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Research article

Free boundary problem for a nonlocal time-periodic diffusive competition model

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Abstract: In this paper we consider a free boundary problem for a nonlocal time-periodic competition model. One species is assumed to adopt nonlocal dispersal, and the other one adopts mixed dispersal, which is a combination of both random dispersal and nonlocal dispersal. We first prove the global well-posedness of solutions to the free boundary problem with more general growth functions, and then discuss the spreading and vanishing phenomena. Moreover, under the weak competition condition, we study the long-time behaviors of solutions for the spreading case.

Keywords: free boundary problem; competition model; time-periodic; mixed dispersal; spreading and vanishing

1. Introduction

In this paper, we study the following free boundary problem for a nonlocal time-periodic competition model

$$
\begin{cases}
\partial_t u = d_1\mathcal{M}_1(u) + u(a(t) - u - c(t)v), \quad t > 0,\ s_1(t) < x < s_2(t),\\
\partial_t v = d_2\mathcal{M}_2(v) + v(b(t) - v - d(t)u), \quad t > 0,\ s_1(t) < x < s_2(t),\\
u(t,x) = v(t,x) = 0, \quad t \geq 0,\ x \geq s_2(t) \text{ or } x \leq s_1(t),\\
s_2'(t) = -\mu v_x(t, s_2(t)) + \rho_1 \int_{s_1(t)}^{s_2(t)}\int_{s_2(t)}^{+\infty} J_1(x-y)u(t,x)dydx\\
\qquad +\rho_2 \int_{s_1(t)}^{s_2(t)}\int_{s_2(t)}^{+\infty} J_2(x-y)v(t,x)dydx, \quad t \geq 0,\\
s_1'(t) = -\mu v_x(t, s_1(t)) - \rho_1 \int_{s_1(t)}^{s_2(t)}\int_{-\infty}^{s_1(t)} J_1(x-y)u(t,x)dydx\\
\qquad -\rho_2 \int_{s_1(t)}^{s_2(t)}\int_{-\infty}^{s_1(t)} J_2(x-y)v(t,x)dydx, \quad t \geq 0,\\
u(0,x) = u_0(x),\ v(0,x) = v_0(x), \quad |x| \leq s_0,\\
s_2(0) = -s_1(0) = s_0,
\end{cases}
\tag{1.1}
$$

where

$$
\mathcal{M}_1(u) := \int_{s_1(t)}^{s_2(t)} J_1(x-y)u(t,y)dy - u(t,x),
$$
$$
\mathcal{M}_2(v) := \tau\partial_x^2 v + (1-\tau)\left(\int_{s_1(t)}^{s_2(t)} J_2(x-y)v(t,y)dy - v(t,x)\right).
$$

We assume that a,b,c,d are T-periodic positive functions, and $a,c \in C([0,T])$, $b,d \in C^{\frac{\gamma}{2}}([0,T])$ with $0 < \gamma < 1$. The kernel functions $J_i : \mathbb{R} \to \mathbb{R}$ ($i = 1,2$) satisfy that

$$
\begin{aligned}
&J_i \text{ is Lipschitz continuous},\ J_i(x) = J_i(-x) \geq 0,\ J_i(0) > 0,\\
&\int_{\mathbb{R}} J_i(x)dx = 1,\ \sup_{\mathbb{R}} J_i < \infty.
\end{aligned}
\tag{\textbf{K}}
$$

The initial functions u_0, v_0 satisfy

$$
\begin{cases}
u_0 \in C^{1-}([-s_0,s_0]), \quad u_0(\pm s_0) = 0, \quad u_0 > 0 \quad \text{in } (-s_0,s_0),\\
v_0 \in C^{2}([-s_0,s_0]), \quad v_0(\pm s_0) = 0, \quad v_0 > 0 \quad \text{in } (-s_0,s_0),
\end{cases}
\tag{1.2}
$$

where $C^{1-}([-s_0,s_0])$ is defined as the Lipschitz continuous function space. s_0, μ and ρ_1 are positive constants, ρ_2 is a nonnegative constant, $\rho_2 > 0$ when $\tau < 1$, and $\rho_2 = 0$ when $\tau = 1$.

Ecologically, (1.1) describes the competing process of two invasion species, which are initially released in the region $[-s_0, s_0]$ and then spread into a new environment with daily or seasonal changes from two sides of $[-s_0, s_0]$. u and v represent the population densities of two competing species, where all individuals in the population u adopt nonlocal dispersal, while, in the population v, a fraction of individuals adopt nonlocal dispersal and the remaining fraction assumes random dispersal. The latter strategy is called mixed dispersal, which was first proposed by Kao et al. [1]. The positive constants d_1, d_2 are dispersal rates, and the constant $0 < \tau \leq 1$ measures the fraction of individuals v adopting random dispersal. $a(t), b(t)$ represent the intrinsic growth rates of species, and $c(t), d(t)$ represent the competition between species. $[s_1(t), s_2(t)]$ is the habitat of species at time $t \geq 0$, and its boundary fronts $s_1(t), s_2(t)$ are called free boundaries. We assume that the expanding speed of the habitat $[s_1(t), s_2(t)]$

is proportional to the outward flux of species across the boundary, which give rise to the free boundary conditions in (1.1).

Problem (1.1) is a variation of the two species competition model studied by Kao et al. [1]:

$$\begin{cases} \partial_t u = d_1 \left(\int_{\mathbb{R}^N} J(x-y)u(t,y)dy - u \right) + u(a(x) - u - v), \\ \partial_t v = d_2 \left[\tau \partial_x^2 v + (1-\tau) \left(\int_{\mathbb{R}^N} J(x-y)v(t,y)dy - v \right) \right] + v(b(x) - u - v). \end{cases}$$

They investigated how the mixed dispersal affects the invasion of a single species and how the mixed dispersal strategies will evolve in a spatially periodic but temporally constant environment. A complete classification of the global dynamics of competition mode with mixed dispersals was studied in [2].

If $\tau = 0$ and a, b, c, d are constants, (1.1) reduces to the nonlocal diffusion system with free boundaries in [3]. The authors proved the global well-posedness of solutions, and obtained criteria for spreading and vanishing. Moreover, for the weak competition case, they determined the long-time asymptotic limit of solutions when spreading occurs. If $\tau = 1$ and a, b, c, d are constants, (1.1) becomes a free boundary problem of the ecological model with nonlocal and local diffusions considered in [4, 5]. They obtained the well-posedness of solutions and spreading-vanishing results. Moreover, Cao et al. [6] considered a free boundary problem for a nonlocal dispersal competition model in a homogeneous environment, where there is a native species distributed in the whole space $\mathbb{R}$. Some free boundary problems for epidemic models with nonlocal dispersals have also been recently studied in [7, 8].

In the absence of the species v (i.e., $v \equiv 0$), and under the condition that $a(t)$ is a constant, (1.1) reduces to the following free boundary problem

$$\begin{cases} \partial_t u = d_1 \mathcal{M}_1(u) + u(a-u), \quad t > 0,\ s_1(t) < x < s_2(t), \\ u(t,x) = 0, \quad t \ge 0,\ x \ge s_2(t) \text{ or } x \le s_1(t), \\ s_2'(t) = \rho_1 \int_{s_1(t)}^{s_2(t)} \int_{s_2(t)}^{+\infty} J_1(x-y)u(t,x)dydx, \quad t \ge 0, \\ s_1'(t) = -\rho_1 \int_{s_1(t)}^{s_2(t)} \int_{-\infty}^{s_1(t)} J_1(x-y)u(t,x)dydx, \quad t \ge 0, \\ u(0,x) = u_0(x), \quad |x| \le s_0, \\ s_2(0) = -s_1(0) = s_0, \end{cases} \tag{1.3}$$

which has been studied in [9]. Problem (1.3) is a natural generalization of the random dispersal model with free boundaries in [10], and similar results including the well-posedness of global solutions and the spreading-vanishing results were established in [9], from which one can see that the nonlocal dispersal brings some essential difficulties in analysis. The spreading speeds of free boundaries for (1.3) were determined in [11] when spreading happens. After the completion of this paper, (1.3) with the assumptions $a(t,x) = \alpha(t) + \beta(x)$ ($\alpha(t)$ is T-periodic) and supp $J_1 \subset [-r_0, r_0]$ was studied in [12].

Based on the work of Du and Lin [10], random dispersal models with free boundary(ies) have been well studied. The model in [10] has been extended to single species models in a heterogeneous environment [13–17], or with advection [18, 19], time delay [20] and general nonlinear terms [21, 22]. We also refer the readers to [23–25] and the references therein. Moreover, two-species competition problems with free boundary(ies) have been considered in a homogeneous environment [26–30] and heterogeneous time-periodic environment [31, 32]. Competition problems with free boundary(ies) and advection were studied in [33, 34]. Free boundary problems for predator-prey problems [35, 36] and epidemic models with random dispersal [37–40] have also been considered recently.

In this paper, we aim to investigate the well-posedness and dynamics of solutions to (1.1). We first prove the global well-posedness of solutions to (1.1) with more general growth functions. To achieve it, we shall establish the maximum principle for linear parabolic equations with mixed dispersal, and prove that the nonlinear parabolic equations with mixed dispersal (see (2.5)) admit a unique positive solution under the assumption that $s_1'(t)$, $s_2'(t)$ and $u(t,x)$ are merely continuous functions by using the approximation method, which plays an important role in the application of the fixed point theorem (see the proof of Lemma 2.5). Then we establish the dichotomy and criteria for spreading and vanishing. To discuss the spreading and vanishing, we need to consider the existence and asymptotic properties of principal eigenvalues for time-periodic parabolic-type eigenvalue problems with random/mixed dispersal. Since the intrinsic growth rates do not contain the spatial variable, we can transform the parabolic-type eigenvalue problems into elliptic-type eigenvalue problems. This transformation is also used in a discussion on the asymptotic behavior of the solution (see the proof of Theorem 4.4). Moreover, by the comparison principle established in Lemma 3.3, we discuss the asymptotic stability and uniqueness of periodic solutions to the nonlocal and mixed dispersal equations in $\mathbb{R}$ (Lemma 4.6), which are used to investigate the long-time behaviors of the solution for the spreading case under the weak competition condition (Theorem 4.7).

The rest of this paper is organized as follows. In Section 2, we prove the global existence and uniqueness of solutions to problem (1.1) with more general growth functions. The comparison principle and discussions on eigenvalue problems are given in Section 3. In the last section, we investigate the spreading and vanishing of species.

2. Well-posedness

In this section, we give the global well-posedness of solutions to (1.1) with more general growth functions. More precisely, we assume that the nonlinearities satisfy the following assumptions:

(f1) $f_1(t,x,0,v), f_2(t,x,u,0) \equiv 0$, $f_1(t,x,u,v) < 0$ for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$, $u > K$ and $v \geq 0$, and $f_2(t,x,u,v) < 0$ for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$, $u \geq 0$ and $v > K$ with some constant $K > 0$;

(f2) For any given $T_0, l, K_1, K_2 > 0$, there exists a constant $L = L(T_0, l, K_1, K_2) > 0$ such that

$$\|f_2(\cdot, x, u, v)\|_{C^{\frac{\gamma}{2}}([0,T_0])} \leq L$$

for all $(x,u,v) \in [-l,l] \times [0,K_1] \times [0,K_2]$;

(f3) For any given $K_1, K_2 > 0$, there exists a constant $L^* = L^*(K_1,K_2) > 0$ such that

$$|f_i(t,x,u,v) - f_i(t,y,u,v)| \leq L^*|x-y|$$

for all $(t,x,y,u,v) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \times [0,K_1] \times [0,K_2]$;

(f4) $f_i(t,x,u,v)$ is locally Lipschitz in $u, v \in \mathbb{R}_+$ uniformly for $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$, i.e., for any $K_1, K_2 > 0$, there exists a constant $\hat{L} = \hat{L}(K_1,K_2) > 0$ such that

$$|f_i(t,x,u_1,v_1) - f_i(t,x,u_2,v_2)| \leq \hat{L}(|u_1-u_2| + |v_1-v_2|)$$

for all $(t,x,u_i,v_i) \in \mathbb{R}_+ \times \mathbb{R} \times [0,K_1] \times [0,K_2]$ $(i = 1,2)$.

Obviously, the growth functions in (1.1) satisfy the conditions (**f1**) – (**f4**). We consider the following problem

$$\begin{cases} \partial_t u = d_1\mathcal{M}_1(u) + f_1(t,x,u,v), \quad t>0,\ s_1(t)<x<s_2(t), \\ \partial_t v = d_2\mathcal{M}_2(v) + f_2(t,x,u,v), \quad t>0,\ s_1(t)<x<s_2(t), \\ u(t,x)=v(t,x)=0, \quad t\ge 0,\ x\ge s_2(t) \text{ or } x\le s_1(t), \\ s_2'(t) = -\mu v_x(t,s_2(t)) + \rho_1\int_{s_1(t)}^{s_2(t)}\int_{s_2(t)}^{+\infty} J_1(x-y)u(t,x)dydx \\ \qquad +\rho_2\int_{s_1(t)}^{s_2(t)}\int_{s_2(t)}^{+\infty} J_2(x-y)v(t,x)dydx, \quad t\ge 0, \\ s_1'(t) = -\mu v_x(t,s_1(t)) - \rho_1\int_{s_1(t)}^{s_2(t)}\int_{-\infty}^{s_1(t)} J_1(x-y)u(t,x)dydx \\ \qquad -\rho_2\int_{s_1(t)}^{s_2(t)}\int_{-\infty}^{s_1(t)} J_2(x-y)v(t,x)dydx, \quad t\ge 0, \\ u(0,x)=u_0(x),\ v(0,x)=v_0(x), \quad |x|\le s_0, \\ s_2(0)=-s_1(0)=s_0, \end{cases} \tag{2.1}$$

in which the assumptions on parameters and functions are the same as for (1.1).

Notations. Throughout the paper, we denote $\Omega_{T_0}^{s_1,s_2} = \{(t,x) : t\in(0,T_0], x\in(s_1(t),s_2(t))\}$, $\Omega_\infty^{s_1,s_2} = \{(t,x) : t\in(0,+\infty), x\in(s_1(t),s_2(t))\}$, $\overline{\Omega}_\infty^{s_1,s_2} = \{(t,x) : t\in[0,+\infty), x\in[s_1(t),s_2(t)]\}$, $D_{T_0} = (0,T_0]\times(-1,1)$ and $a_T = \frac{1}{T}\int_0^T a(t)dt$. Under the transform $x(t,z) = \frac{(s_2(t)-s_1(t))z+s_2(t)+s_1(t)}{2}$, we always denote $\tilde{f}(t,z) = f(t,x(t,z)) = f(t,\frac{(s_2(t)-s_1(t))z+s_2(t)+s_1(t)}{2})$. $C^{1,1-}(\overline{\Omega}_{T_0}^{s_1,s_2})$ denotes the class of functions that are C^1 in t and Lipschitz continuous in x.

Theorem 2.1. *For any given* (u_0,v_0) *satisfying* (1.2)*,* (2.1) *has a unique global solution* $(u,v;s_1,s_2)\in C^{1,1-}(\overline{\Omega}_{T_0}^{s_1,s_2})\times C^{1+\frac{\gamma}{2},2+\gamma}(\Omega_{T_0}^{s_1,s_2})\times[C^{1+\frac{\gamma}{2}}([0,T_0])]^2$ *for any* $0<T_0<+\infty$ *and*

$$\begin{aligned} &0<u\le K_1,\ 0<v\le K_2, \quad \forall\,(t,x)\in\Omega_{T_0}^{s_1,s_2}, \\ &0<-v_x(t,s_2(t)),\ v_x(t,s_1(t))\le K_3, \quad 0<t\le T_0, \end{aligned} \tag{2.2}$$

where

$$K_1 := \max\{\|u_0\|_{L^\infty}, K\}, \quad K_2 := \max\{\|v_0\|_{L^\infty}, K\},$$
$$K_3 := 2K_2\max\left\{\sqrt{\frac{\hat{L}+d_2(1-\tau)}{2d_2\tau}}, \frac{4\|v_0\|_{C^1([-s_0,s_0])}}{3K_2}\right\}$$

and $\hat{L}=\hat{L}(K_1,K_2)$ *is the Lipschitz constant defined in (**f4**).*

To prove Theorem 2.1, we first establish the maximum principle for linear parabolic equations with mixed dispersal. For some s_0 and T_0, we define

$$\mathbb{S}_{2,T_0}^{s_0} := \{s_2\in C^1([0,T_0]) : s_2(0)=s_0,\ 0<s_2'(t)\le R(t)\},$$
$$\mathbb{S}_{1,T_0}^{s_0} := \{s_1\in C^1([0,T_0]) :\ -s_1\in\mathbb{S}_{2,T_0}^{s_0}\}$$

with

$$R(t) := \mu K_3 + 2(s_0\rho_1K_1 + s_0\rho_2K_2 + \mu K_3)e^{(\rho_1K_1+\rho_2K_2)t}.$$

Lemma 2.2. *Assume that* $(s_1, s_2) \in \mathbb{S}^{s_0}_{1,T_0} \times \mathbb{S}^{s_0}_{2,T_0}$. *If* $v \in C^{1,2}(\Omega^{s_1,s_2}_{T_0}) \cap C(\overline{\Omega}^{s_1,s_2}_{T_0})$ *satisfies, for some* $g \in L^\infty(\Omega^{s_1,s_2}_{T_0})$,

$$\begin{cases} \partial_t v \geq d_2\mathcal{M}_2(v) + g(t,x)v, & (t,x) \in \Omega^{s_1,s_2}_{T_0}, \\ v(t, s_1(t)), v(t, s_2(t)) \geq 0, & t \in (0, T_0], \\ v(0,x) \geq 0, & x \in [-s_0, s_0], \end{cases} \tag{2.3}$$

then $v \geq 0$ *in* $\overline{\Omega}^{s_1,s_2}_{T_0}$. *If we further assume that* $v(0,x) \not\equiv 0$ *in* $[-s_0, s_0]$, *then* $v > 0$ *in* $\Omega^{s_1,s_2}_{T_0}$.

Proof. (i) Let $k > 0$ be a large constant satisfying $-k + g(t,x) < 0$ for all $(t,x) \in \Omega^{s_1,s_2}_{T_0}$. Then $\vartheta(t,x) := e^{-kt}v(t,x)$ satisfies

$$\partial_t\vartheta \geq d_2\mathcal{M}_2(\vartheta) + [-k - d_2(1-\tau) + g(t,x)]\vartheta.$$

Next we prove that $\vartheta \geq 0$ in $\overline{\Omega}^{s_1,s_2}_{T_0}$.

