Research article

An eco-epidemic model for assessing the application of integrated pest management strategies

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Abstract: Mathematical models have become indispensable tools for analyzing pest control strategies. However, in the realm of pest control studies, the consideration of a plant population being affected by a model that incorporates pests, natural enemies and disease in the pest population has been relatively limited. Therefore, this paper aims to formulate and investigate a hybrid impulsive eco-epidemic model that incorporates disease in the pest population. Initially, we examine the existence and stability of the pest-eradication periodic solution. Subsequently, to explore the impact of chemical and biological control methods, we propose an updated eco-epidemic model that incorporates varying frequencies of pesticide sprays and the release of both infected pests and natural enemies for pest control. We establish threshold values for the susceptible pest eradication periodic solution under different scenarios, illustrating the global attractiveness of this solution. Finally, we discuss the obtained results and suggest potential avenues for future research in this field.

Keywords: eco-epidemic model; intermittent control strategy; susceptible and infective pests; integrated pest management

1. Introduction

Controlling insects and other arthropods plays a crucial role in agriculture due to the potential ecological and economic consequences of pest outbreaks [1–5]. Integrated pest management (IPM) offers a highly effective and safer approach by employing a combination of pest control techniques including biological, cultural and chemical methods. This strategy involves carefully selecting and implementing tactics that are specifically tailored to the unique characteristics of the cropping systems, pest complexes and biological environments involved [6–9].

The main objective of IPM is to keep pest populations below the economic injury level (EIL) [10]

rather than eradicating them entirely. The selection of suitable tactics is based on the pest densities reaching the economic threshold (ET) [11], which minimizes the negative impact of insecticides on non-target pests and helps maintain environmental quality. The effectiveness of IPM has been demonstrated in various applications [12–15].

Biological control, employing microbial pathogens that target pest species, is a crucial pest control technique that has received significant attention in both experimental and theoretical studies. Drawing inspiration from the classical SIR (susceptible-infected-recovered) model introduced by Kermack and McKendrick [16], the pest population is divided into susceptible and infected pests. As a result, epidemiological models have attracted substantial interest from researchers in the field [17–22].

However, conventional ecological models primarily focus on two-species interactions, with many studies emphasizing predator-prey dynamics using Lotka-Volterra type functional responses. Limited attention has been given to IPM models involving a three-species food web. Gakkhar and Naji [23] investigated the dynamics of a tri-trophic food web comprising a generalist-specialist-prey system, incorporating a modified Holling type II functional response. Apreutesei [24] explored the necessary optimality conditions for a tri-trophic food web system. In 2013, Priyadarshi and Gakkhar [25] analyzed the dynamics of a tri-trophic food web system that included a Leslie-Gower type generalist predator. In 2016, Liang et al. [26] proposed an eco-epidemiological model for a plant-pest-predator system, considering the impulsive effect. These studies highlight the importance of considering three-species interactions in IPM models and their ecological implications.

To achieve effective pest control, it is vital to have a thorough understanding of the ecological dynamics within the cropping system, encompassing the pests, their predators, the environment, and their intricate interrelationships. However, certain control strategies, such as pesticide spraying, natural enemy releases and infected pest releases are implemented instantaneously or over relatively short periods. Hence, determining the optimal timing and frequency of these actions becomes crucial. In this study, we employ impulsive differential equations to model human interventions and develop a novel mathematical model that captures the dynamics of these actions and their impact on pest control.

The paper is organized as follows. In the following section, we will present a novel three-species mathematical model along with its main biological assumptions. Additionally, we will explore the stability of the susceptible pest-eradication periodic solution in Section 3. Section 4 presents an updated eco-epidemic model that incorporates various frequencies of pesticide sprays, releases of infected pests and releases of natural enemies. The threshold value for achieving the susceptible pest-eradication periodic solution is provided for different scenarios, showcasing the global attractivity of this solution. Subsequently, a concise discussion of the obtained results is presented, followed by potential directions for future research.

2. Eco-epidemiological model with IPM strategies at a fixed moment

Recently, there has been a growing interest in investigating IPM models using a three-species food web, which includes a predator (the natural enemy of the pest), the pest itself and a basal producer (a renewable biotic resource such as plants or vegetation). Several papers have delved into this topic [23–28]. In this context, let R represent the state of the basal producer, while P and N denote the densities of the pest and natural enemy, respectively. In 2006, Apreutesei [24] investigated the following pest-

natural enemy-plant model

$$
\begin{cases}
P'(t) &= P(t)[\alpha_P - \beta_N N(t) + \gamma_R R(t)], \\
N'(t) &= N(t)[-\alpha_N + \beta_P P(t)], \\
R'(t) &= R(t)[\alpha_R - \beta_R P(t)],
\end{cases} \tag{2.1}
$$

where α_P is the intrinsic growth rate for the pest, α_N is the death rate for the natural enemy and α_R is the growth rate for the plant. β_N describes the search rate of the natural enemy, β_P denotes the efficiency rate with which captured pests are converted to the new natural enemy, β_R describes the search rate of the pest and γ_R denotes the efficiency rate. Assume that all the coefficients are positive constants.

A novel eco-epidemiological model is proposed, which incorporates the utilization of microbial control through pathogenic agents and the release of natural enemies in integrated pest management. Building upon the research conducted by Xiao et al. [17–19] and Apreutesei [24], the pest species P is classified into two categories: susceptible pests and infected pests, with the introduction of a pest pathogen acting as a biotic insecticide within the pest population. Additionally, the following model takes into account the presence of a food source for the pests

$$
\begin{cases}
S'(t) &= r_S S(t)\left[1 - \frac{S(t)+I(t)}{K}\right] + \frac{\mu_S S(t)R(t)}{a_R+R^2(t)} - \beta S(t)I(t) - \frac{S(t)N(t)}{a_S+S^2(t)}, \\
I'(t) &= \beta S(t)I(t) - \delta_I I(t), \\
N'(t) &= \frac{\mu_N S(t)N(t)}{a_S+S^2(t)} - \delta_N N(t), \\
R'(t) &= r_R R(t) - \frac{S(t)R(t)}{a_R+R^2(t)},
\end{cases} \tag{2.2}
$$

the parameter definitions for model (2.2) are summarized in Table 1. Specifically, in this model, $R(t)$ represents a basal producer, which corresponds to a renewable biotic resource such as plants or vegetation [26]. Additionally, the non-monotonic (sigmoidal) saturation function is defined as follows

$$
\Psi(X) \doteq \frac{X}{a_X + X^2}, \qquad X = S(t), R(t).
$$

This function is a simplified form of the Monod-Haldane or Holling IV function proposed by Sokol and Howell [29] and it describes the collective defense behavior of the prey population when their density is relatively high. Here, $a_X > 0$ denotes the half-saturation constant of species X. It is important to note that only the susceptible pests have the ability to feed on the plants with a Holling type-IV response function, while the infected pests are assumed to be too weak to hunt for food.

Building upon the pest-natural enemy model and the implementation of IPM strategies, the combined approach of periodic pesticide spraying, release of infected pests and release of natural enemies is employed to effectively control the pest population. The reformulation of model (2.2) is as follows

$$
\begin{cases}
\begin{aligned}
S'(t) &= r_S S(t)\left[1 - \frac{S(t)+I(t)}{K}\right] + \frac{\mu_S S(t)R(t)}{a_R+R^2(t)} - \beta S(t)I(t) - \frac{S(t)N(t)}{a_S+S^2(t)}, \\
I'(t) &= \beta S(t)I(t) - \delta_I I(t), \\
N'(t) &= \frac{\mu_N S(t)N(t)}{a_S+S^2(t)} - \delta_N N(t), \\
R'(t) &= r_R R(t) - \frac{S(t)R(t)}{a_R+R^2(t)},
\end{aligned} & \bigg\} \; t \neq nT, \\[4pt]
\begin{aligned}
S(t^+) &= (1 - p_S)S(t), \\
I(t^+) &= (1 + q_I)I(t) + \theta_I, \\
N(t^+) &= (1 + q_N)N(t) + \theta_N, \\
R(t^+) &= (1 + q_R)R(t) + \theta_R,
\end{aligned} & \bigg\} \; t = nT.
\end{cases} \tag{2.3}
$$

The parameter definitions for model (2.3) are summarized in Table 1. It is important to emphasize here that model (2.3) incorporates the impulsive effect at fixed time intervals T, representing the simultaneous application of periodic pesticide sprays, release of infected pests and release of natural enemies to control the pest population. At time points $t = nT$, the susceptible pest population experiences an instantaneous killing rate p_S due to pesticide spraying. Importantly, following pesticide spraying, the susceptible pests become weakened and more susceptible to predation by their natural enemies.

Table 1. Parameter definitions for models (2.2) and (2.3).

