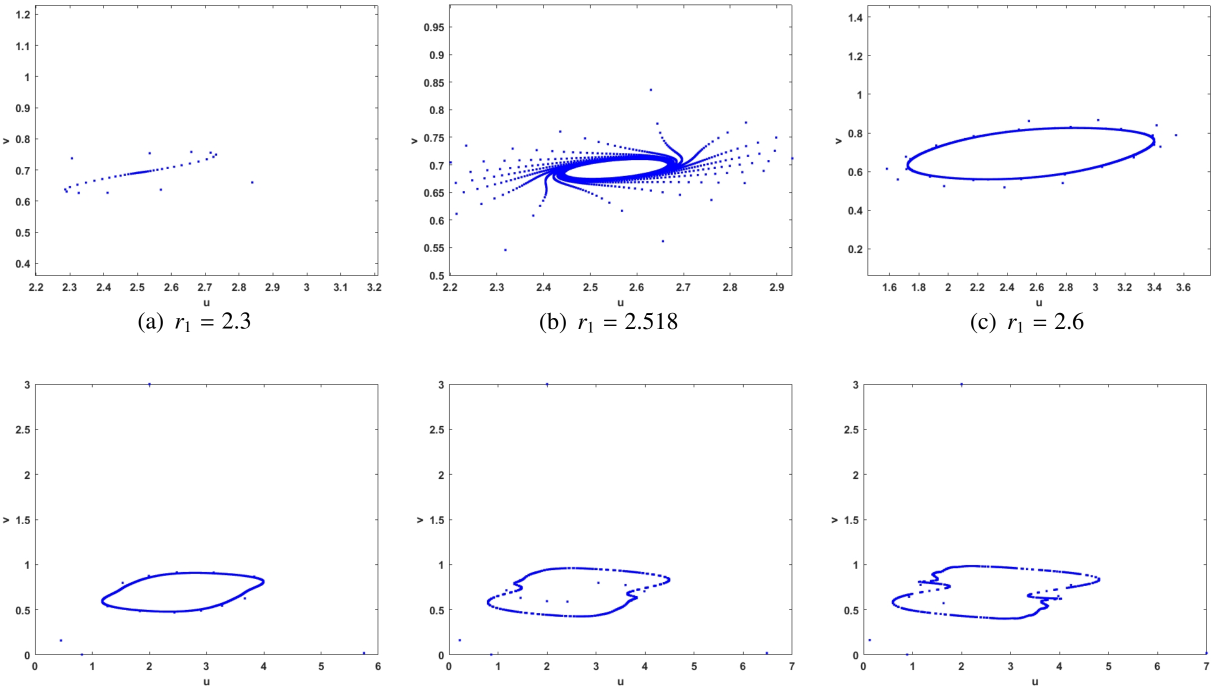


(a) (b) (c)

Figure 6. Neimark-Sacker bifurcation and MLE diagram.

To verify the chaotic control theory, we analyze Figure 6 and its numerical simulation parameters. In Figure 6(c) when the bifurcation parameter $r_1 = 3.1$, the maximum Lyapunov exponent value is greater than zero, system (1.1) will produce chaos. When the q_1 and q_2 are controlled in the triangular region surrounded by three straight lines L', L'', and L''' (see Figure 8), the chaos generated by system

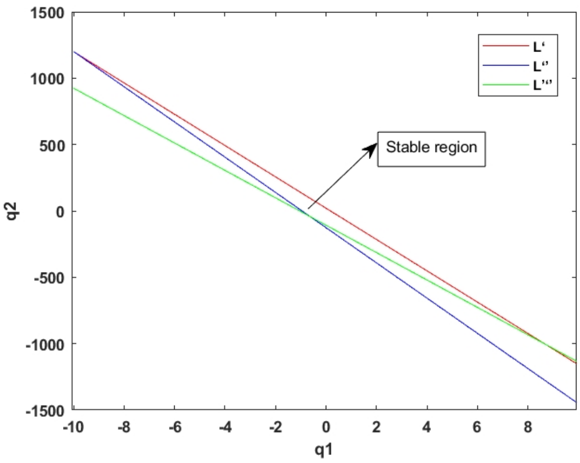
(4.1) will be controlled near the equilibrium point and become asymptotically stable state.



(a) $r_1 = 2.3$ (b) $r_1 = 2.518$ (c) $r_1 = 2.6$

(d) $r_1 = 2.75$ (e) $r_1 = 2.9$ (f) $r_1 = 3$

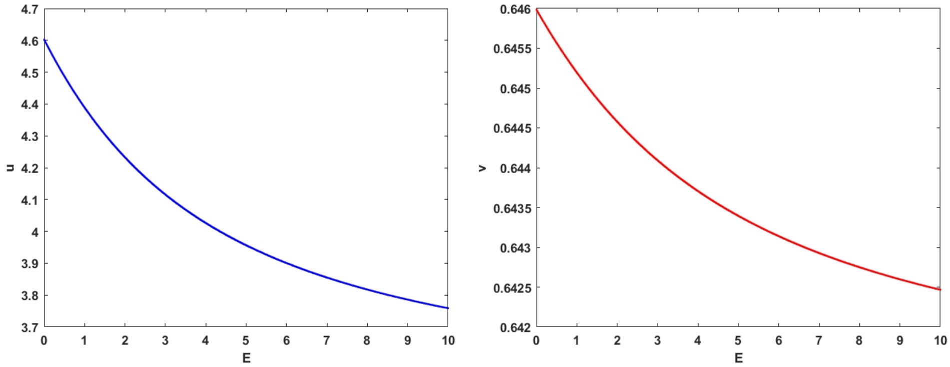
Figure 7. Phase diagram related to Figure 6 when r_1 takes various values.



(a)

Figure 8. The bounded region for the eigenvalues related to the controlled system (4.1) in the (q_1, q_2) plane.

Considering the parameter values $(a, b, c, d, r_1, K, r_2, q, m_1, m_2, h_2) = (0.95, 1.5, 1.2, 0.3, 0.7, 5, 0.5, 0.15, 1, 1, 0.1)$ with the initial value is (3, 2), and E as the bifurcation parameter. At this time, the bifurcation phenomenon of system (1.1) will not occur. As the degree of capture effort E increases, the population density of prey and predator will continue to decrease and will not tend to 0 (see Figure 9).



(a) (b)

Figure 9. Bifurcation diagram.

6. Conclusions

In this paper, we study the stability and bifurcation of equilibrium points in a discrete predator-prey model with Michaelis-Menten type harvesting. The stability analysis indicates that the model has a trivial equilibrium point, two positive boundary equilibrium points and the boundary equilibrium point is always unstable. The bifurcation analysis shows that when $r_1 = 1.5$, the boundary equilibrium point E_2 will have a transcritical bifurcation, and when the coexistence equilibrium E^* exists and loses stability, system (1.1) will have a flip bifurcation (see Figure 1). System (1.1) has, in addition, Neimark-Sacker bifurcation occur at the interior equilibrium point E^* when bifurcation parameter r_1 changes in A_{NS} small ranges (see Figure 6). Numerical simulation reveals that when the internal growth rate of prey r_1 gradually increases, system (1.1) will produce periodic, quasi-periodic windows and chaos.

Finally, we analyze chaos control theory and the existence of bioeconomic equilibrium points. In order to maximize profit in a finite time, we built an optimal control problem with the harvest effort as the control parameter, and theoretically obtained the optimal value of the control variable (harvest effort). As a result, we detected that harvesting efforts for prey and predator populations had a specific value that maximizes net income.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest regarding the publication of this paper.

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