Suppose that $\vartheta_{\inf} := \inf_{(t,x)\in\overline{\Omega}^{s_1,s_2}_{T_0}} \vartheta(t,x) < 0$. By the boundary conditions in (2.3), we know that $\vartheta_{\inf} = \vartheta(t^*, x^*) < 0$ for some $(t^*, x^*) \in \Omega^{s_1,s_2}_{T_0}$. Since $\partial_t\vartheta(t^*, x^*) \leq 0$ and $\partial_x^2\vartheta(t^*, x^*) \geq 0$, we have

$$\begin{aligned} \partial_t\vartheta(t^*, x^*) &\geq d_2\left[\tau\partial_x^2\vartheta(t^*, x^*) + (1-\tau)\int_{s_1(t^*)}^{s_2(t^*)} J_2(x^* - y)\vartheta(t^*, y)dy\right] \\ &\quad + [-k - d_2(1-\tau) + g(t^*, x^*)]\vartheta(t^*, x^*) \\ &\geq d_2\tau\partial_x^2\vartheta(t^*, x^*) + d_2(1-\tau)\vartheta_{\inf}\int_{\mathbb{R}} J_2(x^* - y)dy \\ &\quad + [-k - d_2(1-\tau) + g(t^*, x^*)]\vartheta_{\inf} \\ &= d_2\tau\partial_x^2\vartheta(t^*, x^*) + [-k + g(t^*, x^*)]\vartheta_{\inf}, \end{aligned}$$

which is a contradiction since $[-k + g(t^*, x^*)]\vartheta_{\inf} > 0$. Thus, $\vartheta \geq 0$ in $\overline{\Omega}^{s_1,s_2}_{T_0}$, which implies that

$$v \geq 0 \quad \text{in } \overline{\Omega}^{s_1,s_2}_{T_0}. \tag{2.4}$$

(ii) Now assume that $v(0,x) \not\equiv 0$ in $[-s_0, s_0]$. By (2.4) and the assumption $J_2(x) \geq 0$, we have

$$\partial_t v \geq d_2\tau\partial_x^2 v + [g(t,x) - d_2(1-\tau)]v.$$

Define the transform

$$x(t,z) = \tfrac{(s_2(t) - s_1(t))z + s_2(t) + s_1(t)}{2}, \quad \text{that is,} \quad z(t,x) = \tfrac{2x - s_1(t) - s_2(t)}{s_2(t) - s_1(t)}.$$

Let $\tilde{v}(t,z) = v(t, x(t,z))$ and $\tilde{g}(t,z) = g(t, x(t,z))$. Then, $\tilde{v}(t,z)$ satisfies

$$\begin{cases} \partial_t\tilde{v} \geq d_2\tau\xi(t)\partial_z^2\tilde{v} + \zeta(t,z)\partial_z\tilde{v} + [\tilde{g}(t,z) - d_2(1-\tau)]\tilde{v}, & (t,z) \in D_{T_0}, \\ \tilde{v}(t, \pm 1) \geq 0, & t \in (0, T_0], \\ \tilde{v}(0,z) = v(0, s_0 z) \geq 0, & z \in [-1, 1], \end{cases}$$

where

$$\xi(t) = \tfrac{4}{(s_2(t) - s_1(t))^2}, \quad \zeta(t,z) = \tfrac{s_2'(t) + s_1'(t)}{s_2(t) - s_1(t)} + \tfrac{(s_2'(t) - s_1'(t))z}{s_2(t) - s_1(t)}.$$

From the classical maximum principle, we know that $\tilde{v} > 0$ in D_{T_0}. Thus, $v > 0$ in $\Omega^{s_1,s_2}_{T_0}$. This completes the proof.

Next, we prove that nonlinear parabolic equations with mixed dispersal (see (2.5)) admit a unique positive strong solution for any given continuous function $u(t,x)$ and C^1-functions $(s_1(t), s_2(t)) \in \mathbb{S}^{s_0}_{1,T_0} \times \mathbb{S}^{s_0}_{2,T_0}$. The proof is divided into two parts. First, in Lemma 2.3 we establish the existence and uniqueness results on positive classical solutions by applying the upper-lower solutions method, under the assumptions that u is Hölder continuous and $(s_1(t), s_2(t)) \in \widehat{\mathbb{S}}^{s_0}_{1,T_0} \times \widehat{\mathbb{S}}^{s_0}_{2,T_0}$ with

$$\begin{aligned}
\widehat{\mathbb{S}}^{s_0}_{2,T_0} &:= \{s_2 \in C^{1+\frac{\gamma}{2}}([0,T_0]) : s_2(0) = s_0,\ 0 < s_2'(t) \le R(t)\},\\
\widehat{\mathbb{S}}^{s_0}_{1,T_0} &:= \{s_1 \in C^{1+\frac{\gamma}{2}}([0,T_0]) :\ -s_1 \in \widehat{\mathbb{S}}^{s_0}_{2,T_0}\}.
\end{aligned}$$

Then, we obtain the existence result in Lemma 2.4 by the approximation method.

Lemma 2.3. *If* $(s_1, s_2) \in \widehat{\mathbb{S}}^{s_0}_{1,T_0} \times \widehat{\mathbb{S}}^{s_0}_{2,T_0}$, $u \in C^{\frac{\gamma}{2},\gamma}(\overline{\Omega}^{s_1,s_2}_{T_0})$, f_2 *satisfies (**f1**)–(**f4**) and* v_0 *satisfies* (1.2)*, then for any* $T_0 > 0$*, the problem*

$$\begin{cases}
\partial_t v = d_2\mathcal{M}_2(v) + f_2(t,x,u,v), & (t,x) \in \Omega^{s_1,s_2}_{T_0},\\
v(t,s_1(t)) = v(t,s_2(t)) = 0, & t \in (0,T_0],\\
v(0,x) = v_0(x), & x \in [-s_0,s_0]
\end{cases} \tag{2.5}$$

has a unique solution $v \in C^{1+\frac{\gamma}{2},2+\gamma}(\Omega^{s_1,s_2}_{T_0})$. *Moreover,* v *satisfies*

$$\begin{aligned}
&0 < v \le K_2 \quad \text{in } \Omega^{s_1,s_2}_{T_0},\\
&0 < -v_x(t,s_2(t)), v_x(t,s_1(t)) \le K_3 \quad \text{for } t \in (0,T_0].
\end{aligned} \tag{2.6}$$

Proof. For the existence and uniqueness, we mainly adopt the classical upper-lower solutions method. Assume that $\bar{v}, \underline{v}$ are respectively nonnegative upper and lower solutions of (2.5). Since $u \in C^{\frac{\gamma}{2},\gamma}(\overline{\Omega}^{s_1,s_2}_{T_0})$ and $\bar{v}, \underline{v} \in C(\overline{\Omega}^{s_1,s_2}_{T_0})$, we know that $0 \le u, \bar{v}, \underline{v} \le M$ with some constant $M > 0$ for all $(t,x) \in \overline{\Omega}^{s_1,s_2}_{T_0}$. By (**f4**), we have, for some constant $k > d_2(1-\tau)$,

$$|f_2(t,x,u,v_1) - f_2(t,x,u,v_2)| \le [k - d_2(1-\tau)]|v_1 - v_2|$$

for any $(t,x) \in \overline{\Omega}^{s_1,s_2}_{T_0}$ and $u, v_1, v_2 \in [0,M]$.

For any $\vartheta \in C(\overline{\Omega}^{s_1,s_2}_{T_0})$ satisfying $\vartheta \in [0,M]$, we define a mapping Φ by $v = \Phi\vartheta$, where $v \in C^{\frac{1+\gamma}{2},1+\gamma}(\overline{\Omega}^{s_1,s_2}_{T_0})$ is the unique solution of

$$\begin{cases}
\partial_t v - d_2\tau\partial_x^2 v + kv = d_2(1-\tau)\left(\int_{s_1(t)}^{s_2(t)} J_2(x-y)\vartheta(t,y)dy - \vartheta\right) + f_2(t,x,u,\vartheta) + k\vartheta, & (t,x) \in \Omega^{s_1,s_2}_{T_0},\\
v(t,s_1(t)), v(t,s_2(t)) = 0, & t \in (0,T_0],\\
v(0,x) = v_0(x), & x \in [-s_0,s_0].
\end{cases} \tag{2.7}$$

The existence and uniqueness of $v \in C^{\frac{1+\gamma}{2},1+\gamma}(\overline{\Omega}^{s_1,s_2}_{T_0})$ is guaranteed by the L^p theory for linear parabolic equations and the Sobolev embedding theorem. It is easy to check that Φ is monotone in the sense that if any $\vartheta_1, \vartheta_2 \in C(\overline{\Omega}^{s_1,s_2}_{T_0})$ satisfy $0 \le \vartheta_1, \vartheta_2 \le M$ and $\vartheta_2 \ge \vartheta_1$, then $\Phi\vartheta_2 \ge \Phi\vartheta_1$.

We then construct two sequences $\{v^{(n)}\}$ and $\{w^{(n)}\}$ by defining $v^{(1)} = \Phi\bar{v}$, $v^{(n)} = \Phi v^{(n-1)}$, $w^{(1)} = \Phi\underline{v}$, $w^{(n)} = \Phi w^{(n-1)}$, $n \geq 2$. Thus, $\underline{v} \leq w^{(1)} \leq w^{(2)} \leq \cdots \leq w^{(n)} \leq v^{(n)} \leq \cdots \leq v^{(2)} \leq v^{(1)} \leq \bar{v}$. We conclude that the pointwise limits

$$(w^*(t,x), v^*(t,x)) = (\lim_{n\to\infty} w^{(n)}(t,x), \lim_{n\to\infty} v^{(n)}(t,x))$$

exist at each point in $\Omega_{T_0}^{s_1,s_2}$ and

$$\underline{v} \leq w^* \leq v^* \leq \bar{v} \quad \text{in } \Omega_{T_0}^{s_1,s_2}.$$

Similar to the proof of Theorem 2.4.6 in [41], we can show that v^*, w^* are classical solutions of (2.5) and satisfy $v^* = w^*$. Moreover, the solution in $[\underline{v}, \bar{v}]$ is unique.

Clearly, $\underline{v} = 0$ and $\bar{v} = K_2$ are lower and upper solutions of (2.5), respectively. Then there exists a unique solution v between 0 and K_2. Note that f_2 satisfies the assumption (**f4**). Lemma 2.2 implies that v is the unique solution of (2.5).

Define

$$\Omega := \left\{(t,x) : 0 < t \leq T_0,\ s_2(t) - M^{-1} < x < s_2(t)\right\}$$

and construct an auxiliary function

$$\psi(t,x) = K_2[2M(s_2(t) - x) - M^2(s_2(t) - x)^2].$$

We will choose M such that $\psi \geq v$ holds over Ω.

Direct calculations show that

$$\partial_t\psi = 2K_2Ms_2'(t)(1 - M(s_2(t) - x)) \geq 0,\ \ -\partial_{xx}\psi = 2K_2M^2,\ f_2(t,x,u,v) \leq \hat{L}v.$$

Then,

$$\begin{aligned}\partial_t\psi - d_2\mathcal{M}_2(\psi) &\geq 2d_2\tau K_2M^2 - d_2(1-\tau)K_2\int_{s_1(t)}^{s_2(t)} J_2(x-y)dy\\ &\geq 2d_2\tau K_2M^2 - d_2(1-\tau)K_2 \geq \hat{L}K_2\\ &\geq \hat{L}v \geq \partial_t v - d_2\mathcal{M}_2(v) \quad \text{in } \Omega,\end{aligned}$$

if $M^2 \geq \frac{\hat{L}+d_2(1-\tau)}{2d_2\tau}$. On the other hand,

$$\psi(t, s_2(t) - M^{-1}) = K_2 \geq v(t, s_2(t) - M^{-1}), \quad \psi(t, s_2(t)) = 0 = v(t, s_2(t)).$$

Choosing

$$M := \max\left\{\sqrt{\frac{\hat{L}+d_2(1-\tau)}{2d_2\tau}}, \frac{4\|v_0\|_{C^1([-s_0,s_0])}}{3K_2}\right\},$$

we can prove that $v_0(x) \leq \psi(0,x)$ for $x \in [s_0 - M^{-1}, s_0]$. By applying Lemma 2.2 to $\psi - v$ over Ω, we have $v(t,x) \leq \psi(t,x)$ for $(t,x) \in \Omega$. Then $v_x(t, s_2(t)) \geq -2K_2M$. Moreover, since $v(t, s_2(t)) = 0$ and $v > 0$ in $\Omega_{T_0}^{s_1,s_2}$, we get $v_x(t, s_2(t)) < 0$. The estimates for $v_x(t, s_1(t))$ can be obtained similarly.

Now, by the approximation method we get the unique strong solution of (2.5) provided that $s_1'(t), s_2'(t)$ and u are merely continuous functions, which plays an important role in the proof of Lemma 2.5 later.

Lemma 2.4. *If* $(s_1, s_2) \in \mathbb{S}^{s_0}_{1,T_0} \times \mathbb{S}^{s_0}_{2,T_0}$, $u \in C(\overline{\Omega}^{s_1,s_2}_{T_0})$, f_2 *satisfies* ***(f1)–(f4)*** *and* v_0 *satisfies* (1.2)*, then* (2.5) *has a unique solution* $v \in W^{1,2}_p(\Omega^{s_1,s_2}_{T_0}) \cap C^{\frac{1+\gamma}{2},1+\gamma}(\overline{\Omega}^{s_1,s_2}_{T_0})$ *with any* $p > 3$*. Moreover, v satisfies* (2.6)*.*

Proof. Step 1. (Uniqueness) Let

$$\tilde{v}(t,z) = v(t, x(t,z)), \quad \tilde{f}_2(t,z,\tilde{u},\tilde{v}) = f_2(t, x(t,z), u(t, x(t,z)), v(t, x(t,z))),$$

then the problem becomes

$$\begin{cases} \partial_t \tilde{v} = d_2\tau\xi(t)\partial_z^2\tilde{v} + \zeta(t,z)\partial_z\tilde{v} + d_2(1-\tau)\left(\frac{s_2(t)-s_1(t)}{2}\int_{-1}^1 J_2(\frac{s_2(t)-s_1(t)}{2}(z-s))\tilde{v}(t,s)ds - \tilde{v}\right) \\ \qquad +\tilde{f}_2(t,z,\tilde{u},\tilde{v}), \quad (t,z) \in D_{T_0}, \\ \tilde{v}(t,\pm 1) = 0, \quad t \in (0, T_0], \\ \tilde{v}(0,z) = v_0(s_0 z), \quad z \in [-1,1]. \end{cases} \tag{2.8}$$

Assume that $v_i(t,x) \in W^{1,2}_p(\Omega^{s_1,s_2}_{T_0}) \cap C^{\frac{1+\gamma}{2},1+\gamma}(\overline{\Omega}^{s_1,s_2}_{T_0})$, $i = 1,2$, are two solutions of (2.5), then $\tilde{v}_i(t,z) = v_i(t, x(t,z)) \in W^{1,2}_p(D_{T_0}) \cap C^{\frac{1+\gamma}{2},1+\gamma}(\overline{D}_{T_0})$ are two solutions of (2.8). Let $\tilde{w} = \tilde{v}_1 - \tilde{v}_2$, then $\tilde{w}$ satisfies the following problem

$$\begin{cases} \partial_t \tilde{w} = d_2\tau\xi(t)\partial_z^2\tilde{w} + \zeta(t,z)\partial_z\tilde{w} + d_2(1-\tau)\left(\frac{s_2(t)-s_1(t)}{2}\int_{-1}^1 J_2(\frac{s_2(t)-s_1(t)}{2}(z-s))\tilde{w}(t,s)ds - \tilde{w}\right) \\ \qquad +\tilde{f}_2(t,z,\tilde{u},\tilde{v}_1) - \tilde{f}_2(t,z,\tilde{u},\tilde{v}_2), \quad (t,z) \in D_{T_0}, \\ \tilde{w}(t,\pm 1) = 0, \quad t \in (0, T_0], \\ \tilde{w}(0,z) = 0, \quad z \in [-1,1]. \end{cases} \tag{2.9}$$

Multiplying the first equation in (2.9) by $\tilde{w}\chi_{[0,t]}$, where $\chi_{[0,t]}$ is the characteristic function in $[0,t]$ with any $0 < t \le T_0$, and then integrating over $(0, T_0] \times [-1,1]$ gives

$$\begin{aligned} \frac{1}{2}\int_{-1}^1 \tilde{w}^2(t,z)\Big|_0^t dz &= -d_2\tau\int_0^t\int_{-1}^1 \xi(t)(\partial_z\tilde{w})^2 dzdt + \int_0^t\int_{-1}^1 \zeta(t,z)\tilde{w}\partial_z\tilde{w}dzdt \\ &\quad + d_2(1-\tau)\int_0^t\int_{-1}^1\left(\frac{s_2(t)-s_1(t)}{2}\int_{-1}^1 J_2(\frac{s_2(t)-s_1(t)}{2}(z-s))\tilde{w}(t,s)ds - \tilde{w}\right)\tilde{w}dzdt \\ &\quad + \int_0^t\int_{-1}^1[\tilde{f}_2(t,z,\tilde{u},\tilde{v}_1) - \tilde{f}_2(t,z,\tilde{u},\tilde{v}_2)]\tilde{w}dzdt. \end{aligned}$$

By Young's inequality with $0 < \varepsilon < \frac{4d_2\tau}{(s_2(T_0)-s_1(T_0))^2}$,

$$\int_0^t\int_{-1}^1 \zeta(t,z)\tilde{w}\partial_z\tilde{w}dzdt \le \varepsilon\int_0^t\int_{-1}^1(\partial_z\tilde{w})^2dzdt + C(\varepsilon)\int_0^t\int_{-1}^1\tilde{w}^2dzdt.$$

By Hölder inequality and the continuity of J_2,

$$\begin{aligned} &d_2(1-\tau)\int_0^t\int_{-1}^1\left(\frac{s_2(t)-s_1(t)}{2}\int_{-1}^1 J_2(\frac{s_2(t)-s_1(t)}{2}(z-s))\tilde{w}(t,s)ds - \tilde{w}(t,z)\right)\tilde{w}(t,z)dzdt \\ &\le d_2(1-\tau)C\int_0^t(\int_{-1}^1|\tilde{w}(t,z)|dz)^2dt - d_2(1-\tau)\int_0^t\int_{-1}^1\tilde{w}^2dzdt \\ &\le d_2(1-\tau)C_1\int_0^t\int_{-1}^1\tilde{w}^2dzdt. \end{aligned}$$

By the Lipschitz continuity of f_2 with respect to $\tilde{v}$,

$$\int_0^t \int_{-1}^1 [\tilde{f}_2(t,z,\tilde{u},\tilde{v}_1) - \tilde{f}_2(t,z,\tilde{u},\tilde{v}_2)]\tilde{w}(t,z)dzdt \leq L\int_0^t \int_{-1}^1 \tilde{w}^2 dzdt.$$

Combining the above estimates, we have

$$\int_{-1}^1 \tilde{w}^2(t,z)dz \leq C\int_0^t \int_{-1}^1 \tilde{w}^2 dzdt.$$

By Gronwall's inequality, we know $\int_0^t \int_{-1}^1 \tilde{w}^2 dzdt = 0$, which implies that $\tilde{w} = 0$, a.e. in $(0,t] \times [-1,1]$. Since $t \in (0,T_0]$ is arbitrary and $\tilde{w} \in C(\overline{D}_{T_0})$, we can obtain that $\tilde{w} = 0$ for all (t,z) in $[0,T_0] \times [-1,1]$, which implies the uniqueness of the solution.

Step 2. (Existence) For any $(s_1,s_2) \in \mathbb{S}^{s_0}_{1,T_0} \times \mathbb{S}^{s_0}_{2,T_0}$, we can find some sequences $(s_{1,n},s_{2,n}) \in \widehat{\mathbb{S}}^{s_0}_{1,T_0} \times \widehat{\mathbb{S}}^{s_0}_{2,T_0}$ such that $s_{1,n} \to s_1$ and $s_{2,n} \to s_2$ in $C^1([0,T_0])$. Moreover, for every $u(t,x) \in C(\overline{\Omega}^{s_1,s_2}_{T_0})$, we can obtain $\tilde{u}(t,z) = u(t,x(t,z)) \in C(\overline{D}_{T_0})$ and find some sequence $\tilde{u}_n \in C^{\frac{\gamma}{2},\gamma}(\overline{D}_{T_0})$ such that $\tilde{u}_n \to \tilde{u}$ in $C(\overline{D}_{T_0})$. Taking $u_n(t,x) = \tilde{u}_n(t, \frac{2x-s_{1,n}(t)-s_{2,n}(t)}{s_{2,n}(t)-s_{1,n}(t)})$, we know $u_n \in C^{\frac{\gamma}{2},\gamma}(\overline{\Omega}^{s_{1,n},s_{2,n}}_{T_0})$.

