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Research article

Estimation of stress-strength reliability from unit-Burr III distribution under records data

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Abstract: This paper explores estimation of stress-strength reliability based on upper record values. When the strength and stress variables follow unit-Burr III distributions, a generalized inferential approach is proposed for estimating stress-strength reliability (SSR). Under the common strength and stress parameter case, two types of pivotal quantities are constructed respectively, and then the generalized point and interval estimates for SSR are proposed in consequence, where the associated Monte-Carlo sampling approach is provided for computation. In addition, when strength and stress variables feature unequal model parameters, different generalized point and confidence interval estimates are also established in this regard. Extensive simulation studies are conducted to examine the behavior of proposed methods. Finally, a real-life data example is presented for illustration.

Keywords: stress-strength model; Unit-Burr III distribution; pivotal quantities; upper records data; generalized estimation

1. Introduction

The stress-strength model is of great significance in lifetime studies and engineering applications. In the context, the stress-strength model could be described as the probability of reliability of a system or unit where its strength variable X overcomes the associated stress variable Y upon it. Therefore, stress-strength reliability (SSR) is defined as $R = P(Y < X)$. The notion of stress-strength model was initially illustrated in Birnbaum [1] and further explored by Birnbaum and McCarty [2]. The applications of stress-strength model are reported in many fields such as reliability engineering, oceanography, hydrology, economics and survival analysis. For example, in medical science, when X and Y represent the life span of patients in the treatment group and the control group respectively, the probability $R = P(Y < X)$ is used as a measure of the therapeutic effect. Due to its theoretical importance and applicability, extensive works of on stress-strength model have been done in literature. For example, Baklizi [3] considered the problem of estimating SSR when the available data is in the form of record values from exponential distribution. Krishnamoorthy and Yin [4] studied interval estimation problem of the stress-strength reliability of two independent Weibull distributions, and the interval estimation method based on the generalized variable (GV) method was presented. Ahmed and Batah [5] estimated SSR of a single reliability system when the strength and stress variables come from power Rayleigh distributions, and different inferential methods are proposed in consequence. The SSR under Weibull distrubution is discussed by Pak et al. [6] with inter-record times, when strength and stress shape parameters are assumed to be known. Kumari et al. [7] considered estimation of the reliability in a stress-strength model from classical likelihood and Bayesian methods when generalized exponential distributions are implemented. Classical and Bayesian estimation procedures for SSR $R = P(Y < X)$ for Lomax distribution is discussed by Yadav et al. [8] where the sample information is Type-II hybrid censored. For more discussions, one could also refer to some recent constributions of Jafari and Bafekri [9], Kumari et al. [10], Luo et. al [11], Safariyan et al. [12], Hamad [13] as well as the references therein. Interested readers may also refer to monograph of Kotz and Pensky [14] for more details.

In life-testing and engineering, due to practical limitations of time and cost constraints as well as internal operating mechanisms, complete failure times are difficult to collect, and observations often appeared as censored data in practice. In practice, there are many censoring schemes implemented in experiments for collecting failure data, and some familiar ones include Type-I censoring, Type-II censoring and progressive censoring as well as hybrid censoring scenarios. In this regard, extensive studies have been discussed by many authors, to name a few, see some recent contributions of Abushal [15], Hu and Chen [16], Kohansal [17], Okasha et al. [18], Roy et al. [19] and Singh and Tripathi [20], Zhuang et al. [21] among others. One may also refer to the monographes of Lawless [22] and Balakrishnan and Cramer [23] for a comprehensive review. Besides the aforementioned censored data, record value is also another common appeared data type in many practical situations such as reliability engineering, hydrology, economics, mining and meteorology. For example, a wooden beam breaks when sufficient perpendicular force is applied to it, an electronic component ceases to function in an environment of too high temperature, and a battery dies under the stress of time. Thus, in such experiments, measurements may be performed sequentially, recording only values larger than all previous values. The mathematical study of records is initially introduced by Chandler [24]. Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables, an observation X_j is called a upper record if $X_j > X_i$ for every $i < j$. Therefore, the record time sequence is defined as $T_1 = 1$ and $T_n = \min\left\{j : X_j > X_{T_{n-1}}\right\}$, and the upper record values $R_0, R_1, \ldots, R_n, \ldots$ are observed as $R_n = X_{T_n}, n = 1, 2, \ldots$. Suppose the record values $(R_1, R_2, \ldots, R_m)$ of size m from population with cumulative distribution function (CDF) $F(t)$ and probability density function (PDF) $f(t)$, then the joint density function can be expressed as

$$f(r_1, r_2, \cdots, r_m) = f(r_m) \prod_{i=1}^{m-1} \frac{f(r_i)}{1 - F(r_i)}, \tag{1.1}$$

where $(r_1, r_2, \cdots, r_m)$ is the observations of $(R_1, R_2, \ldots, R_m)$. After proposed by Chandler [24], record values have attracted wide attention and been discussed by many authors. See, for example, some recent contributions of Hassan et al. [25], Kizilaslan [26], Pak [6], Tarvirdizade and Ahmadpour [27] as well as the reference therein. One could also refer to the monograps of Ahsanullah [28].

In data analysis, various lifetime distributions have been proposed for modeling failure samples in both theoretical and practical perspectives, and some famous distributions include exponential, gamma, Weibull, normal etc. One feature of such models is that they all have infinity support, however, data collected in practice sometimes is often appeared as bounded data within specified ranges. Therefore, common lifetime distributoins with infinite supports may not provide good enough fitting under such situations. Thus, the bounded data frequently appears in various fields of economy, measurements, finance, engineering among others. Therefore, distributions with bounded support may be implemented in this regard that may provide higher weight to the bounded data and give better fitting effect in data analysis. Specifically, distributions with unit bound within $(0, 1)$ have attracted considerable attention, some famous ones includes beta, Kumaraswamy, Topp-Lenoey distributions, among other (e.g., Jha et al. [29], Kohansal [17], Mazucheli et al. [30] and Sultana et al. [31]). Specially, Korkmaz et al. have also systemly discussed the importance of unit distributions on their series papers, and interested readers may refer to references [32], [33], [34] and [35] for more details. In addition, distributions with unit bound have a wide application in stress-strength analysis. Jha et al. [36] estimated the multicomponent reliability by assuming the unit-Gompertz strength and stress variables. Cruz et al. [37] proposed a novel estimation procedure of stress-strength reliability in the case of two independent unit-half-normal distributions which could fit asymmetrical data with either positive or negative skew. More applications could be found in the works of Alotaibi et al. [5], Dey and Wang [38] and Jha et al. [39] as well as the reference therein. Recently, Modi and Gill [40] introduced a new unit-Burr III distribution with unit support in $(0, 1)$, which possesses a flexible density and bathtub shape hazard rate curves. Hereafter, the unit-Burr III model with parameters α and β is denoted by UB(α,β) for concision. Let T be the random variable of the UB distribution with parameters α and β, the associated CDF, PDF and hazard rate function (HRF) of T are given respectively by