Parameter	Definition	Parameter	Definition
$S(t)$	density of the susceptible pest	$I(t)$	density of the infected pest
$N(t)$	density of the natural enemy population	$R(t)$	renewable biotic resource [26]
r_S	intrinsic growth rate for $S(t)$	r_R	intrinsic growth rate for $R(t)$
K	carrying capacity for $S(t)$	δ_I	death rate of $I(t)$
δ_N	death rate of $N(t)$	β	transmission coefficient
T	pulse period	μ_S	efficiency rate for $S(t)$
μ_N	efficiency rate for $N(t)$	a_S	half-saturation constant
p_S	killing rate for $S(t)$ as pesticide is applied	a_R	half-saturation constant
p_N	killing rate for $N(t)$ as pesticide is applied	p_I	killing rate for $I(t)$ as pesticide is applied
q_I	proportional increase rate of $I(t)$	θ_I	release amount of $I(t)$
q_N	proportional increase rate of $N(t)$	θ_N	release amount of $N(t)$
q_R	proportional increase rate of $R(t)$	θ_R	release amount of $R(t)$

3. The stability of susceptible pest-eradication periodic solution

Since it is well-known that the infective pest does not cause harm to the plant, our main objective in this section is to examine the stability of the periodic solution for eradicating the susceptible pest and establish the conditions for its global attractivity.

In the following, we investigate the existence and stability of of the susceptible pest-eradication solution, denoted as $S(t) \doteq 0$ for all $t \geq 0$. We derive three subsystems from model (2.3):

$$\begin{cases} I'(t) & = & -\delta_I I(t), & t \neq nT, \\ I(t^+) & = & (1 + q_I)I(t) + \theta_I, & t = nT, \end{cases} \tag{3.1}$$

$$\begin{cases} N'(t) & = & -\delta_N N(t), & t \neq nT, \\ N(t^+) & = & (1 + q_N)N(t) + \theta_N, & t = nT, \end{cases} \tag{3.2}$$

and

$$\begin{cases} R'(t) & = & r_R R(t), & t \neq nT, \\ R(t^+) & = & (1 + q_R)R(t) + \theta_R, & t = nT. \end{cases} \tag{3.3}$$

It follows from the first equation of model (3.1) that $I(t) = \exp(-\delta_I(t - nT))I(nT^+)$ for $t \in (nT, (n + 1)T]$. Consequently, we can conclude that $I((n + 1)T) = \exp(-\delta_I T)I(nT^+)$.

Now, combining the second equation of model (3.1) with the previously mentioned equations results in the following

$$I((n + 1)T^+) = (1 + q_I) \exp(-\delta_I T) I(nT^+) + \theta_I.$$

By denoting $I((n + 1)T^+)$ as $I(n + 1)$, we can rewrite the above equation as the following difference equation

$$I(n + 1) = F(I(n)) \doteq (1 + q_I) \exp(-\delta_I T) I(nT^+) + \theta_I. \qquad (3.4)$$

Equation (3.4) represents the stroboscopic map of model (3.1). It captures the relationship between the density of infected pests at two consecutive pulse points. Specifically, the presence of a positive steady state in Eq (3.4) indicates the existence of a positive periodic solution in model (3.1). Thus, our initial investigation focuses on determining the positive steady state of the stroboscopic map (3.4). The derivative of $F(I(n))$ with respect to $I(n)$ is

$$F'(I(n)) = (1 + q_I) \exp(-\delta_I T).$$

According to the stability theory of differential equations, it is necessary to ensure that the condition $|F'(I(n))| < 1$ holds true, which is equivalent to the inequality

$$\lambda_I = (1 + q_I) \exp(-\delta_I T) < 1.$$

Furthermore, we continue to investigate the non-negative fixed point of the stroboscopic map (3.4). For simplicity, we denote the fixed point as $\tilde{I}$. It can be deduced from Eq (3.4) that $\tilde{I} = (1 + q_I) \exp(-\delta T)\tilde{I} + R_I$. This implies that the stroboscopic map (3.4) possesses a single non-negative fixed point, given by

$$\tilde{I} = \frac{\theta_I}{1 - (1 + q_I) \exp(-\delta_I T)}.$$

Analogously, we can obtain the non-negative fixed points of the corresponding stroboscopic map for subsystems (3.2) and (3.3)

$$\tilde{N} = \frac{\theta_N}{1 - (1 + q_N) \exp(-\delta_N T)}, \quad \tilde{R} = \frac{\theta_R}{1 - (1 + q_R) \exp(r_R T)},$$

given that

$$\lambda_N = (1 + q_N) \exp(-\delta_N T) < 1, \quad \lambda_R = (1 - q_R) \exp(r_R T) < 1$$

for $t \in (nT, (n + 1)T]$, respectively. Hence, we can conclude the following result.

Lemma 3.1. *Model* (3.1) *exists a positive periodic solution* I^*. *Furthermore, for every positive solution* $I(t)$ *of model* (3.1), *it holds that* $\lim_{t \to +\infty} |I(t) - I^*(t)| = 0$. *Here,* $I^*(t)$ *is defined as* $I^*(t) = \exp(-\delta_I(t - nT))\tilde{I}$, *where*

$$\tilde{I} = \frac{\theta_I}{1 - (1 + q_I) \exp(-\delta_I T)}$$

for $t \in (nT, (n + 1)T)$ *and* $n \in \mathbb{Z}$.

Similarly, we can deduce that

$$N^*(t) = \exp(-\delta_N(t - nT))\tilde{N} = \frac{\theta_N \exp(-\delta_N(t - nT))}{1 - (1 + q_N) \exp(-\delta_N T)}$$

and

$$R^*(t) = \exp(r_R(t - nT))\tilde{R} = \frac{\theta_R \exp(r_R(t - nT))}{1 - (1 + q_R)\exp(r_R T)}$$

for all $n \in \mathbb{Z}$.

In order to analyze the global asymptotic stability of the periodic solution $(0, I^*(t), N^*(t), R^*(t))$ of model (2.3), we establish the following two results.

Theorem 3.2. *If* $\min\{\lambda_S, \lambda_I, \lambda_R, \lambda_N\} < 1$, *then the susceptible pest-eradication periodic solution* $(0, I^*(t), N^*(t), R^*(t))$ *of model (2.3) is locally asymptotically stable, where*

$$\lambda_S = (1 - p_S)\exp\left\{r_S\left\{T + \frac{(1+K\beta)[\exp(-\delta T)-1]\tilde{I}}{K\delta_I} + \frac{[\exp(-\delta_N T)-1]\tilde{N}}{a_S \delta_N}\right\} + \frac{r_S \mu_S}{\sqrt{a_R} r_R}\left[\arctan\left(\frac{\tilde{R}\exp(r_R T)}{\sqrt{a_R}}\right) - \arctan\left(\frac{\tilde{R}}{\sqrt{a_R}}\right)\right]\right\}.$$

Theorem 3.3. *The periodic solution* $(0, I^*(t), N^*(t), R^*(t))$ *of model (2.3) is globally asymptotically stable provided that*

$$\mathcal{R}_S = (1 - p_S)\exp\left\{r_S T + \frac{\beta[\exp(-\delta_I T) - 1]\tilde{I}}{\delta_I} + \frac{[\exp(-\delta_N T) - 1]\tilde{N}}{(a_S + K^2)\delta_N} + \frac{[\exp(r_S T) - 1]\tilde{R}}{a_R r_R}\right\} < 1.$$

The proofs for Theorems 3.2 and 3.3 can be found in Appendix A. It is worth noting that in practical pest control, the main objective is to manage the population size of pests within a certain range rather than completely eradicating them. This approach helps maintain the balance of the ecosystem and aligns with the fundamental principles of IPM. Therefore, the pest-eradication periodic solution only exists in theory for the system.

4. Eco-epidemiological model with optimum pulse timing of IPM strategies at different fixed moment

We now extend model (2.3) by incorporating periodic spraying of pesticides and the release of infected pests and natural enemies at specific fixed time intervals, as suggested by Tang et al. [30]. Consequently, we obtain the following revised model.

$$\begin{cases} \begin{aligned} S'(t) &= r_S S(t)\left[1 - \frac{S(t)+I(t)}{K}\right] + \frac{\mu_S S(t)R(t)}{a_R + R^2(t)} - \beta S(t)I(t) - \frac{S(t)N(t)}{a_S + S^2(t)}, \\ I'(t) &= \beta S(t)I(t) - \delta_I I(t), \\ N'(t) &= \frac{\mu_N S(t)N(t)}{a_S + S^2(t)} - \delta_N N(t), \\ R'(t) &= r_R R(t) - \frac{S(t)R(t)}{a_R + R^2(t)}, \end{aligned} \quad \left.\begin{matrix} \\ \\ \\ \\ \end{matrix}\right\} t \neq \tau_n, \quad t \neq \lambda_m, \\ \begin{aligned} S(t^+) &= (1 - p_S)S(t), \\ I(t^+) &= (1 - p_I)I(t), \\ N(t^+) &= (1 - p_N)N(t), \\ R(t^+) &= R(t), \end{aligned} \quad \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} t = \tau_n, \\ \begin{aligned} I(t^+) &= (1 + q_I)I(t) + \theta_I, \\ N(t^+) &= (1 + q_N)N(t) + \theta_N, \\ R(t^+) &= R(t) + \theta_R, \end{aligned} \quad \left.\begin{matrix} \\ \\ \end{matrix}\right\} t = \lambda_m, \end{cases} \tag{4.1}$$

We assume that the application of pesticides has an impact on the populations of pests and natural enemies. Some parameters remain the same as in model (2.3), other paramaters see Table 1 for details.