Consider the approximate problem

$$\begin{cases} \partial_t v = d_2\left[\tau\partial_x^2 v + (1-\tau)\left(\int_{s_{1,n}(t)}^{s_{2,n}(t)} J_2(x-y)v(t,y)dy - v\right)\right] + f_2(t,x,u_n,v), & (t,x) \in \Omega^{s_{1,n},s_{2,n}}_{T_0}, \\ v(t,s_{1,n}(t)) = v(t,s_{2,n}(t)) = 0, & t \in (0,T_0], \\ v(0,x) = v_0(x), & x \in [-s_0,s_0]. \end{cases} \tag{2.10}$$

By Lemma 2.3, we know that (2.10) has a unique classical solution $v_n \in C^{1+\frac{\gamma}{2},2+\gamma}(\Omega^{s_{1,n},s_{2,n}}_{T_0})$, and

$$\begin{aligned} &0 < v_n \leq K_2 \quad \text{in } \Omega^{s_{1,n},s_{2,n}}_{T_0}, \\ &0 < -\partial_x v_n(t,s_{2,n}(t)), \partial_x v_n(t,s_{1,n}(t)) \leq K_3 \quad \text{for } t \in (0,T_0]. \end{aligned}$$

Let $\tilde{v}_n(t,z) = v_n(t,x_n(t,z))$ and

$$\tilde{f}_2(t,z,\tilde{u}_n,\tilde{v}_n) = f_2(t,x_n(t,z),u_n(t,x_n(t,z)),v_n(t,x_n(t,z)))$$

with $x_n(t,z) = \frac{(s_{2,n}(t)-s_{1,n}(t))z+s_{2,n}(t)+s_{1,n}(t)}{2}$, then $\tilde{v}_n(t,z) \in C^{1+\frac{\gamma}{2},2+\gamma}(D_{T_0})$ is the unique solution of

$$\begin{cases} \partial_t \tilde{v}_n = d_2\tau\xi_n(t)\partial_z^2\tilde{v}_n + \zeta_n(t,z)\partial_z\tilde{v}_n \\ \qquad + d_2(1-\tau)\left(\frac{s_{2,n}(t)-s_{1,n}(t)}{2}\int_{-1}^1 J_2(\frac{s_{2,n}(t)-s_{1,n}(t)}{2}(z-s))\tilde{v}_n(t,s)ds - \tilde{v}_n\right) \\ \qquad + \tilde{f}_2(t,z,\tilde{u}_n,\tilde{v}_n), \quad (t,z) \in D_{T_0}, \\ \tilde{v}_n(t,\pm 1) = 0, \quad t \in (0,T_0], \\ \tilde{v}_n(0,z) = v_0(s_0 z), \quad z \in [-1,1], \end{cases} \tag{2.11}$$

and it satisfies

$$\begin{aligned} &0 < \tilde{v}_n \leq K_2 \quad \text{in } D_{T_0}, \\ &0 < -\frac{2}{s_{2,n}(t)-s_{1,n}(t)}\partial_z\tilde{v}_n(t,1), \ \frac{2}{s_{2,n}(t)-s_{1,n}(t)}\partial_z\tilde{v}_n(t,-1) \leq K_3 \quad \text{for } t \in (0,T_0]. \end{aligned} \tag{2.12}$$

Let

$$g(t,z) := d_2(1-\tau)\left(\frac{s_{2,n}(t)-s_{1,n}(t)}{2}\int_{-1}^{1} J_2(\frac{s_{2,n}(t)-s_{1,n}(t)}{2}(z-s))\tilde{v}_n(t,s)ds\right) + \tilde{f}_2(t,z,\tilde{u}_n,\tilde{v}_n),$$

we have $g \in L^\infty(D_{T_0})$. Applying the L^p theory for linear parabolic equations to (2.11), we know that the solution $\tilde{v}_n$ satisfies $\|\tilde{v}_n\|_{W_p^{1,2}(D_{T_0})} \leq C$, where C is independent of n. By the weak compactness of the bounded set in $W_p^{1,2}(D_{T_0})$ and $\mathring{W}_p^{1,1}(D_{T_0})$, and the compactly embedding theorem ($W_p^{1,1}(D_{T_0}) \hookrightarrow\hookrightarrow L^p(D_{T_0})$), there exists a subsequence, still denoted by $\{\tilde{v}_n\}$, such that $\tilde{v}_n \rightharpoonup \tilde{v}$ in $W_p^{1,2}(D_{T_0}) \cap \mathring{W}_p^{1,1}(D_{T_0})$, $\partial_z\tilde{v}_n \to \partial_z\tilde{v}$ in $L^p(D_{T_0})$ and $\tilde{v}_n \to \tilde{v}$ in $L^p(D_{T_0})$, which implies that $\tilde{v} \in W_p^{1,2}(D_{T_0}) \cap \mathring{W}_p^{1,1}(D_{T_0})$ is the strong solution of (2.8). By the Sobolev embedding theorem, $\tilde{v} \in C^{\frac{1+\gamma}{2},1+\gamma}(\overline{D}_{T_0})$.

Note that $\tilde{v}_n$ satisfies (2.12). From the fact $\partial_z\tilde{v}_n \to \partial_z\tilde{v}$, $\tilde{v}_n \to \tilde{v}$ in $L^p(D_{T_0})$ (then a.e. in D_{T_0}) and $\tilde{v} \in C^{\frac{1+\gamma}{2},1+\gamma}(\overline{D}_{T_0})$, we have that $0 < \tilde{v} \leq K_2$ in D_{T_0} and $0 < -\frac{2}{s_2(t)-s_1(t)}\partial_z\tilde{v}(t,1), \frac{2}{s_2(t)-s_1(t)}\partial_z\tilde{v}(t,-1) \leq K_3$ for $t \in (0,T_0]$. Thus, $v(t,x) = \tilde{v}(t,z(t,x))$ satisfies (2.6), which completes the proof.

In the following lemma, we prove the well-posedness for (2.1) with any fixed $(s_1,s_2) \in \mathbb{S}_{1,T_0}^{s_0} \times \mathbb{S}_{2,T_0}^{s_0}$ by using the fixed point theorem. Denote

$$\begin{aligned}
\mathbb{X}_{T_0}^1 &:= \left\{u \in C(\overline{\Omega}_{T_0}^{s_1,s_2}) :\ 0 \leq u \leq K_1, u(0,x) = u_0(x), u(t,s_1(t)) = u(t,s_2(t)) = 0\right\},\\
\mathbb{X}_{T_0}^2 &:= \left\{v \in C(\overline{\Omega}_{T_0}^{s_1,s_2}) :\ 0 \leq v \leq K_2, v(0,x) = v_0(x), v(t,s_1(t)) = v(t,s_2(t)) = 0\right\},\\
\mathbb{X}_{T_0}^{s_1,s_2} &:= \mathbb{X}_{T_0}^1 \times \mathbb{X}_{T_0}^2.
\end{aligned}$$

Lemma 2.5. *For any $T_0 > 0$ and $(s_1,s_2) \in \mathbb{S}_{1,T_0}^{s_0} \times \mathbb{S}_{2,T_0}^{s_0}$, the problem*

$$\begin{cases}
\partial_t u = d_1\mathcal{M}_1(u) + f_1(t,x,u,v),\ (t,x) \in \Omega_{T_0}^{s_1,s_2},\\
\partial_t v = d_2\mathcal{M}_2(v) + f_2(t,x,u,v),\quad (t,x) \in \Omega_{T_0}^{s_1,s_2},\\
u(t,s_1(t)) = u(t,s_2(t)) = 0,\quad t \in [0,T_0],\\
v(t,s_1(t)) = v(t,s_2(t)) = 0,\quad t \in [0,T_0],\\
u(0,x) = u_0(x), v(0,x) = v_0(x),\quad x \in [-s_0,s_0]
\end{cases} \tag{2.13}$$

has a unique solution $(u,v) \in \mathbb{X}_{T_0}^{s_1,s_2}$, and it satisfies

$$\begin{aligned}
&0 < u \leq K_1,\ 0 < v \leq K_2 \quad \text{in } \Omega_{T_0}^{s_1,s_2},\\
&0 < -v_x(t,s_2(t)), v_x(t,s_1(t)) \leq K_3 \quad \text{in } (0,T_0].
\end{aligned} \tag{2.14}$$

Moreover, $v \in W_p^{1,2}(\Omega_{T_0}^{s_1,s_2}) \cap C^{\frac{1+\gamma}{2},1+\gamma}(\overline{\Omega}_{T_0}^{s_1,s_2})$ with any $p > 3$.

Proof. For $u^* \in \mathbb{X}_{\tilde{T}}^1$ with $0 < \tilde{T} \leq T_0$, from Lemma 2.4 we know that the initial-boundary value problem (2.5) with (u,T_0) replaced by $(u^*,\tilde{T})$ admits a unique solution $v \in \mathbb{X}_{\tilde{T}}^2$. For such $v \in \mathbb{X}_{\tilde{T}}^2$, we consider

$$\begin{cases}
\partial_t u = d_1\mathcal{M}_1(u) + f_1(t,x,u,v),\quad (t,x) \in \Omega_{T_0}^{s_1,s_2},\\
u(t,s_1(t)) = u(t,s_2(t)) = 0,\quad t \in [0,T_0],\\
u(0,x) = u_0(x),\quad x \in [-s_0,s_0].
\end{cases}$$

By Lemma 2.3 in [9], it has a unique solution $u \in \mathbb{X}^1_{\tilde{T}}$. We define a mapping $\mathcal{F}_{\tilde{T}} : \mathbb{X}^1_{\tilde{T}} \to \mathbb{X}^1_{\tilde{T}}$ by $\mathcal{F}_{\tilde{T}} u^* = u$. If $\mathcal{F}_{\tilde{T}} u^* = u^*$, then (u^*, v) solves (2.13) with T_0 replaced by $\tilde{T}$.

Next, we shall show that $\mathcal{F}_{\tilde{T}}$ has a fixed point in $\mathbb{X}^1_{\tilde{T}}$ for small $\tilde{T}$. We assume that $u_i^* \in \mathbb{X}^1_{\tilde{T}}$, $u_i = \mathcal{F}_{\tilde{T}} u_i^*$, and v_i is the unique solution of (2.5) with (u, T_0) replaced by $(u_i^*, \tilde{T})$. Denote $\theta^* = u_1^* - u_2^*$, $\theta = u_1 - u_2$ and $w = v_1 - v_2$. Note that w satisfies

$$\begin{cases} \partial_t w = d_2 \mathcal{M}_2(w) + a_0(t,x) w + b_0(t,x)\theta^*, \quad (t,x) \in \Omega^{s_1,s_2}_{\tilde{T}}, \\ w(t, s_1(t)) = w(t, s_2(t)) = 0, \quad t \in [0, \tilde{T}], \\ w(0,x) = 0, \quad x \in [-s_0, s_0], \end{cases}$$

where

$$\begin{aligned} a_0(t,x) &= \int_0^1 f_{2,v}(t, x, u_1^*, v_2 + (v_1 - v_2)\tau) d\tau, \\ b_0(t,x) &= \int_0^1 f_{2,u}(t, x, u_2^* + (u_1^* - u_2^*)\tau, v_2) d\tau. \end{aligned}$$

Let $\tilde{\theta}^*(t,z) = \theta^*(t, x(t,z))$, $\tilde{w}(t,z) = w(t, x(t,z))$, $\tilde{a}_0(t,z) = a_0(t, x(t,z))$ and $\tilde{b}_0(t,z) = b_0(t, x(t,z))$. It is easy to see that $\tilde{w}$ satisfies the following:

$$\begin{cases} \partial_t \tilde{w} = d_2 \tau \xi(t) \partial_z^2 \tilde{w} + \zeta(t,z) \partial_z \tilde{w} + [\tilde{a}_0(t,z) - d_2(1-\tau)]\tilde{w} \\ \qquad + d_2(1-\tau) \frac{s_2(t)-s_1(t)}{2} \int_{-1}^1 J_2(\frac{s_2(t)-s_1(t)}{2}(z-s)) \tilde{w}(t,s) ds + \tilde{b}_0(t,z)\tilde{\theta}^*, \quad (t,z) \in D_{\tilde{T}}, \\ \tilde{w}(t, \pm 1) = 0, \quad t \in [0, \tilde{T}], \\ \tilde{w}(0,z) = 0, \quad z \in [-1, 1]. \end{cases}$$

Similar to the proof of Theorem 2.1 (Step 2) in [42], we can extend $s_2(t)$, $s_1(t)$, $\tilde{w}(t,z)$, $\tilde{a}_0(t,z)$, $\tilde{b}_0(t,z)$ and $\tilde{\theta}^*(t,z)$ by defining $s_2(t) = s_2(\tilde{T})$, $s_1(t) = s_1(\tilde{T})$, $\tilde{w}(t,z) = \tilde{w}(\tilde{T}, z)$, $\tilde{a}_0(t,z) = \tilde{a}_0(\tilde{T}, z)$, $\tilde{b}_0(t,z) = \tilde{b}_0(\tilde{T}, z)$ and $\tilde{\theta}^*(t,z) = \tilde{\theta}^*(\tilde{T}, z)$ for $t \in [\tilde{T}, T_1]$ with some $T_1 \le T_0$. Consider the above equation on D_{T_1}. By the L^p theory for linear parabolic equations, we have

$$\begin{aligned} \|\tilde{w}\|_{W^{1,2}_p(D_{T_1})} &\le C\Big(\Big\| \frac{s_2(t)-s_1(t)}{2} \int_{-1}^1 J_2(\frac{s_2(t)-s_1(t)}{2}(z-s)) \tilde{w}(t,s) ds \Big\|_{L^p(D_{T_1})} + \|\tilde{\theta}^*\|_{L^p(D_{T_1})} \Big) \\ &\le C\Big(\|\tilde{w}\|_{C(\overline{D}_{T_1})} \Big\| \int_{\frac{s_2(t)-s_1(t)}{2}(z-1)}^{\frac{s_2(t)-s_1(t)}{2}(z+1)} J_2(y) dy \Big\|_{L^p(D_{T_1})} + \|\tilde{\theta}^*\|_{L^p(D_{T_1})} \Big) \\ &\le C (2T_1)^{\frac{1}{p}} (\|\tilde{w}\|_{C(\overline{D}_{T_1})} + \|\tilde{\theta}^*\|_{C(\overline{D}_{T_1})}) \end{aligned}$$

with some positive constant $C = C(T_1)$. By the Sobolev embedding theorem, the Hölder semi-norm $[\tilde{w}]_{C^{\frac{\gamma}{2},\gamma}(\overline{D}_{T_1})} \le C' \|\tilde{w}\|_{W^{1,2}_p(D_{T_1})}$ for some positive constant $C' = C'(\frac{1}{T_1})$. Note that

$$|\tilde{w}(t,z)| = |\tilde{w}(t,z) - \tilde{w}(0,z)| \le [\tilde{w}]_{C^{\frac{\gamma}{2},\gamma}(\overline{D}_{\tilde{T}})} t^{\frac{\gamma}{2}} \le [\tilde{w}]_{C^{\frac{\gamma}{2},\gamma}(\overline{D}_{\tilde{T}})} \tilde{T}^{\frac{\gamma}{2}}, \quad \forall (t,z) \in D_{\tilde{T}}.$$

It follows that

$$\begin{aligned} \|\tilde{w}\|_{C(\overline{D}_{\tilde{T}})} &\le [\tilde{w}]_{C^{\frac{\gamma}{2},\gamma}(\overline{D}_{\tilde{T}})} \tilde{T}^{\frac{\gamma}{2}} = [\tilde{w}]_{C^{\frac{\gamma}{2},\gamma}(\overline{D}_{T_1})} \tilde{T}^{\frac{\gamma}{2}} \le C' \|\tilde{w}\|_{W^{1,2}_p(D_{T_1})} \tilde{T}^{\frac{\gamma}{2}} \\ &\le CC' \tilde{T}^{\frac{\gamma}{2}} (2T_1)^{\frac{1}{p}} (\|\tilde{w}\|_{C(\overline{D}_{T_1})} + \|\tilde{\theta}^*\|_{C(\overline{D}_{T_1})}) \\ &= CC' \tilde{T}^{\frac{\gamma}{2}} (2T_1)^{\frac{1}{p}} (\|\tilde{w}\|_{C(\overline{D}_{\tilde{T}})} + \|\tilde{\theta}^*\|_{C(\overline{D}_{\tilde{T}})}). \end{aligned}$$

Choosing $\tilde{T}$ small such that $CC'\tilde{T}^{\frac{\gamma}{2}}(2T_1)^{\frac{1}{p}} < \frac{1}{2}$, we have

$$\|\tilde{w}\|_{C(\overline{D}_{\tilde{T}})} \leq \|\tilde{\theta}^*\|_{C(\overline{D}_{\tilde{T}})}.$$

Similar to the proof of Lemma 2.3 (Step 3) in [5], we deduce

$$\|\theta\|_{C(\overline{\Omega}_{\tilde{T}}^{s_1,s_2})} \leq \frac{1}{2}\|\theta^*\|_{C(\overline{\Omega}_{\tilde{T}}^{s_1,s_2})},$$

for sufficiently small $\tilde{T}$. By applying the contraction mapping theorem, we can show that $\mathcal{F}_{\tilde{T}}$ has a unique fixed point $u \in \mathbb{X}^1_{\tilde{T}}$.

Following the arguments in the proof of Lemma 2.3 (Step 5) in [5], we can prove that the unique solution (u, v) of (2.13) can be extended to $\Omega^{s_1,s_2}_{T_0}$ and $(u, v) \in \mathbb{X}^{s_1,s_2}_{T_0}$. The estimates of $v_x(t, s_2(t)), v_x(t, s_1(t))$ and the regularity of v have been established in Lemma 2.4.

Proof of Theorem 2.1**.** By Lemma 2.5, for any $T_0 > 0$ and $(s_1, s_2) \in \mathbb{S}^{s_0}_{1,T_0} \times \mathbb{S}^{s_0}_{2,T_0}$, we can find a unique $(u, v) \in \mathbb{X}^{s_1,s_2}_{T_0}$ that solves (2.13), and (2.14) holds. For $0 < t \leq T_0$, define the mapping

$$\mathcal{G}(s_1, s_2) = (\tilde{s}_1, \tilde{s}_2)$$

by

$$\begin{aligned}
\tilde{s}_2(t) &= s_0 - \mu\int_0^t v_x(\tau, s_2(\tau))d\tau + \rho_1\int_0^t\int_{s_1(\tau)}^{s_2(\tau)}\int_{s_2(\tau)}^{+\infty} J_1(x-y)u(\tau,x)dydxd\tau \\
&\quad +\rho_2\int_0^t\int_{s_1(\tau)}^{s_2(\tau)}\int_{s_2(\tau)}^{+\infty} J_2(x-y)v(\tau,x)dydxd\tau, \\
\tilde{s}_1(t) &= -s_0 - \mu\int_0^t v_x(\tau, s_1(\tau))d\tau - \rho_1\int_0^t\int_{s_1(\tau)}^{s_2(\tau)}\int_{-\infty}^{s_1(\tau)} J_1(x-y)u(\tau,x)dydxd\tau \\
&\quad -\rho_2\int_0^t\int_{s_1(\tau)}^{s_2(\tau)}\int_{-\infty}^{s_1(\tau)} J_2(x-y)v(\tau,x)dydxd\tau.
\end{aligned}$$

To prove this theorem, we need to prove that if T_0 is sufficiently small, then $\mathcal{G}$ maps a closed subset Σ_{T_0} of $\mathbb{S}^{s_0}_{1,T_0} \times \mathbb{S}^{s_0}_{2,T_0}$ into itself and is a contraction mapping. The proof can be completed by using similar arguments as that of Theorem 2.1 in [3,5]. Here we omit the details.

3. Comparison principle and some eigenvalue problems

In this section, we first give two comparison principles, and then investigate the existence and asymptotic properties of principal eigenvalues for some eigenvalue problems.

3.1. The comparison principle

In this subsection, we discuss the comparison principle for (1.1).