$$\begin{aligned} f(t;\alpha,\beta) &= \alpha\beta\frac{(\log(1/t))^{-\alpha-1}}{t(1+(\log(1/t))^{-\alpha})^{\beta+1}}, 0<t<1,\\ F(t;\alpha,\beta) &= 1-(1+(\log(1/t))^{-\alpha})^{-\beta}, 0<t<1,\\ H(t) &= \frac{\alpha\beta(\log(1/t))^{-\alpha-1}}{t(1+(\log(1/t))^{-\alpha})}, 0<t<1, \end{aligned} \tag{1.2}$$

where $\alpha > 0$ and $\beta > 0$ are both shape parameters. It is noted that the parameters α and β affect the geometric shape of the density distribution curve and the steepness of the density curve. The UB distribution has been also discussed by several authors. For example, Singh et al. [41] studied the problem of estimating multicomponent stress-strength reliability under progressive Type-II censoring when stress and strength variables follow UB distributions with common shape parameter. Dey and Wang [38] used classical estimation method to estimate the parameters of UB distribution. For illustration, some plots of PDF and HRF of the UB model are shown in Figure 1, and one could observe in visual that the UB has very flexible fitting ability and may be used as an alternative to the conventional unit bound models.

Inference for the SSR is of considerable interest and practical significance in reliability applications and lifetime studies. However, sometimes sample size heavily affects the inferential accuracy of SSR estimation especially when there are not enough strength and stress observations due to practical time and cost constriants. In this regard, sample size may has such a strong effect on the validity of analytical results and the estimate sometimes is even misunderstood. In addition, it is frequently observed that the record values appear rarely in practical data collection process, therefore, as data sets consisting of record values often lack sufficient data for statistical inference. Motivated by such reasons and due to the potential theoretical and practical applications of the UB model, this paper proposes generalized inferential approach to estimate the SSR when record values for strength and stress variables are available.

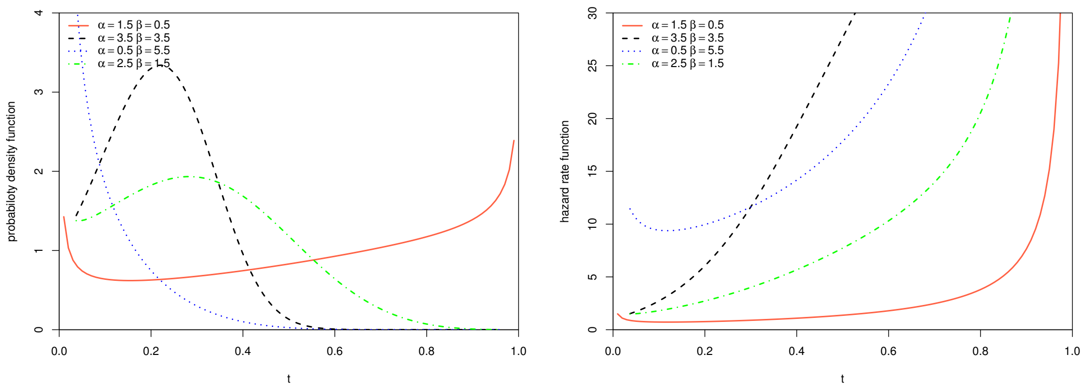

Figure 1. Plots of PDF and HRF of unit-Burr III with different parameters.

The rest parts of this paper is organized as follows. Under common unit-Burr III strength and stress parameter case, Section 2 explores generalized SSR estimation common parameter case. In addition, generalized point and interval estimates are also proposed in Section 3. Section 4 provides some numerical studies to investigate the performance of differnt results. Finally, some concluding remarks are given in Section 5.

2. SSR estimation under common parameter case

In this section, pivotal quantities based generalized inferential approach is proposed for SSR under common parameter case with $\alpha_1 = \alpha_2 = \alpha$. Suppose that $X = (X_1, X_2, \ldots, X_m)$ are strength upper record values of size m from population UB(α,β_1), and the associated stress upper record values $Y = (Y_1, Y_2, \ldots, Y_m)$ from population UB(α,β_2). Under this case, the SSR is obtained as

$$R = P(Y < X) = \int_0^1 \left[\int_0^x f(y; \beta_2, \alpha) dy \right] f(x; \beta_1, \alpha) dx$$
$$= \frac{\beta_2}{\beta_1 + \beta_2}. \tag{2.1}$$

It is noted from (2.1) that when the strength and stress variables share common parameters with $\alpha_1 = \alpha_2 = \alpha$, the SSR just characterized by parameters β_1 and β_2 being free of α.

2.1. CC based generalized estimation

In this part, generalized estimator for the SSR is constructed based on two chi-square pivotal quantities, and the result under this approach is called as CC based generalized estimates for concision.

Theorem 1. *Consider pivotal quantities*

$$A_1^X(\alpha) = 2 \sum_{j=1}^{m-1} \log \left[\frac{\log(1 + (-\log(x_m))^{-\alpha})}{\log(1 + (-\log(x_j))^{-\alpha})} \right] \textit{ and } B_1^X(\beta_1, \alpha) = 2\beta_1 \log(1 + (-\log(x_m))^{-\alpha}).$$

Then $A_1^X(\alpha)$ and $B_1^X(\beta_1, \alpha)$ are statistically independent and follow chi-square distributions with $2(m-1)$ and $2m$ degrees of freedom, respectively.

Proof. See Appendix A. □

Following similar approach of Theorem 1, one further has following results and the proof is omitted for concision.

Theorem 2. *Consider pivotal quantities*

$$A_1^Y(\alpha) = 2\sum_{j=1}^{m-1} \log\left[\frac{\log(1+(-\log(y_m))^{-\alpha})}{\log(1+(-\log(y_j))^{-\alpha})}\right] \text{ and } B_1^Y(\beta_2, \alpha) = 2\beta_2 \log(1+(-\log(y_m))^{-\alpha})).$$

Then A_1^Y and B_1^Y follow chi-square distributions with $2(m-1)$ and m degrees of freedom, and are also statistically independent.

Lemma 1. *For arbitrary constants c and d with $0 < c < d < 1$, let $R(h) = \frac{\log(1+(-\log(d))^{-h})}{\log(1+(-\log(c))^{-h})}$, $h > 0$. Then following results hold*

- *$R(h)$ increases in h.*
- $\lim_{h\to 0} R(h) = 1$ *and* $\lim_{h\to\infty} R(h) = \infty$.

Proof. See Appendix B □

Due to Lemma 1, following results could be obtained in consequence, and the detailed proof is omitted for concision.