Due to the implementation of different strategies for insecticide applications and releases of infected pests in model (4.1) [30, 31], we need to consider two cases regarding the timing of IPM applications.
Case 1. Spraying pesticides more frequently than releasing infected pests and natural enemies.
Case 2. Releases of infected pests and natural enemies more frequent than spraying pesticides.

4.1. Threshold conditions of Case 1

To facilitate the analysis, we assume that the infected pests and natural enemies are released periodically, with a period denoted as T_N. We define $\lambda_{m+1} - \lambda_m = T_N$, where $m \in \mathbb{Z}$ represents an integer. Additionally, we have $\tau_{n+k_p} = T_N + \tau_n$.

Now, we define some variables: $\Delta_i = \tau_{i+1} - \tau_i$ $E_i^h = (hT_N + \tau_i, hT_N + \tau_{i+1}]$, where $i = 0, 1, 2, \cdots, k_p$ $\Delta_0 = \tau_1, \Delta_{k_p} = T_N - \tau_{k_p}, \tau_0 = 0, \tau_{k_p+1} = T_N$ h represents a non-negative number. Furthermore, we analyze model (4.1) by dividing it into three subsystems where $S(t) = 0$ as follows

$$
\begin{cases}
I'(t) &= -\delta_I I(t), & t \neq \tau_n, \quad t \neq \lambda_m, \\
I(t^+) &= (1 - p_I)I(t), & t = \tau_n, \\
I(t^+) &= (1 + q_I)I(t) + \theta_I, & t = \lambda_m,
\end{cases}
\tag{4.2}
$$

$$
\begin{cases}
N'(t) &= -\delta_N N(t), & t \neq \tau_n, \quad t \neq \lambda_m, \\
N(t^+) &= (1 - p_N)N(t), & t = \tau_n, \\
N(t^+) &= (1 + q_N)N(t) + \theta_N, & t = \lambda_m,
\end{cases}
\tag{4.3}
$$

and

$$
\begin{cases}
R'(t) &= r_R R(t), & t \neq \tau_n, \quad t \neq \lambda_m, \\
R(t^+) &= R(t), & t = \tau_n, \\
R(t^+) &= R(t) + \theta_R, & t = \lambda_m.
\end{cases}
\tag{4.4}
$$

From the first equation of model (4.2) within the interval E_0^h, we can deduce that

$$
I(t) = I(hT_N^+) \exp[-\delta_I(t - hT_N)]
$$

and

$$
I(t) = I[(hT_N + \tau_i)^+] \exp[-\delta_I(t - hT_N - \tau_i)], t \in E_i^h, i = 1, 2, \cdots, k_p - 1.
$$

Furthermore, we get

$$
I(t) = I[(hT_N + \tau_{k_p})^+] \exp[-\delta_I(t - hT_N - \tau_{k_p})], t \in E_{k_p}^h.
$$

At the moment $hT_N + \tau_i$, where $i = 1, 2, \cdots, k_p$, corresponding to the application of a pesticide, we observe

$$
I[(hT_N + \tau_i)^+] = (1 - p_I)^i I(hT_N^+) \exp[-\delta_I(\sum_{j=0}^{i-1} \Delta_j)]
$$

and

$$
I[[(h + 1)T_N]^+] = (1 + q_I)(1 - p_I)^{k_p} I(hT_N^+) \exp[-\delta_I(\sum_{j=0}^{k_p-1} \Delta_j)] \exp(-\delta_I \Delta_{k_p}) + \theta_I.
$$

Denote $I[(hT_N)^+] = I(h)$, we can get the following difference equation

$$I(h + 1) = (1 + q_I)(1 - p_I)^{k_p} \exp(-\delta_I T_N)I(h) + \theta_I.$$

It is evident that the linear difference equation mentioned above possesses a unique equilibrium

$$I_1^* = \frac{\theta_I}{1 - (1 + q_I)(1 - p_I)^{k_p} \exp(-\delta_I T_N)}.$$

This equilibrium is considered stable globally, given that $\lambda_I^1 = (1 + q_I)(1 - p_I)^{k_p} \exp(-\delta_I T_N) < 1$. Therefore, model (4.2) has a globally stable periodic solution denoted as $\tilde{I}_{N_1}(t)$ and

$$\tilde{I}_{N1}(t) = \begin{cases} I_1^* \exp(-\delta_I(t - hT_N)), & t \in E_0^h, \\ I((\tau_i)^+) \exp(-\delta_I(t - hT_N - \tau_i)), & t \in E_i^h, \end{cases}$$

where

$$I[(\tau_i)^+] = I_1^*(1 - p_I)^i \exp[-\delta_I(\sum_{j=0}^{i-1} \Delta_j)], i = 1, 2, \cdots, k_p,$$

particularly, $I[(\tau_0)^+] = I_1^*$.

Analogously, we can obtain a globally stable periodic solution denoted as $\tilde{N}_{N_1}(t)$, for model (4.3). This is achieved under the condition that $\lambda_N^1 = (1 + q_N)(1 - p_N)^{k_p} \exp(-\delta_N T_N) < 1$ and

$$\tilde{N}_{N1}(t) = \begin{cases} N_1^* \exp(-\delta_N(t - hT_N)), & t \in E_0^h, \\ N((\tau_i)^+) \exp(-\delta_N(t - hT_N - \tau_i)), & t \in E_i^h, \end{cases}$$

where

$$N[(\tau_i)^+] = N_1^*(1 - p_N)^i \exp[-\delta_N(\sum_{j=0}^{i-1} \Delta_j)], i = 1, 2, \cdots, k_p$$

and

$$N[\tau_0)^+] = N_1^* = \frac{\theta_N}{1 - (1 + q_N)(1 - p_N)^{k_p} \exp(-\delta_N T_N)}.$$

Model (4.3) has a globally stable periodic solution, denoted as $\tilde{R}_{N_1}(t)$, under the condition that $\lambda_R^1 = \exp(r_R T_N) < 1$ and

$$\tilde{R}_{N1}(t) = \begin{cases} r_S^* \exp(r_R(t - hT_N)), & t \in E_0^h, \\ R((\tau_i)^+) \exp(r_R(t - hT_N - \tau_i)), & t \in E_i^h, \end{cases}$$

where

$$R[(\tau_i)^+] = r_S^* \exp[r_R(\sum_{j=0}^{i-1} \Delta_j)], i = 1, 2, \cdots, k_p. \qquad R[(\tau_0)^+] = r_S^* = \frac{\theta_R}{1 - \exp(r_R T_N)}.$$

To analyze the local and global asymptotic stability of the periodic solution $(0, I^*(t), N^*(t), R^*(t))$ of model (4.1), we derive the following two results.

Theorem 4.1. *If* $\min\{\lambda_S^1, \lambda_I^1, \lambda_R^1, \lambda_N^1\} < 1$, *then the periodic solution* $(0, I^*(t), N^*(t), R^*(t))$ *is locally asymptotically stable for model (4.1), where*

$$\lambda_S^1 = (1 - p_S) \exp\left\{ r_S T_N + \sum_{i=0}^{k_p} \left\{ \frac{r_S(1 + K\beta)\exp(-\delta_1 \tau_{i+1}) - \exp(-\delta_I \tau_i)I((\tau_i)^+)}{K\delta_I} \right.\right.$$

$$+ \frac{r_S\{\exp(-\delta_N \tau_{i+1}) - \exp(-\delta_N \tau_i)\}N((\tau_i)^+)}{a_S \delta_N} + \frac{r_S \mu_S}{\sqrt{a_R}r_R}\left[\arctan\left(\frac{R((\tau_{i+1})^+) \, \exp(r_R \tau_{i+1})}{\sqrt{a_R}} \right) - \arctan\left(\frac{R((\tau_i)^+) \exp(r_R \tau_{i+1})}{\sqrt{a_R}} \right)\right]\left.\left.\right\}\right\}.$$

Theorem 4.2. *The periodic solution* $(0, I^*(t), N^*(t), R^*(t))$ *is globally asymptotically stable for model (4.1) provided that*

$$\mathcal{R}_S^1 = (1 - p_S)^{k_p} \exp\left\{ r_S T_N + \sum_{i=0}^{k_p} \left\{ \frac{\beta\{\exp(-\delta_1 \tau_{i+1}) - \exp(-\delta_I \tau_i)\}I((\tau_i)^+)}{\delta_I} \right.\right.$$

$$+ \frac{[\exp(-\delta_N \tau_{i+1}) - \exp(-\delta_N \tau_i)]N((\tau_i)^+)}{(a_S + K^2)\delta_N} + \frac{\mu_S [\exp(r\tau_{i+1}) - \exp(r\tau_i)]R((\tau_i)^+)}{a_R r_R}\left.\left.\right\}\right\} < 1.$$