Lemma 3.1. *Suppose that* $T_0 \in (0, \infty)$*,* $\bar{s}_1, \bar{s}_2 \in C^1([0, T_0])$*,* $\bar{u} \in C(\overline{\Omega}^{\bar{s}_1,\bar{s}_2}_{T_0})$*,* $\bar{v} \in C^{1,2}(\Omega^{\bar{s}_1,\bar{s}_2}_{T_0}) \cap C(\overline{\Omega}^{\bar{s}_1,\bar{s}_2}_{T_0})$*,* *and* $(\bar{u}, \bar{v}; \bar{s}_1, \bar{s}_2)$ *satisfies the following:*

$$\begin{cases} \partial_t \bar{u} \geq d_1 \left(\int_{\bar{s}_1(t)}^{\bar{s}_2(t)} J_1(x-y)\bar{u}(t,y)dy - \bar{u} \right) + \bar{u}(a(t) - \bar{u} - c(t)\bar{v}), \quad (t,x) \in \Omega_{T_0}^{\bar{s}_1,\bar{s}_2}, \\ \partial_t \bar{v} \geq d_2 \left[\tau \partial_x^2 \bar{v} + (1-\tau) \left(\int_{\bar{s}_1(t)}^{\bar{s}_2(t)} J_2(x-y)\bar{v}(t,y)dy - \bar{v} \right) \right] + \bar{v}(b(t) - \bar{v} - d(t)\bar{u}), \quad (t,x) \in \Omega_{T_0}^{\bar{s}_1,\bar{s}_2}, \\ \bar{s}_2'(t) \geq -\mu \bar{v}_x(t, \bar{s}_2(t)) + \rho_1 \int_{\bar{s}_1(t)}^{\bar{s}_2(t)} \int_{\bar{s}_2(t)}^{+\infty} J_1(x-y)\bar{u}(t,x)dydx \\ \qquad +\rho_2 \int_{\bar{s}_1(t)}^{\bar{s}_2(t)} \int_{\bar{s}_2(t)}^{+\infty} J_2(x-y)\bar{v}(t,x)dydx, \quad 0 < t \leq T_0, \\ \bar{s}_1'(t) \leq -\mu \bar{v}_x(t, \bar{s}_1(t)) - \rho_1 \int_{\bar{s}_1(t)}^{\bar{s}_2(t)} \int_{-\infty}^{\bar{s}_1(t)} J_1(x-y)\bar{u}(t,x)dydx \\ \qquad -\rho_2 \int_{\bar{s}_1(t)}^{\bar{s}_2(t)} \int_{-\infty}^{\bar{s}_1(t)} J_2(x-y)\bar{v}(t,x)dydx, \quad 0 < t \leq T_0, \\ \bar{u}(0,x) \geq u_0(x), \bar{v}(0,x) \geq v_0(x), \quad |x| \leq s_0, \\ \bar{s}_2(0) \geq s_0, \ \bar{s}_1(0) \leq -s_0. \end{cases} \tag{3.1}$$

Moreover, $\bar{u}(t, \bar{s}_1(t)), \bar{u}(t, \bar{s}_2(t)) \geq 0$ *and* $\bar{v}(t, \bar{s}_1(t)) = \bar{v}(t, \bar{s}_2(t)) = 0$ *for* $0 < t \leq T_0$. *Then, the solution* $(u, v; s_1, s_2)$ *of* (1.1) *satisfies*

$$s_1(t) \geq \bar{s}_1(t), \ s_2(t) \leq \bar{s}_2(t) \quad \textit{in } (0, T_0],$$
$$u(t,x) \leq \bar{u}(t,x), \ v(t,x) \leq \bar{v}(t,x) \quad \textit{for } (t,x) \in \overline{\Omega}_{T_0}^{s_1,s_2}.$$

Remark 3.2. We should mention that the condition $\bar{v}(t, \bar{s}_1(t)) = \bar{v}(t, \bar{s}_2(t)) = 0$ is necessary in the proof. If $\tau = 0$, as considered in [3], then the expressions of $s_2'(t)$, $s_1'(t)$ in (1.1) and $\bar{s}_2'(t)$, $\bar{s}_1'(t)$ in (3.1) do not include the terms $-\mu v_x(t, s_2(t))$, $-\mu v_x(t, s_1(t))$ and $-\mu \bar{v}_x(t, \bar{s}_2(t))$, $-\mu \bar{v}_x(t, \bar{s}_1(t))$, respectively. In such a case, the conditions $\bar{v}(t, \bar{s}_1(t)) = \bar{v}(t, \bar{s}_2(t)) = 0$ can be weakened into $\bar{v}(t, \bar{s}_1(t)), \bar{v}(t, \bar{s}_2(t)) \geq 0$.

In what follows, we establish a comparison principle for the following nonlocal evolution equation

$$\begin{cases} u_t = d_1[\int_\Omega J_1(x-y)u(t,y)dy - u] + u(a(t) - u), \quad (t,x) \in \mathbb{R} \times \overline{\Omega}, \\ u(0,x) = u(T,x), \quad x \in \overline{\Omega}, \end{cases} \tag{3.2}$$

where Ω is a bounded, connected open interval in $\mathbb{R}$. Define the function spaces $\mathbb{Y}_\Omega, \mathbb{Y}_\Omega^+, \mathbb{Y}_\Omega^{++}$:

$$\mathbb{Y}_\Omega = \left\{ \Phi \in C^{1,0}(\mathbb{R} \times \overline{\Omega}) : \Phi(t+T, x) = \Phi(t,x) \text{ for any } (t,x) \in \mathbb{R} \times \overline{\Omega} \right\},$$
$$\mathbb{Y}_\Omega^+ = \left\{ \Phi \in \mathbb{Y}_\Omega : \Phi \geq 0 \text{ in } \mathbb{R} \times \overline{\Omega} \right\},$$
$$\mathbb{Y}_\Omega^{++} = \left\{ \Phi \in \mathbb{Y}_\Omega : \Phi > 0 \text{ in } \mathbb{R} \times \overline{\Omega} \right\},$$

where $C^{1,0}(\mathbb{R} \times \overline{\Omega})$ denotes the class of functions that are C^1 in t and continuous in x. We call a T-periodic function $\bar{u} \in \mathbb{Y}_\Omega^{++}$ an upper solution of (3.2) when $\bar{u} \in \mathbb{Y}_\Omega^{++}$ satisfies

$$\bar{u}_t \geq d_1[\int_\Omega J_1(x-y)\bar{u}(t,y)dy - \bar{u}] + \bar{u}(a(t) - \bar{u})$$

for $t \in \mathbb{R}$ and $x \in \overline{\Omega}$. The lower solution can be defined by reversing the inequality.

Lemma 3.3. *Let* $\underline{u} \in \mathbb{Y}_\Omega^+$ *and* $\bar{u} \in \mathbb{Y}_\Omega^{++}$ *be a lower and an upper solution to* (3.2)*, respectively. Then,* $\underline{u} \leq \bar{u}$ *in* $\mathbb{R} \times \overline{\Omega}$.

Proof. The proof follows some ideas of Section 6.3 in [43], where the nonlocal stationary problem was considered. Define

$$\alpha^* := \inf\{\alpha > 0 : \ \alpha\bar{u} \geq \underline{u} \text{ in } \mathbb{R}\times\overline{\Omega}\}.$$

We shall prove $\alpha^* \leq 1$. If it does not hold, then

$$\begin{aligned}&(\alpha^*\bar{u})_t - d_1[\int_\Omega J_1(x-y)\alpha^*\bar{u}(t,y)dy - \alpha^*\bar{u}] - \alpha^*\bar{u}(a(t)-\alpha^*\bar{u})\\ &\geq \alpha^*\bar{u}(a(t)-\bar{u}) - \alpha^*\bar{u}(a(t)-\alpha^*\bar{u}) = \alpha^*(\alpha^*-1)\bar{u}^2 > 0.\end{aligned} \tag{3.3}$$

Since $[0,T]\times\overline{\Omega}$ is compact, we know that α^* is attainable, i.e., there exists $(t_0,x_0)\in[0,T]\times\overline{\Omega}$ such that $\alpha^*\bar{u}(t_0,x_0) = \underline{u}(t_0,x_0)$.

(i) If $(t_0,x_0)\in(0,T)\times\overline{\Omega}$, then $\partial_t(\alpha^*\bar{u}-\underline{u})(t_0,x_0) = 0$, since (t_0,x_0) is a minimum point of $\alpha^*\bar{u}-\underline{u}$.

(ii) If $(t_0,x_0)\in\{0,T\}\times\overline{\Omega}$, by the T-periodicity and C^1-smoothness of $\bar{u},\underline{u}$ in t, we can also deduce $\partial_t(\alpha^*\bar{u}-\underline{u})(t_0,x_0) = 0$.

Thus, the following holds

$$\begin{aligned}&(\alpha^*\bar{u})_t(t_0,x_0) - d_1[\int_\Omega J_1(x_0-y)\alpha^*\bar{u}(t_0,y)dy - \alpha^*\bar{u}(t_0,x_0)] - \alpha^*\bar{u}(t_0,x_0)(a(t_0)-\alpha^*\bar{u}(t_0,x_0))\\ &= \underline{u}_t(t_0,x_0) - d_1[\int_\Omega J_1(x_0-y)\alpha^*\bar{u}(t_0,y)dy - \underline{u}(t_0,x_0)] - \underline{u}(t_0,x_0)(a(t_0)-\underline{u}(t_0,x_0))\\ &\leq d_1\int_\Omega J_1(x_0-y)[\underline{u}(t_0,y)-\alpha^*\bar{u}(t_0,y)]dy \leq 0,\end{aligned}$$

which contradicts (3.3). Therefore, $\alpha^*\leq 1$, which implies that $\underline{u}\leq\bar{u}$ in $[0,T]\times\overline{\Omega}$.

3.2. Some eigenvalue problems

In this subsection, we mainly study the properties of principal eigenvalues for some eigenvalue problems. We always assume Ω to be a bounded, connected open set in $\mathbb{R}$ and define its length by $|\Omega|$.

For $(t,x)\in\mathbb{R}\times\overline{\Omega}$, we define

$$-(L_\Omega+\alpha)[\phi](t,x) = \phi_t - d_1[\int_\Omega J_1(x-y)\phi(t,y)dy - \phi] - \alpha(t)\phi, \tag{3.4}$$

where $\alpha\in C_T(\mathbb{R}) := \{\alpha\in C(\mathbb{R}) : \alpha(t+T) = \alpha(t) > 0, \forall t\in\mathbb{R}\}$.

Let

$$\lambda_1(-(L_\Omega+\alpha)) = \inf\Big\{\Re\lambda : \ \lambda\in\sigma(-(L_\Omega+\alpha))\Big\},$$

where $\sigma(-(L_\Omega+\alpha))$ is the spectrum of $-(L_\Omega+\alpha)$. By Theorem **A**(1) in [44], we know that $\lambda_1(-(L_\Omega+\alpha))$ is the principal eigenvalue of $-(L_\Omega+\alpha)$, which means that there exists an eigenfunction $\phi\in\mathbb{Y}_\Omega^{++}$ satisfying

$$-(L_\Omega+\alpha)[\phi] = \lambda_1(-(L_\Omega+\alpha))\phi.$$

Lemma 3.4. *(see Theorem **B** in [44]) Let $q(t,x;q_0)$ be a solution of*

$$\begin{cases} q_t = d_1[\int_\Omega J_1(x-y)q(t,y)dy - q] + q(\alpha(t)-q), & t>0, x\in\overline{\Omega},\\ q(0,x) = q_0(x), & x\in\overline{\Omega},\end{cases}$$

*where J_1 satisfies (**K**), $\alpha \in C_T(\mathbb{R})$, $q_0 \in C(\overline{\Omega})$ is non-negative and $q_0 \not\equiv 0$.*

(i) *If $\lambda_1(-(L_\Omega + \alpha)) < 0$, then*

$$q_t = d_1[\int_\Omega J_1(x-y)q(t,y)dy - q] + q(\alpha(t) - q), \quad t \in \mathbb{R}, x \in \overline{\Omega} \tag{3.5}$$

has a unique solution $\hat{q} \in \mathbb{Y}_\Omega^{++}$. Moreover,

$$\lim_{t\to+\infty} \|q(t,\cdot;q_0) - \hat{q}(t,\cdot)\|_{C(\overline{\Omega})} = 0.$$

(ii) *If $\lambda_1(-(L_\Omega + \alpha)) > 0$, then* (3.5) *admits no solution in $\mathbb{Y}_\Omega^+ \setminus \{0\}$ and*

$$\lim_{t\to+\infty} \|q(t,\cdot;u_0)\|_{C(\overline{\Omega})} = 0.$$

Remark 3.5. For the case $\lambda_1(-(L_\Omega + \alpha)) = 0$, it has been shown in [44] that (3.5) admits no solution in $\mathbb{Y}_\Omega^+ \setminus \{0\}$, but the global dynamical behavior was not provided. Since $\alpha(t)$ is independent of the spatial variable in this paper, we can also get $\|q(t,\cdot;q_0)\|_{C(\overline{\Omega})} \to 0$. More details can be seen in the proof of Theorem 4.4.

In what follows, we present some further properties of λ_1.

Lemma 3.6. *Let J_1 satisfies (**K**) and $\alpha \in C_T(\mathbb{R})$. Then*

(i) *$\lambda_1(-(L_\Omega + \alpha))$ is continuous, strictly decreasing in $|\Omega|$;*

(ii) $\lim_{|\Omega|\to+\infty} \lambda_1(-(L_\Omega + \alpha)) = -\alpha_T$*, where $\alpha_T = \frac{1}{T}\int_0^T \alpha(t)dt$;*

(iii) $\lim_{|\Omega|\to 0} \lambda_1(-(L_\Omega + \alpha)) = d_1 - \alpha_T$.

Proof. Let $\phi \in \mathbb{Y}_\Omega^{++}$ be an eigenfunction of $-(L_\Omega+\alpha)$ associated with the principal eigenvalue $\lambda_1(-(L_\Omega+\alpha))$. We define

$$\psi(t,x) = e^{-\int_0^t(\alpha(s)-\alpha_T)ds}\phi(t,x).$$

Obviously, $\psi \in \mathbb{Y}_\Omega^{++}$.

Multiplying the equation $-(L_\Omega+\alpha)[\phi] = \lambda_1(-(L_\Omega+\alpha))\phi$ by the function $t \mapsto e^{-\int_0^t(\alpha(s)-\alpha_T)ds}$, we have

$$-\psi_t + d_1[\int_\Omega J_1(x-y)\psi(t,y)dy - \psi] + \alpha_T\psi + \lambda_1(-(L_\Omega + \alpha))\psi = 0.$$

Integrating both sides over $[0,T]$ with respect to t, and taking $\Psi(x) = \frac{1}{T}\int_0^T \psi(t,x)dt$, we obtain

$$d_1[\int_\Omega J_1(x-y)\Psi(y)dy - \Psi] + \alpha_T\Psi + \lambda_1(-(L_\Omega + \alpha))\Psi = 0.$$

Denote by $\lambda_1(-(\mathcal{L}_\Omega + \alpha_T))$ the principal eigenvalue of

$$-(\mathcal{L}_\Omega + \alpha_T)[\Psi](x) = -d_1[\int_\Omega J_1(x-y)\Psi(y)dy - \Psi] - \alpha_T\Psi = \lambda\Psi \quad \text{in } \Omega, \tag{3.6}$$

then we have

$$\lambda_1(-(L_\Omega + \alpha)) = \lambda_1(-(\mathcal{L}_\Omega + \alpha_T)). \tag{3.7}$$

According to Proposition 3.4 in [9], we can prove that $\lambda_1(-(L_\Omega + \alpha))$ satisfies the properties (i)–(iii).

Now, we consider another periodic-parabolic eigenvalue problem

$$\begin{cases} -(\tilde{L}_\Omega + \alpha)[\varphi](t,x) = \varphi_t - d_2[\tau\varphi_{xx} + (1-\tau)(\int_\Omega J_2(x-y)\varphi(t,y)dy - \varphi)] - \alpha(t)\varphi \\ \qquad\qquad\qquad\quad = \lambda\varphi \quad \text{in } [0,T]\times\Omega, \\ \varphi(t,x) = 0 \quad \text{on } [0,T]\times\partial\Omega, \\ \varphi(0,x) = \varphi(T,x) \quad \text{in } \Omega. \end{cases} \tag{3.8}$$

As showed in Section II.14 of [45], based on the Krein-Rutman theorem, we can prove that (3.8) admits a principal eigenvalue $\lambda_1(-(\tilde{L}_\Omega + \alpha))$ with the principal eigenfunction φ.

For later applications, we give the following lemma.

Lemma 3.7. *Let J_2 satisfies (**K**) and $\alpha \in C_T(\mathbb{R})$. Then*
(*i*) *$\lambda_1(-(\tilde{L}_\Omega + \alpha))$ is a strictly decreasing continuous function in $|\Omega|$. Moreover, $\lim_{|\Omega|\to 0}\lambda_1(-(\tilde{L}_\Omega + \alpha)) = +\infty$ and $\lim_{|\Omega|\to+\infty}\lambda_1(-(\tilde{L}_\Omega + \alpha)) = -\alpha_T$. Then, $\lambda_1(-(\tilde{L}_\Omega + \alpha)) = 0$ has a unique root $|\Omega| = s^*$;*
(*ii*) *if $\lambda_1(-(\tilde{L}_\Omega + \alpha)) < 0$, then the problem*

$$\begin{cases} \varphi_t - d_2[\tau\varphi_{xx} + (1-\tau)(\int_\Omega J_2(x-y)\varphi(t,y)dy - \varphi)] = \varphi(\alpha(t) - \varphi) \quad \text{in } (0,+\infty)\times\Omega, \\ \varphi = 0 \quad \text{on } (0,+\infty)\times\partial\Omega \end{cases}$$

admits a unique T-periodic positive solution φ^*, and φ^* is globally asymptotically stable.

Proof. (*i*) Let φ be an eigenfunction of (3.8) associated with the principal eigenvalue $\lambda_1(-(\tilde{L}_\Omega + \alpha))$. Define

$$\psi(t,x) = e^{-\int_0^t(\alpha(s)-\alpha_T)ds}\varphi(t,x), \quad \forall (t,x) \in \mathbb{R}\times\overline{\Omega}.$$

Similar to (3.6), $\lambda_1(-(\tilde{L}_\Omega + \alpha))$ is the principal eigenvalue of

$$\begin{cases} -(\tilde{\mathcal{L}}_\Omega + \alpha_T)[\Psi](t,x) = -d_2[\tau\Psi_{xx} + (1-\tau)(\int_\Omega J_2(x-y)\Psi(y)dy - \Psi)] - \alpha_T\Psi = \lambda\Psi \quad \text{in } \Omega, \\ \Psi(x) = 0 \quad \text{on } \partial\Omega \end{cases} \tag{3.9}$$

with an eigenfunction $\Psi(x) = \frac{1}{T}\int_0^T \psi(t,x)dt$. Denote by $\lambda_1(-(\tilde{\mathcal{L}}_\Omega + \alpha_T))$ the principal eigenvalue of (3.9), then we have

$$\lambda_1(-(\tilde{L}_\Omega + \alpha)) = \lambda_1(-(\tilde{\mathcal{L}}_\Omega + \alpha_T)). \tag{3.10}$$

The continuity of $\lambda_1(-(\tilde{\mathcal{L}}_\Omega + \alpha_T))$ with respect to $|\Omega|$ can be obtained by using a simple re-scaling argument for the spatial variable x. Note that $\lambda_1(-(\tilde{\mathcal{L}}_\Omega + \alpha_T))$ can be expressed in a variational formulation:

$$\lambda_1(-(\tilde{\mathcal{L}}_\Omega + \alpha_T)) = \inf_{0\not\equiv\Psi\in H_0^1(\Omega)} \frac{d_2\tau\int_\Omega \Psi_x^2(x)dx - d_2(1-\tau)\int_\Omega\int_\Omega J_2(x-y)\Psi(y)\Psi(x)dydx}{\int_\Omega \Psi^2(x)dx} + [d_2(1-\tau) - \alpha_T].$$

By the zero extension of the principal eigenfunction, we can get the monotonicity of $\lambda_1(\tilde{\mathcal{L}}_\Omega + \alpha_T)$ from the variational formulation of the principal eigenvalue.