Corollary 1. *Pivotal quantity $A_1^X(\alpha)$ and $A_1^Y(\alpha)$ increase in α with range $(0, +\infty)$.*

According to the additivity of chi-square distribution, the quantity $A_1(\alpha) = A_1^X(\alpha) + A_1^Y(\alpha)$ follows chi-square distribution with $4(m-1)$ degrees of freedom. It is also conducted from Lemma 1 that $A_1(\alpha)$ increses in α with range $(0, \infty)$. Therefore, for a given $a_1 \sim \chi^2_{4(m-1)}$, equation $A_1(\alpha) = a_1$ has an unique solution, denoted as $\hat{\alpha} = \hat{\alpha}_1(A_1; x, y)$, where the bisection method can be used to solve the equation. Further, using substitution method of Weerahandi [42] and substituting $\hat{\alpha}_1$ for α, generalized pivotal quantities for β_1 and β_2 can be constructed respectively as

$$S_1^X = \frac{B_1^X}{H_1^X[\hat{\alpha}]} \text{ with } H_1^X[\alpha] = 2\log(1+(-\log(x_m))^{-\alpha}) \text{ and } B_1^X \sim \chi^2_{2m}$$

and

$$S_1^Y = \frac{B_1^Y}{H_1^Y[\hat{\alpha}]} \text{ with } H_1^Y[\alpha] = 2\log(1+(-\log(y_m))^{-\alpha}) \text{ and } B_1^Y \sim \chi^2_{2m}.$$

Consequently, a generalized pivotal quantity for SSR R defined in (2.1) can be constructed by using the substitution method of Weerahandi [42] as follows

$$R_1^{CC} = \frac{\frac{B_1^Y}{H_1^Y[\hat{\alpha}]}}{\frac{B_1^X}{H_1^X[\hat{\alpha}]} + \frac{B_1^Y}{H_1^Y[\hat{\alpha}]}}.$$

Furthermore, a Monte Carlo procedure called Algorithm 1 is provided to obtain the CC based generalized point estimate (GPE) and generlized confidence interval (GCI) of SSR under common parameter case.

Algorithm 1: CC based generalized estimation of SSR under common parameter case.

Step 1 Generate $a_1 \sim \chi^2_{4(m-1)}$, and then obtain $\hat{\alpha}$ from the equation $A_1(\alpha) = a_1$.

Step 2 Generate B_1^X and B_1^Y from χ^2_{2m} and calculate SSR using R_1^{CC}.

Step 3 Repeat Steps 1 and 2 M times, and M estimates of SSR are obtained as $R^{[1]}, R^{[2]}, \cdots, R^{[M]}$ showing in ascending order.

Step 4 A natural GPE of SSR R is constructed as

$$\hat{R} = \frac{1}{M} \sum_{j=1}^{M} R^{[j]}$$

Step 5 Based on $R^{[1]}, R^{[2]}, \cdots, R^{[M]}$ and for $0 < \mu < 1$, a series of $100(1-\mu)\%$ confidence interval of R can be expressed as

$$(R^{[j]}, R^{[j+M-[M\mu+1]]}),\ j = 1, 2, \cdots, [M\mu]$$

where $[h]$ is the integer function. Therefore, the $100(1-\mu)\%$ GCI of R can be selected as $j^{*}th$ one satisfying

$$R^{[j^*+M-[M\mu+1]]} - R^{[j^*]} = \min_{j=1}^{[M\mu]} (R^{[j+M-[M\mu+1]]} - R^{[j]}).$$

2.2. RC pivotal based generalized estimation

In this part, the RG distributed and chi-square distributed pivotal quantities are constructed, and then generalized estimate is proposed for SSR under common parameter case. For simplicity, estimate obtained using these two pivotal quantities is called as RC based generalized estimate for short.

Before proceeding, the previous mentioned RG model is introduced as follows.

Definition 1. *For independent variables $Q_1 \sim F(a, b)$ and $Q_2 \sim F(c, d)$, then the quantity $\frac{ac}{bd} Q_1 Q_2$ is RG distributed with parameters a, c, b, d, i.e. $\frac{ac}{bd} Q_1 Q_2 \sim RG(a, c, b, d)$.*

Theorem 3. *Denote quantities*

$$A_2^X(\alpha) = \frac{(m-2)\log(1 + (-\log(x_1))^{-\alpha})}{\log(1 + (-\log(x_m))^{-\alpha}) - \log(1 + (-\log(x_1))^{-\alpha})} \text{ and } B_2^X(\beta_1, \alpha) = 2\beta_1 \log(1 + (-\log(x_m))^{-\alpha}).$$

Then $A_2^X(\alpha)$ follows an F distribution with $(2, 2m-2)$ degrees of freedom, whereas $B_2^X(\beta_1, \alpha)$ follows chi-square distribution with $2m$ degrees of freedom, and $A_2^X(\alpha)$ and $B_2^X(\beta_1, \alpha)$ statistically independent.

Proof. See Appendix C. □

Similarly, one also has the following results and the proof is omotted for brevity.

Theorem 4. *Quantities*

$$A_2^Y(\alpha) = \frac{(m-2)\log(1+(-\log(y_1))^{-\alpha})}{\log(1+(-\log(y_m))^{-\alpha}) - \log(1+(-\log(y_1))^{-\alpha})} \text{ and } B_2^Y(\beta_2,\alpha) = 2\beta_2 \log(1+(-\log(y_m))^{-\alpha})$$

follow F distribution with $(2, 2m-2)$ *degrees of freedom and chi-square distribution with* $2m$ *degrees of freedom respectively. In addition,* $A_2^Y(\alpha)$ *and* $B_2^Y(\beta_2,\alpha)$ *are statistically independent.*

Lemma 2. *For arbitrary constants* e *and* f *with* $0 < e < f < 1$, *let* $G(h) = \frac{\log(1+(-\log(e))^{-h})}{\log(1+(-\log(f))^{-h})-\log(1+(-\log(e))^{-h})}$, $h > 0$. *Then one has following results.*

- $G(h)$ *decreases in h.*
- $\lim_{h\to 0} G(h) = \infty$ *and* $\lim_{h\to\infty} G(h) = 0$.

Proof. See Appendix D. □

Corollary 2. *Pivotal quantity* $A_2^X(\alpha)$ *and* $A_2^Y(\alpha)$ *decrease in* α *with range* $(0, +\infty)$.

It is noted from Definition 1 that quantity $A_2(\alpha) = \frac{4}{(2m-2)^2} A_2^X A_2^Y \sim RG(2, 2, 2m-2, 2m-2)$, then for a given $a_2 \sim RG(2, 2, 2m-2, 2m-2)$, it is conducted from Corollary 2 that equation $A_2(\alpha) = a_2$ has an unique solution with respect to α that is denoted as $\acute{\alpha} = \acute{\alpha}_(A_2; x, y)$ that can be also computed using the bisection method. Similarly, substituting $\acute{\alpha}$ for α and using the substitution method of Weerahandi [42], the generalized pivotal quantities for parameters β_1 and β_2 can be constructed respectively from Theorems 3 and 4 as

$$S_2^X = \frac{B_2^X}{H_2^X[\acute{\alpha}]} \text{ with } H_2^X[\alpha] = 2\log(1+(-\log(x_m))^{-\alpha}) \text{ and } B_2^X \sim \chi^2_{2m}$$

and

$$S_2^Y = \frac{B_2^Y}{H_2^Y[\acute{\alpha}]} \text{ with } H_2^Y[\alpha] = 2\log(1+(-\log(y_m))^{-\alpha}) \text{ and } B_2^Y \sim \chi^2_{2m}.$$

Therefore, the RC pivotal quantity based generalized estimator for SSR under common parameter case can be expressed as

$$R_1^{RC} = \frac{\frac{B_2^Y}{H_2^Y[\acute{\alpha}]}}{\frac{B_2^X}{H_2^X[\acute{\alpha}]} + \frac{B_2^Y}{H_2^Y[\acute{\alpha}]}}.$$

Consequently, a Monte-Carlo procedure called Algorithm 2 is provided for obtaining the RC based GPE and GCI of SSR under common parameter case.