The proofs for Theorems 4.1 and 4.2 can be found in Appendix B.
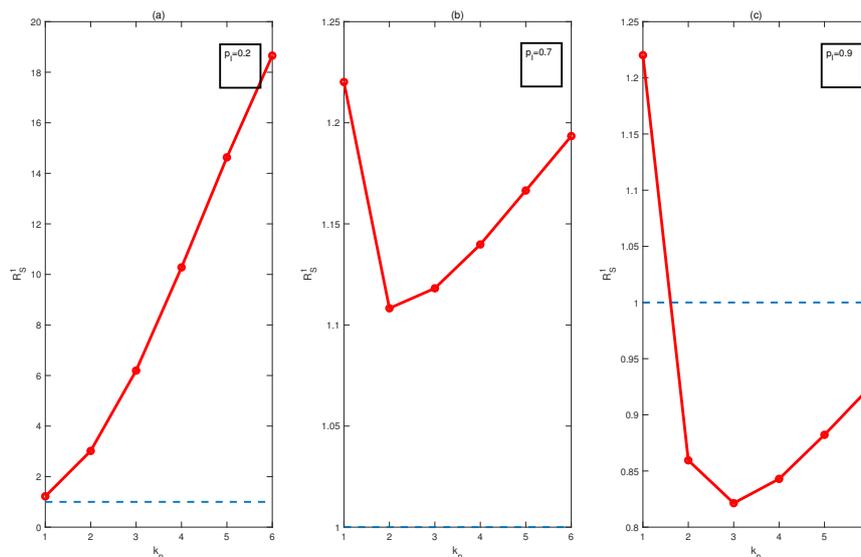
Figure 1. The effect of the killing rate p_I on $\mathcal{R}_S^1$. Parameters are: $r_S = 0.04, T_N = 5, \beta = 0.2, \delta_I = 0.2, \delta_N = 0.08, r_R = 0.02, a_R = 2, a_S = 2, K = 6, \Delta = T_N/k_p, \theta_I = 5, \theta_R = 0.8, \theta_N = 2, p_I = 0.7, p_N = 0.3, p_R = 0.5, p_S = 0.001, \mu_1 = 0.8, q_I = 0.3, q_N = 0.4$.

Next, we study the impact of important and critical paramters on the threshold $\mathcal{R}_S^1$. To accomplish this, we select the killing rate p_I for the infected pest, and the pesticide application frequency k_p as bifurcation parameters, while keeping all other parameters fixed as shown in Figure 1. In fact, the effects of k_p and p_I on $\mathcal{R}_S^1$ are depicted in Figure 1. The results indicate that if the pesticide kills the infected pests with a relatively lower killing rate p_I (such as $p_2 = 0.2$). Then the threshold value $\mathcal{R}_S^1$ is a monotonically increasing function with respect to k_p, as illustrated in Figure 1(a). If

p_I is increased (in this case, $p_2 = 0.7, 0.9$), Figure 1(b),(c) demonstrates that the threshold value $\mathcal{R}_S^1$ becomes nonmonotonic with respect to k_p. This indicates that there exists an optimal number of pesticide applications within the releasing period T_N. It can be observed that in this case, the optimal control strategy is to apply pesticides two or three times over the period T_N. In this scenario, the threshold value $\mathcal{R}_S^1$ consistently remains greater than 1, as depicted in Figure 1(b). However, Figure 1(c) illustrates that $\mathcal{R}_S^1$ is less than 1 if pesticide applications are applied twice. This demonstrates that if pesticide applications lead to significant effects on the infected pests, then the number of pesticide applications should be increased.

Furthermore, we can conduct two-parameter bifurcation analyses for $\mathcal{R}_S^1$, as illustrated in Figure 2. In each subplot, we vary two key parameters simultaneously and observe their impact on $\mathcal{R}_S^1$. All simulation results presented in Figure 2 indicate that $\mathcal{R}_S^1$ is highly sensitive to slight variations in the release constants θ_I, θ_S, θ_N, the releasing period T_N and the killing rates p_I, p_S and p_N. Moreover, these findings emphasize that for a given releasing period, the number of pesticide applications within the period T_N, the releasing constants θ_I, θ_S, θ_N and the killing rates p_I, p_S, p_N are crucial for effective pest control. This information can assist pest control experts in designing and determining the optimal timing for spray applications and the appropriate release rates.
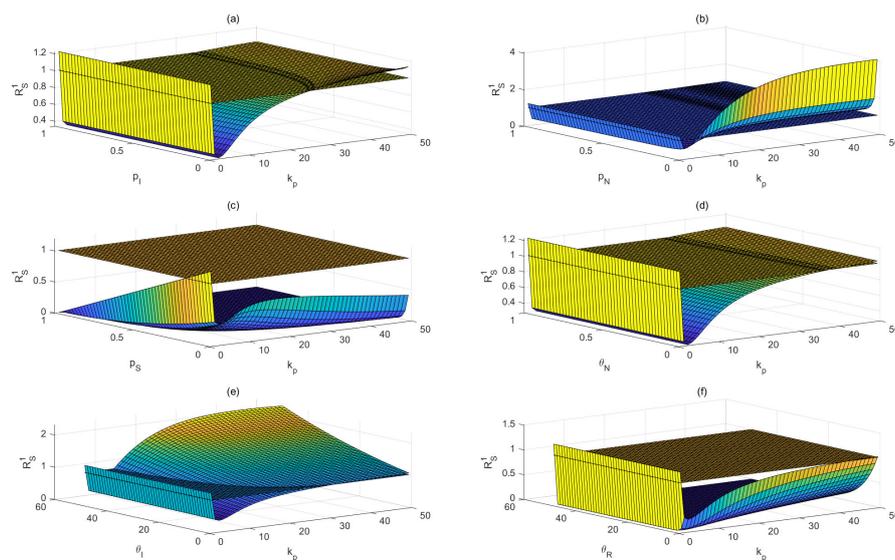
Figure 2. The effect of two parameters on $\mathcal{R}_S^1$. Parameters are: $r_S = 0.04, T_N = 5, \beta = 0.02, \delta_I = 0.9, \delta_N = 0.08, r_R = 0.02, a_R = 1, a_S = 2, K = 10, \Delta = T_N/k_p, q_I = 0.3, q_N = 0.4, \mu_I = 0.8$. (a) $\theta_I = 1, \theta_R = 0.8, \theta_N = 2, p_N = 0.3, p_S = 0.001$; (b) $\theta_I = 1, \theta_R = 0.8, \theta_N = 2, p_I = 0.2, p_S = 0.001$; (c) $\theta_I = 1, \theta_R = 0.8, p_I = 0.2, p_N = 0.3$; (d) $\theta_I = 1, \theta_R = 0.8, p_I = 0.2, p_N = 0.3$; (e) $\theta_N = 2, p_I = 0.2, p_N = 0.3, p_S = 0.001$; (f) $\theta_I = 1, \theta_N = 2, p_I = 0.2, p_N = 0.3, p_S = 0.001$.

Although we mentioned in Section 3 that the pest-eradication periodic solution only exist in theory and are difficult to achieve in practical pest control, we have conducted numerical studies to investigate the existence of the pest-eradication periodic solution, which means that the susceptible pest-

eradication periodic solution is globally attractive, as shown in Figure 3. The theoretical existence of periodic solutions is indeed correct. In fact, as long as the parameters of the model satisfy the threshold conditions by changing the control strategy in practical pest control, the corresponding pest-eradication periodic solution of system can be obtained.
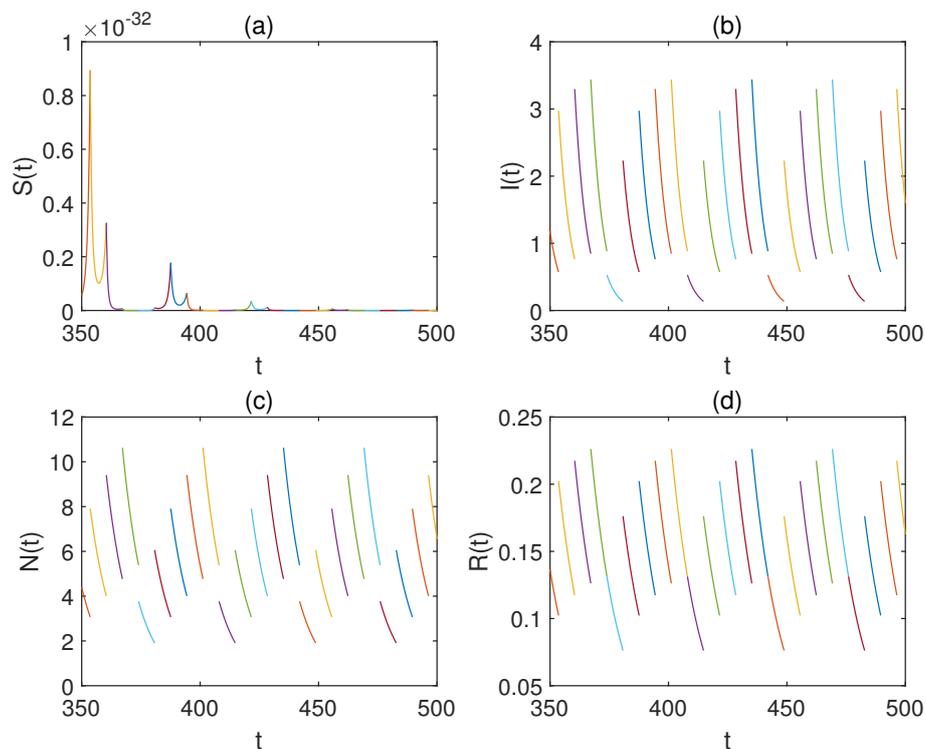
Figure 3. Pest-eradication periodic solution of model (4.1). Parameters are: $r_S = 2, T_N = 5, \beta = 0.2, \delta_I = 0.1, \delta_N = 0.1, r_R = 0.02, a_R = 0.2, a_S = 3, K = 10, \Delta = T_N/k_p, \theta_I = 2, \theta_R = 0.1, \theta_N = 3, p_I = 0.4, p_N = 0.3, p_R = 0.5, p_S = 0.9, \mu_S = 0.6, \mu_N = 0.5, q_I = 0.7, q_N = 0.4, k_p = 5.$