Now we prove the asymptotic limits of $\lambda_1(-(\tilde{L}_\Omega + \alpha))$. We may assume that $\Omega = (0, \ell)$. Since

$$\int_0^\ell \int_0^\ell J_2(x-y)\Psi(y)\Psi(x)dydx \le \int_0^\ell \int_0^\ell J_2(x-y)\frac{\Psi^2(y)+\Psi^2(x)}{2}dydx \le \int_0^\ell \Psi^2(x)dx,$$

we have

$$\lambda_1(-(\tilde{\mathcal{L}}_{(0,\ell)} + \alpha_T)) \ge \inf_{0\not\equiv\Psi\in H_0^1((0,\ell))} \frac{d_2\tau \int_0^\ell \Psi_x^2(x)dx}{\int_0^\ell \Psi^2(x)dx} - \alpha_T.$$

By the fact that

$$\inf_{0\not\equiv\Psi\in H_0^1((0,\ell))} \frac{\int_0^\ell \Psi_x^2(x)dx}{\int_0^\ell \Psi^2(x)dx} = \frac{\pi^2}{4l^2},$$

we know

$$\lim_{\ell\to 0} \lambda_1(-(\tilde{\mathcal{L}}_{(0,\ell)} + \alpha_T)) = +\infty \tag{3.11}$$

and

$$\liminf_{\ell\to+\infty} \lambda_1(-(\tilde{\mathcal{L}}_{(0,\ell)} + \alpha_T)) \ge -\alpha_T. \tag{3.12}$$

On the other hand, by (**K**), for any fixed $0 < \varepsilon \ll 1$, we have

$$\int_{-L}^L J_2(x)dx > 1 - \varepsilon$$

with some $L = L(\varepsilon) > 0$. For any large $\ell > 3L$, we choose the test function $\varphi_\varepsilon(x)$ defined as follows

$$\varphi_\varepsilon(x) = \begin{cases} \frac{x}{\varepsilon}, & x \in [0, \varepsilon], \\ 1, & x \in [\varepsilon, \ell - \varepsilon], \\ \frac{\ell - x}{\varepsilon}, & x \in [\ell - \varepsilon, \ell]. \end{cases}$$

It is easy to check that $\varphi_\varepsilon \in H_0^1((0,\ell))$ satisfies $\int_0^\ell \varphi_\varepsilon^2(x)dx = \ell - \frac{4}{3}\varepsilon$ and $\int_0^\ell (\partial_x\varphi_\varepsilon)^2(x)dx = \frac{2}{\varepsilon}$. Thus,

$$\begin{aligned}
&\lambda_1(-(\tilde{\mathcal{L}}_{(0,\ell)} + \alpha_T)) \\
&\le \frac{d_2\tau \int_0^\ell (\partial_x\varphi_\varepsilon)^2(x)dx - d_2(1-\tau)\int_0^\ell \int_0^\ell J_2(x-y)\varphi_\varepsilon(y)\varphi_\varepsilon(x)dydx}{\int_0^\ell \varphi_\varepsilon^2(x)dx} + [d_2(1-\tau) - \alpha_T] \\
&\le \frac{\frac{2d_2\tau}{\varepsilon} - d_2(1-\tau)\int_{L+\varepsilon}^{\ell-L-\varepsilon}\int_\varepsilon^{\ell-\varepsilon} J_2(x-y)dydx}{\ell - \frac{4}{3}\varepsilon} + [d_2(1-\tau) - \alpha_T] \\
&\le \frac{\frac{2d_2\tau}{\varepsilon} - d_2(1-\tau)\int_{L+\varepsilon}^{\ell-L-\varepsilon}\int_{-L}^{L} J_2(\xi)d\xi dx}{\ell - \frac{4}{3}\varepsilon} + [d_2(1-\tau) - \alpha_T] \\
&\le \frac{\frac{2d_2\tau}{\varepsilon} - d_2(1-\tau)(\ell - 2L - 2\varepsilon)(1-\varepsilon)}{\ell - \frac{4}{3}\varepsilon} + [d_2(1-\tau) - \alpha_T] \\
&\to -d_2(1-\tau)(1-\varepsilon) + [d_2(1-\tau) - \alpha_T] \quad \text{as } \ell \to +\infty.
\end{aligned}$$

Since ε is arbitrary, it follows that

$$\limsup_{\ell\to+\infty} \lambda_1(-(\tilde{\mathcal{L}}_{(0,\ell)} + \alpha_T)) \le -\alpha_T,$$

which together with (3.12) imply that

$$\lim_{\ell\to+\infty} \lambda_1(-(\tilde{\mathcal{L}}_{(0,\ell)} + \alpha_T)) = -\alpha_T. \tag{3.13}$$

From (3.10), (3.11) and (3.13), we know that $\lambda_1(-(\tilde{L}_{(0,\ell)} + \alpha)) = 0$ has a unique root.

(*ii*) By using similar arguments as in the proofs of Lemma 3.3 in [16] and Theorem 28.1 in [45], we can prove the result.

4. Spreading and vanishing for (1.1)

We mainly investigate the spreading-vanishing dichotomy and criteria for spreading and vanishing in this section. In view of (2.2), we see that the free boundaries $s_2(t), -s_1(t)$ are strictly increasing functions of t. Thus, $s_{2,\infty} := \lim_{t\to+\infty} s_2(t)$ and $s_{1,\infty} := \lim_{t\to+\infty} s_1(t)$ are well-defined. Clearly, $s_{2,\infty}, -s_{1,\infty} \le +\infty$.

Similar to the proof of Proposition 3.1 in [35], we can prove the following result.

Lemma 4.1. *Let $d, \mu, s_0 \in \mathbb{R}_+$, $C \in \mathbb{R}$ and $\alpha \in (0,1)$. Assume that $\varphi_0 \in C^2([-s_0, s_0])$ satisfies $\varphi_0(\pm s_0) = 0$ and $\varphi_0 > 0$ in $(-s_0, s_0)$, $(s_1, s_2, \varphi) \in [C^{1+\frac{\gamma}{2}}([0,+\infty), \mathbb{R}_+)]^2 \times C^{1+\frac{\gamma}{2}, 2+\gamma}(\Omega_\infty^{s_1,s_2}, \mathbb{R}_+)$ satisfies $-\infty < \lim_{t\to+\infty} s_1(t) < \lim_{t\to+\infty} s_2(t) < +\infty$, $\lim_{t\to+\infty} s_1'(t) = \lim_{t\to+\infty} s_2'(t) = 0$ and $\|\varphi\|_{C^1([s_1(t), s_2(t)])} \le K$ $(\forall t > 1)$ with some constant $K > 0$. If (φ, s_1, s_2) satisfies the following:*

$$\begin{cases} \varphi_t - d\varphi_{xx} \ge C\varphi, \quad t > 0,\ s_1(t) < x < s_2(t), \\ \varphi = 0, \quad t \ge 0,\ x = s_1(t) \text{ or } x = s_2(t), \\ s_1'(t) \le -\mu\varphi_x(t, s_1(t)),\ s_2'(t) \ge -\mu\varphi_x(t, s_2(t)), \quad t > 0, \\ -s_1(0) = s_2(0) = s_0, \\ \varphi(0, x) = \varphi_0(x), \quad -s_0 < x < s_0, \end{cases}$$

then $\lim_{t\to+\infty} \max_{s_1(t)\le x\le s_2(t)} \varphi(t, x) = 0$.

Next, we give an estimate for v. The proof is a simple modification of that for Lemma 3.2 in [4], so we omit it here.

Lemma 4.2. *If $s_{2,\infty} - s_{1,\infty} < +\infty$, then*

$$\|v\|_{C^{\frac{1+\gamma}{2}, 1+\gamma}(\overline{\Omega}_\infty^{s_1,s_2})} \le C \tag{4.1}$$

for some $C > 0$, and hence

$$\|v_x(\cdot, s_1(\cdot))\|_{C^{\frac{\gamma}{2}}(\overline{\mathbb{R}}_+)} + \|v_x(\cdot, s_2(\cdot))\|_{C^{\frac{\gamma}{2}}(\overline{\mathbb{R}}_+)} \le C. \tag{4.2}$$

Lemma 4.3. *If $s_{2,\infty} - s_{1,\infty} < +\infty$, then $\lim_{t\to+\infty} s_1'(t) = \lim_{t\to+\infty} s_2'(t) = 0$.*

Proof. Obviously, $-\infty < s_{1,\infty} < s_{2,\infty} < +\infty$. From (2.2), we know that $s_1'(t)$ and $s_2'(t)$ defined in (1.1) are bounded. Let

$$\begin{aligned} &\varphi_1(t) = v_x(t, s_2(t)),\ \varphi_2(t) = \int_{s_1(t)}^{s_2(t)} \int_{s_2(t)}^{+\infty} J_1(x-y)u(t,x)dydx, \\ &\varphi_3(t) = \int_{s_1(t)}^{s_2(t)} \int_{s_2(t)}^{+\infty} J_2(x-y)v(t,x)dydx. \end{aligned}$$

By (4.2), we get that $|\varphi_1(t) - \varphi_1(t_0)| \le C_1|t - t_0|^{\frac{\gamma}{2}}$ for any $t, t_0 > 0$. We may assume $t > t_0$. For φ_2, we have $s_2(t) > s_2(t_0)$ and $s_1(t) < s_1(t_0)$. Then,

$$\begin{aligned} \varphi_2(t) - \varphi_2(t_0) &= \int_{s_1(t)}^{s_2(t)} \int_{s_2(t)}^{+\infty} J_1(x-y)u(t,x)dydx - \int_{s_1(t_0)}^{s_2(t_0)} \int_{s_2(t_0)}^{+\infty} J_1(x-y)u(t_0,x)dydx \\ &= \int_{s_1(t_0)}^{s_2(t_0)} \int_{s_2(t)}^{+\infty} J_1(x-y)[u(t,x) - u(t_0,x)]dydx + \int_{s_1(t)}^{s_1(t_0)} \int_{s_2(t)}^{+\infty} J_1(x-y)u(t,x)dydx \\ &\quad + \int_{s_2(t_0)}^{s_2(t)} \int_{s_2(t)}^{+\infty} J_1(x-y)u(t,x)dydx - \int_{s_1(t_0)}^{s_2(t_0)} \int_{s_2(t_0)}^{s_2(t)} J_1(x-y)u(t_0,x)dydx \\ &\le \|\partial_t u\|_{L^\infty(\overline{\Omega}_\infty^{s_1,s_2})}(t - t_0)(s_2(t_0) - s_1(t_0)) + \|u\|_{L^\infty(\overline{\Omega}_\infty^{s_1,s_2})}(s_1(t_0) - s_1(t)) \\ &\quad + 2\|u\|_{L^\infty(\overline{\Omega}_\infty^{s_1,s_2})}(s_2(t) - s_2(t_0)) \\ &\le C_2(t - t_0), \end{aligned}$$

where $\|\partial_t u\|_{L^\infty(\overline{\Omega}_\infty^{s_1,s_2})}$ is obtained by the first equation in (1.1) and the bound of u. Thus,

$$|\varphi_2(t) - \varphi_2(t_0)| \leq C_2|t - t_0|.$$

For φ_3, it follows from (4.1) that $|v(t, x) - v(t_0, x)| \leq C|t - t_0|^{\frac{1+\gamma}{2}}$ for any $x \in [s_1(t_0), s_2(t_0)]$. Similar to φ_2, we have

$$|\varphi_3(t) - \varphi_3(t_0)| \leq C_3|t - t_0|.$$

Therefore, $s_2'(t) = -\mu\varphi_1 + \rho_1\varphi_2 + \rho_2\varphi_3$ is uniformly continuous in $[0, +\infty)$. From $\lim_{t\to+\infty} s_2(t) = s_{2,\infty} < +\infty$, we know $\lim_{t\to+\infty} s_2'(t) = 0$. Similarly, we can get $\lim_{t\to+\infty} s_1'(t) = 0$.

Theorem 4.4. *If $s_{2,\infty} - s_{1,\infty} < +\infty$, then the solution of* (1.1) *satisfies*

$$\|u(t, \cdot)\|_{C([s_1(t),s_2(t)])},\ \|v(t, \cdot)\|_{C([s_1(t),s_2(t)])} \to 0 \quad \text{as } t \to +\infty.$$

Proof. Since $J_2 \geq 0$ and $v > 0$, from the second equation in (1.1), we deduce

$$\partial_t v - d_2\tau\partial_x^2 v \geq Cv$$

for some constant $C > 0$. According to Lemma 4.1, we have

$$\|v(t, \cdot)\|_{C([s_1(t),s_2(t)])} \to 0 \quad \text{as } t \to +\infty.$$

We first show

$$\lambda_1(-(L_{(s_{1,\infty},s_{2,\infty})} + a)) \geq 0, \tag{4.3}$$

where $-(L_{(s_{1,\infty},s_{2,\infty})} + a)$ is defined in (3.4).

For convenient, we define $s_{2,\infty}^{\pm\varepsilon} := s_{2,\infty} \pm \varepsilon$, $s_{1,\infty}^{\pm\varepsilon} := s_{1,\infty} \pm \varepsilon$ for any $\varepsilon > 0$. Assume that (4.3) does not hold, there exists $\varepsilon_1 > 0$ such that $\lambda_1(-(L_{(s_{1,\infty}^{+\varepsilon},s_{2,\infty}^{-\varepsilon})} + a(t) - c(t)\varepsilon)) < 0$ for all $\varepsilon \in (0, \varepsilon_1)$. For such $\varepsilon > 0$, there exists $T_\varepsilon > 0$ such that, for $t > T_\varepsilon$,

$$s_2(t) > s_{2,\infty}^{-\varepsilon},\ s_1(t) < s_{1,\infty}^{+\varepsilon},\ \|v(t, \cdot)\|_{C([s_1(t),s_2(t)])} < \varepsilon.$$

Then, u satisfies the following:

$$\begin{cases} u_t \geq d_1 \int_{s_{1,\infty}^{+\varepsilon}}^{s_{2,\infty}^{-\varepsilon}} J_1(x-y)u(t,y)dy - d_1u + u(a(t) - u - c(t)\varepsilon), & t > T_\varepsilon,\ x \in [s_{1,\infty}^{+\varepsilon}, s_{2,\infty}^{-\varepsilon}], \\ u(T_\varepsilon, x) = u(T_\varepsilon, x), & x \in [s_{1,\infty}^{+\varepsilon}, s_{2,\infty}^{-\varepsilon}]. \end{cases}$$

Since $\lambda_1(-(L_{(s_{1,\infty}^{+\varepsilon},s_{2,\infty}^{-\varepsilon})} + a(t) - c(t)\varepsilon)) < 0$, by Lemma 3.4($i$) we know that the solution $q_\varepsilon(t, x)$ of problem

$$\begin{cases} q_t = d_1 \int_{s_{1,\infty}^{+\varepsilon}}^{s_{2,\infty}^{-\varepsilon}} J_1(x-y)q(t,y)dy - d_1q + q(a(t) - q - c(t)\varepsilon), & t > T_\varepsilon,\ x \in [s_{1,\infty}^{+\varepsilon}, s_{2,\infty}^{-\varepsilon}], \\ q(T_\varepsilon, x) = u(T_\varepsilon, x), & x \in [s_{1,\infty}^{+\varepsilon}, s_{2,\infty}^{-\varepsilon}] \end{cases}$$

converges to $\hat{q}_\varepsilon(t,x)$ uniformly in $[s_{1,\infty}^{+\varepsilon}, s_{2,\infty}^{-\varepsilon}]$ when $t \to +\infty$, and $\hat{q}_\varepsilon(t,x) \in \mathbb{Y}_\varepsilon^{++}$ is the unique periodic solution of

$$q_t = d_1 \int_{s_{1,\infty}^{+\varepsilon}}^{s_{2,\infty}^{-\varepsilon}} J_1(x-y)q(t,y)dy - d_1 q + q(a(t) - q - c(t)\varepsilon), \quad t \in \mathbb{R},\ x \in [s_{1,\infty}^{+\varepsilon}, s_{2,\infty}^{-\varepsilon}].$$

By Lemma 3.3 in [9], we get

$$u \geq q_\varepsilon \quad \text{in } (T_\varepsilon, +\infty) \times [s_{1,\infty}^{+\varepsilon}, s_{2,\infty}^{-\varepsilon}].$$

Hence, there exist two constants $\tilde{T}_\varepsilon > T_\varepsilon$ and $C > 0$ such that

$$u(t,x) \geq \frac{1}{2}\hat{q}_\varepsilon(t,x) \geq C > 0, \quad \forall\, t > \tilde{T}_\varepsilon,\ x \in [s_{1,\infty}^{+\varepsilon}, s_{2,\infty}^{-\varepsilon}].$$

From the assumption (**K**) (the Lipschitz continuity of J_1 and $J_1(0) > 0$), we deduce that there exist constants $\bar{\epsilon} \in (0, \frac{h_0}{4})$ and $\eta_0 > 0$ such that $J_1(x-y) > \eta_0$ if $|x-y| < \bar{\epsilon}$. It follows that, for $0 < \varepsilon < \min\{\varepsilon_1, \frac{\bar{\epsilon}}{2}\}$ and $t > \tilde{T}_\varepsilon$,

$$\begin{aligned} s_2'(t) &\geq \rho_1 \int_{s_1(t)}^{s_2(t)} \int_{s_2(t)}^{+\infty} J_1(x-y)u(t,x)dydx \geq \rho_1 \int_{s_{1,\infty}^{+\varepsilon}}^{s_{2,\infty}^{-\varepsilon}} \int_{s_{2,\infty}}^{+\infty} J_1(x-y)u(t,x)dydx \\ &\geq \rho_1 \int_{s_{2,\infty}^{-\frac{\bar{\epsilon}}{2}}}^{s_{2,\infty}^{-\varepsilon}} \int_{s_{2,\infty}}^{s_{2,\infty}^{-\frac{\bar{\epsilon}}{2}}} \eta_0 \frac{1}{2}\hat{q}_\varepsilon(t,x)dydx \geq \rho_1 \int_{s_{2,\infty}^{-\frac{\bar{\epsilon}}{2}}}^{s_{2,\infty}^{-\varepsilon}} \int_{s_{2,\infty}}^{s_{2,\infty}^{-\frac{\bar{\epsilon}}{2}}} \eta_0 C dydx \\ &= \rho_1 (\frac{\bar{\epsilon}}{2} - \varepsilon)\frac{\bar{\epsilon}}{2}\eta_0 C. \end{aligned}$$

It follows that $s_{2,\infty} = +\infty$, which is a contradiction. Then, (4.3) is proved.

Now we prove that the solution U of

$$\begin{cases} U_t = d_1 \int_{s_{1,\infty}}^{s_{2,\infty}} J_1(x-y)U(t,y)dy - d_1 U + U(a(t) - U), \quad t > 0, x \in [s_{1,\infty}, s_{2,\infty}], \\ U(0,x) = u_0(x),\ x \in [-s_0, s_0], \\ U(0,x) = 0,\ x \in [s_{1,\infty}, s_{2,\infty}] \setminus [-s_0, s_0] \end{cases}$$

satisfies $\lim_{t\to+\infty} U(t,x) = 0$ uniformly for $x \in [s_{1,\infty}, s_{2,\infty}]$. Since (4.3) holds, we divide the discussion into two cases:

(i) for the case $\lambda_1(-(L_{(s_{1,\infty},s_{2,\infty})} + a)) > 0$, the result can be directly deduced from Lemma 3.4(ii).

(ii) for the case $\lambda_1(-(L_{(s_{1,\infty},s_{2,\infty})} + a)) = 0$, we define

$$\tilde{U}(t,x) = e^{-\int_0^t [a(\theta) - a_T]d\theta} U(t,x).$$

Then, $\tilde{U}$ satisfies

$$\begin{cases} \tilde{U}_t = d_1 \int_{s_{1,\infty}}^{s_{2,\infty}} J_1(x-y)\tilde{U}(t,y)dy - d_1 \tilde{U} + \tilde{U}(a_T - e^{\int_0^t [a(\theta)-a_T]d\theta}\tilde{U}),\ t > 0, x \in [s_{1,\infty}, s_{2,\infty}], \\ \tilde{U}(0,x) = u_0(x),\ x \in [-s_0, s_0], \\ \tilde{U}(0,x) = 0,\ x \in [s_{1,\infty}, s_{2,\infty}] \setminus [-s_0, s_0]. \end{cases}$$

For any $t > 0$, we can write $t = mT + \tau_0$ with $0 \le \tau_0 < T$, and then

$$e^{\int_0^t [a(\theta)-a_T]d\theta} = e^{\int_0^{mT+\tau_0} [a(\theta)-a_T]d\theta} = e^{\int_0^{\tau_0} [a(\theta)-a_T]d\theta},$$

which together with the continuity of $a(t)$ imply that $M_1 \le e^{\int_0^t [a(\theta)-a_T]d\theta} \le M_2$ for some positive constants M_1 and M_2. By Lemma 3.3 in [9], we know $\tilde{U}(t,x) \le \hat{U}(t,x)$, where $\hat{U}(t,x)$ is the unique solution of

$$\begin{cases} \hat{U}_t = d_1 \int_{s_{1,\infty}}^{s_{2,\infty}} J_1(x-y)\hat{U}(t,y)dy - d_1\hat{U} + \hat{U}(a_T - M_1\hat{U}), & t > 0, x \in [s_{1,\infty}, s_{2,\infty}], \\ \hat{U}(0,x) = u_0(x), \ x \in [-s_0, s_0], \\ \hat{U}(0,x) = 0, \ x \in [s_{1,\infty}, s_{2,\infty}] \setminus [-s_0, s_0]. \end{cases}$$

Recall that from (3.7) we get $\lambda_1(-(\mathcal{L}_{(s_{1,\infty},s_{2,\infty})} + a_T)) = \lambda_1(-(L_{(s_{1,\infty},s_{2,\infty})} + a)) = 0$. By Proposition 3.5 in [9] (see also [43, 46]), we know that $\lim_{t\to+\infty} \hat{U}(t,x) = 0$ uniformly for $x \in [s_{1,\infty}, s_{2,\infty}]$. Thus, $\tilde{U}(t,x)$ converges to 0 uniformly in $[s_{1,\infty}, s_{2,\infty}]$ as $t \to +\infty$, which implies that $\lim_{t\to+\infty} U(t,x) = \lim_{t\to+\infty} e^{\int_0^t [a(\theta)-a_T]d\theta}\tilde{U}(t,x) = 0$ uniformly for $x \in [s_{1,\infty}, s_{2,\infty}]$.