Algorithm 2: RC based generalized estimation of SSR under common parameter case.

Step 1 Generate a sample a_2 from $GR(2,2,2m-2,2m-2)$ distribution, and obtain an observation of $\acute{\alpha}$ from the equation $A_2(\alpha) = a_2$.

Step 2 Generate samples of B_2^X and B_2^Y from χ^2_{2m}, and then obtain an estimate of SSR using R_1^{RC}.

Step 3 Step 1 and Step 2 are repeated M times and obtain M estimates of SSR.

Step 4 Using same Steps 4 and 5 shown in Algrithm 1, the RC based GPE and GCI are obtained in this manner.

3. SSR estimation under unequal parameter case

In this section, pivotal quantities based generalized inferential approach is proposed for SSR when the strength and stress parameters are totally unequal with $\alpha_1 \neq \alpha_2$ and $\beta_1 \neq \beta_2$. In this regard, suppose that $(X_1, X_2, \ldots, X_m)$ are upper strength variables from UB(α_1,β_1), and independent stress upper record values $(Y_1, Y_2, \ldots, Y_m)$ follows UB(α_2,β_2) distribution. Under this case, the SSR is obtained as

$$\begin{aligned} R = P(Y < X) &= \int_0^1 \left[\int_0^x f(y;\beta_2,\alpha_2)dy \right] f(x;\beta_1,\alpha_1)dx \\ &= 1 - \alpha_1\beta_1 \int_0^1 \frac{(-\log x)^{-\alpha_1-1}}{x(1+(-\log x)^{-\alpha_2})^{\beta_2}(1+(-\log x)^{-\alpha_1})^{\beta_1+1}}dx. \end{aligned} \tag{3.1}$$

Clearly, it is seen from (3.1) that under arbitrary unequal parameter case, the SSR is associated with all strength and stress parameters α_1,β_1 and α_2,β_2. Consequently, the generalized result of R will be obtained by using the substitution method after the generalized estimates of model parameters are derived.

3.1. CC based generalized estimation

In this part, CC based generalized estimator for SSR is conducted under uneuqal parameter case.

Theorem 5. *Denote pivotal quantities*

$$C_1^X(\alpha_1) = 2\sum_{j=1}^{m-1} \log\left[\frac{\log(1+(-\log(x_m))^{-\alpha_1})}{\log(1+(-\log(x_j))^{-\alpha_1})}\right], \ D_1^X(\beta_1,\alpha_1) = 2\beta_1\log(1+(-\log(x_m))^{-\alpha_1})$$

and

$$C_1^Y(\alpha_2) = 2\sum_{j=1}^{m-1} \log\left[\frac{\log(1+(-\log(y_m)^{-\alpha_2}))}{\log(1+(-\log(y_j)^{-\alpha_2}))}\right], \ D_1^Y(\beta_2,\alpha_2) = 2\beta_2\log(1+(-\log(y_m)^{-\alpha_2})).$$

Then, following results is observed as

- $C_1^X(\alpha_1) \sim \chi^2_{2(m-1)}$, $D_1^X(\beta_1,\alpha_1) \sim \chi^2_{2m}$, $C_1^X(\alpha_1)$ *and* $D_1^X(\beta_1,\alpha_1)$ *are statistically independent.*
- $C_1^Y(\alpha_2) \sim \chi^2_{2(m-1)}$, $D_1^Y(\beta_2,\alpha_2) \sim \chi^2_{2m}$,$C_1^Y(\alpha_2)$ *and* $D_1^Y(\beta_2,\alpha_2)$*are statistically independent.*

Proof. Such results could be conducted following similar approach as Theorems 1 and 2, and the detailed proof is ommited for concision. □

Based on Theorem 5, let $\breve{\alpha}_1 = \breve{\alpha}_1(C_1^X; x)$ and $\breve{\alpha}_2 = \breve{\alpha}_2(C_1^Y; y)$ be the unique solutions of equations $C_1^X(\alpha_1) = c_1^X$ and $C_1^Y(\alpha_2) = c_1^Y$ respectively, where $c_1^X \sim \chi^2_{2(m-1)}$ and $c_1^Y \sim \chi^2_{2(m-1)}$ being arbitray generalied data form the chi-square distributons. Using the substitution method of Weerahandi [42] and substituting $\breve{\alpha}_1$ and $\breve{\alpha}_2$ for α_1 and α_2, then generalized pivotal quantities for β_1 and β_2 are constructed from Theorem 5 as

$$S_1^X = \frac{D_1^X}{H_1^X[\breve{\alpha}_1]} \text{ with } H_1^X[\alpha_1] = 2\log(1 + (-\log(x_m))^{-\alpha_1}) \text{ and } D_1^X \sim \chi^2_{2m}$$

and

$$S_1^Y = \frac{D_1^Y}{H_1^Y[\breve{\alpha}_2]} \text{ with } H_1^Y[\alpha_2] = 2\log(1 + (-\log(y_m))^{-\alpha_2}) \text{ and } D_1^Y \sim \chi^2_{2m}.$$

Therefore, based the substitution method of Weerahandi [42], the CC pivotal quantities based generalized estimator for SSR under unequal parameter case can be constructed as

$$R_2^{CC} = 1 - \frac{\breve{\alpha}_1 B_1^X}{H_1^X[\breve{\alpha}_1]} \int_0^1 \frac{(-\log x)^{-\breve{\alpha}_1 - 1}}{x(1 + (-\log x)^{-\breve{\alpha}_2})^{\frac{B_1^Y}{H_1^Y[\breve{\alpha}_2]}} (1 + (-\log x)^{-\breve{\alpha}_2})^{\frac{B_1^X}{H_1^X[\breve{\alpha}_1]}+1}} dx.$$

Further, another Monte-Carlo sampling termed as Algorithom 3 is presented to evaluate the GPE and GCI of SSR under CC based generalized estimation in all unknown parameters case.

Algorithm 3: CC based SSR estimation under unequal parameter case.

Step 1 Generate samples c_1^X and c_1^Y from $\chi^2_{2(m-1)}$, and obtain $\breve{\alpha}_1$ and $\breve{\alpha}_2$ from the equations $C_1^X(\alpha_1) = c_1^X$ and $C_1^Y(\alpha_2) = c_1^Y$, respectively.

Step 2 Generate D_1^X and D_1^Y from χ^2_{2m} and obtain an estimate of SSR based on R_2^{CC}.

Step 3 Repeat Step 1 and 2 M times, and then obtain M estimates of SSR.

Step 4 Using same Steps 4 and 5 shown in Algrithm 1, the CC based GPE and GCI are obtained in this manner.

3.2. FC based generalized estimation

In this part, generalized estimator for SSR is constructed based on F and chi-square distributed pivotal quantities, and the result under this approach is called as FC based generalized estimates for concision.