4.2. Threshold conditions of Case 2

We assume that the pesticide is sprayed periodically, with the period recorded as T_p. This means that $\tau_{n+1} - \tau_n = T_p$ for every n belonging to the set of integers $\mathbb{Z}$. During each pesticide spraying period of T_p, susceptible pests and natural enemies are introduced k_p times. Consequently, we have the relationship

$$\lambda_{n+k_p} = T_p + \lambda_n.$$

Define

$$\Delta_i = \lambda_{i+1} - \lambda_i, F_i^h = (hT_p + \lambda_i, hT_p + \lambda_{i+1}], i = 0, 1, 2, \cdots, k_q,$$

where $\Delta_0 = \lambda_R, \Delta_{k_q} = T_p - \lambda_{k_q}, \tau_0 = 0, \tau_{k_q+1} = T_p, h$ is non-negative number, based on these definitions,

we can conclude

$$\begin{cases} I'(t) & = & -\delta_I I(t), & t \neq \tau_n, \quad t \neq \lambda_m, \\ I(t^+) & = & (1 - p_I)I(t), & t = \tau_n, \\ I(t^+) & = & (1 + q_I)I(t) + \theta_I, & t = \lambda_m, \end{cases} \tag{4.5}$$

$$\begin{cases} N'(t) & = & -\delta_N N(t), & t \neq \tau_n, \quad t \neq \lambda_m, \\ N(t^+) & = & (1 - p_N)N(t), & t = \tau_n, \\ N(t^+) & = & (1 + q_N)N(t) + \theta_N, & t = \lambda_m, \end{cases} \tag{4.6}$$

and

$$\begin{cases} R'(t) & = & r_R R(t), & t \neq \tau_n, \quad t \neq \lambda_m, \\ R(t^+) & = & R(t), & t = \tau_n, \\ R(t^+) & = & R(t) + \theta_R, & t = \lambda_m. \end{cases} \tag{4.7}$$

It follows from model (4.1) that

$$I([(h + 1)T_p]^+) = (1 - p_I)[c_1 I((hT_p)^+) + c_2] = (1 - p_I)c_1 I((hT_p)^+) + (1 - p_I)c_2,$$

where

$$c_1 = (1 + q_I)^{k_p} \exp(-\delta_I T_p), c_2 = \theta_I \sum_{i=0}^{k_q-1} \exp\left(-(1 + q_I)^i \delta_I \sum_{j=k_q-i}^{k_q} \Delta_j\right).$$

Denote $y([(h + 1)T_p]^+)$ as $y(h + 1)$. Then, the above equation can be rewritten as the following difference equation

$$I(h + 1) = (1 - p_I)c_1 I(h) + (1 - p_I)c_2.$$

It is evident that the above equation has a unique global equilibrium given by

$$I_2^* = \frac{(1 - p_I)\theta_I \sum_{i=0}^{k_q-1} \exp\left(-(1 + q_I)^i \delta_I \sum_{j=k_q-i}^{k_q} \Delta_j\right)}{1 - (1 - p_I)(1 + q_I)^{k_q} \exp(-\delta_I T_p)}.$$

This equilibrium is considered unique and globally stable under the condition that

$$\lambda_I^2 = (1 - p_I)(1 + q_I)^{k_p} \exp(-\delta_I T_p) < 1.$$

Therefore, model (4.5) has a globally stable periodic solution

$$\tilde{I}_{N_2}(t) = \begin{cases} I_2^* \exp(-\delta_I(t - hT_p)), & t \in F_0^h, \\ I_2[(\lambda_k)^+] \exp(-\delta_I(t - hT_p - \tau_i)), & t \in F_i^h, \end{cases}$$

where

$$I_2[(\lambda_k)^+] = I_2^*(1 + q_I)^k \exp(-\delta_I \sum_{i=0}^{k-1} \Delta_i) + \theta_I\left(\sum_{i=1}^{K}[(1 + q_I)^i \exp(-\delta_I \sum_{j=k-i}^{k-1} \Delta_j)] + 1\right),$$

$k = 1, 2, \cdots, k_p$, particularly $I_2[(\lambda_0)^+] = I_2^*$.

Analogously, model (4.6) has a a globally stable periodic solution

$$\tilde{N}_{N_2}(t) = \begin{cases} N_2^* \exp(-\delta_I(t - hT_p)), & t \in F_0^h, \\ N_2[(\lambda_k)^+] \exp(-\delta_N(t - hT_p - \tau_i)), & t \in F_i^h \end{cases}$$

as $\lambda_N^2 = (1 - p_N)(1 + q_N)^k \exp(-\delta_N T_p) < 1$. Here,

$$N_2[(\lambda_k)^+] = I_2^*(1 + q_N)^k \exp(-\delta_N \sum_{i=0}^{k-1} \Delta_i) + \theta_N \Big(\sum_{i=1}^{K} [(1 + q_N)^i \exp(-\theta_S \sum_{j=k-i}^{k-1} \Delta_j)] + 1 \Big),$$

$k = 1, 2, \cdots, k_p$ and

$$N_2[(\lambda_0)^+] = N_2^* = \frac{(1 - p_N)\theta_N \sum_{i=0}^{k_q-1} \exp\Big(-(1 + q_N)^i \delta_I \sum_{j=k_q-i}^{k_q} \Delta_j \Big)}{1 - (1 - p_N)(1 + q_N)^{k_q} \exp(-\delta_N T_p)}.$$

Model (4.7) has a a globally stable periodic solution

$$\tilde{R}_{N_2}(t) = \begin{cases} r_R^* \exp(r_R(t - hT_p)), & t \in F_0^h, \\ r_R[(\lambda_k)^+] \exp(r_R(t - hT_p - \tau_i)), & t \in F_i^h \end{cases}$$

as $\lambda_R^2 = (1 - q_R) \exp(r_R T_P) < 1$. Here,

$$r_R[(\lambda_k)^+] = r_R^* \exp(r_R \sum_{i=0}^{k-1} \Delta_i) + \theta_R \Big(\sum_{i=1}^{K} [\exp(r_R \sum_{j=k-i}^{k-1} \Delta_j)] + 1 \Big), \quad k = 1, 2, \cdots, k_p$$

and

$$r_R[(\lambda_0)^+] = r_R^* = \frac{(1 - p_I)\theta_R \sum_{i=0}^{k_q-1} \exp\Big(r_R \sum_{j=k_q-i}^{k_q} \Delta_j \Big)}{1 - \exp(r_R T_p)}.$$

Hence, we have determined the local and global stability conditions for the periodic solution of model (4.1).

Theorem 4.3. *If $\min\{\lambda_S^2, \lambda_I^2, \lambda_R^2, \lambda_N^2\} < 1$, then the periodic solution $(0, I^*(t), N^*(t), R^*(t))$ of model (4.1) is locally asymptotically stable, where*

$$\lambda_S^2 = (1 - p_S) \exp \Bigg\{ r_S T_p + \sum_{i=0}^{k_p} \Bigg\{ \frac{r_S(1+K\beta)[\exp(-\delta_I \lambda_{i+1}) - \exp(-\delta_I \lambda_i)]I_2[(\lambda_i)^+]}{K\delta_I}$$

$$+ \frac{r_S[\exp(-\delta_N \lambda_{i+1}) - \exp(-\delta_N \lambda_i)]N_2[(\lambda_i)^+]}{a_S \delta_N} + \frac{r_S \mu_S}{\sqrt{a_R} r_R} \Big[\arctan\Big(\frac{r_R[(\lambda_i)^+]\exp(r_R \lambda_{i+1})}{\sqrt{a_R}} \Big) - \arctan\Big(\frac{r_R[(\lambda_i)^+]\exp(r_R \lambda_i)}{\sqrt{a_R}} \Big) \Big] \Bigg\} \Bigg\} < 1.$$