On the other hand, it is easy to know that

$$\begin{cases} U_t \ge d_1 \int_{s_1(t)}^{s_2(t)} J_1(x-y)U(t,y)dy - d_1U + U(a(t) - U), & (t,x) \in \Omega_\infty^{s_1,s_2}, \\ U(t, s_1(t)) \ge 0, \ U(t, s_2(t)) \ge 0, & t > 0, \\ U(0,x) = u_0(x), \ x \in [-s_0, s_0]. \end{cases}$$

By Lemma 2.2 in [9], we know that $u \le U$ in $\overline{\Omega}_\infty^{s_1,s_2}$. Thus, $\lim_{t\to+\infty} \|u(t,\cdot)\|_{C([s_1(t),s_2(t)])} = 0$.

By Theorem 4.4, we have the spreading-vanishing dichotomy.

Corollary 4.5. *Let* $(u, v; s_1, s_2)$ *be the solution of* (1.1). *Then, the following alternative holds: either* (*i*) *spreading:* $\lim_{t\to+\infty}(s_2(t) - s_1(t)) = +\infty$, *or* (*ii*) *vanishing:* $(s_{1,\infty}, s_{2,\infty}) := \lim_{t\to+\infty}(s_1(t), s_2(t))$ *is a bounded set and*

$$\lim_{t\to+\infty} \max_{s_1(t)\le x\le s_2(t)} u(t,x) = \lim_{t\to+\infty} \max_{s_1(t)\le x\le s_2(t)} v(t,x) = 0.$$

If we further assume the weak competition condition

$$\min_{[0,T]} a(t) > \max_{[0,T]} c(t) \cdot \max_{[0,T]} b(t), \quad \min_{[0,T]} b(t) > \max_{[0,T]} d(t) \cdot \max_{[0,T]} a(t), \tag{4.4}$$

then we can establish the asymptotic estimates of the solution when spreading occurs. To achieve it, we first give a lemma concerning the asymptotic stability of time-periodic solutions for the equations with nonlocal/mixed dispersal in $\mathbb{R}$.

Lemma 4.6. *Assume that* $d > 0$, $0 \le \theta \le 1$, $\alpha \in C_T(\mathbb{R})$ *and* J *satisfies* (***K***).

(*i*) *The* T*-periodic problem*

$$\begin{cases} \partial_t q = d\left[\theta\partial_x^2 q + (1-\theta)\left(\int_{\mathbb{R}} J(x-y)q(t,y)dy - q\right)\right] + q(\alpha(t) - q), & t \in [0,T], \ x \in \mathbb{R}, \\ q(0,x) = q(T,x), & x \in \mathbb{R} \end{cases} \tag{4.5}$$

has a unique positive solution, which satisfies

$$q' = q(\alpha(t) - q), \quad q(0) = q(T).$$

(ii) *For any bounded, uniformly continuous initial value* q_0 *with* $\inf_{x\in\mathbb{R}} q_0 > 0$*, the unique solution* $q(t, x; q_0)$ *of*

$$\begin{cases} \partial_t q = d\left[\theta\partial_x^2 q + (1-\theta)\left(\int_{\mathbb{R}} J(x-y)q(t,y)dy - q\right)\right] + q(\alpha(t) - q), & t > 0,\ x \in \mathbb{R}, \\ q(0,x) = q_0(x), \quad x \in \mathbb{R} \end{cases} \tag{4.6}$$

satisfies $\lim_{t\to+\infty} \|q(t,\cdot;q_0) - q^*(t)\|_{L^\infty(\mathbb{R})} = 0$*, where* $q^*(t)$ *is the positive solution of* (4.5).

Proof. (i) We first consider the case $\theta = 0$.

Step 1. we give a lower bound estimate of any bounded positive solutions of $(4.5)_{\theta=0}$.

Consider the stationary problem

$$-d\left(\int_{-\ell}^{\ell} J(x-y)q(y)dy - q\right) = q(\min_{t\in[0,T]} \alpha(t) - q), \quad -\ell < x < \ell. \tag{4.7}$$

From Proposition 3.6 in [9], (4.7) has a unique positive bounded solution $q^\ell(x)$ for sufficiently large ℓ, and $q^\ell(x) \to \min_{t\in[0,T]} \alpha(t)$ in $C([-L, L])$ for any $L > 0$ as $\ell \to +\infty$. For any positive solution $\hat{q}(t, x)$ of $(4.5)_{\theta=0}$, by Lemma 3.3, we have that $\hat{q} \geq q^\ell$ on $[0, T] \times [-\ell, \ell]$. Letting $\ell \to +\infty$, we get $\hat{q} \geq \min_{t\in[0,T]} \alpha(t) > 0$.

Step 2. we briefly show that $(4.5)_{\theta=0}$ has a minimal positive solution.

Consider the problem

$$\begin{cases} \partial_t q = d\left(\int_{-\ell}^{\ell} J(x-y)q(t,y)dy - q\right) + q(\alpha(t) - q), & 0 \leq t \leq T,\ -\ell < x < \ell, \\ q(0,x) = q(T,x), \quad -\ell \leq x \leq \ell. \end{cases} \tag{4.8}$$

For sufficiently large ℓ, by Lemma 3.4(i), (4.8) admits a unique positive solution q_*^ℓ. By Lemma 3.3, we can show that q_*^ℓ is increasing in ℓ and $q_*^\ell \leq \hat{q}$ in $[0, T] \times [-\ell, \ell]$ for any positive solution $\hat{q}(t, x)$ of $(4.5)_{\theta=0}$ and any $\ell > 0$. Thus, the limit function $q_* = \lim_{\ell\to+\infty} q_*^\ell$ is exactly a minimal positive solution of $(4.5)_{\theta=0}$.

Step 3. we prove the uniqueness by using a technique introduced by Marcus and Véron [47].

Arguing indirectly, we assume that $(4.5)_{\theta=0}$ has a positive bounded solution $\hat{q}$ such that $\hat{q} \not\equiv q_*$. Then there exists a constant $k > 1$ such that $q_* \leq \hat{q} \leq kq_*$ in $[0, T] \times \mathbb{R}$. By the strong maximum principle (see Definition 1.4 and Theorem **F** in [44]), we have $q_* < \hat{q}$. Define $\bar{q} = q_* - (2k)^{-1}(\hat{q} - q_*)$. By direct calculations, we get

$$q_* > \bar{q} \geq \frac{k+1}{2k} q_*, \quad \frac{2k}{2k+1}\bar{q} + \frac{1}{2k+1}\hat{q} = q_*. \tag{4.9}$$

By the convexity of $f(x) = x^2$ in x, we have $q_*^2 \leq \frac{2k}{2k+1}\bar{q}^2 + \frac{1}{2k+1}\hat{q}^2$. Then

$$\begin{aligned} \partial_t \bar{q} &\geq d\left(\int_{\mathbb{R}} J(x-y)\bar{q}(t,y)dy - \bar{q}\right) + \bar{q}(\alpha(t) - \bar{q}) \\ &\geq d\left(\int_{-\ell}^{\ell} J(x-y)\bar{q}(t,y)dy - \bar{q}\right) + \bar{q}(\alpha(t) - \bar{q}), \quad 0 \leq t \leq T,\ x \in [-\ell, \ell], \end{aligned}$$

and $\bar{q}(0,x) = \bar{q}(T,x)$ for $x \in [-\ell,\ell]$. By Lemma 3.3, we have that $q_*^\ell \le \bar{q}$ in $[0,T]\times[-\ell,\ell]$. Since $q_*^\ell \to q_*$ in $C^{1,0}([0,T]\times[-L,L])$ for any $L>0$ as $\ell\to+\infty$, it follows that $q_* \le \bar{q}$ in $[0,T]\times\mathbb{R}$, which contradicts (4.9). Thus, the positive solution of $(4.5)_{\theta=0}$ is unique.

In proving the uniqueness of the positive solution to $(4.5)_{\theta\in(0,1]}$, we need to replace the auxiliary problems (4.7) and (4.8) with

$$\begin{cases} -d\left[\theta\partial_x^2 q + (1-\theta)\left(\int_{-\ell}^{\ell} J(x-y)q(y)dy - q\right)\right] = q(\min_{t\in[0,T]}\alpha(t) - q), \quad -\ell < x < \ell, \\ q(\pm\ell) = 0 \end{cases}$$

and

$$\begin{cases} \partial_t q = d\left[\theta\partial_x^2 q + (1-\theta)\left(\int_{-\ell}^{\ell} J(x-y)q(t,y)dy - q\right)\right] \\ \qquad\quad +q(\alpha(t)-q), \quad 0\le t\le T, \ -\ell < x < \ell, \\ q(t,\pm\ell) = 0, \quad 0\le t\le T, \\ q(0,x) = q(T,x), \quad -\ell\le x\le\ell, \end{cases}$$

respectively, which have null Dirichlet boundary conditions. The proof is similar to that of the case $\theta = 0$. Here we omit the details.

(*ii*) For the case $\theta = 0$, the proof can be seen in that of Theorem 2.3(3) in [48]. The case $0<\theta\le 1$ can be proved similarly by using Lemma 2.3 in [49] (the comparison principle). Here we omit the details.

Theorem 4.7. *If* (4.4) *holds and* $s_{2,\infty} - s_{1,\infty} = +\infty$, *then*

$$\begin{aligned} q_*(t) &\le \liminf_{n\to\infty} u(t+nT,x) \le \limsup_{n\to\infty} u(t+nT,x) \le q^*(t), \\ p_*(t) &\le \liminf_{n\to\infty} v(t+nT,x) \le \limsup_{n\to\infty} v(t+nT,x) \le p^*(t) \end{aligned}$$

uniformly in $[0,T]\times[-\ell,\ell]$ *for any* $\ell>0$, *where* $q^*(t)$, $p^*(t)$, $p_*(t)$ *and* $q_*(t)$ *are positive solutions of*

$$\begin{aligned} \tfrac{dq^*}{dt} &= q^*(a(t)-q^*), \quad q^*(0) = q^*(T), \\ \tfrac{dp^*}{dt} &= p^*(b(t)-p^*), \quad p^*(0) = p^*(T), \\ \tfrac{dp_*}{dt} &= p_*\big(b(t)-p_*-d(t)q^*(t)\big), \quad p_*(0) = p_*(T) \end{aligned}$$

and

$$\tfrac{dq_*}{dt} = q_*\big(a(t)-q_*-c(t)p^*(t)\big), \quad q_*(0) = q_*(T),$$

respectively.

Proof. In Theorem 3.2 of [32], similar results have been obtained for the random dispersal case. Since the nonlocal dispersal is considered here, we give the details.

Step 1. For any $\ell>0$, $\limsup_{n\to\infty} u(t+nT,x)\le q^*(t)$ and $\limsup_{n\to\infty} v(t+nT,x)\le p^*(t)$ uniformly in $[0,T]\times[-\ell,\ell]$.

Let $q(t,x)$ be the positive solution of

$$\begin{cases} \partial_t q = d_1\left(\int_{\mathbb{R}} J_1(x-y)q(t,y)dy - q\right) + q(a(t)-q), \quad t>0, \ x\in\mathbb{R}, \\ q(0,x) = \|u_0\|_{L^\infty([-s_0,s_0])} > 0, \quad x\in\mathbb{R}. \end{cases}$$

By Lemma 4.6(ii), we know that $\lim_{n\to\infty} q(t+nT,x) \to q^*(t)$ uniformly for $(t,x) \in [0,T]\times[-\ell,\ell]$. Moreover, since q satisfies

$$\partial_t q \geq d_1 \left(\int_{s_1(t)}^{s_2(t)} J_1(x-y)q(t,y)dy - q \right) + q(a(t)-q), \quad (t,x) \in \Omega_\infty^{s_1,s_2},$$

by Lemma 2.2 in [9] we have that $u \leq q$ in $\Omega_\infty^{s_1,s_2}$. Thus, $\limsup_{n\to\infty} u(t+nT,x) \leq q^*(t)$ uniformly in $[0,T]\times[-\ell,\ell]$.

Similarly, by applying Lemma 4.6(ii) and Lemma 2.2, we can prove the result for v.

Step 2. For any given $\ell > 0$, $\liminf_{n\to\infty} v(t+nT,x) \geq p_*(t)$ uniformly in $[0,T]\times[-\ell,\ell]$.

By the assumption (4.4) and the fact that $q^* \leq \max_{[0,T]} a(t)$, we know that there exists $\varepsilon_0 > 0$ such that

$$b_\varepsilon(t) := b(t) - d(t)(q^*(t)+\varepsilon) \geq \min_{[0,T]} b(t) - \max_{[0,T]} d(t)\cdot(\max_{[0,T]} a(t)+\varepsilon) > 0$$

for any $0 < \varepsilon \leq \varepsilon_0$. For such a fixed ε, from Lemma 3.7(i) we can deduce that there exists $L_\varepsilon > \ell$ such that $\lambda_1(-(\tilde{L}_{(-l,l)} + b_\varepsilon)) < 0$ for all $l \geq L_\varepsilon$. Since $s_{2,\infty} - s_{1,\infty} = +\infty$ and $\limsup_{n\to\infty} u(t+nT,x) \leq q^*(t)$ locally uniformly in $[0,T]\times\mathbb{R}$, for any $\varepsilon \in (0,\varepsilon_0)$ and $l > L_\varepsilon$ there exists $m \in \mathbb{N}$ such that

$$s_1(t) < -l,\ s_2(t) > l,\ u(t,x) < q^*(t)+\varepsilon, \quad \forall t \geq mT,\ |x| \leq l.$$

Let p_l^ε be the unique positive solution of

$$\begin{cases} \partial_t p = d_2 \left[\tau \partial_x^2 p + (1-\tau)\left(\int_{-l}^{l} J_2(x-y)p(t,y)dy - p \right) \right] \\ \qquad\qquad +p(b_\varepsilon(t)-p), \quad t > mT,\ |x| < l, \\ p(t,\pm l) = 0, \quad t > mT, \\ p(mT,x) = p(mT,x), \quad |x| < l. \end{cases}$$

By the comparison principle derived from Lemma 2.2, $v(t,x) \geq p_l^\varepsilon(t,x)$ for $t \geq mT$ and $x \in [-l,l]$. Since $\lambda_1(-(\tilde{L}_{(-l,l)} + b_\varepsilon)) < 0$, by Lemma 3.7($ii$), we deduce that $\lim_{n\to\infty} p_l^\varepsilon(t+nT,x) = P_l^\varepsilon(t,x)$ in $C^{1,2}([0,T]\times[-l,l])$, where $P_l^\varepsilon(t,x)$ is the positive solution of

$$\begin{cases} \partial_t P = d_2 \left[\tau \partial_x^2 P + (1-\tau)\left(\int_{-l}^{l} J_2(x-y)P(t,y)dy - P \right) \right] \\ \qquad\qquad +P(b_\varepsilon(t)-P), \quad 0 \leq t \leq T,\ |x| < l, \\ P(t,\pm l) = 0, \quad 0 \leq t \leq T, \\ P(0,x) = P(T,x), \quad |x| < l. \end{cases}$$

Since P_l^ε is increasing with respect to l, we know that

$$\lim_{l\to+\infty} P_l^\varepsilon = P^\varepsilon \quad \text{in } C^{1,2}([0,T]\times[-\ell,\ell]),$$

where P^ε is the positive solution of

$$\begin{cases} \partial_t P = d_2 \left[\tau \partial_x^2 P + (1-\tau)\left(\int_{\mathbb{R}} J_2(x-y)P(t,y)dy - P \right) \right] \\ \qquad\qquad +P(b_\varepsilon(t)-P), \quad t \in [0,T],\ x \in \mathbb{R}, \\ P(0,x) = P(T,x), \quad x \in \mathbb{R}. \end{cases}$$

By Lemma 4.6(i), P^ε satisfies

$$\frac{dP}{dt} = P(b_\varepsilon(t) - P), \quad P(0) = P(T).$$

Thus, $\lim_{n\to\infty} v(t + nT, x) \geq P^\varepsilon(t, x)$ uniformly for $(t, x) \in [0, T] \times [-\ell, \ell]$. Letting $\varepsilon \to 0$, we can deduce the result.

Step 3. For any given $\ell > 0$, $\liminf_{n\to\infty} u(t + nT, x) \geq q_*(t)$ uniformly in $[0, T] \times [-\ell, \ell]$.

By the assumption (4.4) and the fact that $p^* \leq \max_{[0,T]} b(t)$, we know that there exists $\varepsilon_1 > 0$ such that

$$a_\varepsilon(t) := a(t) - c(t)(p^*(t) + \varepsilon) \geq \min_{[0,T]} a(t) - \max_{[0,T]} c(t) \cdot (\max_{[0,T]} b(t) + \varepsilon) > 0$$

for any $0 < \varepsilon \leq \varepsilon_1$. For such a fixed ε, from Lemma 3.6 we can deduce that there exists $l_\varepsilon > \ell$ such that $\lambda_1(-(L_{(-l,l)} + a_\varepsilon)) < 0$ for all $l \geq l_\varepsilon$. Since $s_{2,\infty} - s_{1,\infty} = +\infty$ and $\limsup_{n\to\infty} v(t + nT, x) \leq p^*(t)$ locally uniformly in $[0, T] \times \mathbb{R}$, for any $\varepsilon \in (0, \varepsilon_1)$ and $l > l_\varepsilon$ there exists $m_1 \in \mathbb{N}$ such that

$$s_1(t) < -l,\ s_2(t) > l,\ v(t, x) < p^*(t) + \varepsilon, \quad \forall t \geq m_1 T,\ -l \leq x \leq l.$$

Let q_l^ε be the positive solution of

$$\begin{cases} \partial_t q = d_1 \left(\int_{-l}^{l} J_1(x - y) q(t, y) dy - q \right) + q(a_\varepsilon(t) - q), & t > m_1 T,\ -l < x < l, \\ q(m_1 T, x) = u(m_1 T, x), & -l < x < l. \end{cases}$$

Since $\lambda_1(-(L_{(-l,l)} + a_\varepsilon)) < 0$, by Lemma 3.4, we know that $\lim_{n\to\infty} q_l^\varepsilon(t + nT, x) = Q_l^\varepsilon(t, x)$ in $C^{1,0}([0, T] \times [-l, l])$, where $Q_l^\varepsilon(t, x)$ is the positive solution of

$$\begin{cases} \partial_t Q = d_1 \left(\int_{-l}^{l} J_1(x - y) Q(t, y) dy - Q \right) + Q(a_\varepsilon(t) - Q), & 0 \leq t \leq T,\ -l < x < l, \\ Q(0, x) = Q(T, x), & -l < x < l. \end{cases}$$

By Lemma 3.3, $Q_l^\varepsilon(t, x)$ is increasing in l. Thus,

$$\lim_{l\to+\infty} Q_l^\varepsilon = Q^\varepsilon \quad \text{in } C^{1,0}([0, T] \times [-\ell, \ell]),$$

where $Q^\varepsilon(t, x)$ is the positive solution of

$$\begin{cases} \partial_t Q = d_1 \left(\int_{\mathbb{R}} J_1(x - y) Q(t, y) dy - Q \right) + Q(a_\varepsilon(t) - Q), & t \in [0, T],\ x \in \mathbb{R}, \\ Q(0, x) = Q(T, x), & x \in \mathbb{R}. \end{cases}$$

By Lemma 4.6(i), Q^ε satisfies

$$\frac{dQ}{dt} = Q(a_\varepsilon(t) - Q), \quad Q(0) = Q(T).$$

Thus, $\lim_{n\to\infty} u(t + nT, x) \geq Q^\varepsilon(t, x)$ uniformly for $(t, x) \in [0, T] \times [-\ell, \ell]$. Letting $\varepsilon \to 0$, we can get the result.