Theorem 6. *Denote pivotal quantities*

$$C_2^X(\alpha_1) = \frac{(m-2)\log(1 + (-\log(x_1))^{-\alpha_1})}{\log(1 + (-\log(x_m))^{-\alpha_1}) - \log(1 + (-\log(x_1))^{-\alpha_1})}, \quad D_2^X(\beta_1, \alpha_1) = 2\beta_1 \log(1 + (-\log(x_m))^{-\alpha_1})$$

and

$$C_2^Y(\alpha_2) = \frac{(m-2)\log(1 + (-\log(y_1))^{-\alpha_2})}{\log(1 + (-\log(y_m))^{-\alpha_2}) - \log(1 + (-\log(y_1))^{-\alpha_2})}, \quad D_2^Y(\beta_2, \alpha_2) = 2\beta_2 \log(1 + (-\log(y_m))^{-\alpha_2}).$$

It is noted that

- $C_2^X(\alpha_1) \sim F_{(2,2m-2)}, D_2^X(\beta_1, \alpha_1) \sim \chi^2_{2m}$, $C_2^X(\alpha_1), D_2^X(\beta_1, \alpha_1)$ *are statistically independent.*
- $C_2^Y(\alpha_2) \sim F_{(2,2m-2)}, D_2^Y(\beta_2, \alpha_2) \sim \chi^2_{2m}$, $C_2^Y(\alpha_2), D_2^Y(\beta_2, \alpha_2)$ *are statistically independent.*

Proof. The proof is similar as Theorems 3 and 4, and the details are omitted for concision. □

For given $c_2^X \sim F(2, 2m-2)$ and $c_2^Y \sim F(2, 2m-2)$, let $\grave{\alpha}_1 = \grave{\alpha}_2^X(C_2^X; x)$ and $\grave{\alpha}_2 = \grave{\alpha}_2^Y(C_2^Y; y)$ be the unique solutions of equations $C_2^X(\alpha_1) = c_2^X$ and $C_2^Y(\alpha_2) = c_2^Y$, respectively. Further, using the substitution method of Weerahandi [42] and substituting $\grave{\alpha}_1$ and $\grave{\alpha}_2$ for α_1 and α_2, the generalized pivotal quantities for β_1 and β_2 are constructed respectively as

$$S_2^X = \frac{D_2^X}{H_2^X[\grave{\alpha}_1]} \text{ with } H_2^X[\alpha_1] = 2\log(1 + (-\log(x_m))^{-\alpha_1}) \text{ and } D_2^X \sim \chi^2_{2m}$$

and

$$S_2^Y = \frac{D_2^Y}{H_2^Y[\grave{\alpha}_2]} \text{ with } H_2^Y[\alpha_2] = 2\log(1 + (-\log(y_m))^{-\alpha_2}) \text{ and } D_2^Y \sim \chi^2_{2m}.$$

Consequently, the pivotal quantities based generalized estimator for SSR can be established as follows

$$R_2^{FC} = 1 - \frac{\grave{\alpha}_1 B_2^X}{H_2^X[\grave{\alpha}_1]} \int_0^1 \frac{(-\log x)^{-\grave{\alpha}_1 - 1}}{x(1 + (-\log x)^{-\grave{\alpha}_2})^{\frac{B_2^Y}{H_2^Y[\grave{\alpha}_2]}} (1 + (-\log x)^{-\grave{\alpha}_2})^{\frac{B_2^X}{H_2^X[\grave{\alpha}_1]} + 1}} dx$$

In addition, another Monte-Carlo procedure termed as Algorithom 4 is presented to estimate GPE and GCI of SSR in this manner under unequal parameters case.

Algorithm 4: FC based generalized estimation of SSR under unequal parameter case.

Step 1 Generate sample c_2^X and c_2^Y from $F(2, 2m-2)$, and obtain the unique solution $\grave{\alpha}_1$ and $\grave{\alpha}_2$ from equations $C_2^X(\alpha_1) = c_2^X$ and $C_2^Y(\alpha_2) = c_2^Y$.

Step 2 Generate D_2^X and D_2^Y from χ^2_{2m}, and then obtain an estimate of R using R_2^{FC}.

Step 3 Repeat Steps 1 and 2 M times, and obtain M estimates of SSR.

Step 4 Using same Steps 4 and 5 shown in Algrithm 1, the FC based GPE and GCI are obtained in this manner.

Remark 1. *It is worth mentioning that different inferential approaches are conducted in previous two sections based on the common and unequal strength and stress parameters, respectively. Therefore, one may interest in whether these parameters are equal or not in practice. For solving this problem, traditional likelihood ratio testing technique could be implemented for comparing the eqiuvalence of different strength and stress parameters.*

4. Numerical analysis

4.1. Simulation studies

In this subsection, simulation experiments are conducted to examine the performance of the proposed methods for SSR. The estimates are evaluated on the basis of mean square error (MSE) and average bias (AB) values for point estimates, and in terms of coverage probability (CP) and average width (AW) for interval estimates, respectively.

Before proceeding, a sampling approach termed as Algorithm 5 to generate a group of upper record values.

Algorithm 5: Generation of upper record values

Step 1 Generate a group of i.i.d. samples, namely $Z_1, Z_2, \cdots, Z_n$ from uniform distribution $U(0,1)$.

Step 2 Make transformation $Y_i = -\log(1-Z_i)$, then $Y_i, i = 1, 2, \cdots, n$ are i.i.d. samples from standard exponential distribution with mean one.

Step 3 Let $W_i = \sum_{k=1}^{i} Y_k, i = 1, 2, \ldots, n$, then sequences $W_1, W_2, \ldots, W_n$ are record values from the standard exponential distribution.

Step 4 Denote $U_i = 1 - e^{-W_i}, i = 1, 2, \ldots, n$, sequence $U_1, U_2, \ldots, U_n$ are record values from the uniform distribution $U(0,1)$.

Step 5 For unit-Burr distribution $F(x;\alpha,\beta)$, let $X_i = F^{-1}(U_i) = \exp\left\{-((1-u)^{-\frac{1}{\beta}} - 1)^{-\frac{1}{\alpha}}\right\}$, then $X_1, X_2, \ldots, X_n$ are record values from unit-Burr distribution, where $F^{-1}(\cdot)$ is the inverse function of $F(\cdot)$.