Theorem 4.4. *The periodic solution $(0, I^*(t), N^*(t), R^*(t))$ of model (4.1) is globally asymptotically stable provided that*

$$\mathcal{R}_S^2 = \exp \Bigg\{ r_S T_P + \sum_{i=0}^{k_p} \Bigg\{ \frac{\beta[\exp(-\delta_1 \lambda_{i+1}) - \exp(-\delta_I \lambda_i)]I_2[(\lambda_i)^+]}{\delta_I} + \frac{[\exp(-\delta_N \lambda_{i+1}) - \exp(-\delta_N \lambda_i)]N_2[(\lambda_i)^+]}{(a_S + K^2)\delta_N} +$$

$$\frac{\mu_S[\exp(r_R \lambda_{i+1}) - \exp(r_R \lambda_i)]R[(\lambda_i)^+]}{a_R r_R} \Bigg\} \Bigg\} < 1.$$

By employing the same methods as those used in Section 4.1, we can investigate the impact of key parameters on the threshold value $\mathcal{R}_S^2$ and discuss its biological implications. Figure 4 provides detailed information on how different release constants θ_I affect the threshold value $\mathcal{R}_S^2$. Regarding the parameters δ_I, δ_S, r_R, Figure 4 suggests that even small changes can significantly influence $\mathcal{R}_S^2$, indicating that a slight increase in the release constant θ_I of infected pests can considerably reduce the threshold parameter $\mathcal{R}_S^2$. The larger the release rate θ_I of infective pests, the smaller the resulting threshold value, thereby benefiting pest control efforts.
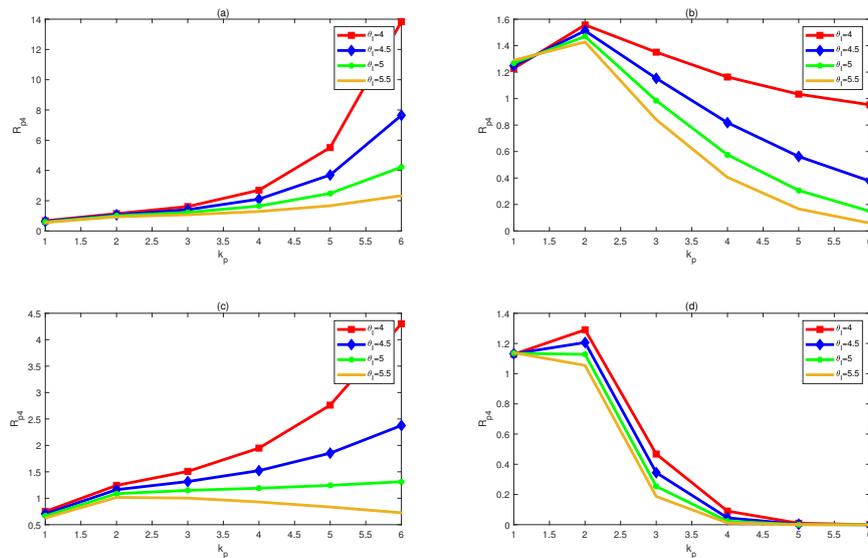
Figure 4. The effects of k_p and θ_I on $\mathcal{R}_S^2$. Parameters are: $r_S = 0.14, T_N = 5, \beta = 0.2, T_p = 10, a_R = 10, a_S = 2, K = 5, \Delta = T_p/k_q, \theta_R = 0.2, \theta_N = 2, p_I = 0.6, p_N = 0.3, p_S = 0.001, \mu_S = 0.8, q_I = 0.003, q_N = 0.004$. (a) $\delta_I = 0.2, \delta_S = 0.08, r_R = 0.13$; (b) $\delta_I = 0.3, \delta_S = 0.08, r_R = 0.13$; (c) $\delta_I = 0.2, \delta_S = 0.1 r_R = 0.13$; (d) $\delta_I = 0.2, \delta_S = 0.08, r_R = 0.2$.

5. Concluding remarks

The objective of this paper is to introduce and analyze an eco-epidemiological pest-natural enemy-plant model that investigates the synergistic combination of chemical and biological control methods. Our model examines the effects of varying frequencies of pesticide sprays and releases of infected pests and natural enemies on the suppression of pest populations. Through the use of impulsive differential equations, we establish threshold conditions for achieving a periodic solution that eradicates the susceptible pest population under different scenarios.

We examine two distinct strategies for releasing natural enemies and two different tactics for pest control. One strategy involves more frequent pesticide sprays, while the other focuses on more frequent releases of susceptible pests. We establish sufficient conditions for the global attraction of the periodic solution that eradicates the susceptible pest population. This finding implies that complete elimination of the pest population is achievable under certain conditions regarding the release quantities of infected pests and natural enemies. In fact, in practical pest control, it is difficult to achieve the complete eradication of pests if the implemented control strategies cannot precisely bring the relevant parameter set to meet the theoretical threshold. In other words, the complete eradication of pests is a challenging goal to achieve.

The theoretical findings presented in Section 4 offer several pest management control strategies. Nevertheless, complete eradication of the pest population is often unattainable in practice. Instead, the primary objective is to prevent the pest population from reaching harmful levels and to maintain its density below the economic threshold. To address this goal, we propose the development of a

state-dependent impulsive differential equation, which will serve as a threshold control model. This approach will be explored as a potential future direction for our research.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

No potential conflict of interest was reported by the authors.

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Appendix

Appendix A

The proof of Theorem 3.2. To establish the local stability of the periodic solution, we employ the small amplitude perturbation method. Let

$$S(t) = s(t), I(t) = i(t) + I^*(t), N(t) = n(t) + N^*(t), R(t) = r(t) + R^*(t),$$

where $s(t), i(t), n(t)$, and $r(t)$ represent small perturbations. By employing Taylor expansions and disregarding higher-order terms, we can linearize model (2.3), resulting in the following set of linearized equations

$$
\left\{
\begin{array}{rcl}
\left.
\begin{array}{rcl}
s'(t) & = & s(t)\mathcal{A}(t), \\
i'(t) & = & \beta s(t)I^*(t) - \delta_I i(t), \\
n'(t) & = & \frac{\mu_N s(t)N^*(t)}{a_S} - \delta_N n(t), \\
r'(t) & = & r_R r(t) - \frac{s(t)R^*(t)}{a_R + (R^*(t))^2},
\end{array}
\right\} & t \neq nT, \\
\left.
\begin{array}{rcl}
s(t^+) & = & (1 - p_S)s(t), \\
i(t^+) & = & (1 + q_I)i(t), \\
n(t^+) & = & (1 + q_N)n(t), \\
r(t^+) & = & (1 + q_R)r(t),
\end{array}
\right\} & t = nT.
\end{array}
\right.
\tag{A.1}
$$

where

$$\mathcal{A} = r_S \left\{ 1 - \frac{(1 + K\beta)}{K} + \frac{\mu_S R^*(t)}{a_R + (R^*(t))^2} - \frac{N^*(t)}{a_S} \right\}.$$

Let $\Phi(t)$ be the fundamental solution matrix of (A.1), $\Phi(t)$ must satisfy

$$
\frac{d\Phi(t)}{dt} = \begin{pmatrix}
\mathcal{A} & 0 & 0 & 0 \\
\beta I^*(t) & -\delta_I & 0 & 0 \\
\frac{\mu_N I^*(t)}{a_S} & 0 & -\delta_N & 0 \\
\frac{R^*(t)}{a_R + (R^*(t))^2} & 0 & 0 & r_R
\end{pmatrix} \Phi(t),
$$

$\Phi(0) = I_4$ is the identical matrix. For all $t \in (nT, (n+1)T]$, the fundamental solution matrix is

$$\Phi(t) = \begin{pmatrix} \exp\left[\int_{nT}^{t} \mathcal{A}(\tau)d\tau\right] & 0 & 0 & 0 \\ * & \exp(-\delta_I(t-nT)) & 0 & 0 \\ * & * & \exp(-\delta_N(t-nT)) & 0 \\ * & * & * & \exp(r_R(t-nT)) \end{pmatrix}.$$

Furthermore, the last four equations of model (A.1) can be expressed as follows

$$\begin{pmatrix} s[(n+1)T^+] \\ i[(n+1)T^+] \\ n[(n+1)T^+] \\ r[(n+1)T^+] \end{pmatrix} = diag\{1-p_S, 1+q_I, 1+q_N, 1+q_R\} \begin{pmatrix} s[(n+1)T] \\ i[(n+1)T] \\ n[(n+1)T] \\ r[(n+1)T] \end{pmatrix}.$$

Hence, the eigenvalues of

$$M = diag\{1-p_S, 1+q_I, 1+q_N, 1+q_R\}\Phi((n+1)T)$$

are

$$\lambda_I = (1+q_I)\exp(-\delta_I T), \qquad \lambda_N = (1+q_N)\exp(-\delta_N T), \qquad \lambda_R = (1+q_R)\exp(r_R T),$$

$$\lambda_S = (1-p_S)\exp\left(\int_{nT}^{(n+1)T} \mathcal{A}(\tau)d\tau\right)$$
$$= (1-p_S)\exp\left\{r_S\left\{T + \frac{(1+K\beta)[\exp(-\delta T)-1]\tilde{I}}{K\delta_I} + \frac{[\exp(-\delta_N T)-1]\tilde{N}}{a_S\delta_N}\right\} + \frac{r_S\mu_S}{\sqrt{a_R}r_R}\left[\arctan\left(\frac{\tilde{R}\exp(r_R T)}{\sqrt{a_R}}\right) - \arctan\left(\frac{\tilde{R}}{\sqrt{a_R}}\right)\right]\right\}.$$

Since $\min\{\lambda_S, \lambda_I, \lambda_R, \lambda_N\} < 1$, then the periodic solution is locally stable. The proof is complete.