In what follows, we will provide some sufficient conditions for spreading and vanishing.

Theorem 4.8. *If* $s_{2,\infty} - s_{1,\infty} < +\infty$*, then* $s_{2,\infty} - s_{1,\infty} \le s^*$*, where* $|\Omega| = s^*$ *is the unique root of* $\lambda_1(-(\tilde{L}_\Omega + b)) = 0$ *with* $-(\tilde{L}_\Omega + b)$ *defined as in* (3.8).

Proof. Assume that the conclusion is not true, there exist $0 < \varepsilon \ll 1$ and $\mathcal{T}_0 \gg 1$ such that

$$\begin{aligned} & s_{2,\infty}^{-\varepsilon} - s_{1,\infty}^{+\varepsilon} = s_{2,\infty} - s_{1,\infty} - 2\varepsilon > s_\varepsilon^*, \\ & s_1(\mathcal{T}_0) < s_{1,\infty}^{+\varepsilon},\ s_2(\mathcal{T}_0) > s_{2,\infty}^{-\varepsilon}, \\ & 0 \le u < \varepsilon \quad \text{in } [\mathcal{T}_0, +\infty) \times [s_{1,\infty}^{+\varepsilon}, s_{2,\infty}^{-\varepsilon}], \end{aligned}$$

where $|\Omega| = s_\varepsilon^*$ is the unique root of $\lambda_1(-(\tilde{L}_\Omega + b(t) - d(t)\varepsilon)) = 0$. Then, v satisfies the following:

$$\begin{cases} v_t \ge d_2\left[\tau v_{xx} + (1-\tau)\left(\int_{s_{1,\infty}^{+\varepsilon}}^{s_{2,\infty}^{-\varepsilon}} J_2(x-y)v(t,y)dy - v\right)\right] \\ \qquad +v(b(t) - d(t)\varepsilon - v), \quad t > \mathcal{T}_0,\ x \in (s_{1,\infty}^{+\varepsilon}, s_{2,\infty}^{-\varepsilon}), \\ v(t, s_{1,\infty}^{+\varepsilon}),\ v(t, s_{2,\infty}^{-\varepsilon}) > 0,\ t \ge \mathcal{T}_0, \\ v(\mathcal{T}_0, x) > 0,\ x \in (s_{1,\infty}^{+\varepsilon}, s_{2,\infty}^{-\varepsilon}). \end{cases}$$

Let p be the positive solution of

$$\begin{cases} p_t = d_2\left[\tau p_{xx} + (1-\tau)\left(\int_{s_{1,\infty}^{+\varepsilon}}^{s_{2,\infty}^{-\varepsilon}} J_2(x-y)p(t,y)dy - p\right)\right] \\ \qquad +p(b(t) - d(t)\varepsilon - p), \quad t > \mathcal{T}_0,\ x \in (s_{1,\infty}^{+\varepsilon}, s_{2,\infty}^{-\varepsilon}), \\ p(t, s_{1,\infty}^{+\varepsilon}) = p(t, s_{2,\infty}^{-\varepsilon}) = 0,\ t \ge \mathcal{T}_0, \\ p(\mathcal{T}_0, x) = v(\mathcal{T}_0, x),\ x \in (s_{1,\infty}^{+\varepsilon}, s_{2,\infty}^{-\varepsilon}). \end{cases}$$

By Lemma 2.2, we have

$$p \le v \quad \text{in } [\mathcal{T}_0, +\infty) \times [s_{1,\infty}^{+\varepsilon}, s_{2,\infty}^{-\varepsilon}].$$

Since $s_{2,\infty}^{-\varepsilon} - s_{1,\infty}^{+\varepsilon} = s_{2,\infty} - s_{1,\infty} - 2\varepsilon > s_\varepsilon^*$, we have $\lambda_1(-(\tilde{L}_{(s_{1,\infty}^{+\varepsilon}, s_{2,\infty}^{-\varepsilon})} + b(t) - d(t)\varepsilon)) < 0$, and then Lemma 3.7(*ii*) implies that $p(t + nT, x) \to P(t, x)$ as $n \to \infty$ uniformly in the compact subset of $(s_{1,\infty}^{+\varepsilon}, s_{2,\infty}^{-\varepsilon})$, where $P(t, x)$ is the positive solution of

$$\begin{cases} P_t = d_2\left[\tau P_{xx} + (1-\tau)\left(\int_{s_{1,\infty}^{+\varepsilon}}^{s_{2,\infty}^{-\varepsilon}} J_2(x-y)P(t,y)dy - P\right)\right] \\ \qquad +P(b(t) - d(t)\varepsilon - P), \quad t \in [0, T],\ x \in (s_{1,\infty}^{+\varepsilon}, s_{2,\infty}^{-\varepsilon}), \\ P(t, s_{1,\infty}^{+\varepsilon}) = P(t, s_{2,\infty}^{-\varepsilon}) = 0, \quad t \in [0, T], \\ P(0, x) = P(T, x), \quad x \in (s_{1,\infty}^{+\varepsilon}, s_{2,\infty}^{-\varepsilon}). \end{cases}$$

Therefore, $\liminf_{n\to\infty} v(t + nT, x) \ge \lim_{n\to\infty} p(t + nT, x) = P(t, x) > 0$ for all $x \in (s_{1,\infty}^{+\varepsilon}, s_{2,\infty}^{-\varepsilon})$.

Recall that, in Theorem 4.4, we have proved that $s_{2,\infty} - s_{1,\infty} < +\infty$ implies

$$\lim_{t\to+\infty} \|u(t,\cdot)\|_{C([s_1(t), s_2(t)])} = \lim_{t\to+\infty} \|v(t,\cdot)\|_{C([s_1(t), s_2(t)])} = 0,$$

which is a contradiction. This completes the proof.

Corollary 4.9. *If* $s_0 \geq \frac{1}{2}s^*$*, then spreading occurs, that is,* $s_{2,\infty} - s_{1,\infty} = +\infty$.

If $a_T \geq d_1$, then Lemma 3.6 implies that $\lambda_1(-(L_\Omega + a)) < 0$ for all $\ell := |\Omega| > 0$. Thus, the vanishing can not happen by the proof of Theorem 4.4, which means that $s_{2,\infty} - s_{1,\infty} = +\infty$ always holds.

Theorem 4.10. *If* $a_T \geq d_1$*, then spreading always happens.*

On the other hand, if $a_T < d_1$, then Lemma 3.6 implies that $\lambda_1(-(L_\Omega + a)) > 0$ for $0 < |\Omega| \ll 1$, and $\lambda_1(-(L_\Omega + a)) < 0$ for $|\Omega| \gg 1$. Since $\lambda_1(-(L_\Omega + a))$ is strictly decreasing in $|\Omega|$, there exists some $\ell^* > 0$ such that $\lambda_1(-(L_\Omega + a)) = 0$ for $|\Omega| = \ell^*$, $\lambda_1(-(L_\Omega + a)) > 0$ for $|\Omega| < \ell^*$ and $\lambda_1(-(L_\Omega + a)) < 0$ for $|\Omega| > \ell^*$. From the proof of (4.3), we know that if $s_{2,\infty} - s_{1,\infty} < +\infty$ then $s_{2,\infty} - s_{1,\infty} \leq \ell^*$. Therefore, if $s_0 \geq \frac{\ell^*}{2}$ then we have $s_{2,\infty} - s_{1,\infty} = +\infty$.

Theorem 4.11. *Assume that* $a_T < d_1$ *and* $s_0 < \frac{1}{2}\min\{\ell^*, s^*\}$*. Then there exists* $M_0 > 0$ *such that* $s_{2,\infty} - s_{1,\infty} < +\infty$ *when* $\mu + \rho_1 + \rho_2 \leq M_0$.

Proof. Since $\lambda_1(-(L_{(-s_0,s_0)} + a)) > 0$, we can choose $s_0 < \tilde{s}_0 < \frac{\ell^*}{2}$ such that $\lambda := \lambda_1(-(L_{(-\tilde{s}_0,\tilde{s}_0)} + a)) > 0$. Let φ be the eigenfunction associated with λ, and assume that φ satisfies $\|\varphi\|_{L^\infty([0,T]\times[-\tilde{s}_0,\tilde{s}_0])} = 1$, that is,

$$-\left(L_{(-\tilde{s}_0,\tilde{s}_0)} + a\right)[\varphi] = \lambda\varphi.$$

Define $\omega(t,x) = Ke^{-\frac{\lambda t}{2}}\varphi(t,x)$ for some $K > 0$, then ω satisfies

$$\begin{aligned}
&\omega_t - d_1 \int_{-\tilde{s}_0}^{\tilde{s}_0} J_1(x-y)\omega(t,y)dy + d_1\omega - a(t)\omega \\
&= Ke^{-\frac{\lambda t}{2}}\left(\varphi_t - d_1 \int_{-\tilde{s}_0}^{\tilde{s}_0} J_1(x-y)\varphi(t,y)dy + d_1\varphi - a(t)\varphi - \frac{\lambda}{2}\varphi\right) \\
&= \frac{1}{2}\lambda Ke^{-\frac{\lambda t}{2}}\varphi > 0
\end{aligned}$$

in $(0,+\infty)\times[-\tilde{s}_0,\tilde{s}_0]$. Choose $K > 0$ large such that $\omega(0,x) = K\varphi(0,x) > u_0(x)$ in $[-\tilde{s}_0,\tilde{s}_0]$. Assume that $\bar{u}$ is the solution of

$$\begin{cases}
\bar{u}_t = d_1 \int_{-\tilde{s}_0}^{\tilde{s}_0} J_1(x-y)\bar{u}(t,y)dy - d_1\bar{u} + a(t)\bar{u}, \quad t > 0, x \in [-\tilde{s}_0,\tilde{s}_0], \\
\bar{u}(0,x) = u_0(x), \quad |x| \leq s_0, \\
\bar{u}(0,x) = 0, \quad s_0 < |x| \leq \tilde{s}_0.
\end{cases}$$

Applying Lemma 3.3 in [9], we have

$$\bar{u} \leq \omega = Ke^{-\frac{\lambda t}{2}}\varphi \leq Ke^{-\frac{\lambda t}{2}}$$

in $(0,+\infty)\times[-\tilde{s}_0,\tilde{s}_0]$.

On the other hand, since $s_0 < \frac{s^*}{2}$, we have $\lambda_1(-(\tilde{L}_{(-s_0,s_0)} + b)) > 0$. Then, there exists $0 < \varepsilon_0 \ll 1$ such that for any $0 < \varepsilon \leq \varepsilon_0$, the following eigenvalue problem has a postive principle eigenvalue $\tilde{\lambda}_1 > 0$:

$$\begin{cases}
\varphi_t - d_2\Big[\tau\varphi_{xx} + (1-\tau)\Big((1+\varepsilon)\int_\Omega J_2(x-y)\varphi(t,y)dy - \varphi\Big)\Big] - b(t)\varphi = \lambda\varphi \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in } [0,T]\times(-s_0,s_0), \\
\varphi(t,\pm s_0) = 0 \quad \text{in } [0,T], \\
\varphi(0,x) = \varphi(T,x) \quad \text{in } [-s_0,s_0].
\end{cases}$$

Let $\tilde{\varphi}(t,x)$ be the normalized eigenfunction associated with $\tilde{\lambda}_1$. Since $\tilde{\varphi}_x(t,s_0)<0, \tilde{\varphi}_x(t,-s_0)>0$ in $[0,T]$, we have

$$x\tilde{\varphi}_x(t,x)\leq \alpha\tilde{\varphi}(t,x), \quad \forall (t,x)\in[0,T]\times[-s_0,s_0]$$

with some constant $\alpha>0$.

For any $(t,x)\in[0,+\infty)\times[-s(t),s(t)]$, we define

$$s(t)=s_0\varsigma(t), \quad \varsigma(t)=1+2\delta-\delta e^{-\sigma t}, \quad \bar{v}(t,x)=ke^{-\sigma t}\tilde{\varphi}(\xi(t),\eta(t,x))$$

with

$$\xi(t)=\int_0^t \frac{1}{\varsigma^2(\theta)}d\theta, \quad \eta(t,x)=\frac{s_0}{s(t)}x=\frac{x}{\varsigma(t)},$$

where $k>0$, $\sigma>0$ and $0<\delta<\frac{1}{2}(\frac{\tilde{s}_0}{s_0}-1)$ are positive constants to be determined later. Then, $\bar{v}(t,x)$ satisfies

$$\begin{aligned}
&\bar{v}_t(t,x)-d_2[\tau\bar{v}_{xx}+(1-\tau)(\int_{-s(t)}^{s(t)}J_2(x-y)\bar{v}(t,y)dy-\bar{v}(t,x))]-\bar{v}(t,x)(b(t)-\bar{v}(t,x))\\
&=ke^{-\sigma t}\Big[-\sigma\tilde{\varphi}(\xi,\eta)-\frac{\varsigma'(t)}{\varsigma(t)}\eta\tilde{\varphi}_\eta(\xi,\eta)+d_2(1-\tau)\Big(\frac{1+\varepsilon}{\varsigma^2(t)}\int_{-s_0}^{s_0}J_2(\eta-\tilde{\eta})\tilde{\varphi}(\xi,\tilde{\eta})d\tilde{\eta}\\
&\qquad -\varsigma(t)\int_{-s_0}^{s_0}J_2(\varsigma(t)\eta-\varsigma(t)\tilde{\eta})\tilde{\varphi}(\xi,\tilde{\eta})d\tilde{\eta}\Big)+d_2(1-\tau)(1-\frac{1}{\varsigma^2(t)})\tilde{\varphi}(\xi,\eta)\\
&\qquad +(\frac{1}{\varsigma^2(t)}b(\xi)-b(t))\tilde{\varphi}(\xi,\eta)+\frac{1}{\varsigma^2(t)}\tilde{\lambda}_1\tilde{\varphi}(\xi,\eta)+ke^{-\sigma t}\tilde{\varphi}^2(\xi,\eta)\Big]\\
&\geq ke^{-\sigma t}\Big[\Big(-\sigma-\sigma\alpha+d_2(1-\tau)(1-\frac{1}{\varsigma^2(t)})+\frac{1}{\varsigma^2(t)}\tilde{\lambda}_1+(\frac{1}{\varsigma^2(t)}b(\xi)-b(t))\Big)\tilde{\varphi}(\xi,\eta)\\
&\quad +d_2(1-\tau)\Big(\frac{1+\varepsilon}{\varsigma^2(t)}\int_{-s_0}^{s_0}J_2(\eta-\tilde{\eta})\tilde{\varphi}(\xi,\tilde{\eta})d\tilde{\eta}-\varsigma(t)\int_{-s_0}^{s_0}J_2(\varsigma(t)\eta-\varsigma(t)\tilde{\eta})\tilde{\varphi}(\xi,\tilde{\eta})d\tilde{\eta}\Big)\Big].
\end{aligned}$$

Define

$$G(t,\xi,\eta)=\frac{1+\varepsilon}{\varsigma^2(t)}\int_{-s_0}^{s_0}J_2(\eta-\tilde{\eta})\tilde{\varphi}(\xi,\tilde{\eta})d\tilde{\eta}-\varsigma(t)\int_{-s_0}^{s_0}J_2(\varsigma(t)\eta-\varsigma(t)\tilde{\eta})\tilde{\varphi}(\xi,\tilde{\eta})d\tilde{\eta}.$$

Obviously, $G(t,\xi,\eta)$ is a T-periodic function of ξ. Similar to the proof of Theorem 3.3 in [50], we can show that

$$\begin{aligned}
G(t,\xi,\eta) \geq\ &\frac{\varepsilon}{\varsigma^2(t)}\int_{-s_0}^{s_0}J_2(\eta-\tilde{\eta})\tilde{\varphi}(\xi,\tilde{\eta})d\tilde{\eta}-\varsigma(t)\int_{-s_0}^{s_0}\Big|J_2(\eta-\tilde{\eta})-J_2(\varsigma(t)\eta-\varsigma(t)\tilde{\eta})\Big|d\tilde{\eta}\\
&-\delta(\delta^2+3\delta+3).
\end{aligned}$$

Let

$$m=\frac{\varepsilon}{4}\min_{\xi\in[0,T]}\min_{\eta\in[-s_0,s_0]}\int_{-s_0}^{s_0}J_2(\eta-\tilde{\eta})\tilde{\varphi}(\xi,\tilde{\eta})d\tilde{\eta}>0.$$

By (**K**), there exists $\delta^*\in(0,\frac{1}{2})$ such that for any $0<\delta\leq\delta^*$,

$$\varsigma(t)\int_{-s_0}^{s_0}\Big|J_2(\eta-\tilde{\eta})-J_2(\varsigma(t)\eta-\varsigma(t)\tilde{\eta})\Big|d\tilde{\eta}\leq\frac{m}{2}.$$

It follows that for any $0<\delta\leq\min\{\delta^*,\frac{m}{10}\}$,

$$G(t,\xi,\eta)\geq 0, \quad \forall(t,\xi,\eta)\in[0,+\infty)\times[0,T]\times[-s_0,s_0].$$

By the fact that $\varsigma(t) \to 1$ as $\delta \to 0$, we can choose $0 < \sigma, \delta \ll 1$ such that, for $(t, x) \in [0, +\infty) \times (-s(t), s(t))$,

$$\begin{aligned}
&\bar{v}_t(t, x) - d_2[\tau \bar{v}_{xx} + (1-\tau)(\textstyle\int_{-s(t)}^{s(t)} J_2(x-y)\bar{v}(t, y)dy - \bar{v}(t, x))] - \bar{v}(t, x)(b(t) - \bar{v}(t, x)) \\
&\geq ke^{-\sigma t}\left(-\sigma - \sigma\alpha + \tfrac{1}{\varsigma^2(t)}\tilde{\lambda}_1 + (\tfrac{1}{\varsigma^2(t)}b(\xi) - b(t))\right)\tilde{\varphi}(\xi, \eta) \\
&> 0.
\end{aligned}$$

Moreover, we choose k large enough such that

$$\bar{v}(0, x) = k\tilde{\varphi}(0, \frac{x}{1+2\delta}) \geq v_0(x), \quad \forall x \in [-s_0, s_0].$$

Since $s(t) < s_0(1 + 2\delta) < \tilde{s}_0$, we know that

$$\bar{u}_t \geq d_1 \int_{-s(t)}^{s(t)} J_1(x-y)\bar{u}(t, y)dy - d_1\bar{u} + \bar{u}(a(t) - \bar{u}), \quad t > 0, x \in (-s(t), s(t)).$$

Note that

$$\begin{aligned}
&-\bar{v}_x(t, s(t)) = -\tfrac{k}{\varsigma(t)}e^{-\sigma t}\tilde{\varphi}_\eta(\xi(t), s_0) \leq \tfrac{k}{1-\delta}e^{-\sigma t}\|\tilde{\varphi}\|_{C^1([0,T]\times[-s_0,s_0])}, \\
&\int_{-s(t)}^{s(t)}\int_{s(t)}^{+\infty} J_2(x-y)\bar{v}(t, x)dydx \leq 2ks_0(1+2\delta)e^{-\sigma t}, \\
&\int_{-s(t)}^{s(t)}\int_{s(t)}^{+\infty} J_1(x-y)\bar{u}(t, x)dydx \leq 2Ks_0(1+2\delta)e^{-\frac{\lambda t}{2}}.
\end{aligned}$$

Since $0 < \sigma \ll 1$, we may further assume that $\sigma < \frac{\lambda}{2}$. Suppose that

$$0 < \mu + \rho_1 + \rho_2 \leq \tfrac{s_0\delta\sigma}{A}$$

with

$$A := \max\left\{\tfrac{k}{1-\delta}\|\tilde{\varphi}\|_{C^1([0,T]\times[-s_0,s_0])}, 2ks_0(1+2\delta), 2Ks_0(1+2\delta)\right\},$$

we have

$$\begin{aligned}
s'(t) &= s_0\delta\sigma e^{-\sigma t} \geq A(\mu + \rho_1 + \rho_2)e^{-\sigma t} \\
&\geq \tfrac{k}{1-\delta}\mu e^{-\sigma t}\|\tilde{\varphi}\|_{C^1([0,T]\times[-s_0,s_0])} + 2ks_0(1+2\delta)\rho_2 e^{-\sigma t} + 2Ks_0(1+2\delta)\rho_1 e^{-\sigma t} \\
&\geq \tfrac{k}{1-\delta}\mu e^{-\sigma t}\|\tilde{\varphi}\|_{C^1([0,T]\times[-s_0,s_0])} + 2ks_0(1+2\delta)\rho_2 e^{-\sigma t} + 2Ks_0(1+2\delta)\rho_1 e^{-\frac{\lambda t}{2}} \\
&\geq -\mu\bar{v}_x(t, s(t)) + \rho_1 \int_{-s(t)}^{s(t)}\int_{s(t)}^{+\infty} J_1(x-y)\bar{u}(t, x)dydx \\
&\quad +\rho_2 \int_{-s(t)}^{s(t)}\int_{s(t)}^{+\infty} J_2(x-y)\bar{v}(t, x)dydx.
\end{aligned}$$

Similarly, we can prove

$$\begin{aligned}
-s'(t) &\leq -\mu\bar{v}_x(t, -s(t)) - \rho_1 \int_{-s(t)}^{s(t)}\int_{-\infty}^{-s(t)} J_1(x-y)\bar{u}(t, x)dydx \\
&\quad -\rho_2 \int_{-s(t)}^{s(t)}\int_{-\infty}^{-s(t)} J_2(x-y)\bar{v}(t, x)dydx.
\end{aligned}$$

Applying Lemma 3.1, we get that $s_2(t) \leq s(t)$ and $s_1(t) \geq -s(t)$, which implies $s_{2,\infty} - s_{1,\infty} \leq 2\tilde{s}_0 < +\infty$.