Table 1. ABs(MSEs) and CPs(ALs) of UB stress-strength parameters with $(\alpha, \beta_1, \beta_2) = (2, 1.5, 1)$.

m	parameters	GPE		GCI	
		CC	RC	CC	RC
4	α	0.8229[1.1929]	0.7100[0.8040]	5.0652[0.9730]	4.5762[0.9409]
	β_1	1.0632[3.6400]	1.2723[3.5756]	5.8383[0.9820]	6.5546[0.9800]
	β_2	0.7989[2.9311]	1.0364[0.9670]	4.4065[0.9750]	5.2035[0.9670]
	R	0.1236[0.0241]	0.1160[0.0215]	0.5874[0.9730]	0.5991[0.9820]
5	α	0.8348[1.2233]	0.7021[0.7754]	5.0402[0.9590]	4.5261[0.9359]
	β_1	0.9842[1.8093]	1.1653[2.3588]	5.3671[0.9780]	5.9968[0.9660]
	β_2	0.7986[1.4719]	1.0082[1.9205]	4.2393[0.9610]	4.8533[0.9580]
	R	0.1154[0.0210]	0.1114[0.0198]	0.5389[0.9660]	0.5440[0.9760]
6	α	0.8221[1.1568]	0.6896[0.7619]	5.0593[0.9610]	4.5054[0.9400]
	β_1	0.9802[1.8067]	1.1449[2.2443]	5.4484[0.9710]	5.8754[0.9650]
	β_2	0.9183[0.9609]	0.8453[1.2547]	4.0419[0.9720]	4.4379[0.9660]
	R	0.0996[0.0156]	0.0979[0.0150]	0.4981[0.9650]	0.4996[0.9730]
7	α	0.8554[1.2627]	0.7157[0.8091]	5.3265[0.9630]	4.6262[0.9510]
	β_1	0.9621[1.7218]	1.0733[2.0935]	5.4388[0.9680]	5.7120[0.9580]
	β_2	0.7155[1.0447]	0.8045[1.2442]	3.9559[0.9590]	4.2045[0.9560]
	R	0.0993[0.0152]	0.0985[0.0150]	0.4624[0.9560]	0.4634[0.9560]

Table 2. ABs(MSEs) and CPs(ALs) of UB stress-strength parameters with $(\alpha, \beta_1, \beta_2) = (2.2, 0.5, 1.5)$.

m	parameters	GPE		GCI	
		CC	RC	CC	RC
4	α	0.8815[1.2776]	0.8321[0.9741]	5.6114[0.9670]	5.0658[0.9389]
	β_1	0.5733[0.6422]	0.8120[1.0502]	3.0928[0.9680]	3.8560[0.9530]
	β_2	1.2448[7.4208]	1.5761[12.2446]	6.8284[0.9710]	7.8299[0.9670]
	R	0.0992[0.0162]	0.0967[0.0156]	0.5375[0.9680]	0.5577[0.9770]
5	α	0.8904[1.2696]	0.7778[0.8989]	5.7103[0.9660]	5.1718[0.9510]
	β_1	0.5335[0.5378]	0.6766[0.7794]	2.9014[0.9690]	3.3536[0.9500]
	β_2	1.2102[3.6247]	1.4420[4.3719]	6.3595[0.9650]	6.9773[0.9580]
	R	0.0930[0.0140]	0.0919[0.0137]	0.4729[0.9530]	0.4828[0.9650]
6	α	0.8958[1.2915]	0.7919[0.8920]	5.7603[0.9690]	5.1388[0.9419]
	β_1	0.5136[0.4697]	0.6051[0.6331]	2.8171[0.9770]	3.0788[0.9560]
	β_2	1.1424[2.3675]	1.3184[2.8475]	5.2213[0.9710]	6.6142[0.9580]
	R	0.0837[0.0114]	0.0830[0.0112]	0.4355[0.9680]	0.4401[0.9660]
7	α	0.8831[1.2722]	0.7496[0.8749]	5.8189[0.9650]	5.1400[0.9550]
	β_1	0.5089[0.4830]	0.5502[0.5378]	2.7543[0.9630]	2.8318[0.9540]
	β_2	1.1639[2.3066]	1.2521[2.4578]	6.3595[0.9730]	6.5225[0.9720]
	R	0.0774[0.0102]	0.0774[0.0102]	0.4005[0.9570]	0.4005[0.9229]

Table 3. ABs(MSEs) and CPs(ALs) of UB stress-strength parameters with $(\alpha_1, \alpha_2, \beta_1, \beta_2) = (1.5, 2.5, 1, 2.5)$.

m	parameters	GPE		GCI	
		CC	FC	CC	FC
4	α_1	0.6044[0.5700]	0.5857[0.4841]	4.1094[0.9379]	3.7952[0.9319]
	α_2	1.0038[1.6691]	0.8316[1.0189]	6.1951[0.9720]	6.1465[0.9640]
	β_1	1.1301[2.1974]	1.3584[2.5398]	5.9356[0.9770]	6.4808[0.9730]
	β_2	2.1404[7.7400]	2.1209[8.3022]	13.1700[0.9750]	13.7354[0.9820]
	R	0.1903[0.0465]	0.2059[0.0512]	0.8901[0.9940]	0.9103[0.9910]
5	α_1	0.5913[0.5295]	0.8221[0.4877]	4.1169[0.9389]	3.4418[0.9100]
	α_2	0.8910[1.3209]	0.7992[0.9098]	5.9124[0.9720]	5.6916[0.9400]
	β_1	1.1499[1.6356]	1.7105[3.2579]	5.9308[0.9800]	7.2143[0.9660]
	β_2	1.2338[5.4290]	1.5255[5.4320]	7.5757[0.9800]	8.8648[0.9820]
	R	0.1965[0.0484]	0.2445[0.0698]	0.8928[0.9930]	0.9139[0.9920]
6	α_1	0.5899[0.5401]	0.6953[0.5676]	3.9401[0.9279]	3.1012[0.8819]
	α_2	0.9210[1.3615]	0.8266[0.9396]	5.7032[0.9720]	5.3478[0.9389]
	β_1	1.2743[2.0341]	2.1162[4.8848]	6.1050[0.9750]	8.0335[0.9459]
	β_2	1.1581[4.0446]	1.8750[6.4406]	7.3349[0.9760]	9.4623[0.9660]
	R	0.2091[0.0548]	0.2780[0.0845]	0.8899[0.8970]	0.9239[0.9940]
7	α_1	0.6042[0.5225]	0.7498[0.6303]	4.0987[0.9379]	2.9550[0.8799]
	α_2	0.8554[1.1800]	0.8478[0.9731]	5.7339[0.9680]	5.1827[0.9259]
	β_1	1.3278[2.1405]	2.4889[6.5570]	6.2263[0.9740]	8.9111[0.9379]
	β_2	1.0883[2.5727]	2.2221[6.6183]	7.2329[0.9730]	9.9989[0.9469]
	R	0.2156[0.0582]	0.3062[0.0998]	0.8930[0.9940]	0.9320[0.9940]

Under different choices of parameter values $\alpha_1, \alpha_2, \beta_1, \beta_2$ and sample sizes m, the simulation procedure are conducted based on 1000 times repetitions, and then the performance of SSR under both common and unequal parameter cases as well as the strength and stress parameters are obtained via criteria quantities. The generlized point estimate and generlized confidence interval based on two chi-square pivotal quantites are called as GPE(CC) and GCI(CC) respectively. GPE(FC) and GCI(FC) denote the generlized point estimate and generlized confidence interval based on F pivotal quantitey and chi-square pivotal quantitey. The significance level is 0.95 for interval estimates. The results are presented in Tables 1 to 4, respectively.