The proof of Theorem 3.3. The periodic solution $(0, I^*(t), N^*(t), R^*(t))$ exhibits local asymptotic stability, and we are primarily concerned with its global attractiveness. Let

$$\phi(t, \varepsilon) \doteq r_S - \beta(I^*(t) - \varepsilon_1) + \frac{\mu_S[R^*(t) + \varepsilon_1]}{a_R} - \frac{N^*(t) - \varepsilon_1}{a_S + K^2},$$

where $\varepsilon_1 > 0$. Furthermore, we have $I'(t) = \beta S(t)I(t) - \delta_I I(t) \geq -\delta_I I(t)$. By utilizing the comparison principle and Lemma (3.1), we can establish the existence of n_1 such that $I(t) \geq I^*(t) - \varepsilon_1$ for all $t \geq n_1 T$. Similarly, we can find a n_2 (here $n_2 > n_1$) such that $N(t) \geq N^*(t) - \varepsilon_1$ for all $t \geq n_2 T$. Additionally, there exists a n_3 (here $n_3 > n_2$) such that $R(t) \leq R^*(t) - \varepsilon_1$ for all $t \geq n_3 T$. Consequently, we can conclude that

$$\begin{aligned} S'(t) &= r_S S(t)\left[1 - \frac{S(t)+I(t)}{K}\right] + \frac{\mu_S S(t)R(t)}{a_R+R^2(t)} - \beta S(t)I(t) - \frac{S(t)N(t)}{a_S+S^2(t)} \\ &\leq r_S S(t) - \beta S(t)I(t) + \frac{\mu_S S(t)R(t)}{a_R} - \frac{S(t)N(t)}{a_S+K^2} \\ &\leq S(t)\left\{r_S - \beta(I^*(t) - \varepsilon_1) + \frac{\mu_S[R^*(t)+\varepsilon_1]}{a_R} - \frac{N^*(t)-\varepsilon_1}{a_S+K^2}\right\} \end{aligned}$$

for $t \geq n_3 T$. The first equation of model (2.3) can be expressed as follows

$$\begin{cases} S'(t) &\leq \phi(t, \varepsilon_1)S(t), & t \neq nT, \\ S(t^+) &= (1-p_S)S(t), & t \neq nT. \end{cases}$$

By integrating over the interval $(n_3+k)T$ to $(n_3+k+1)T$, we can obtain

$$\begin{aligned} S[(n_3+k+1)T^+] &\leq S[(n_3+k)T^+](1-p_S)\exp\left(\int_{(n_3+k)T}^{(n_3+k+1)T} \phi(t, \varepsilon_1)dt\right) \\ &= S[(n_3+k)T^+]\mathcal{R}_5 \\ &= S[T^+]\mathcal{R}_5^{n_3+k}, \end{aligned}$$

where

$$\mathcal{R}_5 = (1 - p_S) \exp\Big(\int_{(n_3+k)T}^{(n_3+k+1)T} \phi(t, \varepsilon_1) dt \Big).$$

Since $\mathcal{R}_5 < 1$, we get $S(t) \to 0$ as $k \to \infty$. Thus, for an arbitrary positive constant ε_2 small enough, there exists a n_3 $(n_3 > n_2)$ such that $S(t) < \varepsilon_2$ for all $t \geq n_3 T$.

Next, we will demonstrate that $I(t)$ approaches $I^*(t)$. For $0 < \varepsilon_2 < \delta_I/\beta$, there exists $0 < S(t) < \varepsilon_1$ such that $0 < S(t) < \varepsilon_1$ for $t > n_4 T$ (here $n_4 > n_3$). By considering the second equation of model (2.3), we can conclude that

$$(-\beta\varepsilon_2 - \delta_I)I(t) \leq I'(t) = \beta S(t)I(t) - \delta_I I(t) < (\beta\varepsilon_2 - \delta_I)I(t).$$

Let $I_1(t)$ and $I_2(t)$ be the solutions of the following equations, respectively

$$\begin{cases} I'(t) & = & -(\beta\varepsilon_2 + \delta_I)I(t), & t \neq nT, \\ I(t^+) & = & (1 + q_I)I(t) + \theta_I, & t \neq nT, \end{cases}$$

and

$$\begin{cases} I'(t) & = & -(-\beta\varepsilon_2 + \delta_I)I(t), & t \neq nT, \\ I(t^+) & = & (1 + q_I)I(t) + \theta_I, & t \neq nT. \end{cases}$$

From here it is easy to show that

$$I_1^*(t) = \frac{\theta_R \exp\big[-(\beta\varepsilon_2 + \delta_I)(t - nT)\big]}{1 + (1 + q_I)\exp(-\delta_I T)}, \qquad I_2^*(t) = \frac{\theta_R \exp\big[-(-\beta\varepsilon_2 + \delta_I)(t - nT)\big]}{1 + (1 + q_I)\exp(-\delta_I T)}.$$

It can be easily deduced that $I(t)$ approaches $I^*(t)$ as t tends to infinity.

Similarly, we can prove that $N(t)$ tends to $N^*(t)$ as t approaches infinity. For $R(t)$, the following inequality holds

$$R(t)\Big(r_R - \frac{\varepsilon_2}{a_R}\Big) < \dot{R}(t) < r_R R(t)$$

for $t > n_5 T$ (here $n_5 > n_4$). By combining the solution of model (3.3) with the comparison theorem, we obtain the following result

$$\exp[(r_R - \frac{\varepsilon_2}{a_R}(t - nT))]\tilde{R} < R(t) < \exp[r_R(t - nT)]\tilde{R}.$$

Therefore, as ε_2 approaches 0, we have $R(t) \to R^*(t)$. In fact, if $\mathcal{R}_5 < 1$, then $I(t) \to I^*(t)$, $N(t) \to N^*(t)$, and $R(t) \to R^*(t)$ as $t \to 0$. This implies that the periodic solution $(0, I^*(t), N^*(t), R^*(t))$ is globally asymptotically stable. Thus, the proof is complete.

Appendix B

The proof of Theorem 4.1. We define

$$S(t) = s(t), I(t) = i(t) + I^*(t), N(t) = n(t) + N^*(t), R(t) = r(t) + R^*(t),$$

where $s(t), i(t), n(t)$, and $r(t)$ represent small perturbations. By linearizing model (4.1) through Taylor expansions and neglecting higher-order terms, the resulting linearized equations are given by

$$
\left.
\begin{cases}
\left.
\begin{array}{rcl}
s'(t) & = & s(t)\mathcal{A}_1(t), \\
i'(t) & = & \beta s(t)I^*(t) - \delta_I i(t), \\
n'(t) & = & \frac{\mu_N s(t)N^*(t)}{a_S} - \delta_N n(t), \\
r'(t) & = & r_R r(t) - \frac{s(t)R^*(t)}{a_R+[R^*(t)]^2},
\end{array}
\right\} \; t \neq \tau_n, \quad t \neq \lambda_m, \\[2em]
\left.
\begin{array}{rcl}
s(t^+) & = & (1 - p_S)s(t), \\
i(t^+) & = & (1 - p_I)i(t), \\
n(t^+) & = & (1 - p_N)n(t), \\
r(t^+) & = & r(t),
\end{array}
\right\} \; t = \tau_n, \\[2em]
\left.
\begin{array}{rcl}
s(t^+) & = & s(t), \\
i(t^+) & = & (1 + q_I)i(t), \\
n(t^+) & = & (1 + q_N)n(t), \\
r(t^+) & = & r(t),
\end{array}
\right\} \; t = \lambda_m,
\end{cases}
\right. \tag{B.1}
$$

where

$$
\mathcal{A}_1(t) = r_S \left\{ 1 - \frac{(1+K\beta)\tilde{I}_{N_1}(t)}{K} + \frac{\mu_S \tilde{R}_{N_1}(t)}{a_R + [\tilde{R}_{N_1}(t)]^2} - \frac{\tilde{N}_{N_1}(t)}{a_S} \right\}.
$$

Let $\Phi(t)$ be the fundamental solution matrix of (B.1). In this case, $\Phi(t)$ must satisfy the following

$$
\frac{d\Phi(t)}{dt} =
\begin{pmatrix}
\mathcal{A}_1(t) & 0 & 0 & 0 \\
\beta I^*(t) & -\delta_I & 0 & 0 \\
\frac{\mu_N I^*(t)}{a_S} & 0 & -\delta_N & 0 \\
\frac{R^*(t)}{a_R+[R^*(t)]^2} & 0 & 0 & r_R
\end{pmatrix}
\Phi(t),
$$

where $\Phi(0) = I_4$ is the identical matrix. For all $t \in E_i^h$, the fundamental matrix is

$$
\Phi(t) =
\begin{pmatrix}
\exp(\int_{hT_p}^{t} \mathcal{A}_1(\tau)d\tau) & 0 & 0 & 0 \\
* & \exp(-\delta_I(t - hT_p)) & 0 & 0 \\
* & * & \exp(-\delta_N(t - hT_p)) & 0 \\
* & * & * & \exp(r_R(t - nT_p))
\end{pmatrix}.
$$