To establish the criteria for spreading and vanishing, we give an abstract lemma which can be proved by similar arguments as the proof of Lemma 3.2 in [51]. Here we omit the details of the proof.

Lemma 4.12. *Assume that* (***K***) *holds and* $C \in \mathbb{R}_+$. *For any* $\mathcal{K}, r_0 \in \mathbb{R}_+$ *with* $\mathcal{K} > r_0$, *and any* $q_0 \in C^2([-r_0, r_0])$ *satisfying* $q_0(\pm r_0) = 0$ *and* $q_0 > 0$ *in* $(-r_0, r_0)$, *there exist* $\mu^0 > 0$ *and* $\rho_2^0 > 0$ *such that if either* $\mu \geq \mu^0$ *or* $\rho_2 \geq \rho_2^0$ *holds, and* $(q; \alpha, \beta)$ *satisfies*

$$\begin{cases} \partial_t q \geq d_2\left[\tau\partial_x^2 q + (1-\tau)\left(\int_{\alpha(t)}^{\beta(t)} J_2(x-y)q(t,y)dy - q\right)\right] - Cq, \quad t > 0,\ \alpha(t) < x < \beta(t), \\ q(t,\alpha(t)) = q(t,\beta(t)) = 0, \quad t \geq 0, \\ \beta'(t) \geq -\mu q_x(t,\beta(t)) + \rho_2 \int_{\alpha(t)}^{\beta(t)} \int_{\beta(t)}^{+\infty} J_2(x-y)q(t,x)dydx, \quad t \geq 0, \\ \alpha'(t) \leq -\mu q_x(t,\alpha(t)) - \rho_2 \int_{\alpha(t)}^{\beta(t)} \int_{-\infty}^{\alpha(t)} J_2(x-y)q(t,x)dydx, \quad t \geq 0, \\ q(0,x) = q_0(x), \quad |x| \leq r_0, \\ \beta(0) = -\alpha(0) = r_0, \end{cases} \tag{4.10}$$

then we have $\lim_{t\to+\infty} \alpha(t) \leq -\mathcal{K}$ *and* $\lim_{t\to+\infty} \beta(t) \geq \mathcal{K}$.

Theorem 4.13. *Let* $|\Omega| = \ell^*$ *and* $|\Omega| = s^*$ *be the unique roots of* $\lambda_1(-(L_\Omega + a)) = 0$ *and* $\lambda_1(-(\tilde{L}_\Omega + b)) = 0$, *respectively.*
(i) *For* $d_1 \leq a_T$, *the spreading always happens.*
(ii) *For* $d_1 > a_T$,
$(ii.1)$ *if* $s_0 \geq \frac{1}{2}\min\{\ell^*, s^*\}$, *then the spreading happens;*
$(ii.2)$ *if* $s_0 < \frac{1}{2}\min\{\ell^*, s^*\}$, *then there exist* $M^* > M_* > 0$ *such that the vanishing happens when* $\mu + \rho_1 + \rho_2 \leq M_*$ *and the spreading happens when* $\mu + \rho_1 + \rho_2 \geq M^*$.

Proof. From Theorem 4.10, we can get (i). Now we consider the case $d_1 > a_T$.

$(ii.1)$ For the case $s_0 \geq \frac{1}{2}\ell^*$, we have $s_{2,\infty} - s_{1,\infty} > 2s_0 \geq \ell^*$, and then $\lambda_1(-(L_{(s_{1,\infty}, s_{2,\infty})} + a)) < 0$. However, from (4.3) we deduce $\lambda_1(-(L_{(s_{1,\infty}, s_{2,\infty})} + a)) \geq 0$ for the vanishing case. Thus, spreading happens. If $s_0 \geq \frac{1}{2}s^*$, then Corollary 4.9 implies that the spreading always occurs.

$(ii.2)$ From (2.2), we have

$$\begin{cases} s_2'(t) > -\mu v_x(t, s_2(t)) + \rho_2 \int_{s_1(t)}^{s_2(t)} \int_{s_2(t)}^{+\infty} J_2(x-y)v(t,x)dydx, \\ s_1'(t) < -\mu v_x(t, s_1(t)) - \rho_2 \int_{s_1(t)}^{s_2(t)} \int_{-\infty}^{s_1(t)} J_2(x-y)v(t,x)dydx \end{cases}$$

and

$$\begin{cases} s_2'(t) > \rho_1 \int_{s_1(t)}^{s_2(t)} \int_{s_2(t)}^{+\infty} J_1(x-y)u(t,x)dydx, \\ s_1'(t) < -\rho_1 \int_{s_1(t)}^{s_2(t)} \int_{-\infty}^{s_1(t)} J_1(x-y)u(t,x)dydx. \end{cases} \tag{4.11}$$

Since u, v are positive and bounded, we have

$$v(b(t) - v - d(t)u) \geq -Cv$$

and

$$u(a(t) - u - c(t)v) \geq -Cu \tag{4.12}$$

with some constant $C > 0$. Thus, $(v; s_1, s_2)$ satisfies (4.10). For any given constant $\mathcal{K} > \frac{1}{2}\min\{\ell^*, s^*\}$, by Lemma 4.12 there exist $\mu^0, \rho_2^0 > 0$ such that

$$s_{2,\infty} - s_{1,\infty} \geq 2\mathcal{K} \tag{4.13}$$

for any $\mu \geq \mu^0$ or $\rho_2 \geq \rho_2^0$. Moreover, since $(u; s_1, s_2)$ satisfies (4.11) and (4.12), from Lemma 4.2 in [4] we can deduce that there exists ρ_1^0 such that (4.13) still holds for any $\rho_1 \geq \rho_1^0$.

Taking $M^0 = \mu^0 + \rho_1^0 + \rho_2^0$, by $(ii.1)$ we know that $s_{2,\infty} - s_{1,\infty} = +\infty$ for $\mu + \rho_1 + \rho_2 \geq M^0$. Note that in Theorem 4.11 we have that $s_{2,\infty} - s_{1,\infty} < +\infty$ for $\mu + \rho_1 + \rho_2 \leq M_0$. Applying the continuity method, we can get the desired results.

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Conflict of interest

The authors declare there is no conflict of interest.

References

1. C.Y. Kao, Y. Lou, W. X. Shen, Evolution of mixed dispersal in periodic environments, *Discrete Contin. Dyn. Syst. B*, **17** (2012), 2047–2072. https://doi.org/10.3934/dcdsb.2012.17.2047
2. X. L. Bai, F. Li, Classification of global dynamics of competition models with nonlocal dispersals I: Symmetric kernels, *Calc. Var. Partial Differ. Equations*, **57** (2018), 144. https://doi.org/10.1007/s00526-018-1419-6
3. Y. H. Du, M. X. Wang, M. Zhao, Two species nonlocal diffusion systems with free boundaries, *Discrete Contin. Dyn. Syst.*, **42** (2022), 1127–1162. https://doi.org/10.3934/dcds.2021149
4. J. P. Wang, M. X. Wang, Free boundary problems with nonlocal and local diffusions II: Spreading-vanishing and long-time behavior, *Discrete Contin. Dyn. Syst. B*, **25** (2020), 4721–4736. https://doi.org/10.3934/dcdsb.2020121
5. J. P. Wang, M. X. Wang, Free boundary problems with nonlocal and local diffusions I: Global solution, *J. Math. Anal. Appl.*, **490** (2020), 123974. https://doi.org/10.1016/j.jmaa.2020.123974
6. J. F. Cao, W. T. Li, J. Wang, M. Zhao, The Dynamics of a Lotka-Volterra competition model with nonlocal diffusion and free boundaries, *Adv. Differ. Equations*, **26** (2021), 163–200. https://doi.org/10.57262/ade026-0304-163
7. Y. H. Du, W. J. Ni, Analysis of a West Nile virus model with nonlocal diffusion and free boundaries, *Nonlinearity*, **33** (2020), 4407–4448. https://doi.org/10.1088/1361-6544/ab8bb2

8. M. Zhao, Y. Zhang, W. T. Li, Y. H. Du, The dynamics of a degenerate epidemic model with nonlocal diffusion and free boundaries, *J. Differ. Equations*, **269** (2020), 3347–3386. https://doi.org/10.1016/j.jde.2020.02.029
9. J. F. Cao, Y. Du, F. Li, W. T. Li, The dynamics of a Fisher-KPP nonlocal diffusion model with free boundaries, *J. Funct. Anal.*, **277** (2019), 2772–2814. https://doi.org/10.1016/j.jfa.2019.02.013
10. Y. H. Du, Z. G. Lin, Spreading-vanishing dichotomy in the diffusive logistic model with a free boundary, *SIAM J. Math. Anal.*, **42** (2010), 377–405. https://doi.org/10.1137/090771089
11. Y. H. Du, F. Li, M. L. Zhou, Semi-wave and spreading speed of the nonlocal Fisher-KPP equation with free boundaries, *J. Math. Pures Appl.*, **154** (2021), 30–66. https://doi.org/10.1016/j.matpur.2021.08.008
12. W. Y. Zhang, Z. H. Liu, L. Zhou, Dynamics of a nonlocal diffusive Logistic model with free boundaries in time periodic environment, *Discrete Contin. Dyn. Syst. B*, **26** (2021), 3767–3784. https://doi.org/10.3934/dcdsb.2020256
13. W. W. Ding, R. Peng, L. Wei, The diffusive logistic model with a free boundary in a heterogeneous time-periodic environment, *J. Differ. Equations*, **263** (2017), 2736–2779. https://doi.org/10.1016/j.jde.2017.04.013
14. Y. H. Du, Z. M. Guo, R. Peng, A diffusive logistic model with a free boundary in time-periodic environment, *J. Funct. Anal.*, **265** (2013), 2089–2142. https://doi.org/10.1016/j.jfa.2013.07.016
15. C. X. Lei, Z. G. Lin, Q. Y. Zhang, The spreading front of invasive species in favorable habitat or unfavorable habitat, *J. Differ. Equations*, **257** (2014), 145–166. https://doi.org/10.1016/j.jde.2014.03.015
16. M. X. Wang, A diffusive logistic equation with a free boundary and sign-changing coefficient in time-periodic environment, *J. Funct. Anal.*, **270** (2016), 483–508. https://doi.org/10.1016/j.jfa.2015.10.014
17. P. Zhou, D. M. Xiao, The diffusive logistic model with a free boundary in heterogeneous environment, *J. Differ. Equations*, **256** (2014), 1927–1954. https://doi.org/10.1016/j.jde.2013.12.008
18. H. Gu, Z. G. Lin, B. D. Lou, Different asymptotic spreading speeds induced by advection in a diffusion problem with free boundaries, *Proc. Am. Math. Soc.*, **143** (2015), 1109–1117. https://doi.org/10.1090/S0002-9939-2014-12214-3
19. C. H. Wu, Biased movement and the ideal free distribution in some free boundary problems, *J. Differ. Equations*, **265** (2018), 4251–4282. https://doi.org/10.1016/j.jde.2018.06.002
20. N. K. Sun, J. Fang, Propagation dynamics of Fisher-KPP equation with time delay and free boundaries, *Calc. Var. Partial Differ. Equations*, **58** (2019), 148. https://doi.org/10.1007/s00526-019-1599-8
21. Y. H. Du, B. D. Lou, Spreading and vanishing in nonlinear diffusion problems with free boundaries, *J. Eur. Math. Soc.*, **17** (2015), 2673–2724. https://doi.org/10.4171/JEMS/568
22. J. L. Ren, D. D. Zhu, On a reaction-advection-diffusion equation with double free boundaries and mth-order Fisher non-linearity, *IMA J. Appl. Math.*, **84** (2019), 197–227. https://doi.org/10.1093/imamat/hxy057

23. G. Bunting, Y. H. Du, K. Krakowski, Spreading speed revisited: Analysis of a free boundary model, *Networks Heterogen. Media*, **7** (2012), 583–603. https://doi.org/10.3934/nhm.2012.7.583
24. X. Liu, B. D. Lou, On a reaction-diffusion equation with Robin and free boundary conditions, *J. Differ. Equations*, **259** (2015), 423–453. https://doi.org/10.1016/j.jde.2015.02.012
25. Y. Kaneko, Spreading and vanishing behaviors for radially symmetric solutions of free boundary problems for reaction-diffusion equations, *Nonlinear Anal. Real World Appl.*, **18** (2014), 121–140. https://doi.org/10.1016/j.nonrwa.2014.01.008
26. Y. H. Du, Z. G. Lin, The diffusive competition model with a free boundary: Invasion of a superior or inferior competitor, *Discrete Contin. Dyn. Syst. B*, **19** (2014), 3105–3132. https://doi.org/10.3934/dcdsb.2014.19.3105
27. Y. H. Du, M. X. Wang, M. Zhou, Semi-wave and spreading speed for the diffusive competition model with a free boundary, *J. Math. Pures Appl.*, **107** (2017), 253–287. https://doi.org/10.1016/j.matpur.2016.06.005
28. J. S. Guo, C. H. Wu, On a free boundary problem for a two-species weak competition system, *J. Dynam. Differ. Equations*, **24** (2012), 873–895. https://doi.org/10.1007/s10884-012-9267-0
29. J. S. Guo, C. H. Wu, Dynamics for a two-species competition-diffusion model with two free boundaries, *Nonlinearity*, **28** (2015), 1–27. https://doi.org/10.1088/0951-7715/28/1/1
30. J. Wang, L. Zhang, Invasion by an inferior or superior competitor: A diffusive competition model with a free boundary in a heterogeneous environment, *J. Math. Anal. Appl.*, **423** (2015), 377–398. https://doi.org/10.1016/j.jmaa.2014.09.055
31. Q. L. Chen, F. Q. Li, F. Wang, A reaction-diffusion-advection competition model with two free boundaries in heterogeneous time-periodic environment, *IMA J. Appl. Math.*, **82** (2017), 445–470. https://doi.org/10.1093/imamat/hxw059
32. M. X. Wang, Y. Zhang, The time-periodic diffusive competition models with a free boundary and sign-changing growth rates, *Z. Angew. Math. Phys.*, **67** (2016), 132. https://doi.org/10.1007/s00033-016-0729-9
33. C. R. Tian, S. G. Ruan, On an advection-reaction-diffusion competition system with double free boundaries modeling invasion and competition of Aedes albopictus and Aedes aegypti mosquitoes, *J. Differ. Equations*, **265** (2018), 4016–4051. https://doi.org/10.1016/j.jde.2018.05.027
34. L. Zhou, S. Zhang, Z. H. Liu, An evolutional free-boundary problem of a reaction-diffusion-advection system, *Proc. R. Soc. Edinburgh Sect. A: Math.*, **147** (2017), 615–648. https://doi.org/10.1017/S0308210516000226
35. M. X. Wang, On some free boundary problems of the prey-predator model, *J. Differ. Equations*, **256** (2014), 3365–3394. https://doi.org/10.1016/j.jde.2014.02.013
36. M. X. Wang, Y. Zhang, Dynamics for a diffusive prey-predator model with different free boundaries, *J. Differ. Equations*, **264** (2018), 3527–3558. https://doi.org/10.1016/j.jde.2017.11.027
37. J. F. Cao, W. T. Li, J. Wang, F. Y. Yang, A free boundary problem of a diffusive SIRS model with nonlinear incidence, *Z. Angew. Math. Phys.*, **68** (2017), 39. https://doi.org/10.1007/s00033-017-0786-8

38. J. Ge, K. Kim, Z. G. Lin, H. P. Zhu, A SIS reaction-diffusion-advection model in a low-risk and high-risk domain, *J. Differ. Equations*, **259** (2015), 5486–5509. https://doi.org/10.1016/j.jde.2015.06.035

39. Z. G. Lin, H. P. Zhu, Spatial spreading model and dynamics of West Nile virus in birds and mosquitoes with free boundary, *J. Math. Biol.*, **75** (2017), 1381–1409. https://doi.org/10.1007/s00285-017-1124-7

40. Q. L. Chen, F. Q. Li, Z. D. Teng, F. Wang, Global dynamics and asymptotic spreading speeds for a partially degenerate epidemic model with time-delay and free boundaries, *J. Dyn. Differ. Equations*, **34** (2022), 1209–1236. https://doi.org/10.1007/s10884-020-09934-4

41. Q. X. Ye, Z. Y. Li, M. X. Wang, Y. P. Wu, *Introduction to Reaction Diffusion Equations (in Chinese)*, 2nd edition, Science Press, Beijing, 2011.

42. R. Wang, Y. H. Du, Long-time dynamics of a diffusive epidemic model with free boundaries, *Discrete Contin. Dyn. Syst. Ser. B*, **26** (2021), 2201–2238. https://doi.org/10.3934/dcdsb.2020360

43. J. Coville, On a simple criterion for the existence of a principal eigenfunction of some nonlocal operators, *J. Differ. Equations*, **249** (2010), 2921–2953. https://doi.org/10.1016/j.jde.2010.07.003

44. Z. W. Shen, H. Vo, Nonlocal dispersal equations in time-periodic media: Principal spectral theory, limiting properties and long-time dynamics, *J. Differ. Equations*, **267** (2019), 1423–1466. https://doi.org/10.1016/j.jde.2019.02.013

45. P. Hess, *Periodic-Parabolic Boundary Value Problems and Positivity*, Harlow, Essex: Longman Scientific and Technical, 1991.

46. P. Bates, G. Zhao, Existence, uniqueness and stability of the stationary solution to a nonlocal evolution equation arising in population dispersal, *J. Math. Anal. Appl.*, **332** (2007), 428–440. https://doi.org/10.1016/j.jmaa.2006.09.007

47. M. Marcus, L. Véron, The boundary trace of positive solutions of semilinear elliptic equations: The subcritical case, *Arch. Ration. Mech. Anal.*, **144** (1998), 201–231. https://doi.org/10.1007/s002050050116

48. J. P. Gao, S. G. Guo, W. X. Shen, Persistence and time periodic positive solutions of doubly nonlocal Fisher-KPP equations in time periodic and space heterogeneous media, *Discrete Contin. Dyn. Syst. B*, **26** (2021), 2645–2676. https://doi.org/10.3934/dcdsb.2020199

49. F. X. Chen, Almost periodic traveling waves of nonlocal evolution equations, *Nonlinear Anal.*, **50** (2002), 807–838. https://doi.org/10.1016/S0362-546X(01)00787-8

50. H. M. Huang, M. X. Wang, A nonlocal SIS epidemic problem with double free boundaries, *Z. Angew. Math. Phys.*, **70** (2019), 109. https://doi.org/10.1007/s00033-019-1156-5

51. M. X. Wang, J. F. Zhao, Free boundary problems for a Lotka-Volterra competition system, *J. Dynam. Differ. Equations*, **26** (2014), 655–672. https://doi.org/10.1007/s10884-014-9363-4