Table 4. ABs(MSEs) and CPs(ALs) of UB stress-strength parameters with $(\alpha_1, \alpha_2, \beta_1, \beta_2) = (1.8, 1.3, 0.6, 1.2)$.

m	parameters	GPE		GCI	
		CC	FC	CC	FC
4	α_1	0.8677[0.9183]	0.9242[0.9979]	3.5925[0.7928]	3.3176[0.7928]
	α_2	0.5184[0.4387]	0.4982[0.3482]	3.9129[0.9640]	3.5237[0.9469]
	β_1	1.3396[1.9601]	1.5450[2.5336]	5.6589[0.9710]	6.1842[0.9640]
	β_2	1.0584[1.9267]	1.4772[2.8442]	6.1049[0.9750]	7.2095[0.9690]
	R	0.2119[0.0518]	0.2189[0.0533]	0.8784[0.9980]	0.9019[0.9990]
5	α_1	0.8657[0.9165]	1.0159[1.1420]	3.5912[0.7948]	2.9826[0.7518]
	α_2	0.5410[0.4810]	0.5287[0.3414]	3.9925[0.9660]	3.3232[0.9329]
	β_1	1.4289[2.2340]	1.9109[3.7890]	5.7687[0.9730]	6.9564[0.9489]
	β_2	1.0639[1.6895]	1.7639[3.6214]	6.0434[0.9790]	7.7772[0.9610]
	R	0.2163[0.0546]	0.2501[0.0671]	0.8787[0.9950]	0.9128[0.9970]
6	α_1	0.8879[0.9489]	1.0974[1.2890]	3.5818[0.7666]	2.7624[0.7127]
	α_2	0.5389[0.4863]	0.5526[0.3726]	3.8958[0.9560]	3.0004[0.9049]
	β_1	1.5266[2.5151]	2.2784[5.3119]	5.9287[0.9690]	7.7777[0.9319]
	β_2	1.1287[1.7340]	2.1221[4.9711]	6.1317[0.9820]	8.4908[0.9489]
	R	0.2237[0.0588]	0.2796[0.0821]	0.8799[0.9970]	0.9210[0.9990]
7	α_1	0.8935[0.9617]	1.1685[1.4284]	3.6297[0.7678]	2.6091[0.6807]
	α_2	0.5273[0.4295]	0.5979[0.4253]	3.8719[0.9369]	2.7949[0.8629]
	β_1	1.5846[2.7127]	2.6480[7.1184]	6.0594[0.9429]	8.6587[0.8589]
	β_2	1.2113[1.9432]	2.5066[6.7948]	6.2998[0.9770]	9.2740[0.9339]
	R	0.2261[0.0596]	0.3085[0.0984]	0.8879[0.9980]	0.9269[0.9980]

From the tabulated results, some conclusions could be observed as follows.

- Under both common and unequal parameter cases, the ABs and MSEs of SSR tends to decrease when sample size m increase. Similar phenomenon also appears for the strength and stress parameters. It indicates that the pivotal quantities based estimates feature consistency properties and work satisfactorily under simulation design scenarios.
- In terms of AB and MSE, the GPEs of SSR and model parameters obtained with respect to chi-square pivotal quantities outperform to the results from F distribution.
- For different GCIs, the corresponding CPs of intervals perform well that are close to nominal level under both common and unequal parameter cases, whereas for fixed sample sizes, the interval lengthes with respect to chi-square quantities are relatively smaller than those of other GCIs.

4.2. *Real data illustration*

In this real-life example, the dataset Badar and Priest [43] represents the strength measured in GPa (giga-Pascals) for single carbon fibers, and impregnated 1000-carbon fiber tows. Single fibers were tested under tension at gauge lengths of 20 and 10 mm, respectively. For these carbon fibers data, Kundu and Gupta [44] also discussed stress-strength estimation based on the traditional Weibull distribution. In this illustration, we treat the data of carbon fibers at gauge lengths of 20 and 10 mm as strength and stress variables, and following the similar idea of Kundu and Gupta [44], and rescale the origin data by multiplying 1/3 and 1/5 for making them in interval (0, 1). Therefore, the transoformed carbon fibers data is preented in Table 5.

Table 5. Carbon fibers data at different gauge lengths.

Strength	0.1873	0.4053	0.4913	0.5440	0.6053	0.6733	0.7723	0.1880	0.4157
	0.4967	0.5587	0.6067	0.6743					
Stress	0.2302	0.3408	0.3748	0.4454	0.5028	0.5574	0.6950	0.3764	0.3448
	0.3818	0.4492	0.5044	0.5608					

Before proceeding, we first check if the the UB distribution could be used as a proper model to fit these real-life data. Based the observations shown in Table 5, parameter estimates, Kolmogorov-Simirnov (K-S) distances and associated p-values for strength and stress variables are reported in Table 6. In addition, the empirical cumulative distributions (ECD) plot overlaid with theoretical UB distribution, the probability-probability (P-P) and the quantile-quantile (Q-Q) plots also presented in Figure 2. Therefore, it is conducted that the UB distribution could be used as a reasonable model to fit the real data.

Table 6. Summary of estimates and goodness-of-fit results under carbon fibers data.

Data	α	β	K-S	p-value
Strength	2.9760	0.5812	0.2390	0.3970
Stress	3.6960	0.7742	0.2015	0.6083

From Table 5, following upper carbon fibers record values for strength and stress variables are generated as follows

strength	0.1873	0.4053	0.4913	0.5440	0.6053	0.6733	0.7723
stress	0.2302	0.3408	0.3748	0.4454	0.5028	0.5574	0.6950

In addition, for null hypothesis $H_0 : \alpha_1 = \alpha_2$ and alternative hypothesis $H_1 : \alpha_1 \neq \alpha_2$, likelihood ratio test is also conducted with testing statistic 0.0225 and p-value 0.2335. Therefore, there is no sufficient evident to reject null hypothesis $\alpha_1 = \alpha_2$ at significence level 0.05, and we treat UB strength and stress parameters α_1 and α_2 as equal in our illustration. Therefore, different point and interval estimates are calculated under common parameter case, and the associated results are provided in Table 7.

Table 7. Generalized estimates of stress-strength reliability and model parameters.

parameters	GPE		GCI	
	CC	RC	CC	RC
α	2.1095	3.4869	(0.8791,4.8093)[3.9302]	(1.5692,6.5457)[4.9756]
β_1	2.7772	1.6844	(0.7449,6.7730)[6.0281]	(0.4636,3.8435)[3.3799]
β_2	3.4381	2.1772	(1.0054,8.1146)[7.1092]	(0.6655,4.9692)[4.3036]
R	0.5668	0.5775	(0.3003,0.7803)[0.4860]	(0.2976,0.8049)[0.5072]