Furthermore, system (B.1) can be expressed as

$$
\begin{pmatrix}
s[(h+1)T_p^+] \\
i[(h+1)T_p^+] \\
n[(h+1)T_p^+] \\
r[(h+1)T_p^+]
\end{pmatrix}
= M
\begin{pmatrix}
s[(hT_p)^+] \\
i[(hT_p)^+] \\
n[(hT_p)^+] \\
r[(hT_p)^+]
\end{pmatrix},
$$

where

$$
M = diag\{(1 - p_S)^{k_p}, (1 + q_I)(1 - p_I)^{k_p}, (1 + q_N)(1 - p_N)^{k_p}, 1\}\Phi(h+1)T.
$$

Given that $\min \lambda_S^1, \lambda_I^1, \lambda_R^1, \lambda_N^1 < 1$, indicating that all eigenvalues of matrix M

$$
\lambda_S^1 = \mathcal{R}_9 = (1 - p_s)^{k_p} \exp\left(\int_{nT}^{(n+1)T} \mathcal{A}_1(\tau)d\tau \right) = (1 - p_s) \exp\left\{ r_S T_p + \sum_{i=0}^{k_p} \left\{ \frac{r_S(1+K\beta)\exp(-\delta_I\tau_{i+1})-\exp(-\delta_I\tau_i)I((\tau_i)^+)}{K\delta_I} \right. \right.
$$

$$
+ \frac{r_S\{\exp(-\delta_N\tau_{i+1})-\exp(-\delta_N\tau_i)\}N((\tau_i)^+)}{a_S\delta_N} + \frac{r_S\mu_S}{\sqrt{a_R}r_R}\left[\arctan\left(\frac{R((\tau_{i+1})^+)\exp(r_R\tau_{i+1})}{\sqrt{a_R}} \right) - \arctan\left(\frac{R((\tau_i)^+)\exp(r_R\tau_{i+1})}{\sqrt{a_R}} \right) \right] \right\} \right\} < 1,
$$

$$\lambda_I^1 = (1 + q_I)(1 - p_I)^{k_p} \exp(-\delta_I T) < 1,$$

$$\lambda_N^1 = (1 + q_N)(1 - p_N)^{k_p} \exp(-\delta_N T) < 1, \qquad \lambda_R^1 = \mathcal{R}_8 = \exp(r_R T) < 1,$$

thus, the periodic solution $(0, I^*(t), N^*(t), R^*(t))$ is locally asymptotically stable. Hence, the proof is complete.

The proof of Theorem 4.2. We define

$$\phi(t, \varepsilon_1) \doteq \left\{ r_S - \beta(\tilde{I}_{N_1}(t) - \varepsilon_1) + \frac{\mu_S[\tilde{R}_{N_1}(t) + \varepsilon_1]}{a_R} - \frac{\tilde{N}_{N_1}(t) - \varepsilon_1}{a_S + K^2} \right\},$$

where $\varepsilon_1 > 0$. By utilizing the comparison principle, we can establish the existence of n_1 such that $I(t) \geq I^*(t) - \varepsilon_1$ for all $t \geq n_1 T$. Similarly, there exists a n_2 (here $n_2 > n_1$) such that $N(t) \geq N^*(t) - \varepsilon_1$ for all $t \geq n_2 T$. Additionally, there is a n_3 (here $n_3 > n_2$) such that $R(t) \leq R^*(t) + \varepsilon_1$ for all $t \geq n_3 T$. Hence, we can conclude that

$$
\begin{aligned}
S'(t) &= r_S S(t) + \frac{\mu_S S(t) R(t)}{a_R + R^2(t)} - \beta S(t) I(t) - \frac{S(t) N(t)}{a_S + S^2(t)} \\
&\leq r_S S(t) - \beta S(t) I(t) + \frac{\mu_S S(t) R(t)}{a_R} - \frac{S(t) N(t)}{a_S + K^2} \\
&\leq S(t) \left\{ r_S - \beta(\tilde{I}_{N_1}(t) - \varepsilon_1) + \frac{\mu_S[\tilde{R}_{N_1}(t) + \varepsilon_1]}{a_R} - \frac{\tilde{N}_{N_1}(t) - \varepsilon_1}{a_S + K^2} \right\}
\end{aligned}
$$

for $t \geq n_3 T$. It is follows from model (4.1) that

$$
\begin{cases}
S'(t) & \leq & \varphi(t, \varepsilon_1) S(t), & t \neq \tau_n, \quad t \neq \lambda_m, \\
S(t^+) & = & (1 - p_S) S(t), & t = \tau_n, \\
S(t^+) & = & S(t), & t = \lambda_m,
\end{cases}
$$

Taking integral on $E_i^h, i = 1, 2, \cdots, k_p$ that

$$S[(hT_N + \tau_{i+1})^+] \leq S[(hT_N + \tau_i)^+](1 - p_S) \exp\left(\int_{hT_N + \tau_i}^{hT_N + \tau_{i+1}} \phi(t, \varepsilon_1) dt \right)$$

and

$$S[((h+1)T_N)^+] \leq S[(hT_N)^+](1 - p_S)^{k_p} \exp\left(\int_{hT_N}^{(h+1)T_N} \phi(t, \varepsilon_1) dt \right).$$

We define

$$\lambda_S^1 = (1 - p_S)^{k_p} \exp\left(\int hT_N^{(h+1)T_N} \phi(t, \varepsilon_1) dt \right), t \in ((n_3 + k)T, (n_3 + k + 1)T], k \in \mathbb{Z}^+.$$

If $\lambda_S^1 < 1$, we can observe that $S(t) \to 0$ as $k \to \infty$. Consequently, for any sufficiently small positive constant ε_2, there exists an n_3 (here $n_3 > n_2$) such that $S(t) < \varepsilon_2$ for all $t \geq n_3 T$.

Next, we will prove that $I(t)$ approaches $I^*(t)$. For $0 < \varepsilon_2 < \frac{\delta_I}{\beta}$, there exists ε_1 such that $0 < S(t) < \varepsilon_1$ for $t > n_4 T$ (here $n_4 > n_3$). Consequently, we have

$$(-\beta \varepsilon_2 - \delta_I) I(t) \leq I'(t) = \beta S(t) I(t) - \delta_I I(t) < (-\beta \varepsilon_2 - \delta_I) I(t).$$

Let $I_1(t)$ and $I_2(t)$ denote the solutions of the following equations, respectivelyd

$$
\begin{cases}
I'(t) & = & -(\beta \varepsilon_2 + \delta_I) I(t), & t \neq \tau_n, \quad t \neq \lambda_m \\
I(t^+) & = & (1 - p_I) I(t), & t = \tau_n, \\
I(t^+) & = & (1 + q_I) I(t) + \theta_I, & t = \lambda_m,
\end{cases}
$$

and

$$\begin{cases} I'(t) & = & -(\delta_I - \beta\varepsilon_2)I(t) & t \neq \tau_n, \quad t \neq \lambda_m, \\ I(t^+) & = & (1 - p_I)I(t), & t = \tau_n, \\ I(t^+) & = & (1 + q_I)I(t) + \theta_I, & t = \lambda_m. \end{cases}$$

Therefore, we can conclude that

$$I_1^{**}(t) = \frac{\theta_I}{1 - (1 + q_I)(1 - p_I)^{k_p} \exp[-(\beta\varepsilon_2 + \delta_I)T_N]},$$

$$I_2^{**}(t) = \frac{\theta_R}{1 - (1 + q_I)(1 - p_I)^{k_p} \exp[-(\delta_I - \beta\varepsilon_2)T_N]}.$$

So it gives that $\lim_{\varepsilon_2 \to 0} I_i^{**}(t) = I^*(t)$ from $\lim_{t \to \infty} I(t) = I^*(t)$.

Analogously, we can prove that $N(t) \to N^*(t)$ as $t \to \infty$.

For $R(t)$, it gives that

$$R(t)\left(r_R - \frac{\varepsilon_2}{a_R}\right) < \dot{R}(t) < r_R R(t)$$

for $t > n_5 T$ (here $n_5 > n_4$), and we have

$$\exp[(r_R - \frac{\varepsilon_2}{a_R}(t - nT_N))]\tilde{R} < R(t) < \exp[r_R(t - hT_N)]\tilde{R}.$$

Therefore, we can conclude that $\lim_{\varepsilon_2 \to 0} R(t) = R^*(t)$. In summary, if $\lambda_S^1 < 1$, it follows that $I(t) \to I^*(t)$, $N(t) \to N^*(t)$, and $R(t) \to R^*(t)$ as $t \to 0$. This implies that the periodic solution $(0, I^*(t), N^*(t), R^*(t))$ is globally asymptotically stable. Hence, the proof is complete.