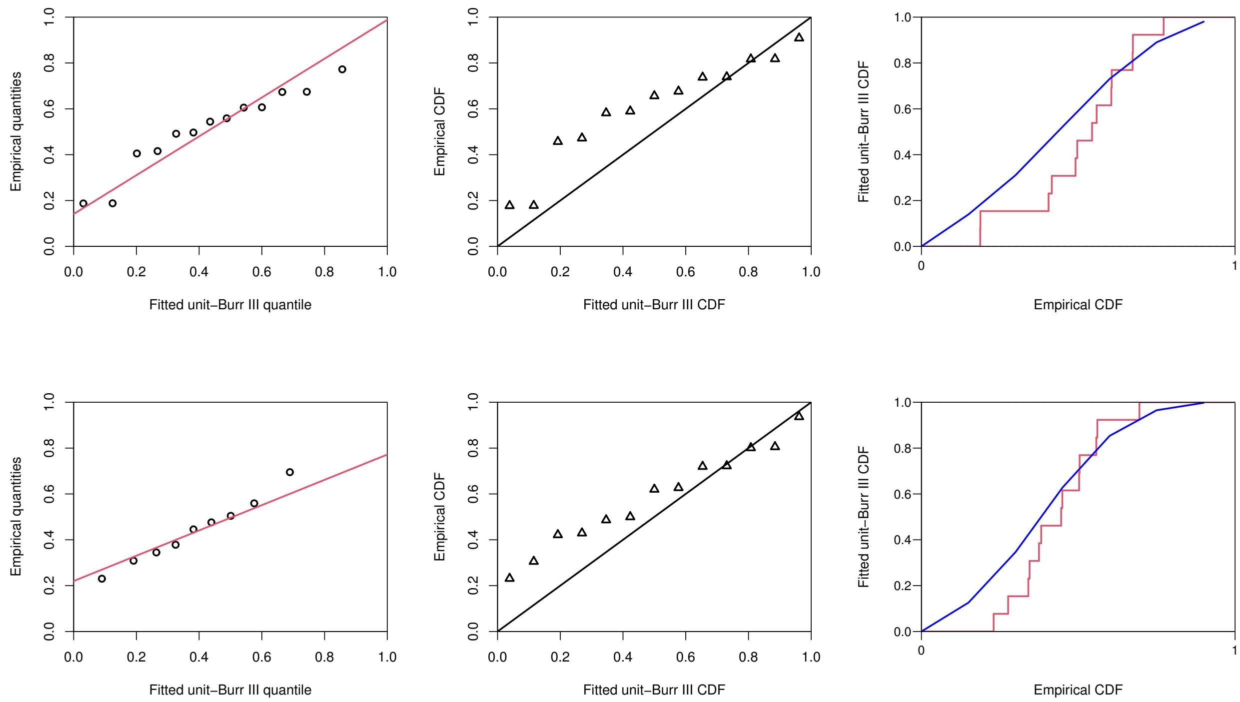

Figure 2. Q-Q, P-P and ECD plots for carbon fiber data.

5. Conclusions

In this paper, the stress-strength model is analyzed when stress and strength variables follow the unit-Burr III distributions and the failure times are upper record values. Various generalized inferential approaches are proposed under both common and unequal strength and stress parameters cases, and the generalized point and interval estimators of stress-strength reliability are constructed based on proposed pivotal quantities, where associated Monte-Carlo sampling algorithm is also provided. Extensive simulation studies and real-life example are further presented for investigating the performance of our methods, and the results indicate that the estimates work satisfactorily. Although the discussions are established for SSR $R = P(Y < X)$, the results could be extended for a more general stress-strength reliability $R = P(Y < X < Z)$, which has attracted much attention recently (e.g., Pan et al. [45], Karam and Yousif [46]). Following similar approaches, the generalized inferential estimation could be also established when there are three strength and stress variables considered in analysis, which will be discussed in future.

A. Proof of Theorem 1

For $j = 1, 2, \cdots, m$, $X_1, X_2, \cdots, X_m$ denote m record data from UB distribution. So $Z_j = \beta_1 \log(1 + (-\log(X_j))^{-\alpha})$, $j = 1, 2, \cdots, m$ are record samples from one parameter exponential distribution with average being one. It is observed that

$$L_1 = Z_1, L_2 = Z_2 - Z_1, \ldots, L_{m-1} = Z_{m-1} - Z_{m-2}, L_m = Z_m - Z_{m-1},$$

are i.i.d. standard exponential distribution.

Consider $W_j = \sum_{i=1}^{j} L_j = Z_j = \beta_1 \log(1 + (-\log(X_j))^{-\alpha})$, $j = 1, 2, \cdots m$, one further gets that

$$U_1 = \frac{W_1}{W_m},\ U_2 = \frac{W_2}{W_m},\ \cdots,\ U_{m-1} = \frac{W_{m-1}}{W_m},$$

are order statistics from standard uniform distribution of size $m-1$. Further $U_1 < U_2 < \cdots < U_{m-1}$ are independent such that $W_m = \beta_1 \log(1 + (-\log(X_m))^{-\alpha})$.

Furthermore, it is seen that

$$A_1^X(\alpha) = -2\sum_{j=1}^{m-1} \log(U_j) = 2\sum_{j=1}^{m-1} \log\left[\frac{\log(1 + (-\log(X_m))^{-\alpha})}{\log(1 + (-\log(X_j))^{-\alpha})}\right],$$

has chi-squared distribution with $2(m-1)$ degrees of freedom. This above quantity is independent of the folllowing variable

$$B_1^X(\beta_1, \alpha) = 2W_m = 2\beta_1 \log(1 + (-\log(X_m))^{-\alpha}),$$

which follows chi-squared distribution with $2m$ degrees of freedom.

B. Proof of Lemma 1

For $R(h)$, it is noted that

$$\frac{dR(h)}{dh} = \frac{\log(d)\log(1 + (-\log(c))^{-h})}{\log(1 + (-\log(d))^{h})} - \frac{\log(c)\log(1 + (-\log(d))^{-h})}{\log(1 + (-\log(c))^{h})}.$$

Since $0 < c < d < 1$, one directly has

$$\frac{\log(d)\log(1 + (-\log(c))^{-h})}{\log(1 + (-\log(d))^{h})} > \frac{\log(c)\log(1 + (-\log(d))^{-h})}{\log(1 + (-\log(c))^{h})}.$$

Thus the proof of the first result is completed. In addition, by direct computation,the limitations are obtained. Therefore, the assertion is completed.

C. Proof of Theorem 3

For $X_1, X_2, \cdots, X_m$ being the record values from UB distribution, using similar notations in proof of Theorem 1, it is noted that $L_1, L_2, \ldots, L_{m-1}, L_m$ are i.i.d. standard exponential distribution. In addition, it is observed that

$$K_1 = 2L_1 \ \text{and}\ K_2 = 2\sum_{i=2}^{m} L_i,$$

follow the chi-square distribution with 2 and $2(m-1)$ degrees of freedom, respectively. Additionally K_1 and K_2 are statistically independent. Further, using sampling distribution theory, one has $A_2^X(\alpha) = \frac{K_1/2}{K_2/2(m-2)}$ follows the F distribution with $(2, 2m-2)$ degrees of freedom. Similarly, quantity $B_2^X(\beta_1, \alpha) = K_1 + K_2$ follows the chi-square distribution with $2m$ degrees of freedom. Therefore, the assertion is completed.

D. Proof of Lemma 2

Since function $G(h)$ can be rewritten as

$$\frac{1}{G(h)} = \frac{\log(1 + (-\log(f))^{-h})}{\log(1 + (-\log(e))^{-h})} - 1.$$

Therefore, using the results of Lemma 1, the monotone and limitation results of $G(h)$ could be established in consequence, and the assertions is completed.

Conflict of interest

The authors declare that there is no conflict of interest.

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